

# Asymptotic-Möbius maps.

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## Abstract

Roughly speaking, a map between metric spaces is an asymptotic-Möbius map if it almost preserves cross-ratios of points being a large distance apart from each other. We show that under such maps, some large-scale notions of dimension increase: asymptotic dimension for finitely generated nilpotent groups, telescopic dimension for  $CAT(0)$  spaces.

## 1 Introduction

This paper introduces and investigates a class of maps that mimics quasi-Möbius maps in the context of geometric group theory.

In [13] Pierre Pansu introduces a notion of large-scale conformal maps that mimics the infinitesimal behavior of conformal maps. Briefly, large scale conformal maps map families of disjoint balls to families of weakly overlapping quasi-balls. It is a very flexible notion, that includes for example coarse embeddings. This flexibility makes the asymptotic behavior of such maps however less predictable.

In this paper, we introduce a more restrictive and semantically simpler notion that mimics quasi-Möbius behaviors. Roughly speaking, a map between metric spaces is an asymptotic-Möbius map if it almost preserves cross-ratios of points being a large distance apart from each other. In consequence, the results presented in this paper are much stronger than in [13].

### 1.1 Examples

We postpone the precise definition of asymptotic-Möbius ( $AM$ ) maps until later. Sources of examples of  $AM$ -maps are:

1. Quasi-isometric embeddings,
2. Sublinear-Bilipschitz-equivalences (i.e. maps that induce Lipschitz-Equivalences on asymptotic cones [5]),
3. Snowflaking (i.e. replacing a metric by a power of it),
4. Assouad maps from doubling metric spaces to  $\mathbb{R}^N$ .

For instance, every nilpotent Lie group or finitely generated group can be  $AM$ -embedded in Euclidean space of sufficiently high dimension [1]. We shall furthermore encounter infinite dimensional examples.

## 1.2 Results

Our main result is that under  $AM$ -maps some large-scale notion of dimension increases. The relevant notion depends on the class of groups considered.

**Theorem 3.** *Let  $G$  and  $G'$  be finitely generated nilpotent groups and  $f : G \rightarrow G'$  an  $AM$ -map, then  $\text{asdim}(G) \leq \text{asdim}(G')$ . Furthermore, if  $\text{asdim}(G) = \text{asdim}(G')$ , then the asymptotic cones of  $G$  and  $G'$  are isomorphic. Conversely, given nilpotent groups with isomorphic asymptotic cones, there exists an asymptotic-Möbius map between them.*

In the world of  $\text{CAT}(0)$ -spaces, the analogous theorem takes the following form.

**Theorem 4.** *Let  $X$  and  $Y$  be  $\text{CAT}(0)$  spaces. If there exists an  $AM$ -map  $f : X \rightarrow Y$ , then the telescopic dimension increases e.g.  $\text{tele-dim}(X) \leq \text{tele-dim}(Y)$ .*

## 2 Asymptotic-Möbius maps

### 2.1 Quasi-Möbius maps

Let  $X$  be a metric space, and  $x, y, z, w \in X$  a quadruple of distinct points. Their *cross-ratio* is

$$[x, y, z, w] = \frac{d(x, z)d(y, w)}{d(x, w)d(y, z)}.$$

If  $X$  is unbounded, then the cross-ratio extends to the one-point-compactification of  $X$  [14]. A map  $f : X \rightarrow Y$  is *quasi-Möbius* if there exists a homeomorphism  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that for all quadruples of distinct points  $x, y, z, w \in X$ ,

$$[f(x), f(y), f(z), f(w)] \leq \eta([x, y, z, w]).$$

If  $f$  is a homeomorphism, then  $f$  is quasi-Möbius iff its inverse is quasi-Möbius.

Examples of quasi-Möbius maps are

1. the stereographic projection  $\mathbb{R}^n \rightarrow S^n$ ,
2. the Cayley transforms, the complex, quaternionic and octonionic analoga of the stereographic projection,
3. inversions  $x \mapsto \frac{x}{|x|^2}$  in Banach spaces.

### 2.2 Asymptotic-Möbius maps

Given  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ , we say that the points  $x$  and  $y$  are separated in  $X$  by  $u$  seen from a fixed origin  $o$ , if  $d(x, y) > u(d(x, o) + d(o, y))$ . For simplicity we often write  $x, y > u$  and  $|x| = d(o, x)$  for the distance to the origin.

**Definition 1.** A function  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  is admissible if

1. it is non-decreasing,
2.  $u$  grows sublinearly i.e.  $\limsup_{r \rightarrow \infty} \frac{u(r)}{r} = 0$ .

**Definition 2.** A map  $f : X \rightarrow Y$  is an asymptotic-Möbius map or AM-map, if there exists an admissible function  $u$  and a homeomorphism  $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that for every  $x, y, z, w$  in  $X$  such that all 6 distances between  $x, y, z, w > u$ , then

$$[f(x), f(y), f(z), f(w)] < \eta([x, y, z, w]).$$

**Definition 3.** A metric space  $X$  is asymptotically chained, if for every origin  $o \in X$  there exists an admissible function  $v$ , such that for all  $x, y \in X$  there exists a chain  $x_1 = x, \dots, x_{k+1} = y$  that satisfies

$$\max_{i \in 1 \dots k} \{d(x_i, x_{i+1})\} < v(|x| + |y|).$$

Our main technical step will be the following theorem, whose proof, after some preliminary definition, will occupy Section 2.

**Theorem 1.** Let  $f$  be an AM-map from an asymptotically chained space  $X$  to a space  $Y$  that preserves diverging and bounded sequences, then  $f$  induces a continuous, injective, quasi-symmetric map  $g$  between some asymptotic cones of  $X$  and  $Y$ .

## 2.3 Ultralimits and asymptotic-cones

**Definition 4.** A (non-principal) ultrafilter  $\omega$  over  $\mathbb{N}$  is a set of subsets of  $\mathbb{N}$  satisfying the following conditions:

1. If  $A, B \in \omega$  then  $A \cap B \in \omega$ .
2. If  $A \in \omega, A \subset B \subset \mathbb{N}$ , then  $B \in \omega$ .
3. For every  $A \subset \mathbb{N}$  either  $A \in \omega$  or  $\mathbb{N} \setminus A \in \omega$ .
4. No finite subset of  $\mathbb{N}$  is in  $\omega$ .

Equivalently,  $\omega$  is a finitely additive measure on  $\mathbb{N}$  such that each subset has measure either 0 or 1 and every finite subset has measure 0.

If a statement  $A(n)$  holds for all  $n \in B$  with  $B$  belonging to an ultra filter  $\omega$ , we say that  $A(n)$  holds  $\omega$ -almost surely.

**Definition 5.** Let  $\omega$  be a (non-principle) ultrafilter over  $\mathbb{N}$ . An  $\omega$ -limit of a sequence of points  $\{x_n\}$  in a topological space  $X$ , is a point  $x$  in  $X$  such that for every neighborhood  $U$  of  $x$  the relation  $x_n \in U$  holds  $\omega$ -almost surely.

If  $X$  is a Hausdorff-space, then the  $\omega$ -limit of a sequence is unique. We denote this point by  $\lim_{\omega} x_n$ .

**Definition 6.** The ultrapower of a set  $X$  w.r.t. an ultrafilter  $\omega$ , denoted by  $X^{\omega}$ , consists of equivalence classes of sequences  $\{x_n\}$ ,  $x_n \in X$ , where two sequences  $\{x_n\}$  and  $\{y_n\}$  are identified if and only if  $x_n = y_n$   $\omega$ -almost surely.

We adopt the notation  $\{x_n\}_\omega$  for the equivalence class of sequences.

Let  $(X, d)$  be a metric space,  $\omega$  an ultrafilter over  $\mathbb{N}$  and  $o$  an origin in  $X$ . Furthermore, there is given a sequence of numbers  $\{d_n\}$ , with  $\lim_\omega d_n = \infty$ . Then there exists a pseudo-metric on the ultrapower  $X^\omega$  by setting the distance between  $x = \{x_n\}_\omega$  and  $y = \{y_n\}_\omega$  to

$$d_\omega(x, y) = \lim_\omega \frac{d(x_n, y_n)}{d_n}.$$

Let  $X_o^\omega$  be the set of equivalence classes of elements  $x = \{x_n\}_\omega$  in  $X^\omega$  that satisfy  $\lim_\omega \frac{d(x_n, o)}{d_n} < \infty$  and two elements  $x = \{x_n\}_\omega$  and  $y = \{y_n\}_\omega$  of  $X^\omega$  are identified if and only if  $d_\omega(x, y) = 0$ . The metric space  $(X_o^\omega, d_\omega)$  is called an *asymptotic cone* of  $(X, d)$ .

We adopt the notation  $x_\omega$  for points in the asymptotic cone  $(X_o^\omega, d_\omega)$  of  $(X, d)$ .

**Lemma 7.** *If  $x_\omega \neq y_\omega$ , then for all admissible functions  $u$  and all sequences  $\{x_n\}$ ,  $\{y_n\}$  representing  $x_\omega$  and  $y_\omega$  respectively,*

$$d(x_n, y_n) > u(|x_n| + |y_n|), \quad \omega - a.s.$$

**Proof:** If there exists an admissible function  $u$  and sequences  $\{x_n\}$ ,  $\{y_n\}$  such that

$$d(x_n, y_n) \leq u(|x_n| + |y_n|), \quad \omega - a.s.$$

then  $\omega - a.s.$ ,

$$\frac{d(x_n, y_n)}{d_n} \leq \frac{u(|x_n| + |y_n|)}{d_n}.$$

Thus

$$\lim_\omega \frac{d(x_n, y_n)}{d_n} = 0.$$

■

**Lemma 8.** *Let  $X$  be an asymptotically chained metric space. Let  $\{x_n\}$  and  $\{y_n\}$  be two sequences in  $X$  representing the same point  $x_\omega$  different from the origin  $o_\omega$ , and such that  $x_n \neq y_n$   $\omega - a.s.$  Then for every admissible function  $u$ , either*

$$d(x_n, y_n) > u, \quad \omega - a.s.$$

*or there exists a sequence  $\{w_n\}$  representing the same point s.t.*

$$d(x_n, w_n) > u \quad \omega - a.s.$$

*and*

$$d(y_n, w_n) > u \quad \omega - a.s.$$

**Proof:** Suppose  $d(x_n, y_n) \leq u$   $\omega - a.s.$  and by assumption,  $|x_n|, |y_n| \rightarrow +\infty$ .

Since  $X$  is asymptotically chained, there exists an admissible function  $v$  and a chain  $x_1^n = o, \dots, x_{k+1}^n = y_n$  such that

$$\max_{i \in 1, \dots, k} \{d(x_i^n, x_{i+1}^n)\} < v(|y_n|).$$

Assume that  $n$  is large enough such that  $2u(4|y_n|) < |y_n|$ , then at least one point  $x_{i_n}^n$  in the chain satisfies

$$2u(4|y_n|) < d(y_n, x_{i_n}^n) \leq 2u(4|y_n|) + v(|y_n|).$$

Define  $w_n = x_{i_n}^n$ .

For  $n$  large enough,

$$\begin{aligned} 2d(x_n, y_n) &\leq 2u(|x_n| + |y_n|) \\ &\leq |x_n| + |y_n| \\ &\leq 2|y_n| + d(x_n, y_n). \end{aligned} \tag{1}$$

Therefore,

$$d(x_n, y_n) \leq u(4|y_n|).$$

and hence

$$u(4|y_n|) < d(x_n, w_n) \leq 3u(4|y_n|) + v(|y_n|). \tag{2}$$

Pick  $N$  s.t.  $\forall n \geq N$

$$3u(4|y_n|) + v(|y_n|) \leq 2|y_n|.$$

Then

$$\begin{aligned} u(|x_n| + |w_n|) &\leq u(|x_n| + |y_n| + d(y_n, w_n)) \\ &\leq u(2|y_n| + d(x_n, y_n) + 2u(4|y_n|) + v(|y_n|)) \\ &\leq u(2|y_n| + 3u(4|y_n|) + v(|y_n|)) \\ &\leq u(4|y_n|) \\ &< d(x_n, w_n). \end{aligned} \tag{3}$$

That means  $x_n, w_n > u$  and

$$\frac{d(x_n, w_n)}{d_n} \leq \frac{3u(4|y_n|) + v(|y_n|)}{d_n} \rightarrow 0.$$

Similarly,

$$\begin{aligned} u(|y_n| + |w_n|) &\leq u(2|y_n| + d(y_n, w_n)) \\ &\leq u(2|y_n| + 2u(4|y_n|) + v(|y_n|)) \\ &\leq u(4|y_n|) \\ &< d(y_n, w_n). \end{aligned} \tag{4}$$

■

## 2.4 Proof of Theorem 1.

**Proof:** Let  $u$  be an admissible function as given by the definition of  $f$ . Fix an origin  $o \in X$ . Let  $B_n$  be the ball of radius  $n$  around  $o$ . For every  $n$ , pick a point  $z_n$  in  $B_n$  such that

$$d(f(o), f(z_n)) \tag{5}$$

is 'maximal'. More precisely, let  $\epsilon > 0$  and pick  $z_n$  such that

$$d(f(o), f(z_n)) > \sup_{x \in B_n} d(f(o), f(x)) - \epsilon. \quad (6)$$

The supremum on the right-hand-side exists. If not we could construct a bounded sequence in  $B_n$  that is mapped by  $f$  to an unbounded sequence. This contradicts the assumptions.

Furthermore, the four point condition

$$[f(x), f(w), f(y), f(z)] \leq \eta([x, w, y, z]),$$

reduces to the three point condition

$$\frac{d(f(x), f(y))}{d(f(x), f(z))} \leq \eta\left(\frac{d(x, y)}{d(x, z)}\right),$$

by taking  $w$  to a point at  $\infty$ .

In the following we often shorten the notation  $f(x)$  to  $x'$ .

Define

$$d_n = d(o, z_n) \quad (7)$$

and

$$d'_n = d(o', z'_n). \quad (8)$$

By construction and assumption on  $f$ , the sequence  $d'_n$  diverges. Hence, also the sequence  $d_n$  must diverge by assumption on  $f$ .

(1) We show that if  $\frac{d(o, x_n)}{d_n}$  is bounded, then  $\frac{d(o', f(x_n))}{d'_n}$  is bounded as well.

If  $x_\omega = o_\omega$ , then  $d(o, x_n) \leq d_n$  for  $n$  large, therefore  $d(o', x'_n) \leq d'_n$ .

If  $x_\omega \neq z_\omega$ ,  $x_\omega \neq o_\omega$ , then by lemma 7  $\{x_n\}, \{z_n\}, \{o\}$  are separated by  $u$   $\omega$ -a.s. and the Möbius-condition gives

$$\frac{d(o', x'_n)}{d'_n} = \frac{d(o', x'_n)}{d(o', z'_n)} < \eta\left(\frac{d(o, x_n)}{d(o, z_n)}\right) = \eta\left(\frac{d(o, x_n)}{d_n}\right) \quad \omega - a.s.$$

Thus  $\frac{d(o', x'_n)}{d'_n}$  is bounded.

If  $x_\omega = z_\omega$ , by lemma 8, there exists  $\{w_n\}$  representing the same point s.t.  $\{w_n\}, \{x_n\}$  are  $u$  separated and  $\{w_n\}, \{z_n\}$  are  $u$  separated. In particular,  $\{w_n\}, \{x_n\}, \{o\}$  and  $\{w_n\}, \{z_n\}, \{o\}$  are  $u$  separated. Using the Möbius-condition again,

$$\begin{aligned} \frac{d(o', x'_n)}{d'_n} &= \frac{d(o', x'_n)}{d(o', z'_n)} \\ &= \frac{d(o', x'_n)}{d(o', w'_n)} \frac{d(o', w'_n)}{d(o', z'_n)} \\ &< \eta\left(\frac{d(o, x_n)}{d(o, w_n)}\right) \eta\left(\frac{d(o, w_n)}{d(o, z_n)}\right) \\ &= \eta\left(\frac{d(o, x_n)}{d_n} \frac{d_n}{d(o, w_n)}\right) \eta\left(\frac{d(o, w_n)}{d_n}\right) \\ &\rightarrow \eta(1)^2 \end{aligned} \quad (9)$$

Thus  $\frac{d(o', x'_n)}{d'_n}$  is bounded.

(2) The map  $\{x_n\} \rightarrow \{f(x_n)\}$  is well-defined from  $X_\omega(d_n)$  to  $Y_\omega(d'_n)$ .

If  $\{x_n\}$  and  $\{y_n\}$  are two sequences representing the same point different from the origin  $o_\omega$ , then by lemma 8 and lemma 7, either  $\{x_n\}, \{y_n\}, \{o\}$  are separated by  $u$ , or there exists a sequence  $\{w_n\}$  representing the same point s.t.  $\{x_n\}, \{w_n\}, \{o\}$  and  $\{y_n\}, \{w_n\}, \{o\}$  are separated by  $u$ .

In the first case,

$$\begin{aligned} \frac{d(x'_n, y'_n)}{d'_n} &= \frac{d(o', x'_n)}{d'_n} \frac{d(x'_n, y'_n)}{d(x'_n, o')} \\ &\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, y_n)}{d(x_n, o)}\right) \\ &= \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, y_n)}{d_n} \frac{d_n}{d(x_n, o)}\right) \\ &\rightarrow 0 \end{aligned}$$

In the second case,

$$\begin{aligned} \frac{d(x'_n, y'_n)}{d'_n} &\leq \frac{d(x'_n, w'_n)}{d'_n} + \frac{d(w'_n, y'_n)}{d'_n} \\ &\leq \frac{d(o', x'_n)}{d'_n} \frac{d(x'_n, w'_n)}{d(x'_n, o')} + \frac{d(o', y'_n)}{d'_n} \frac{d(y'_n, w'_n)}{d(y'_n, o')} \\ &\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, w_n)}{d(x_n, o)}\right) + \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(y_n, w_n)}{d(y_n, o)}\right) \\ &\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, w_n)}{d_n} \frac{d_n}{d(x_n, o)}\right) + \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(y_n, w_n)}{d_n} \frac{d_n}{d(y_n, o)}\right) \\ &\rightarrow 0 \end{aligned}$$

Hence, in both cases  $x'_\omega = y'_\omega$ .

Let  $\{x_n\}$  be a sequence representing the origin  $o_\omega$ . If  $\{x_n\}$  is a bounded sequence, then by assumption on  $f$  also  $\{x'_n\}$  is bounded and therefore  $x'_\omega = o'_\omega$ .

If  $d(o, x_n)$  diverges, then  $\{x_n\}$  and  $\{o\}$  are  $u$ -separated. We may pick  $y_\omega \neq o_\omega$ , then by lemma 7,  $\{x_n\}, \{y_n\}, \{o\}$  are  $u$ -separated. Thus,

$$\begin{aligned} \frac{d(o', x'_n)}{d'_n} &= \frac{d(o', y'_n)}{d'_n} \frac{d(o', x'_n)}{d(o', y'_n)} \\ &\leq \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(o, x_n)}{d(o, y_n)}\right) \\ &= \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(o, x_n)}{d_n} \frac{d_n}{d(o, y_n)}\right) \\ &\rightarrow 0. \end{aligned} \tag{10}$$

Therefore  $o'_\omega = x'_\omega$ .

(3) The map  $g : \{x_n\} \rightarrow \{f(x_n)\}$  is quasi-symmetric and in particular continuous and injective.

Let  $\{w_n\}, \{x_n\}, \{y_n\}$  be sequences representing three distinct points  $w_\omega, x_\omega$  and  $y_\omega$  in  $X_\omega(d_n)$ . By lemma 7,  $\{w_n\}, \{x_n\}, \{y_n\}$  are pairwise  $u$  separated. Thus

$$\frac{d(x'_n, y'_n)}{d(x'_n, w'_n)} < \eta \left( \frac{d(x_n, y_n)}{d(x_n, w_n)} \right),$$

equivalently

$$\frac{d(x'_n, y'_n)}{d'_n} \frac{d'_n}{d(x'_n, w'_n)} < \eta \left( \frac{d(x_n, y_n)}{d_n} \frac{d_n}{d(x_n, w_n)} \right).$$

Taking the  $\omega$ -limit gives

$$\frac{d_\omega(x'_\omega, y'_\omega)}{d_\omega(x'_\omega, w'_\omega)} \leq \eta \left( \frac{d_\omega(x_\omega, y_\omega)}{d_\omega(x_\omega, w_\omega)} \right).$$

Hence, if  $g : \{x_n\} \mapsto \{f(x_n)\}$  is non-constant, then it is continuous and injective. Notice that  $\{o'\}$  and  $\{f(z_n)\}$  do not represent the same points, so  $g$  is indeed non-constant. ■

## 3 Examples

### 3.1 Sublinearly Lipschitz Equivalences

In [5] Y. Cornulier introduces *sublinearly Lipschitz maps*. A map  $f : X \rightarrow Y$  between metric spaces is a sublinearly Lipschitz map, if there exists an admissible function  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  such that

$$d(f(x), f(y)) \leq Cd(x, y) + C'u(|x| + |y|), \quad \forall x, y \in X,$$

for some constants  $C, C' > 0$ .

Two sublinearly Lipschitz maps  $f, f'$  are equivalent, if there exists an admissible function  $v$  and a constant  $C'' > 0$ , such that

$$d(f(x), f'(x)) \leq C''v(|x|)$$

for all  $x \in X$ .

Sublinearly Lipschitz maps between metric spaces form a category. Taking asymptotic cones yields a functor from the sublinearly Lipschitz category to the Lipschitz category. The sublinearly Lipschitz category is in some sense the maximal category with such a property.

The isomorphisms in the sublinearly Lipschitz category are called sublinearly Lipschitz equivalences, or SBE maps.

**Proposition 9.** *Every SBE-map  $f : X \rightarrow Y$  is an AM-map with linear  $\eta$ .*

**Proof:** If  $f$  is SBE, then  $f$  is bi-Lipschitz except at scales below some admissible function  $u$ . Indeed, let  $x, y \in X$  s.t.

$$d(x, y) \geq u(|x| + |y|).$$

Then

$$\begin{aligned} d(f(x), f(y)) &\leq cd(x, y) + Cu(|x| + |y|) \\ &\leq (c + C)d(x, y), \end{aligned} \tag{11}$$

and

$$\begin{aligned} d(f(x), f(y)) &\geq c'd(x, y) - C'u(|x| + |y|) \\ &\geq (c' - C')d(x, y) \end{aligned} \tag{12}$$

Hence, there exists  $D > 0$ , s.t. for all  $x, y > u$ ,

$$\frac{1}{D}d(x, y) \leq d(f(x), f(y)) \leq Dd(x, y).$$

In particular,  $f$  is an  $AM$ -map with linear  $\eta$ . ■

### 3.2 Assouad-type maps

Assouad's embedding theorem [1] [10] says that every snowflake  $X^\alpha = (X, d^\alpha)$ ,  $0 < \alpha < 1$ , of a doubling metric space admits a bi-Lipschitz embedding into some Euclidean space.

Clearly, the Assouad embedding of a doubling metric space into Euclidean space is an asymptotic-Möbius map.

In the following we construct an example of a Assouad type map from an infinite dimensional Heisenberg group into a Hilbert space. This construction follows straightforwardly from a construction by Lee and Naor in the finite dimensional case [9].

Let  $H$  be an infinite dimensional complex Hilbert space.  $H$  carries the symplectic form  $\Omega(a, b) = \text{Im}(\langle a, b \rangle)$ .

The *infinite dimensional Heisenberg group*  $\mathcal{H}_\Omega$ , is the set of tuples  $(a, t)$  with  $a \in H$ ,  $t \in \mathbb{R}$  and group law

$$(a, t)(a', t') = (a + a', t + t' + 2\Omega(a, a')).$$

Let  $G$  be a group with identity element  $e$ . A group seminorm on  $G$  is a function  $G \rightarrow [0, \infty)$  that satisfies  $N(g^{-1}) = N(g)$  for all  $g \in G$ ,  $N(gh) \leq N(g) + N(h)$  for all  $g, h \in G$  and  $N(e) = 0$ . If in addition  $N(g) = 0$  if and only if  $g = e$ , then  $N$  is a group norm on  $G$ .

The function

$$N(a, t) = \sqrt{\sqrt{\|a\|^4 + t^2} + \|a\|^2},$$

is a group norm on  $\mathcal{H}_\Omega$ . Indeed, if  $N_1$  and  $N_2$  are group seminorms, then  $\sqrt{N_1^2 + N_2^2}$  is a group seminorm. In this case  $N_1 = (\|a\|^4 + t^2)^{\frac{1}{4}}$  is the Koranyi-norm. Given  $N$ ,

$$d_N((a, t), (a', t')) = N((a, t)(a', t')^{-1})$$

is a right-invariant metric on  $\mathcal{H}_\Omega$ .

Let  $G$  be a group, a Hermitian kernel on  $G$  is a complex valued function  $K : G \times G \rightarrow \mathbb{C}$  that satisfies  $K(g, h) = \overline{K(h, g)}$  for all  $g, h \in G$ .

A Hermitian kernel on  $G$  is positive definite if

$$\sum_{i,j}^n K(g_i, g_j) c_i \overline{c_j} \geq 0$$

for all  $g_1, \dots, g_n \in G$  and for all complex numbers  $c_1, \dots, c_n \in \mathbb{C}$ , with equality if and only if the  $c_i$  vanish. A Hermitian kernel on  $G$  is conditionally negative definite if

$$\sum_{i,j}^n K(g_i, g_j) c_i \overline{c_j} \leq 0$$

for all  $g_1, \dots, g_n \in G$  and for all complex numbers  $c_1, \dots, c_n \in \mathbb{C}$  satisfying  $\sum_i^n c_i = 0$  and with equality if and only if the  $c_i$  vanish.

**Theorem 2** (Schönberg). *Let  $G$  be a group,  $K : G \times G \rightarrow \mathbb{R}$  a real valued kernel on  $G$  satisfying  $K(g, g) = 0 \forall g \in G$ . Then  $K$  is negative definite if and only if there exists a Hilbertspace  $H$  and a function  $T : G \rightarrow H$  such that  $\forall g, h \in G : K(g, h) = \|T(g) - T(h)\|^2$ .*

Thus, if  $d_N$  is negative definite, then there exists a Hilbert space  $H$  and an isometry  $T : (\mathcal{H}_\Omega, \sqrt{d_N}) \rightarrow H$ . In particular, the map  $T : (\mathcal{H}_\Omega, d_N) \rightarrow H$  is an  $AM$ -map.

A complex valued function  $F$  on a group  $G$  that satisfies  $F(g^{-1}) = \overline{F(g)}$  for all  $g \in G$  induces a Hermitian kernel on  $G$  by defining  $K_F(g, h) := F(gh^{-1})$ . The function  $F$  is said to be positive definite, if  $K_F$  is a positive definite Hermitian kernel on  $G$ .

**Lemma 10.** *For every  $\lambda \in \mathbb{R}$  the function*

$$\Phi_\lambda(a, t) = e^{-|\lambda| \|a\|^2 + i\lambda t}$$

*is positive definite on  $\mathcal{H}_\Omega$ .*

**Proof:** Indeed the function  $\Phi_\lambda$  satisfies  $\Phi_\lambda((a, t)^{-1}) = \overline{\Phi_\lambda((a, t))}$  and induces the kernel

$$K((a, s), (b, t)) := \exp(-|\lambda| \|a - b\|^2 + i\lambda(s - t - 2\Omega(a, b))).$$

The kernel can be rewritten as product of three exponentials

$$\exp(-|\lambda|(\|a\|^2 + \|b\|^2)) \exp(i\lambda(s - t)) \exp(2|\lambda|(\operatorname{Re}(\langle a, b \rangle) - i \operatorname{sign}(\lambda) \Omega(a, b))).$$

The product of positive definite kernels is positive definite. The first two factors are positive definite since for all  $c_1, \dots, c_n \in \mathbb{C}$ , the matrix  $(c_i \overline{c_j})_{i,j}$  is positive semi-definite. In particular also the matrix  $(\exp(i\lambda(s_i - t_j)) c_i \overline{c_j})_{i,j}$ . The exponential of any positive definite kernel is again a positive definite kernel ([3] Proposition 8.2.). Therefore the kernel  $K$  is positive definite if

$$\operatorname{Re}(\langle a, b \rangle) - i \operatorname{sign}(\lambda) \Omega(a, b)$$

is positive definite. This is  $\langle a, b \rangle$  if  $\lambda < 0$  and  $\overline{\langle a, b \rangle}$  if  $\lambda \geq 0$  and therefore clearly positive definite.  $\blacksquare$

**Proposition 11.**  $d_N : \mathcal{H}_\Omega \times \mathcal{H}_\Omega \rightarrow H$  is conditionally negative definite.

**Proof:** The existence of  $\frac{1}{2}$ -stable distributions implies that for all  $\epsilon > 0$ , there exists a non-negative integrable function  $\varphi_\epsilon : \mathbb{R} \rightarrow [0, \infty)$  s.t. its Fourier-transform  $\hat{\varphi}_\epsilon(t) = e^{-\epsilon\sqrt{|t|}}$ . Notice that  $\frac{1}{2\pi}\hat{\varphi}_\epsilon(x) = \varphi_\epsilon(x)$ .

By the lemma above,

$$F_\epsilon(a, t) = \int_{\mathbb{R}} e^{-|\lambda|\|a\|^2 + i\lambda t} \varphi_\epsilon(\lambda) d\lambda$$

is positive definite on  $\mathcal{H}_\Omega$ . Let

$$h_k(x) = \frac{k}{\pi} \frac{1}{k^2 + x^2}$$

denote the Cauchy-distribution with scale parameter  $k > 0$ . We rewrite

$$\begin{aligned} F_\epsilon(a, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda t} \hat{h}_{\|a\|^2}(\lambda) \hat{\varphi}_\epsilon(\lambda) d\lambda \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda t} \widehat{h_{\|a\|^2} * \varphi_\epsilon}(\lambda) d\lambda \\ &= (h_{\|a\|^2} * \varphi_\epsilon)(t). \end{aligned} \quad (13)$$

Since  $F_\epsilon$  is positive definite,  $\frac{1-F_\epsilon}{\epsilon}$  is conditionally negative definite and

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1 - F_\epsilon(a, t)}{\epsilon} &= \lim_{\epsilon \rightarrow 0} \left[ h_{\|a\|^2} * \frac{1 - \hat{\varphi}_\epsilon}{\epsilon} \right] \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \frac{1 - e^{-\epsilon\sqrt{|x|}}}{\epsilon} h_{\|a\|^2}(t - x) dx \\ &= \frac{\|a\|^2}{\pi} \int_{\mathbb{R}} \frac{\sqrt{|x|}}{\|a\|^4 + (t - x)^2} dx. \end{aligned} \quad (14)$$

We now show that for all  $r, t \in \mathbb{R}$

$$r^2 \int_{\mathbb{R}} \frac{\sqrt{|x|}}{r^4 + (t - x)^2} dx = \pi \sqrt{r^4 + t^2 + r^2}.$$

By change of variables  $x = r^2 y$  and  $s = t/r^2$ , the left-hand-side can be written as

$$\int_0^\infty \left( \frac{1}{1 + (s - y)^2} + \frac{1}{1 + (s + y)^2} \right) \sqrt{y} dy$$

This integral equals

$$\lim_{r \rightarrow 0} \lim_{R \rightarrow \infty} \frac{1}{2} \int_{C_{r,R}} \left( \frac{1}{1 + (s - z)^2} + \frac{1}{1 + (s + z)^2} \right) \sqrt{z} dz,$$

where  $C_{r,R}$  is the keyhole contour with a branch cut along the positive real axis. The integrand has simple poles at  $i \pm s$  and  $-i \pm s$ . With  $\text{Res}(\frac{\sqrt{z}}{1 + (s - z)^2}, i + s) = \frac{\sqrt{i + s}}{2i}$ ,  $\text{Res}(\frac{\sqrt{z}}{1 + (s - z)^2}, -i + s) = \frac{\sqrt{-i + s}}{-2i}$ ,  $\text{Res}(\frac{\sqrt{z}}{1 + (s + z)^2}, i - s) = \frac{\sqrt{i - s}}{2i}$  and  $\text{Res}(\frac{\sqrt{z}}{1 + (s + z)^2}, -i - s) = \frac{\sqrt{-i - s}}{-2i}$ . Now adding all the residues and multiplying by  $2\pi i$ , lets us evaluate the above equation as being

$$\frac{\pi}{2} (\sqrt{i + s} - \sqrt{-i + s} + \sqrt{i - s} - \sqrt{-i - s}).$$

This simplifies further to

$$\begin{aligned}
\pi \operatorname{Re}(\sqrt{i+s} + \sqrt{i-s}) &= \pi (\operatorname{Re}(\sqrt{i+s}) + \operatorname{Im}(\sqrt{i+s})) \\
&= \pi \left( \sqrt{\frac{\sqrt{1+s^2}+s}{2}} + \sqrt{\frac{\sqrt{1+s^2}-s}{2}} \right) \quad (15) \\
&= \pi \sqrt{\sqrt{1+s^2}+1}
\end{aligned}$$

Thus we have seen, that the  $N$ -norm is a limit of conditionally negative definite functions on  $\mathcal{H}_\Omega$ , hence it is conditionally negative definite. ■

## 4 Applications to dimension theory

### 4.1 Dimension theory of finitely-generated groups

In 1993 M. Gromov introduced a notion of asymptotic dimension as a large scale analogue of Lebesgue's covering dimension [6]. The asymptotic dimension of a finitely generated group is a quasi-isometric invariant. Its most prominent application is due to Yu G., who showed that every finitely generated group with finite homotopy type and finite asymptotic dimension, satisfies the Novikov conjecture [15].

**Definition 12.** *Let  $X$  be a metric space. We say that the asymptotic dimension of  $X$  does not exceed  $n$  provided for every uniformly bounded open cover  $\mathcal{V}$  of  $X$  there is a uniformly bounded open cover  $\mathcal{U}$  of  $X$  of multiplicity  $\leq n+1$  so that  $\mathcal{V}$  refines  $\mathcal{U}$ . We write  $\operatorname{asdim} X = n$  if it is true that  $\operatorname{asdim} X \leq n$  and  $\operatorname{asdim} X \not\leq n-1$*

The asymptotic dimension of a finitely generated group  $\Gamma$  has several interesting implications. For example  $\operatorname{asdim} \Gamma = 0$  if and only if  $\Gamma$  is finite (Proposition 65. in [2]) and  $\operatorname{asdim} \Gamma = 1$  if and only if  $\Gamma$  is virtually free (Theorem 66. in [2]).

If  $G$  is a finitely generated nilpotent group, then its asymptotic cone  $G_\omega$  is a Carnot group [11] and the asymptotic dimension of  $G$  agrees with the topological dimension of its asymptotic cone [2].

**Theorem 3.** *Let  $G$  and  $G'$  be a finitely generated nilpotent groups and  $f : G \rightarrow G'$  an AM-map, then  $\operatorname{asdim}(G) \leq \operatorname{asdim}(G')$ . If  $G$  and  $G'$  are nilpotent Lie groups, then  $\dim(G) \leq \dim(G')$ . Furthermore, if  $\operatorname{asdim}(G) = \operatorname{asdim}(G')$ , then the asymptotic cones of  $G$  and  $G'$  are isomorphic graded Lie groups. Conversely, given nilpotent groups with isomorphic asymptotic cones, there exists an asymptotic-Möbius map between them.*

**Proof:** The map  $f$  induces a quasi-symmetric map  $g$  between asymptotic cones  $G_\omega$  and  $G'_\omega$  of  $G$  and  $G'$  respectively.

In particular  $g$  is an injective map from  $G_\omega$  to  $G'_\omega$  and therefore the topological dimensions increase. Since  $G$  and  $G'$  are nilpotent,  $\operatorname{asdim}(G) = \operatorname{top-dim}(G_\omega) \leq \operatorname{top-dim}(G'_\omega) = \operatorname{asdim}(G')$ .

The case for nilpotent Lie groups follows from the fact that for nilpotent Lie groups,  $\operatorname{top-dim}(G) = \operatorname{top-dim}(G_\omega)$ .

If  $\text{asdim}(G) = \text{asdim}(G')$ , the invariance of domain theorem implies that  $g$  is a homeomorphism. Since both  $g$  and  $g^{-1}$  are quasi-symmetric, they are a.e. differentiable by Pansu's theorem and the differential of  $g$  is at almost every point a graded isomorphism of groups [12]. ■

## 4.2 Dimension theory of CAT(0)-spaces

Heinrich Jung observed in early 1900, that the diameter of a set in Euclidean space  $\mathbb{R}^n$  is related to the radius of the minimum enclosing ball by an inequality, whose constant depends only on the dimension  $n$  [7]. Precisely, for any set  $A \subset \mathbb{R}^n$

$$r \leq d(A) \sqrt{\frac{n}{2(n+1)}}, \quad (16)$$

where  $r$  is the radius of the minimum enclosing ball of  $A$  and  $d(A)$  is the diameter of  $A$ .

Caprace and Lytchak observed [4], that a CAT(0)-space has *geometric dimension* less than  $n$  (as defined by B. Kleiner [8]), if and only if Jung's inequality (16) is true.

**Remark 13.** *It turns out that the geometric dimension of a CAT(0)-space  $X$  is equal to the maximal topological dimension of all compact subsets in  $X$  [8].*

The geometric dimension can be turned into a large-scale notion in the following manner. A CAT(0)-space  $X$  has telescopic dimension less than  $n$  if and only if for every  $\delta > 0$ , there exists a  $D > 0$  such that for every set  $A$  in  $X$  with diameter larger than  $D$ ,

$$r \leq \left( \delta + \sqrt{\frac{n}{2(n+1)}} \right) d(A). \quad (17)$$

Alternatively,

**Definition 14.** *A CAT(0)-space  $X$  has telescopic dimension  $\leq n$ , if every asymptotic cone has geometric dimension  $\leq n$ .*

We adopt the notation  $\text{tele-dim}(X)$  for the telescopic dimension of  $X$ .

**Theorem 4.** *Let  $X$  and  $Y$  be CAT(0) spaces. If there exists an AM-map  $f : X \rightarrow Y$ , then the telescopic dimension increases e.g.  $\text{tele-dim}(X) \leq \text{tele-dim}(Y)$ .*

The theorem follows straight forward from Theorem 1, Remark 13 and the fact that the topological dimension increases under injective continuous maps.

As a consequence, there cannot exist an AM-map from Euclidean buildings of finite rank  $r$  and thus telescopic dimension  $r$ , to a Euclidean building of lower rank.

If there exists an AM-map from a CAT(0) space to a hyperbolic metric space, then  $X$  is a hyperbolic metric space as well.

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