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Asymptotic-Möbius maps.

Georg Grützner

Abstract

Roughly speaking, a map between metric spaces is an asymptotic-Möbius map if it almost preserves cross-ratios of points being a large distance apart from each other. We show that under such maps, some large-scale notions of dimension increase: asymptotic dimension for finitely generated nilpotent groups, telescopic dimension for $CAT(0)$ spaces.

1 Introduction

This paper introduces and investigates a class of maps that mimics quasi-Möbius maps in the context of geometric group theory.

In [13] Pierre Pansu introduces a notion of large-scale conformal maps that mimics the infinitesimal behavior of conformal maps. Briefly, large scale conformal maps map families of disjoint balls to families of weakly overlapping quasi-balls. It is a very flexible notion, that includes for example coarse embeddings. This flexibility makes the asymptotic behavior of such maps however less predictable.

In this paper, we introduce a more restrictive and semantically simpler notion that mimics quasi-Möbius behaviors. Roughly speaking, a map between metric spaces is an asymptotic-Möbius map if it almost preserves cross-ratios of points being a large distance apart from each other. In consequence, the results presented in this paper are much stronger than in [13].

1.1 Examples

We postpone the precise definition of asymptotic-Möbius (AM) maps until later. Sources of examples of AM -maps are:

1. Quasi-isometric embeddings,
2. Sublinear-Bilipschitz-equivalences (i.e. maps that induce Lipschitz-Equivalences on asymptotic cones [5]),
3. Snowflaking (i.e. replacing a metric by a power of it),
4. Assouad maps from doubling metric spaces to \mathbb{R}^N .

For instance, every nilpotent Lie group or finitely generated group can be AM -embedded in Euclidean space of sufficiently high dimension [1]. We shall furthermore encounter infinite dimensional examples.

1.2 Results

Our main result is that under AM -maps some large-scale notion of dimension increases. The relevant notion depends on the class of groups considered.

Theorem 3. *Let G and G' be finitely generated nilpotent groups and $f : G \rightarrow G'$ an AM -map, then $\text{asdim}(G) \leq \text{asdim}(G')$. Furthermore, if $\text{asdim}(G) = \text{asdim}(G')$, then the asymptotic cones of G and G' are isomorphic. Conversely, given nilpotent groups with isomorphic asymptotic cones, there exists an asymptotic-Möbius map between them.*

In the world of $\text{CAT}(0)$ -spaces, the analogous theorem takes the following form.

Theorem 4. *Let X and Y be $\text{CAT}(0)$ spaces. If there exists an AM -map $f : X \rightarrow Y$, then the telescopic dimension increases e.g. $\text{tele-dim}(X) \leq \text{tele-dim}(Y)$.*

2 Asymptotic-Möbius maps

2.1 Quasi-Möbius maps

Let X be a metric space, and $x, y, z, w \in X$ a quadruple of distinct points. Their *cross-ratio* is

$$[x, y, z, w] = \frac{d(x, z)d(y, w)}{d(x, w)d(y, z)}.$$

If X is unbounded, then the cross-ratio extends to the one-point-compactification of X [14]. A map $f : X \rightarrow Y$ is *quasi-Möbius* if there exists a homeomorphism $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for all quadruples of distinct points $x, y, z, w \in X$,

$$[f(x), f(y), f(z), f(w)] \leq \eta([x, y, z, w]).$$

If f is a homeomorphism, then f is quasi-Möbius iff its inverse is quasi-Möbius.

Examples of quasi-Möbius maps are

1. the stereographic projection $\mathbb{R}^n \rightarrow S^n$,
2. the Cayley transforms, the complex, quaternionic and octonionic analoga of the stereographic projection,
3. inversions $x \mapsto \frac{x}{|x|^2}$ in Banach spaces.

2.2 Asymptotic-Möbius maps

Given $u : \mathbb{R}_+ \rightarrow \mathbb{R}$, we say that the points x and y are separated in X by u seen from a fixed origin o , if $d(x, y) > u(d(x, o) + d(o, y))$. For simplicity we often write $x, y > u$ and $|x| = d(o, x)$ for the distance to the origin.

Definition 1. A function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is admissible if

1. it is non-decreasing,
2. u grows sublinearly i.e. $\limsup_{r \rightarrow \infty} \frac{u(r)}{r} = 0$.

Definition 2. A map $f : X \rightarrow Y$ is an asymptotic-Möbius map or AM-map, if there exists an admissible function u and a homeomorphism $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for every x, y, z, w in X such that all 6 distances between $x, y, z, w > u$, then

$$[f(x), f(y), f(z), f(w)] < \eta([x, y, z, w]).$$

Definition 3. A metric space X is asymptotically chained, if for every origin $o \in X$ there exists an admissible function v , such that for all $x, y \in X$ there exists a chain $x_1 = x, \dots, x_{k+1} = y$ that satisfies

$$\max_{i \in 1 \dots k} \{d(x_i, x_{i+1})\} < v(|x| + |y|).$$

Our main technical step will be the following theorem, whose proof, after some preliminary definition, will occupy Section 2.

Theorem 1. Let f be an AM-map from an asymptotically chained space X to a space Y that preserves diverging and bounded sequences, then f induces a continuous, injective, quasi-symmetric map g between some asymptotic cones of X and Y .

2.3 Ultralimits and asymptotic-cones

Definition 4. A (non-principal) ultrafilter ω over \mathbb{N} is a set of subsets of \mathbb{N} satisfying the following conditions:

1. If $A, B \in \omega$ then $A \cap B \in \omega$.
2. If $A \in \omega, A \subset B \subset \mathbb{N}$, then $B \in \omega$.
3. For every $A \subset \mathbb{N}$ either $A \in \omega$ or $\mathbb{N} \setminus A \in \omega$.
4. No finite subset of \mathbb{N} is in ω .

Equivalently, ω is a finitely additive measure on \mathbb{N} such that each subset has measure either 0 or 1 and every finite subset has measure 0.

If a statement $A(n)$ holds for all $n \in B$ with B belonging to an ultra filter ω , we say that $A(n)$ holds ω -almost surely.

Definition 5. Let ω be a (non-principle) ultrafilter over \mathbb{N} . An ω -limit of a sequence of points $\{x_n\}$ in a topological space X , is a point x in X such that for every neighborhood U of x the relation $x_n \in U$ holds ω -almost surely.

If X is a Hausdorff-space, then the ω -limit of a sequence is unique. We denote this point by $\lim_{\omega} x_n$.

Definition 6. The ultrapower of a set X w.r.t. an ultrafilter ω , denoted by X^{ω} , consists of equivalence classes of sequences $\{x_n\}$, $x_n \in X$, where two sequences $\{x_n\}$ and $\{y_n\}$ are identified if and only if $x_n = y_n$ ω -almost surely.

We adopt the notation $\{x_n\}_\omega$ for the equivalence class of sequences.

Let (X, d) be a metric space, ω an ultrafilter over \mathbb{N} and o an origin in X . Furthermore, there is given a sequence of numbers $\{d_n\}$, with $\lim_\omega d_n = \infty$. Then there exists a pseudo-metric on the ultrapower X^ω by setting the distance between $x = \{x_n\}_\omega$ and $y = \{y_n\}_\omega$ to

$$d_\omega(x, y) = \lim_\omega \frac{d(x_n, y_n)}{d_n}.$$

Let X_o^ω be the set of equivalence classes of elements $x = \{x_n\}_\omega$ in X^ω that satisfy $\lim_\omega \frac{d(x_n, o)}{d_n} < \infty$ and two elements $x = \{x_n\}_\omega$ and $y = \{y_n\}_\omega$ of X^ω are identified if and only if $d_\omega(x, y) = 0$. The metric space (X_o^ω, d_ω) is called an *asymptotic cone* of (X, d) .

We adopt the notation x_ω for points in the asymptotic cone (X_o^ω, d_ω) of (X, d) .

Lemma 7. *If $x_\omega \neq y_\omega$, then for all admissible functions u and all sequences $\{x_n\}$, $\{y_n\}$ representing x_ω and y_ω respectively,*

$$d(x_n, y_n) > u(|x_n| + |y_n|), \quad \omega - a.s.$$

Proof: If there exists an admissible function u and sequences $\{x_n\}$, $\{y_n\}$ such that

$$d(x_n, y_n) \leq u(|x_n| + |y_n|), \quad \omega - a.s.$$

then $\omega - a.s.$,

$$\frac{d(x_n, y_n)}{d_n} \leq \frac{u(|x_n| + |y_n|)}{d_n}.$$

Thus

$$\lim_\omega \frac{d(x_n, y_n)}{d_n} = 0.$$

■

Lemma 8. *Let X be an asymptotically chained metric space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X representing the same point x_ω different from the origin o_ω , and such that $x_n \neq y_n$ $\omega - a.s.$ Then for every admissible function u , either*

$$d(x_n, y_n) > u, \quad \omega - a.s.$$

or there exists a sequence $\{w_n\}$ representing the same point s.t.

$$d(x_n, w_n) > u \quad \omega - a.s.$$

and

$$d(y_n, w_n) > u \quad \omega - a.s.$$

Proof: Suppose $d(x_n, y_n) \leq u$ $\omega - a.s.$ and by assumption, $|x_n|, |y_n| \rightarrow +\infty$.

Since X is asymptotically chained, there exists an admissible function v and a chain $x_1^n = o, \dots, x_{k+1}^n = y_n$ such that

$$\max_{i \in \{1, \dots, k\}} \{d(x_i^n, x_{i+1}^n)\} < v(|y_n|).$$

Assume that n is large enough such that $2u(4|y_n|) < |y_n|$, then at least one point $x_{i_n}^n$ in the chain satisfies

$$2u(4|y_n|) < d(y_n, x_{i_n}^n) \leq 2u(4|y_n|) + v(|y_n|).$$

Define $w_n = x_{i_n}^n$.

For n large enough,

$$\begin{aligned} 2d(x_n, y_n) &\leq 2u(|x_n| + |y_n|) \\ &\leq |x_n| + |y_n| \\ &\leq 2|y_n| + d(x_n, y_n). \end{aligned} \tag{1}$$

Therefore,

$$d(x_n, y_n) \leq u(4|y_n|).$$

and hence

$$u(4|y_n|) < d(x_n, w_n) \leq 3u(4|y_n|) + v(|y_n|). \tag{2}$$

Pick N s.t. $\forall n \geq N$

$$3u(4|y_n|) + v(|y_n|) \leq 2|y_n|.$$

Then

$$\begin{aligned} u(|x_n| + |w_n|) &\leq u(|x_n| + |y_n| + d(y_n, w_n)) \\ &\leq u(2|y_n| + d(x_n, y_n) + 2u(4|y_n|) + v(|y_n|)) \\ &\leq u(2|y_n| + 3u(4|y_n|) + v(|y_n|)) \\ &\leq u(4|y_n|) \\ &< d(x_n, w_n). \end{aligned} \tag{3}$$

That means $x_n, w_n > u$ and

$$\frac{d(x_n, w_n)}{d_n} \leq \frac{3u(4|y_n|) + v(|y_n|)}{d_n} \rightarrow 0.$$

Similarly,

$$\begin{aligned} u(|y_n| + |w_n|) &\leq u(2|y_n| + d(y_n, w_n)) \\ &\leq u(2|y_n| + 2u(4|y_n|) + v(|y_n|)) \\ &\leq u(4|y_n|) \\ &< d(y_n, w_n). \end{aligned} \tag{4}$$

■

2.4 Proof of Theorem 1.

Proof: Let u be an admissible function as given by the definition of f . Fix an origin $o \in X$. Let B_n be the ball of radius n around o . For every n , pick a point z_n in B_n such that

$$d(f(o), f(z_n)) \tag{5}$$

is 'maximal'. More precisely, let $\epsilon > 0$ and pick z_n such that

$$d(f(o), f(z_n)) > \sup_{x \in B_n} d(f(o), f(x)) - \epsilon. \quad (6)$$

The supremum on the right-hand-side exists. If not we could construct a bounded sequence in B_n that is mapped by f to an unbounded sequence. This contradicts the assumptions.

Furthermore, the four point condition

$$[f(x), f(w), f(y), f(z)] \leq \eta([x, w, y, z]),$$

reduces to the three point condition

$$\frac{d(f(x), f(y))}{d(f(x), f(z))} \leq \eta\left(\frac{d(x, y)}{d(x, z)}\right),$$

by taking w to a point at ∞ .

In the following we often shorten the notation $f(x)$ to x' .

Define

$$d_n = d(o, z_n) \quad (7)$$

and

$$d'_n = d(o', z'_n). \quad (8)$$

By construction and assumption on f , the sequence d'_n diverges. Hence, also the sequence d_n must diverge by assumption on f .

(1) We show that if $\frac{d(o, x_n)}{d_n}$ is bounded, then $\frac{d(o', f(x_n))}{d'_n}$ is bounded as well.

If $x_\omega = o_\omega$, then $d(o, x_n) \leq d_n$ for n large, therefore $d(o', x'_n) \leq d'_n$.

If $x_\omega \neq z_\omega$, $x_\omega \neq o_\omega$, then by lemma 7 $\{x_n\}, \{z_n\}, \{o\}$ are separated by u ω -a.s. and the Möbius-condition gives

$$\frac{d(o', x'_n)}{d'_n} = \frac{d(o', x'_n)}{d(o', z'_n)} < \eta\left(\frac{d(o, x_n)}{d(o, z_n)}\right) = \eta\left(\frac{d(o, x_n)}{d_n}\right) \quad \omega - a.s.$$

Thus $\frac{d(o', x'_n)}{d'_n}$ is bounded.

If $x_\omega = z_\omega$, by lemma 8, there exists $\{w_n\}$ representing the same point s.t. $\{w_n\}, \{x_n\}$ are u separated and $\{w_n\}, \{z_n\}$ are u separated. In particular, $\{w_n\}, \{x_n\}, \{o\}$ and $\{w_n\}, \{z_n\}, \{o\}$ are u separated. Using the Möbius-condition again,

$$\begin{aligned} \frac{d(o', x'_n)}{d'_n} &= \frac{d(o', x'_n)}{d(o', z'_n)} \\ &= \frac{d(o', x'_n)}{d(o', w'_n)} \frac{d(o', w'_n)}{d(o', z'_n)} \\ &< \eta\left(\frac{d(o, x_n)}{d(o, w_n)}\right) \eta\left(\frac{d(o, w_n)}{d(o, z_n)}\right) \\ &= \eta\left(\frac{d(o, x_n)}{d_n} \frac{d_n}{d(o, w_n)}\right) \eta\left(\frac{d(o, w_n)}{d_n}\right) \\ &\rightarrow \eta(1)^2 \end{aligned} \quad (9)$$

Thus $\frac{d(o', x'_n)}{d'_n}$ is bounded.

(2) The map $\{x_n\} \rightarrow \{f(x_n)\}$ is well-defined from $X_\omega(d_n)$ to $Y_\omega(d'_n)$.

If $\{x_n\}$ and $\{y_n\}$ are two sequences representing the same point different from the origin o_ω , then by lemma 8 and lemma 7, either $\{x_n\}, \{y_n\}, \{o\}$ are separated by u , or there exists a sequence $\{w_n\}$ representing the same point s.t. $\{x_n\}, \{w_n\}, \{o\}$ and $\{y_n\}, \{w_n\}, \{o\}$ are separated by u .

In the first case,

$$\begin{aligned} \frac{d(x'_n, y'_n)}{d'_n} &= \frac{d(o', x'_n)}{d'_n} \frac{d(x'_n, y'_n)}{d(x'_n, o')} \\ &\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, y_n)}{d(x_n, o)}\right) \\ &= \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, y_n)}{d_n} \frac{d_n}{d(x_n, o)}\right) \\ &\rightarrow 0 \end{aligned}$$

In the second case,

$$\begin{aligned} \frac{d(x'_n, y'_n)}{d'_n} &\leq \frac{d(x'_n, w'_n)}{d'_n} + \frac{d(w'_n, y'_n)}{d'_n} \\ &\leq \frac{d(o', x'_n)}{d'_n} \frac{d(x'_n, w'_n)}{d(x'_n, o')} + \frac{d(o', y'_n)}{d'_n} \frac{d(y'_n, w'_n)}{d(y'_n, o')} \\ &\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, w_n)}{d(x_n, o)}\right) + \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(y_n, w_n)}{d(y_n, o)}\right) \\ &\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, w_n)}{d_n} \frac{d_n}{d(x_n, o)}\right) + \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(y_n, w_n)}{d_n} \frac{d_n}{d(y_n, o)}\right) \\ &\rightarrow 0 \end{aligned}$$

Hence, in both cases $x'_\omega = y'_\omega$.

Let $\{x_n\}$ be a sequence representing the origin o_ω . If $\{x_n\}$ is a bounded sequence, then by assumption on f also $\{x'_n\}$ is bounded and therefore $x'_\omega = o'_\omega$.

If $d(o, x_n)$ diverges, then $\{x_n\}$ and $\{o\}$ are u -separated. We may pick $y_\omega \neq o_\omega$, then by lemma 7, $\{x_n\}, \{y_n\}, \{o\}$ are u -separated. Thus,

$$\begin{aligned} \frac{d(o', x'_n)}{d'_n} &= \frac{d(o', y'_n)}{d'_n} \frac{d(o', x'_n)}{d(o', y'_n)} \\ &\leq \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(o, x_n)}{d(o, y_n)}\right) \\ &= \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(o, x_n)}{d_n} \frac{d_n}{d(o, y_n)}\right) \\ &\rightarrow 0. \end{aligned} \tag{10}$$

Therefore $o'_\omega = x'_\omega$.

(3) The map $g : \{x_n\} \rightarrow \{f(x_n)\}$ is quasi-symmetric and in particular continuous and injective.

Let $\{w_n\}, \{x_n\}, \{y_n\}$ be sequences representing three distinct points w_ω, x_ω and y_ω in $X_\omega(d_n)$. By lemma 7, $\{w_n\}, \{x_n\}, \{y_n\}$ are pairwise u separated. Thus

$$\frac{d(x'_n, y'_n)}{d(x'_n, w'_n)} < \eta \left(\frac{d(x_n, y_n)}{d(x_n, w_n)} \right),$$

equivalently

$$\frac{d(x'_n, y'_n)}{d'_n} \frac{d'_n}{d(x'_n, w'_n)} < \eta \left(\frac{d(x_n, y_n)}{d_n} \frac{d_n}{d(x_n, w_n)} \right).$$

Taking the ω -limit gives

$$\frac{d_\omega(x'_\omega, y'_\omega)}{d_\omega(x'_\omega, w'_\omega)} \leq \eta \left(\frac{d_\omega(x_\omega, y_\omega)}{d_\omega(x_\omega, w_\omega)} \right).$$

Hence, if $g : \{x_n\} \mapsto \{f(x_n)\}$ is non-constant, then it is continuous and injective. Notice that $\{o'\}$ and $\{f(z_n)\}$ do not represent the same points, so g is indeed non-constant. ■

3 Examples

3.1 Sublinearly Lipschitz Equivalences

In [5] Y. Cornulier introduces *sublinearly Lipschitz maps*. A map $f : X \rightarrow Y$ between metric spaces is a sublinearly Lipschitz map, if there exists an admissible function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$d(f(x), f(y)) \leq Cd(x, y) + C'u(|x| + |y|), \quad \forall x, y \in X,$$

for some constants $C, C' > 0$.

Two sublinearly Lipschitz maps f, f' are equivalent, if there exists an admissible function v and a constant $C'' > 0$, such that

$$d(f(x), f'(x)) \leq C''v(|x|)$$

for all $x \in X$.

Sublinearly Lipschitz maps between metric spaces form a category. Taking asymptotic cones yields a functor from the sublinearly Lipschitz category to the Lipschitz category. The sublinearly Lipschitz category is in some sense the maximal category with such a property.

The isomorphisms in the sublinearly Lipschitz category are called sublinearly Lipschitz equivalences, or SBE maps.

Proposition 9. *Every SBE-map $f : X \rightarrow Y$ is an AM-map with linear η .*

Proof: If f is SBE, then f is bi-Lipschitz except at scales below some admissible function u . Indeed, let $x, y \in X$ s.t.

$$d(x, y) \geq u(|x| + |y|).$$

Then

$$\begin{aligned} d(f(x), f(y)) &\leq cd(x, y) + Cu(|x| + |y|) \\ &\leq (c + C)d(x, y), \end{aligned} \tag{11}$$

and

$$\begin{aligned} d(f(x), f(y)) &\geq c'd(x, y) - C'u(|x| + |y|) \\ &\geq (c' - C')d(x, y) \end{aligned} \tag{12}$$

Hence, there exists $D > 0$, s.t. for all $x, y > u$,

$$\frac{1}{D}d(x, y) \leq d(f(x), f(y)) \leq Dd(x, y).$$

In particular, f is an *AM*-map with linear η . ■

3.2 Assouad-type maps

Assouad's embedding theorem [1] [10] says that every snowflake $X^\alpha = (X, d^\alpha)$, $0 < \alpha < 1$, of a doubling metric space admits a bi-Lipschitz embedding into some Euclidean space.

Clearly, the Assouad embedding of a doubling metric space into Euclidean space is an asymptotic-Möbius map.

In the following we construct an example of a Assouad type map from an infinite dimensional Heisenberg group into a Hilbert space. This construction follows straightforwardly from a construction by Lee and Naor in the finite dimensional case [9].

Let H be an infinite dimensional complex Hilbert space. H carries the symplectic form $\Omega(a, b) = \text{Im}(\langle a, b \rangle)$.

The *infinite dimensional Heisenberg group* \mathcal{H}_Ω , is the set of tuples (a, t) with $a \in H$, $t \in \mathbb{R}$ and group law

$$(a, t)(a', t') = (a + a', t + t' + 2\Omega(a, a')).$$

Let G be a group with identity element e . A group seminorm on G is a function $G \rightarrow [0, \infty)$ that satisfies $N(g^{-1}) = N(g)$ for all $g \in G$, $N(gh) \leq N(g) + N(h)$ for all $g, h \in G$ and $N(e) = 0$. If in addition $N(g) = 0$ if and only if $g = e$, then N is a group norm on G .

The function

$$N(a, t) = \sqrt{\sqrt{\|a\|^4 + t^2} + \|a\|^2},$$

is a group norm on \mathcal{H}_Ω . Indeed, if N_1 and N_2 are group seminorms, then $\sqrt{N_1^2 + N_2^2}$ is a group seminorm. In this case $N_1 = (\|a\|^4 + t^2)^{\frac{1}{4}}$ is the Koranyi-norm. Given N ,

$$d_N((a, t), (a', t')) = N((a, t)(a', t')^{-1})$$

is a right-invariant metric on \mathcal{H}_Ω .

Let G be a group, a Hermitian kernel on G is a complex valued function $K : G \times G \rightarrow \mathbb{C}$ that satisfies $K(g, h) = \overline{K(h, g)}$ for all $g, h \in G$.

A Hermitian kernel on G is positive definite if

$$\sum_{i,j}^n K(g_i, g_j) c_i \overline{c_j} \geq 0$$

for all $g_1, \dots, g_n \in G$ and for all complex numbers $c_1, \dots, c_n \in \mathbb{C}$, with equality if and only if the c_i vanish. A Hermitian kernel on G is conditionally negative definite if

$$\sum_{i,j}^n K(g_i, g_j) c_i \overline{c_j} \leq 0$$

for all $g_1, \dots, g_n \in G$ and for all complex numbers $c_1, \dots, c_n \in \mathbb{C}$ satisfying $\sum_i^n c_i = 0$ and with equality if and only if the c_i vanish.

Theorem 2 (Schönberg). *Let G be a group, $K : G \times G \rightarrow \mathbb{R}$ a real valued kernel on G satisfying $K(g, g) = 0 \forall g \in G$. Then K is negative definite if and only if there exists a Hilbertspace H and a function $T : G \rightarrow H$ such that $\forall g, h \in G : K(g, h) = \|T(g) - T(h)\|^2$.*

Thus, if d_N is negative definite, then there exists a Hilbert space H and an isometry $T : (\mathcal{H}_\Omega, \sqrt{d_N}) \rightarrow H$. In particular, the map $T : (\mathcal{H}_\Omega, d_N) \rightarrow H$ is an AM -map.

A complex valued function F on a group G that satisfies $F(g^{-1}) = \overline{F(g)}$ for all $g \in G$ induces a Hermitian kernel on G by defining $K_F(g, h) := F(gh^{-1})$. The function F is said to be positive definite, if K_F is a positive definite Hermitian kernel on G .

Lemma 10. *For every $\lambda \in \mathbb{R}$ the function*

$$\Phi_\lambda(a, t) = e^{-|\lambda| \|a\|^2 + i\lambda t}$$

is positive definite on \mathcal{H}_Ω .

Proof: Indeed the function Φ_λ satisfies $\Phi_\lambda((a, t)^{-1}) = \overline{\Phi_\lambda((a, t))}$ and induces the kernel

$$K((a, s), (b, t)) := \exp(-|\lambda| \|a - b\|^2 + i\lambda(s - t - 2\Omega(a, b))).$$

The kernel can be rewritten as product of three exponentials

$$\exp(-|\lambda| (\|a\|^2 + \|b\|^2)) \exp(i\lambda(s - t)) \exp(2|\lambda| (\operatorname{Re}(\langle a, b \rangle) - i \operatorname{sign}(\lambda) \Omega(a, b))).$$

The product of positive definite kernels is positive definite. The first two factors are positive definite since for all $c_1, \dots, c_n \in \mathbb{C}$, the matrix $(c_i \overline{c_j})_{i,j}$ is positive semi-definite. In particular also the matrix $(\exp(i\lambda(s_i - t_j)) c_i \overline{c_j})_{i,j}$. The exponential of any positive definite kernel is again a positive definite kernel ([3] Proposition 8.2.). Therefore the kernel K is positive definite if

$$\operatorname{Re}(\langle a, b \rangle) - i \operatorname{sign}(\lambda) \Omega(a, b)$$

is positive definite. This is $\langle a, b \rangle$ if $\lambda < 0$ and $\overline{\langle a, b \rangle}$ if $\lambda \geq 0$ and therefore clearly positive definite. ■

Proposition 11. $d_N : \mathcal{H}_\Omega \times \mathcal{H}_\Omega \rightarrow H$ is conditionally negative definite.

Proof: The existence of $\frac{1}{2}$ -stable distributions implies that for all $\epsilon > 0$, there exists a non-negative integrable function $\varphi_\epsilon : \mathbb{R} \rightarrow [0, \infty)$ s.t. its Fourier-transform $\hat{\varphi}_\epsilon(t) = e^{-\epsilon\sqrt{|t|}}$. Notice that $\frac{1}{2\pi}\hat{\varphi}_\epsilon(x) = \varphi_\epsilon(x)$.

By the lemma above,

$$F_\epsilon(a, t) = \int_{\mathbb{R}} e^{-|\lambda|\|a\|^2 + i\lambda t} \varphi_\epsilon(\lambda) d\lambda$$

is positive definite on \mathcal{H}_Ω . Let

$$h_k(x) = \frac{k}{\pi} \frac{1}{k^2 + x^2}$$

denote the Cauchy-distribution with scale parameter $k > 0$. We rewrite

$$\begin{aligned} F_\epsilon(a, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda t} \hat{h}_{\|a\|^2}(\lambda) \hat{\varphi}_\epsilon(\lambda) d\lambda \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda t} \widehat{h_{\|a\|^2} * \varphi_\epsilon}(\lambda) d\lambda \\ &= (h_{\|a\|^2} * \hat{\varphi}_\epsilon)(t). \end{aligned} \quad (13)$$

Since F_ϵ is positive definite, $\frac{1-F_\epsilon}{\epsilon}$ is conditionally negative definite and

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1 - F_\epsilon(a, t)}{\epsilon} &= \lim_{\epsilon \rightarrow 0} \left[h_{\|a\|^2} * \frac{1 - \hat{\varphi}_\epsilon}{\epsilon} \right] \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \frac{1 - e^{-\epsilon\sqrt{|x|}}}{\epsilon} h_{\|a\|^2}(t-x) dx \\ &= \frac{\|a\|^2}{\pi} \int_{\mathbb{R}} \frac{\sqrt{|x|}}{\|a\|^4 + (t-x)^2} dx. \end{aligned} \quad (14)$$

We now show that for all $r, t \in \mathbb{R}$

$$r^2 \int_{\mathbb{R}} \frac{\sqrt{|x|}}{r^4 + (t-x)^2} dx = \pi \sqrt{\sqrt{r^4 + t^2} + r^2}.$$

By change of variables $x = r^2 y$ and $s = t/r^2$, the left-hand-side can be written as

$$\int_0^\infty \left(\frac{1}{1+(s-y)^2} + \frac{1}{1+(s+y)^2} \right) \sqrt{y} dy$$

This integral equals

$$\lim_{r \rightarrow 0} \lim_{R \rightarrow \infty} \frac{1}{2} \int_{C_{r,R}} \left(\frac{1}{1+(s-z)^2} + \frac{1}{1+(s+z)^2} \right) \sqrt{z} dz,$$

where $C_{r,R}$ is the keyhole contour with a branch cut along the positive real axis. The integrand has simple poles at $i \pm s$ and $-i \pm s$. With $\text{Res}\left(\frac{\sqrt{z}}{1+(s-z)^2}, i+s\right) = \frac{\sqrt{i+s}}{2i}$, $\text{Res}\left(\frac{\sqrt{z}}{1+(s-z)^2}, -i+s\right) = \frac{\sqrt{-i+s}}{-2i}$, $\text{Res}\left(\frac{\sqrt{z}}{1+(s+z)^2}, i-s\right) = \frac{\sqrt{i-s}}{2i}$ and $\text{Res}\left(\frac{\sqrt{z}}{1+(s+z)^2}, -i-s\right) = \frac{\sqrt{-i-s}}{-2i}$. Now adding all the residues and multiplying by $2\pi i$, lets us evaluate the above equation as being

$$\frac{\pi}{2} (\sqrt{i+s} - \sqrt{-i+s} + \sqrt{i-s} - \sqrt{-i-s}).$$

This simplifies further to

$$\begin{aligned}
\pi \operatorname{Re}(\sqrt{i+s} + \sqrt{i-s}) &= \pi (\operatorname{Re}(\sqrt{i+s}) + \operatorname{Im}(\sqrt{i+s})) \\
&= \pi \left(\sqrt{\frac{\sqrt{1+s^2} + s}{2}} + \sqrt{\frac{\sqrt{1+s^2} - s}{2}} \right) \quad (15) \\
&= \pi \sqrt{\sqrt{1+s^2} + 1}
\end{aligned}$$

Thus we have seen, that the N -norm is a limit of conditionally negative definite functions on \mathcal{H}_Ω , hence it is conditionally negative definite. \blacksquare

4 Applications to dimension theory

4.1 Dimension theory of finitely-generated groups

In 1993 M. Gromov introduced a notion of asymptotic dimension as a large scale analogue of Lebesgue's covering dimension [6]. The asymptotic dimension of a finitely generated group is a quasi-isometric invariant. Its most prominent application is due to Yu G., who showed that every finitely generated group with finite homotopy type and finite asymptotic dimension, satisfies the Novikov conjecture [15].

Definition 12. *Let X be a metric space. We say that the asymptotic dimension of X does not exceed n provided for every uniformly bounded open cover \mathcal{V} of X there is a uniformly bounded open cover \mathcal{U} of X of multiplicity $\leq n+1$ so that \mathcal{V} refines \mathcal{U} . We write $\operatorname{asdim} X = n$ if it is true that $\operatorname{asdim} X \leq n$ and $\operatorname{asdim} X \not\leq n-1$*

The asymptotic dimension of a finitely generated group Γ has several interesting implications. For example $\operatorname{asdim} \Gamma = 0$ if and only if Γ is finite (Proposition 65. in [2]) and $\operatorname{asdim} \Gamma = 1$ if and only if Γ is virtually free (Theorem 66. in [2]).

If G is a finitely generated nilpotent group, then its asymptotic cone G_ω is a Carnot group [11] and the asymptotic dimension of G agrees with the topological dimension of its asymptotic cone [2].

Theorem 3. *Let G and G' be a finitely generated nilpotent groups and $f : G \rightarrow G'$ an AM-map, then $\operatorname{asdim}(G) \leq \operatorname{asdim}(G')$. If G and G' are nilpotent Lie groups, then $\dim(G) \leq \dim(G')$. Furthermore, if $\operatorname{asdim}(G) = \operatorname{asdim}(G')$, then the asymptotic cones of G and G' are isomorphic graded Lie groups. Conversely, given nilpotent groups with isomorphic asymptotic cones, there exists an asymptotic-Möbius map between them.*

Proof: The map f induces a quasi-symmetric map g between asymptotic cones G_ω and G'_ω of G and G' respectively.

In particular g is an injective map from G_ω to G'_ω and therefore the topological dimensions increase. Since G and G' are nilpotent, $\operatorname{asdim}(G) = \operatorname{top-dim}(G_\omega) \leq \operatorname{top-dim}(G'_\omega) = \operatorname{asdim}(G')$.

The case for nilpotent Lie groups follows from the fact that for nilpotent Lie groups, $\operatorname{top-dim}(G) = \operatorname{top-dim}(G_\omega)$.

If $\text{asdim}(G) = \text{asdim}(G')$, the invariance of domain theorem implies that g is a homeomorphism. Since both g and g^{-1} are quasi-symmetric, they are a.e. differentiable by Pansu's theorem and the differential of g is at almost every point a graded isomorphism of groups [12]. ■

4.2 Dimension theory of CAT(0)-spaces

Heinrich Jung observed in early 1900, that the diameter of a set in Euclidean space \mathbb{R}^n is related to the radius of the minimum enclosing ball by an inequality, whose constant depends only on the dimension n [7]. Precisely, for any set $A \subset \mathbb{R}^n$

$$r \leq d(A) \sqrt{\frac{n}{2(n+1)}}, \quad (16)$$

where r is the radius of the minimum enclosing ball of A and $d(A)$ is the diameter of A .

Caprace and Lytchak observed [4], that a CAT(0)-space has *geometric dimension* less than n (as defined by B. Kleiner [8]), if and only if Jung's inequality (16) is true.

Remark 13. *It turns out that the geometric dimension of a CAT(0)-space X is equal to the maximal topological dimension of all compact subsets in X [8].*

The geometric dimension can be turned into a large-scale notion in the following manner. A CAT(0)-space X has telescopic dimension less than n if and only if for every $\delta > 0$, there exists a $D > 0$ such that for every set A in X with diameter larger than D ,

$$r \leq \left(\delta + \sqrt{\frac{n}{2(n+1)}} \right) d(A). \quad (17)$$

Alternatively,

Definition 14. *A CAT(0)-space X has telescopic dimension $\leq n$, if every asymptotic cone has geometric dimension $\leq n$.*

We adopt the notation $\text{tele-dim}(X)$ for the telescopic dimension of X .

Theorem 4. *Let X and Y be CAT(0) spaces. If there exists an AM-map $f : X \rightarrow Y$, then the telescopic dimension increases e.g. $\text{tele-dim}(X) \leq \text{tele-dim}(Y)$.*

The theorem follows straight forward from Theorem 1, Remark 13 and the fact that the topological dimension increases under injective continuous maps.

As a consequence, there cannot exist an AM-map from Euclidean buildings of finite rank r and thus telescopic dimension r , to a Euclidean building of lower rank.

If there exists an AM-map from a CAT(0) space to a hyperbolic metric space, then X is a hyperbolic metric space as well.

References

- [1] ASSOUAD, P. Plongements lipschitziens dans \mathbb{R}^n . *Bulletin de la Société Mathématique de France* 111 (1983), 429 – 448.
- [2] BELL, G., AND DRANISHNIKOV, A. Asymptotic dimension. *Topology and its Applications* 155, 12 (2008), 1265–1296.
- [3] BENYAMINI, Y., AND LINDENSTRAUSS, J. *Geometric Nonlinear Functional Analysis*, vol. 1 of *American Mathematical Society colloquium publications*. American Mathematical Soc., 1998.
- [4] CAPRACE, P., AND LYTCHAK, A. At infinity of finite-dimensional $\text{cat}(0)$ spaces. *Math. Ann.* 346 (2010), 1 – 21.
- [5] CORNULIER, Y. On sublinear bilipschitz equivalence of groups. *Annales scientifiques de l'École normale supérieure* 52 (02 2017).
- [6] GROMOV, M. *Asymptotic invariants of infinite groups*, vol. 2. Cambridge Univ. Press, 1993.
- [7] JUNG, H. Ueber die kleinste kugel, die eine räumliche figur einschliesst. *Journal für die reine und angewandte Mathematik (Crelles Journal)* 1901, 123 (1901), 241–257.
- [8] KLEINER, B. The local structure of length spaces with curvature bounded above. *Mathematische Zeitschrift* 231 (1999), 409 – 456.
- [9] LEE, J. R., AND NAOR, A. Lp metrics on the heisenberg group and the goemans-linial conjecture. In *2006 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS'06)* (2006), pp. 99–108.
- [10] NAOR, A., AND NEIMAN, O. Assouad's theorem with dimension independent of the snowflaking. *Rev. Mat. Iberoam.* 28, 4 (2012), 1123 – 1142.
- [11] PANSU, P. Croissance des boules et des géodésiques fermées dans les nilvariétés. *Ergodic Theory and Dynamical Systems* 3, 3 (1983), 415–445.
- [12] PANSU, P. Métriques de carnot-caratheodory et quasiisométries des espaces symétriques de rang un. *Annals of Mathematics* 129, 1 (1989), 1–60.
- [13] PANSU, P. Large scale conformal maps. *Annales Scientifiques de l'École Normale Supérieure : publiées sous les auspices du Ministre de l'Instruction publique* 54, 4 (2021), 831 – 887.
- [14] VÄISÄLÄ, J. Quasi-möbius maps. *J. Analyse Math.* 44 (1984/85), 218 – 234.
- [15] YU, G. The novikov conjecture for groups with finite asymptotic dimension. *Annals of Mathematics* 147, 2 (1998), 325–355.

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