Asymptotic-Möbius maps

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Abstract

Roughly speaking, a map between metric spaces is an asymptotic-Möbius map if it almost preserves cross-ratios of points being a large distance apart from each other. We show that under such maps, some large-scale notions of dimension increases: asymptotic dimension for finitely generated nilpotent groups, telescopic dimension for $CAT(0)$ spaces.

1 Introduction

This paper introduces and investigates a class of maps that mimics quasi-Möbius maps in the context of geometric group theory.

In [13] Pierre Pansu introduces a notion of large-scale conformal maps that mimics the infinitesimal behavior of conformal maps. Briefly, large scale conformal maps map families of disjoint balls to families of weakly overlapping quasi-balls. It is a very flexible notion, that includes for example coarse embeddings. This flexibility makes the asymptotic behavior of such maps however less predictable.

In this paper, we introduce a more restrictive and semantically simpler notion that mimics quasi-Möbius behaviors. Roughly speaking, a map between metric spaces is an asymptotic-Möbius map if it almost preserves cross-ratios of points being a large distance apart from each other. In consequence, the results presented in this paper are much stronger than in [13].

1.1 Examples

We postpone the precise definition of asymptotic-Möbius (AM) maps until later. Sources of examples of AM-maps are:

1. Quasi-isometric embeddings,
2. Sublinear-Bilipschitz-equivalences (i.e. maps that induce Lipschitz-Equivalences on asymptotic cones [5]),
3. Snowflaking (i.e. replacing a metric by a power of it),
4. Assouad maps from doubling metric spaces to $\mathbb{R}^N$.

For instance, every nilpotent Lie group or finitely generated group can be AM-embedded in Euclidean space of sufficiently high dimension [1]. We shall furthermore encounter infinite dimensional examples.

1.2 Results

Our main result is that under AM-maps some large-scale notion of dimension increases. The relevant notion depends on the class of groups considered.

Theorem 3. *Let G and G' be finitely generated nilpotent groups and $f : G \to G'$ an AM-map, then* $\mathrm{asdim}(G) \leq \mathrm{asdim}(G')$. *Futhermore, if* $\mathrm{asdim}(G) = \mathrm{asdim}(G')$, *then the asymptotic cones of G and G' are isomorphic. Conversely, given nilpotent groups with isomorphic asymptotic cones, there exists an asymptotic-Möbius map between them.*

In the world of CAT(0)-spaces, the analogous theorem takes the following form.

Theorem 4. *Let X and Y be* CAT(0) *spaces. If there exists an AM-map $f : X \to Y$, then the telescopic dimension increases e.g.* $\text{tele-dim}(X) \leq \text{tele-dim}(Y)$.

2 Asymptotic-Möbius maps

2.1 Quasi-Möbius maps

Let X be a metric space, and $x, y, z, w \in X$ a quadruple of distinct points. Their *cross-ratio* is

$$[x, y, z, w] = \frac{d(x, z)d(y, w)}{d(x, w)d(y, z)}.$$

If X is unbounded, then the cross-ratio extends to the one-point-compactification of X [14]. A map $f : X \to Y$ is *quasi-Möbius* if there exists a homeomorphism $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all quadruples of distinct points $x, y, z, w \in X$,

$$[f(x), f(y), f(z), f(w)] \leq \eta([x, y, z, w]).$$

If f is a homeomorphism, then f is quasi-Möbius iff its inverse is quasi-Möbius.

Examples of quasi-Möbius maps are

1. the stereographic projection $\mathbb{R}^n \to S^n$,
2. the Cayley transforms, the complex, quaternionic and octonionic analoga of the stereographic projection,
3. inversions $x \mapsto \frac{x}{|x|^2}$ in Banach spaces.

2.2 Asymptotic-Möbius maps

Given $u : \mathbb{R}_+ \to \mathbb{R}$, we say that the points x and y are separated in X by u seen from a fixed origin o, if $d(x, y) > u(d(x, o) + d(o, y))$. For simplicity we often write $x, y > u$ and $|x| = d(o, x)$ for the distance to the origin.

Definition 1. *A function $u : \mathbb{R}_+ \to \mathbb{R}$ is admissible if*

1. *it is non-decreasing,*
2. *u grows sublinearly i.e. $\limsup_{r\to\infty} \frac{u(r)}{r} = 0$.*

Definition 2. *A map $f : X \to Y$ is an asymptotic-Möbius map or AM-map, if there exists an admissible function u and a homeomorphism $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ such that for every x, y, z, w in X such that all 6 distances between $x, y, z, w > u$, then*

$$[f(x), f(y), f(z), f(w)] < \eta([x, y, z, w]).$$

Definition 3. *A metric space X is asymptotically chained, if for every origin $o \in X$ there exists an admissible function v, such that for all $x, y \in X$ there exists a chain $x_1 = x, \ldots, x_{k+1} = y$ that satisfies*

$$\max_{i\in 1\ldots k}\{d(x_i, x_{i+1})\} < v(|x| + |y|).$$

Our main technical step will be the following theorem, whose proof, after some preliminary definition, will occupy Section 2.

Theorem 1. *Let f be an AM-map from an asymptotically chained space X to a space Y that preserves diverging and bounded sequences, then f induces a continuous, injective, quasi-symmetric map g between some asymptotic cones of X and Y.*

2.3 Ultralimits and asymptotic-cones

Definition 4. *A (non-principal) ultrafilter ω over $\mathbb{N}$ is a set of subsets of $\mathbb{N}$ satisfying the following conditions:*

1. *If $A, B \in \omega$ then $A \cap B \in \omega$.*
2. *If $A \in \omega, A \subset B \subset \mathbb{N}$, then $B \in \omega$.*
3. *For every $A \subset \mathbb{N}$ either $A \in \omega$ or $\mathbb{N} \setminus A \in \omega$.*
4. *No finite subset of $\mathbb{N}$ is in ω.*

Equivalently, ω is a finitely additive measure on $\mathbb{N}$ such that each subset has measure either 0 or 1 and every finite subset has measure 0.

If a statement $A(n)$ holds for all $n \in B$ with B belonging to an ultra filter ω, we say that $A(n)$ holds *ω-almost surely.*

Definition 5. *Let ω be a (non-principle) ultrafilter over $\mathbb{N}$. An ω-limit of a sequence of points $\{x_n\}$ in a topological space X, is a point x in X such that for every neighborhood U of x the relation $x_n \in U$ holds ω-almost surely.*

If X is a Hausdorff-space, then the ω-limit of a sequence is unique. We denote this point by $\lim_\omega x_n$.

Definition 6. *The ultrapower of a set X w.r.t. an ultrafilter ω, denoted by X^ω, consists of equivalence classes of sequences $\{x_n\}$, $x_n \in X$, where two sequences $\{x_n\}$ and $\{y_n\}$ are identified if and only if $x_n = y_n$ ω-almost surely.*

We adopt the notation $\{x_n\}_\omega$ for the equivalence class of sequences.

Let (X, d) be a metric space, ω an ultrafilter over $\mathbb{N}$ and o an origin in X. Furthermore, there is given a sequence of numbers $\{d_n\}$, with $\lim_\omega d_n = \infty$. Then there exists a pseudo-metric on the ultrapower X^ω by setting the distance between $x = \{x_n\}_\omega$ and $y = \{y_n\}_\omega$ to

$$d_\omega(x, y) = \lim_\omega \frac{d(x_n, y_n)}{d_n}.$$

Let X_o^ω be the set of equivalence classes of elements $x = \{x_n\}_\omega$ in X^ω that satisfy $\lim_\omega \frac{d(x_n, o)}{d_n} < \infty$ and two elements $x = \{x_n\}_\omega$ and $y = \{y_n\}_\omega$ of X^ω are identified if and only if $d_\omega(x, y) = 0$. The metric space (X_o^ω, d_ω) is called an *asymptotic cone* of (X, d).

We adopt the notation x_ω for points in the asymptotic cone (X_o^ω, d_ω) of (X, d).

Lemma 7. *If $x_\omega \neq y_\omega$, then for all admissible functions u and all sequences $\{x_n\}$, $\{y_n\}$ representing x_ω and y_ω respectively,*

$$d(x_n, y_n) > u(|x_n| + |y_n|), \quad \omega - a.s.$$

Proof: If there exists an admissible function u and sequences $\{x_n\}$, $\{y_n\}$ such that

$$d(x_n, y_n) \leq u(|x_n| + |y_n|), \quad \omega - a.s.$$

then $\omega - a.s.$,

$$\frac{d(x_n, y_n)}{d_n} \leq \frac{u(|x_n| + |y_n|)}{d_n}.$$

Thus

$$\lim_\omega \frac{d(x_n, y_n)}{d_n} = 0.$$

■

Lemma 8. *Let X be an asymptotically chained metric space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X representing the same point x_ω different from the origin o_ω, and such that $x_n \neq y_n$ $\omega - a.s.$ Then for every admissible function u, either*

$$d(x_n, y_n) > u, \quad \omega - a.s.$$

or there exists a sequence $\{w_n\}$ representing the same point s.t.

$$d(x_n, w_n) > u \quad \omega - a.s.$$

and

$$d(y_n, w_n) > u \quad \omega - a.s.$$

Proof: Suppose $d(x_n, y_n) \leq u$ $\omega - a.s.$ and by assumption, $|x_n|, |y_n| \to +\infty$.

Since X is asymptotically chained, there exists an admissible function v and a chain $x_1^n = o, \ldots, x_{k+1}^n = y_n$ such that

$$\max_{i \in 1, \ldots k} \{d(x_i^n, x_{i+1}^n)\} < v(|y_n|).$$

Assume that n is large enough such that $2\,u(4|y_n|) < |y_n|$, then at least one point $x_{i_n}^n$ in the chain satisfies

$$2\,u(4|y_n|) < d(y_n, x_{i_n}^n) \leq 2\,u(4|y_n|) + v(|y_n|).$$

Define $w_n = x_{i_n}^n$.
For n large enough,

$$\begin{aligned} 2d(x_n, y_n) &\leq 2u(|x_n| + |y_n|) \\ &\leq |x_n| + |y_n| \\ &\leq 2|y_n| + d(x_n, y_n). \end{aligned} \tag{1}$$

Therefore,

$$d(x_n, y_n) \leq u(4|y_n|).$$

and hence

$$u(4|y_n|)) < d(x_n, w_n) \leq 3\,u(4|y_n|) + v(|y_n|). \tag{2}$$

Pick N s.t. $\forall n \geq N$

$$3\,u(4|y_n|) + v(|y_n|) \leq 2|y_n|.$$

Then

$$\begin{aligned} u(|x_n| + |w_n|) &\leq u(|x_n| + |y_n| + d(y_n, w_n)) \\ &\leq u(2|y_n| + d(x_n, y_n) + 2\,u(4|y_n|) + v(|y_n|)) \\ &\leq u(2|y_n| + 3\,u(4|y_n|) + v(|y_n|)) \\ &\leq u(4|y_n|) \\ &< d(x_n, w_n). \end{aligned} \tag{3}$$

That means $x_n, w_n > u$ and

$$\frac{d(x_n, w_n)}{d_n} \leq \frac{3\,u(4|y_n|) + v(|y_n|)}{d_n} \to 0.$$

Similarly,

$$\begin{aligned} u(|y_n| + |w_n|) &\leq u(2|y_n| + d(y_n, w_n)) \\ &\leq u(2|y_n| + 2\,u(4|y_n|) + v(|y_n|)) \\ &\leq u(4|y_n|) \\ &< d(y_n, w_n). \end{aligned} \tag{4}$$

■

2.4 Proof of Theorem 1.

Proof: Let u be an admissible function as given by the definition of f. Fix an origin $o \in X$. Let B_n be the ball of radius n around o. For every n, pick a point z_n in B_n such that

$$d(f(o), f(z_n)) \tag{5}$$

is 'maximal'. More precisely, let $\epsilon > 0$ and pick z_n such that

$$d(f(o), f(z_n)) > \sup_{x \in B_n} d(f(o), f(x)) - \epsilon. \tag{6}$$

The supremum on the right-hand-side exists. If not we could construct a bounded sequence in B_n that is mapped by f to an unbounded sequence. This contradicts the assumptions.

Furthermore, the four point condition

$$[f(x), f(w), f(y), f(z)] \leq \eta([x, w, y, z]),$$

reduces to the three point condition

$$\frac{d(f(x), f(y))}{d(f(x), f(z))} \leq \eta(\frac{d(x, y)}{d(x, z)}),$$

by taking w to a point at ∞.

In the following we often shorten the notation $f(x)$ to x'.

Define

$$d_n = d(o, z_n) \tag{7}$$

and

$$d'_n = d(o', z'_n). \tag{8}$$

By construction and assumption on f, the sequence d'_n diverges. Hence, also the sequence d_n must diverge by assumption on f.

(1) We show that if $\frac{d(o,x_n)}{d_n}$ is bounded, then $\frac{d(o',f(x_n))}{d'_n}$ is bounded as well.

If $x_\omega = o_\omega$, then $d(o, x_n) \leq d_n$ for n large, therefore $d(o', x'_n) \leq d'_n$.

If $x_\omega \neq z_\omega$, $x_\omega \neq o_\omega$, then by lemma 7 $\{x_n\}$,$\{z_n\}$,$\{o\}$ are separated by u ω-a.s. and the Möbius-condition gives

$$\frac{d(o', x'_n)}{d'_n} = \frac{d(o', x'_n)}{d(o', z'_n)} < \eta(\frac{d(o, x_n)}{d(o, z_n)}) = \eta(\frac{d(o, x_n)}{d_n}) \quad \omega - a.s.$$

Thus $\frac{d(o',x'_n)}{d'_n}$ is bounded.

If $x_\omega = z_\omega$, by lemma 8, there exists $\{w_n\}$ representing the same point s.t. $\{w_n\}, \{x_n\}$ are u separated and $\{w_n\}$, $\{z_n\}$ are u separated. In particular, $\{w_n\}, \{x_n\}, \{o\}$ and $\{w_n\}, \{z_n\}, \{o\}$ are u separated. Using the Möbius-condition again,

$$\begin{aligned}
\frac{d(o', x'_n)}{d'_n} &= \frac{d(o', x'_n)}{d(o', z'_n)} \\
&= \frac{d(o', x'_n)}{d(o', w'_n)} \frac{d(o', w'_n)}{d(o', z'_n)} \\
&< \eta(\frac{d(o, x_n)}{d(o, w_n)}) \eta(\frac{d(o, w_n)}{d(o, z_n)}) \\
&= \eta(\frac{d(o, x_n)}{d_n} \frac{d_n}{d(o, w_n)}) \eta(\frac{d(o, w_n)}{d_n}) \\
&\to \eta(1)^2
\end{aligned} \tag{9}$$

Thus $\frac{d(o',x'_n)}{d'_n}$ is bounded.

(2) The map $\{x_n\} \to \{f(x_n)\}$ is well-defined from $X_\omega(d_n)$ to $Y_\omega(d'_n)$.

If $\{x_n\}$ and $\{y_n\}$ are two sequences representing the same point different from the origin o_ω, then by lemma 8 and lemma 7, either $\{x_n\}, \{y_n\}, \{o\}$ are separated by u, or there exists a sequence $\{w_n\}$ representing the same point s.t. $\{x_n\}, \{w_n\}, \{o\}$ and $\{y_n\}, \{w_n\}, \{o\}$ are separated by u.

In the first case,

$$\begin{aligned}\frac{d(x'_n,y'_n)}{d'_n} &= \frac{d(o',x'_n)}{d'_n}\frac{d(x'_n,y'_n)}{d(x'_n,o')}\\ &\le \frac{d(o',x'_n)}{d'_n}\eta\Big(\frac{d(x_n,y_n)}{d(x_n,o)}\Big)\\ &= \frac{d(o',x'_n)}{d'_n}\eta\Big(\frac{d(x_n,y_n)}{d_n}\frac{d_n}{d(x_n,o)}\Big)\\ &\to 0\end{aligned}$$

In the second case,

$$\begin{aligned}\frac{d(x'_n,y'_n)}{d'_n} &\le \frac{d(x'_n,w'_n)}{d'_n} + \frac{d(w'_n,y'_n)}{d'_n}\\ &\le \frac{d(o',x'_n)}{d'_n}\frac{d(x'_n,w'_n)}{d(x'_n,o')} + \frac{d(o',y'_n)}{d'_n}\frac{d(y'_n,w'_n)}{d(y'_n,o')}\\ &\le \frac{d(o',x'_n)}{d'_n}\eta\Big(\frac{d(x_n,w_n)}{d(x_n,o)}\Big) + \frac{d(o',y'_n)}{d'_n}\eta\Big(\frac{d(y_n,w_n)}{d(y_n,o)}\Big)\\ &\le \frac{d(o',x'_n)}{d'_n}\eta\Big(\frac{d(x_n,w_n)}{d_n}\frac{d_n}{d(x_n,o)}\Big) + \frac{d(o',y'_n)}{d'_n}\eta\Big(\frac{d(y_n,w_n)}{d_n}\frac{d_n}{d(y_n,o)}\Big)\\ &\to 0\end{aligned}$$

Hence, in both cases $x'_\omega = y'_\omega$.

Let $\{x_n\}$ be a sequence representing the origin o_ω. If $\{x_n\}$ is a bounded sequence, then by assumption on f also $\{x'_n\}$ is bounded and therefore $x'_\omega = o'_\omega$.

If $d(o,x_n)$ diverges, then $\{x_n\}$ and $\{o\}$ are u-separated. We may pick $y_\omega \neq o_\omega$, then by lemma 7, $\{x_n\}, \{y_n\}, \{o\}$ are u-separated. Thus,

$$\begin{aligned}\frac{d(o',x'_n)}{d'_n} &= \frac{d(o',y'_n)}{d'_n}\frac{d(o',x'_n)}{d(o',y'_n)}\\ &\le \frac{d(o',y'_n)}{d'_n}\eta(\frac{d(o,x_n)}{d(o,y_n)})\\ &= \frac{d(o',y'_n)}{d'_n}\eta(\frac{d(o,x_n)}{d_n}\frac{d_n}{d(o,y_n)})\\ &\to 0.\end{aligned} \tag{10}$$

Therefore $o'_\omega = x'_\omega$.

(3) The map $g : \{x_n\} \to \{f(x_n)\}$ is quasi-symmetric and in particular continuous and injective.

Let $\{w_n\}, \{x_n\}, \{y_n\}$ be sequences representing three distinct points w_ω, x_ω and y_ω in $X_\omega(d_n)$. By lemma 7, $\{w_n\}, \{x_n\}, \{y_n\}$ are pairwise u separated. Thus

$$\frac{d(x'_n, y'_n)}{d(x'_n, w'_n)} < \eta(\frac{d(x_n, y_n)}{d(x_n, w_n)}),$$

equivalently

$$\frac{d(x'_n, y'_n)}{d'_n}\frac{d'_n}{d(x'_n, w'_n)} < \eta(\frac{d(x_n, y_n)}{d_n}\frac{d_n}{d(x_n, w_n)}).$$

Taking the ω-limit gives

$$\frac{d_\omega(x'_\omega, y'_\omega)}{d_\omega(x'_\omega, w'_\omega)} \leq \eta(\frac{d_\omega(x_\omega, y_\omega)}{d_\omega(x_\omega, w_\omega)}).$$

Hence, if $g : \{x_n\} \mapsto \{f(x_n)\}$ is non-constant, then it is continuous and injective. Notice that $\{o'\}$ and $\{f(z_n)\}$ do not represent the same points, so g is indeed non-constant.

■

3 Examples

3.1 Sublinearly Lipschitz Equivalences

In [5] Y. Cornulier introduces *sublinearly Lipschitz maps*. A map $f : X \to Y$ between metric spaces is a sublinearly Lipschitz map, if there exists an admissible function $u : \mathbb{R}_+ \to \mathbb{R}$ such that

$$d(f(x), f(y)) \leq Cd(x, y) + C'u(|x| + |y|), \quad \forall x, y \in X,$$

for some constants $C, C' > 0$.

Two sublinearly Lipschitz maps f, f' are equivalent, if there exists an admissible function v and a constant $C'' > 0$, such that

$$d(f(x), f'(x)) \leq C''v(|x|)$$

for all $x \in X$.

Sublinearly Lipschitz maps between metric spaces form a category. Taking asymptotic cones yields a functor from the sublinearly Lipschitz category to the Lipschitz category. The sublinearly Lipschitz category is in some sense the maximal category with such a property.

The isomorphisms in the sublinearly Lipschitz category are called sublinearly Lipschitz equivalences, or SBE maps.

Proposition 9. *Every SBE-map $f : X \to Y$ is an AM-map with linear η.*

Proof: If f is SBE, then f is bi-Lipschitz except at scales below some admissible function u. Indeed, let $x, y \in X$ s.t.

$$d(x, y) \geq u(|x| + |y|).$$

Then

$$\begin{aligned} d(f(x), f(y)) &\leq cd(x, y) + Cu(|x| + |y|) \\ &\leq (c + C)d(x, y), \end{aligned} \tag{11}$$

and

$$\begin{aligned} d(f(x), f(y)) &\geq c'd(x, y) - C'u(|x| + |y|) \\ &\geq (c' - C')d(x, y) \end{aligned} \tag{12}$$

Hence, there exists $D > 0$, s.t. for all $x, y > u$,

$$\frac{1}{D}d(x, y) \leq d(f(x), f(y)) \leq Dd(x, y).$$

In particular, f is an AM-map with linear η. ■

3.2 Assouad-type maps

Assouad's embedding theorem [1] [10] says that every snowflake $X^\alpha = (X, d^\alpha)$, $0 < \alpha < 1$, of a doubling metric space admits a bi-Lipschitz embedding into some Euclidean space.

Clearly, the Assouad embedding of a doubling metric space into Euclidean space is an asymptotic-Möbius map.

In the following we construct an example of a Assouad type map from an infinite dimensional Heisenberg group into a Hilbert space. This construction follows straightforwardly from a construction by Lee and Naor in the finite dimensional case [9].

Let H be an infinite dimensional complex Hilbert space. H carries the symplectic form $\Omega(a, b) = \mathrm{Im}(\langle a, b\rangle)$.

The *infinite dimensional Heisenberg group* $\mathcal{H}_\Omega$, is the set of tuples (a, t) with $a \in H$, $t \in \mathbb{R}$ and group law

$$(a, t)(a', t') = (a + a', t + t' + 2\,\Omega(a, a')).$$

Let G be a group with identity element e. A group seminorm on G is a function $G \to [0, \infty)$ that satisfies $N(g^{-1}) = N(g)$ for all $g \in G$, $N(gh) \leq N(g) + N(h)$ for all $g, h \in G$ and $N(e) = 0$. If in addition $N(g) = 0$ if and only if $g = e$, then N is a group norm on G.

The function

$$N(a, t) = \sqrt{\sqrt{||a||^4 + t^2} + ||a||^2},$$

is a group norm on $\mathcal{H}_\Omega$. Indeed, if N_1 and N_2 are group seminorms, then $\sqrt{N_1^2 + N_2^2}$ is a group seminorm. In this case $N_1 = (||a||^4 + t^2)^{\frac{1}{4}}$ is the Koranyi-norm. Given N,

$$d_N((a, t), (a', t')) = N((a, t)(a', t')^{-1})$$

is a right-invariant metric on $\mathcal{H}_\Omega$.

Let G be a group, a Hermitian kernel on G is a complex valued function $K : G \times G \to \mathbb{C}$ that satisfies $K(g, h) = \overline{K(h, g)}$ for all $g, h \in G$.

A Hermitian kernel on G is positive definite if

$$\sum_{i,j}^{n} K(g_i, g_j) c_i \overline{c_j} \geq 0$$

for all $g_1, \ldots, g_n \in G$ and for all complex numbers $c_1, \ldots, c_n \in \mathbb{C}$, with equality if and only if the c_i vanish. A Hermitian kernel on G is conditionally negative definite if

$$\sum_{i,j}^{n} K(g_i, g_j) c_i \overline{c_j} \leq 0$$

for all $g_1, \ldots, g_n \in G$ and for all complex numbers $c_1, \ldots, c_n \in \mathbb{C}$ satisfying $\sum_i^n c_i = 0$ and with equality if and only if the c_i vanish.

Theorem 2 (Schönberg)**.** *Let G be a group, $K : G \times G \to \mathbb{R}$ a real valued kernel on G satisfying $K(g, g) = 0 \ \forall g \in G$. Then K is negative definite if and only if there exists a Hilberspace H and a function $T : G \to H$ such that $\forall g, h \in G : \quad K(g, h) = \|T(g) - T(h)\|^2$.*

Thus, if d_N is negative definite, then there exists a Hilbert space H and an isometry $T : (\mathcal{H}_\Omega, \sqrt{d_N}) \to H$. In particular, the map $T : (\mathcal{H}_\Omega, d_N) \to H$ is an AM-map.

A complex valued function F on a group G that satisfies $F(g^{-1}) = \overline{F(g)}$ for all $g \in G$ induces a Hermitian kernel on G by defining $K_F(g, h) := F(gh^{-1})$. The function F is said to be positive definite, if K_F is a positive definite Hermitian kernel on G.

Lemma 10. *For every $\lambda \in \mathbb{R}$ the function*

$$\Phi_\lambda(a, t) = e^{-|\lambda| \|a\|^2 + i\lambda t}$$

is positive definite on $\mathcal{H}_\Omega$.

Proof: Indeed the function Φ_λ satisfies $\Phi_\lambda((a, t)^{-1}) = \overline{\Phi_\lambda((a, t))}$ and induces the kernel

$$K((a, s), (b, t)) := \exp(-|\lambda| \|a - b\|^2 + i\lambda(s - t - 2\,\Omega(a, b))).$$

The kernel can be rewritten as product of three exponentials

$$\exp(-|\lambda|(\|a\|^2 + \|b\|^2)) \exp(i\lambda(s - t)) \exp(2|\lambda|(\mathrm{Re}(\langle a, b\rangle) - i \,\mathrm{sign}(\lambda)\, \Omega(a, b))).$$

The product of positive definite kernels is positive definite. The first two factors are positive definite since for all $c_1, \ldots, c_n \in \mathbb{C}$, the matrix $(c_i \overline{c_j})_{i,j}$ is positive semi-definite. In particular also the matrix $(\exp(i\lambda(s_i - t_j)) c_i \overline{c_j})_{i,j}$. The exponential of any positive definite kernel is again a positive definite kernel ([3] Proposition 8.2.). Therefore the kernel K is positive definite if

$$\mathrm{Re}(\langle a, b\rangle) - i \,\mathrm{sign}(\lambda)\, \Omega(a, b)$$

is positive definite. This is $\langle a, b\rangle$ if $\lambda < 0$ and $\overline{\langle a, b\rangle}$ if $\lambda \geq 0$ and therefore clearly positive definite. ∎

Proposition 11. *$d_N : \mathcal{H}_\Omega \times \mathcal{H}_\Omega \to H$ is conditionally negative definite.*

Proof: The existence of $\frac{1}{2}$-stable distributions implies that for all $\epsilon > 0$, there exists a non-negative integrable function $\varphi_\epsilon : \mathbb{R} \to [0, \infty)$ s.t. its Fourier-transform $\hat{\varphi}_\epsilon(t) = e^{-\epsilon\sqrt{|t|}}$. Notice that $\frac{1}{2\pi}\hat{\hat{\varphi}}_\epsilon(x) = \varphi_\epsilon(x)$.
By the lemma above,

$$F_\epsilon(a,t) = \int_{\mathbb{R}} e^{-|\lambda|\|a\|^2 + i\lambda t}\varphi_\epsilon(\lambda)d\lambda$$

is positive definite on $\mathcal{H}_\Omega$. Let

$$h_k(x) = \frac{k}{\pi}\frac{1}{k^2 + x^2}$$

denote the Cauchy-distribution with scale parameter $k > 0$. We rewrite

$$\begin{aligned} F_\epsilon(a,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda t}\hat{h}_{\|a\|^2}(\lambda)\hat{\varphi}_\epsilon(\lambda)d\lambda \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda t}\widehat{h_{\|a\|^2} * \varphi_\epsilon}(\lambda)d\lambda \\ &= (h_{\|a\|^2} * \hat{\varphi}_\epsilon)(t). \end{aligned} \tag{13}$$

Since F_ϵ is positive definite, $\frac{1-F_\epsilon}{\epsilon}$ is conditionally negative definite and

$$\begin{aligned} \lim_{\epsilon\to 0}\frac{1 - F_\epsilon(a,t)}{\epsilon} &= \lim_{\epsilon\to 0}\big[h_{\|a\|^2} * \frac{1-\hat{\varphi}_\epsilon}{\epsilon}\big] \\ &= \lim_{\epsilon\to 0}\int_{\mathbb{R}} \frac{1 - e^{-\epsilon\sqrt{|x|}}}{\epsilon}h_{\|a\|^2}(t-x)dx \\ &= \frac{\|a\|^2}{\pi}\int_{\mathbb{R}}\frac{\sqrt{|x|}}{\|a\|^4 + (t-x)^2}dx. \end{aligned} \tag{14}$$

We now show that for all $r, t \in \mathbb{R}$

$$r^2\int_{\mathbb{R}}\frac{\sqrt{|x|}}{r^4 + (t-x)^2}dx = \pi\sqrt{\sqrt{r^4 + t^2} + r^2}.$$

By change of variables $x = r^2 y$ and $s = t/r^2$, the left-hand-side can be written as

$$\int_0^\infty \Big(\frac{1}{1 + (s-y)^2} + \frac{1}{1 + (s+y)^2}\Big)\sqrt{y}\,dy$$

This integral equals

$$\lim_{r\to 0}\lim_{R\to\infty}\frac{1}{2}\int_{C_{r,R}}\Big(\frac{1}{1 + (s-z)^2} + \frac{1}{1 + (s+z)^2}\Big)\sqrt{z}\,dz,$$

where $C_{r,R}$ is the keyhole contour with a branch cut along the positive real axis. The integrand has simple poles at $i \pm s$ and $-i \pm s$. With $\mathrm{Res}(\frac{\sqrt{z}}{1+(s-z)^2}, i+s) = \frac{\sqrt{i+s}}{2i}$, $\mathrm{Res}(\frac{\sqrt{z}}{1+(s-z)^2}, -i+s) = \frac{\sqrt{-i+s}}{-2i}$, $\mathrm{Res}(\frac{\sqrt{z}}{1+(s+z)^2}, i-s) = \frac{\sqrt{i-s}}{2i}$ and $\mathrm{Res}(\frac{\sqrt{z}}{1+(s+z)^2}, -i-s) = \frac{\sqrt{-i-s}}{-2i}$. Now adding all the residues and multiplying by $2\pi i$, lets us evaluate the above equation as being

$$\frac{\pi}{2}\left(\sqrt{i+s} - \sqrt{-i+s} + \sqrt{i-s} - \sqrt{-i-s}\right).$$

This simplifies further to

$$\begin{aligned}\pi\ \mathrm{Re}(\sqrt{i+s}+\sqrt{i-s}) &= \pi\left(\mathrm{Re}(\sqrt{i+s})+\mathrm{Im}(\sqrt{i+s})\right)\\ &= \pi\left(\sqrt{\frac{\sqrt{1+s^2}+s}{2}}+\sqrt{\frac{\sqrt{1+s^2}-s}{2}}\right) \qquad (15)\\ &= \pi\sqrt{\sqrt{1+s^2}+1}\end{aligned}$$

Thus we have seen, that the N-norm is a limit of conditionally negative definite functions on $\mathcal{H}_\Omega$, hence it is conditionally negative definite. ■

4 Applications to dimension theory

4.1 Dimension theory of finitely-generated groups

In 1993 M. Gromov introduced a notion of asymptotic dimension as a large scale analogue of Lebesgue's covering dimension [6]. The asymptotic dimension of a finitely generated group is a quasi-isometric invariant. Its most prominent application is due to Yu G., who showed that every finitely generated group with finite homotopy type and finite asymptotic dimension, satisfies the Novikov conjecture [15].

Definition 12. *Let X be a metric space. We say that the* asymptotic dimension *of X does not exceed n provided for every uniformly bounded open cover $\mathcal{V}$ of X there is a uniformly bounded open cover $\mathcal{U}$ of X of multiplicity $\leq n+1$ so that $\mathcal{V}$ refines $\mathcal{U}$. We write asdim$X = n$ if it is true that asdim$X \leq n$ and asdim$X \not\leq n-1$*

The asymptotic dimension of a finitely generated group Γ has several interesting implications. For example $\operatorname{asdim}\Gamma = 0$ if and only if Γ is finite (Proposition 65. in [2]) and $\operatorname{asdim}\Gamma = 1$ if and only if Γ is virtually free (Theorem 66. in [2]).

If G is a finitely generated nilpotent group, then its asymptotic cone G_ω is a Carnot group [11] and the asymptotic dimension of G agrees with the topological dimension of its asymptotic cone [2].

Theorem 3. *Let G and G' be a finitely generated nilpotent groups and $f : G \to G'$ an AM-map, then* $\operatorname{asdim}(G) \leq \operatorname{asdim}(G')$. *If G and G' are nilpotent Lie groups, then* $\dim(G) \leq \dim(G')$. *Futhermore, if* $\operatorname{asdim}(G) = \operatorname{asdim}(G')$*, then the asymptotic cones of G and G' are isomorphic graded Lie groups. Conversely, given nilpotent groups with isomorphic asymptotic cones, there exists an asymptotic-Möbius map between them.*

Proof: The map f induces a quasi-symmetric map g between asymptotic cones G_ω and G'_ω of G and G' respectively.

In particular g is an injective map from G_ω to G'_ω and therefore the topological dimensions increase. Since G and G' are nilpotent, $\operatorname{asdim}(G) = \text{top-dim}(G_\omega) \leq \text{top-dim}(G'_\omega) = \operatorname{asdim}(G')$.

The case for nilpotent Lie groups follows from the fact that for nilpotent Lie groups, $\text{top-dim}(G) = \text{top-dim}(G_\omega)$.

If $\mathrm{asdim}(G) = \mathrm{asdim}(G')$, the invariance of domain theorem implies that g is a homeomorphism. Since both g and g^{-1} are quasi-symmetric, they are a.e. differentiable by Pansu's theorem and the differential of g is at almost every point a graded isomorphism of groups [12]. ■

4.2 Dimension theory of $\mathrm{CAT}(0)$-spaces

Heinrich Jung observed in early 1900, that the diameter of a set in Euclidean space $\mathbb{R}^n$ is related to the radius of the minimum enclosing ball by an inequality, whose constant depends only on the dimension n [7]. Precisely, for any set $A \subset \mathbb{R}^n$

$$r \leq d(A)\sqrt{\frac{n}{2(n+1)}}, \tag{16}$$

where r is the radius of the minimum enclosing ball of A and $d(A)$ is the diameter of A.

Caprace and Lytchak observed [4], that a CAT(0)-space has *geometric dimension* less than n (as defined by B. Kleiner [8]), if and only if Jung's inequality (16) is true.

Remark 13. *It turns out that the geometric dimension of a* $\mathrm{CAT}(0)$*-space* X *is equal to the maximal topological dimension of all compact subsets in* X *[8].*

The geometric dimension can be turned into a large-scale notion in the following manner. A CAT(0)-space X has telescopic dimension less then n if and only if for every $\delta > 0$, there exists a $D > 0$ such that for every set A in X with diameter larger then D,

$$r \leq \left(\delta + \sqrt{\frac{n}{2(n+1)}}\right) d(A). \tag{17}$$

Alternatively,

Definition 14. *A* $\mathrm{CAT}(0)$*-space* X *has telescopic dimension* $\leq n$*, if every asymptotic cone has geometric dimension* $\leq n$*.*

We adopt the notation $\text{tele-dim}(X)$ for the telescopic dimension of X.

Theorem 4. *Let* X *and* Y *be* $\mathrm{CAT}(0)$ *spaces. If there exists an* AM*-map* $f : X \to Y$*, then the telescopic dimension increases e.g.* $\text{tele-dim}(X) \leq \text{tele-dim}(Y)$.

The theorem follows straight forward from Theorem 1, Remark 13 and the fact that the topological dimension increases under injective continuous maps.

As a consequence, there cannot exist an AM-map from Euclidean buildings of finite rank r and thus telescopic dimension r, to a Euclidean building of lower rank.

If there exists an AM-map from a CAT(0) space to a hyperbolic metric space, then X is a hyperbolic metric space as well.

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