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# Asymptotic-Möbius maps.

Georg Grütznér

## Abstract

Roughly speaking, a map between metric spaces is asymptotically Möbius if it induces quasi-Möbius maps on asymptotic cones. We show that under such maps, some large-scale notions of dimension increase: asymptotic dimension for finitely generated nilpotent groups, telescopic dimension for  $CAT(0)$  spaces.

## 1 Introduction

### 1.1 Microscopic conformality

Examples of conformal maps appeared very early in history. Already in 1569 discovered the Flemish geographer and cartographer Gerardus Mercator a map to conformally project the sphere onto an infinite cylinder. Today this map is known as the Mercator map. Distances under conformal maps can change dramatically. Nevertheless, the metric distortion can often be estimated. This leads to metric space analogues of conformal maps such as quasi-symmetric maps or quasi-Möbius maps. Grosso modo, these are maps that map balls  $B$  to quasi-balls  $f(B)$ , in the sense that there exists a ball  $B' \subset f(B)$  that covers  $f(B)$  when the radius is increased by a uniform factor. For details see [3], [4], [5].

### 1.2 Mesoscopic conformality

Much later it was discovered that conformality may manifest itself as well in discrete settings. The beginning marks Paul Koebe's circle packing theorem in 1931. A circle packing of the 2-sphere is a collection of interior-disjoint disks. The incidence graph of a circle-packing has a vertex for every circle and an edge between two vertices whenever the corresponding circles touch. Koebe discovered, that every triangulation of the 2-sphere is the incidence graph of a circle-packing which is unique up to Möbius-transformations of the sphere.

### 1.3 A new class of maps

In this paper, we go one step further and define a class of asymptotic-Möbius maps, in short *AM*-maps. Roughly speaking, asymptotic means that our definitions are unaffected by local changes in metric or topology.

Technically, it means that pre- or post-composition of asymptotic-Möbius maps with quasi-isometries are again asymptotic-Möbius maps. Due to this invariance property, asymptotic-Möbius maps form a class of maps, suitable for use in geometric group theory.

## 1.4 Examples

We postpone the precise definition until later. Sources of examples of  $AM$ -maps are:

1. Quasi-isometric embeddings,
2. Sublinear-Bilipschitz-equivalences (i.e. maps that induce Lipschitz-Equivalences on asymptotic cones [2]),
3. Snowflaking (i.e. replacing a metric by a power of it),
4. Assuad maps from doubling metric spaces to  $\mathbb{R}^N$ .

Every nilpotent Lie group or finitely generated group can be  $AM$ -embedded in Euclidean space of sufficiently high dimension [1].

## 1.5 Results

Our main results is that under  $AM$ -maps some large-scale notion of dimension increases. The relevant notion depends on the class of groups considered.

**Theorem 1.** *Let  $G$  and  $G'$  be finitely generated nilpotent groups and  $f : G \rightarrow G'$  an  $AM$ -map, then  $asdim(G) \leq asdim(G')$ . Furthermore, if  $asdim(G) = asdim(G')$ , then the cones of  $G$  and  $G'$  are diffeomorphic.*

In the world of  $CAT(0)$ -spaces, the analogous theorem takes the following form.

**Theorem 2.** *Let  $X$  and  $Y$  be  $CAT(0)$  spaces. If there exists an  $AM$ -map  $f : X \rightarrow Y$ , then the telescopic dimension increases e.g.  $tele-dim(X) \leq tele-dim(Y)$ .*

# 2 Asymptotic-Möbius maps

## 2.1 Quasi-Möbius maps

Let  $X$  be a metric space, and  $x, y, z, w \in X$  a quadruple of distinct points. Their *cross-ratio* is

$$[x, y, z, w] = \frac{d(x, z)d(y, w)}{d(x, w)d(y, z)}.$$

If  $X$  is unbounded, then the cross-ratio extends to the one-point-compactification of  $X$  [6]. A map  $f : X \rightarrow Y$  is *quasi-Möbius* if there exists a homeomorphism  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that for all quadruples of distinct points  $x, y, z, w \in X$ ,

$$[f(x), f(y), f(z), f(w)] \leq \eta([x, y, z, w]).$$

If  $f$  is a homeomorphism, then  $f$  is quasi-Möbius iff its inverse is quasi-Möbius.

Examples of quasi-Möbius maps are

1. the stereographic projection  $\mathbb{R}^n \rightarrow S^n$ ,
2. the Cayley transforms, the complex, quaternionic and octonionic analoga of the stereographic projection,
3. inversions  $x \mapsto \frac{x}{|x|^2}$  in Banach spaces.

## 2.2 Ultralimits and asymptotic-cones

**Definition 1.** A (non-principle) ultrafilter  $\omega$  over  $\mathbb{N}$  is a set of subsets of  $\mathbb{N}$  satisfying the following conditions:

1. If  $A, B \in \omega$  then  $A \cap B \in \omega$ .
2. If  $A \in \omega, A \subset B \subset \mathbb{N}$ , then  $B \in \omega$ .
3. For every  $A \subset \mathbb{N}$  either  $A \in \omega$  or  $\mathbb{N} \setminus A \in \omega$ .
4. No finite subset of  $\mathbb{N}$  is in  $\omega$ .

Equivalently,  $\omega$  is a finitely additive measure on  $\mathbb{N}$  such that each subset has measure either 0 or 1 and every finite subset has measure 0.

If a statement  $A(n)$  holds for all  $n \in B$  with  $B$  belonging to an ultra filter  $\omega$ , we say that  $A(n)$  holds  $\omega$ -almost surely.

**Definition 2.** Let  $\omega$  be a (non-principle) ultrafilter over  $\mathbb{N}$ . An  $\omega$ -limit of a sequence of points  $\{x_n\}$  in a topological space  $X$ , is a point  $x$  in  $X$  such that for every neighborhood  $U$  of  $x$  the relation  $x_n \in U$  holds  $\omega$ -almost surely.

If  $X$  is a Hausdorff-space, then the  $\omega$ -limit of a sequence is unique. We denote this point by  $\lim_{\omega} x_n$ .

**Definition 3.** The ultrapower of a set  $X$  w.r.t. an ultrafilter  $\omega$ , denoted by  $X^{\omega}$ , consists of equivalence classes of sequences  $\{x_n\}$ ,  $x_n \in X$ , where two sequences  $\{x_n\}$  and  $\{y_n\}$  are identified if and only if  $x_n = y_n$   $\omega$ -almost surely.

We adopt the notation  $\{x_n\}_{\omega}$  for the equivalence class of sequences.

Let  $(X, d)$  be a metric space,  $\omega$  an ultrafilter over  $\mathbb{N}$  and  $o$  an origin in  $X$ . Furthermore, there is given a sequence of numbers  $\{d_n\}$ , with  $\lim_{\omega} d_n = \infty$ . Then there exists a pseudo-metric on the ultrapower  $X^{\omega}$  by setting the distance between  $x = \{x_n\}_{\omega}$  and  $y = \{y_n\}_{\omega}$  to

$$d_{\omega}(x, y) = \lim_{\omega} \frac{d(x_n, y_n)}{d_n}.$$

Let  $X_o^{\omega}$  be the set of equivalence classes of elements  $x = \{x_n\}_{\omega}$  in  $X^{\omega}$  that satisfy  $\lim_{\omega} \frac{d(x_n, o)}{d_n} < \infty$  and two elements  $x = \{x_n\}_{\omega}$  and  $y = \{y_n\}_{\omega}$

of  $X^\omega$  are identified if and only if  $d_\omega(x, y) = 0$ . The metric space  $(X_o^\omega, d_\omega)$  is called an *asymptotic cone* of  $(X, d)$ .

We adopt the notation  $x_\omega$  for points in the asymptotic cone  $(X_o^\omega, d_\omega)$  of  $(X, d)$ .

### 2.3 Asymptotic-Möbius maps

Given  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ , we say that the points  $x$  and  $y$  are separated in  $X$  by  $u$  seen from a fixed origin  $o$ , if  $d(x, y) > u(d(x, o) + d(o, y))$ . For simplicity we often write  $x, y > u$  and  $|x| = d(o, x)$  for the distance to the origin.

**Definition 4.** A function  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  is admissible if

1. it is non-decreasing,
2.  $u$  grows sublinearly i.e.  $\limsup_{r \rightarrow \infty} \frac{u(r)}{r} = 0$ .

**Definition 5.** A map  $f : X \rightarrow Y$  is an asymptotic-Möbius map or AM-map, if there exists an admissible map  $u$  and a homeomorphism  $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that for every  $x, y, z, w$  in  $X$  such that all 6 distances between  $x, y, z, w > u$ , then

$$[f(x), f(y), f(z), f(w)] < \eta([x, y, z, w]).$$

**Definition 6.** A pointed metric space  $(X, o)$  is uniformly  $\mathfrak{o}(id)$ -separated, if there exists a function  $v \in \mathfrak{o}(id)$  such that for all  $u \in \mathfrak{o}(id)$  there exists a constant  $C > 0$  such that for all  $|x| \geq C$ , there exists  $w \in X$  that satisfies

$$u(|x|) < d(x, w) \leq u(|x|) + v(|x|).$$

**Lemma 7.** If  $x_\omega \neq y_\omega$ , then for all  $u \in \mathfrak{o}(id)$  and all sequences  $\{x_n\}, \{y_n\}$  representing  $x_\omega$  and  $y_\omega$  respectively,

$$d(x_n, y_n) > u(|x_n| + |y_n|), \quad \omega - a.e.$$

**Proof:** If there exists  $u \in \mathfrak{o}(id)$  and sequences  $\{x_n\}, \{y_n\}$  such that

$$d(x_n, y_n) \leq u(|x_n| + |y_n|), \quad \text{for } \omega - a.e. n$$

then for  $\omega - a.e. n$ ,

$$\frac{d(x_n, y_n)}{d_n} \leq \frac{u(|x_n| + |y_n|)}{d_n}.$$

Thus

$$\lim_\omega \frac{d(x_n, y_n)}{d_n} = 0. \quad \blacksquare$$

**Lemma 8.** Let  $(X, o)$  be a uniformly  $\mathfrak{o}(id)$ -separated space. Let  $\{x_n\}$  and  $\{y_n\}$  be two sequences in  $X$  representing the same point  $x_\omega$  different from the origin  $o_\omega$ , and such that  $x_n \neq y_n$  for  $\omega - a.e. n$ . Then for every  $u \in \mathfrak{o}(id)$ , either

$$d(x_n, y_n) > u,$$

or there exists a sequence  $\{w_n\}$  representing the same point s.t.

$$d(x_n, w_n) > u$$

and

$$d(y_n, w_n) > u$$

for  $\omega - a.e. n$ .

**Proof:** Suppose  $d(x_n, y_n) \leq u$  for  $\omega - a.e. n$ . By assumption,  $|x_n|, |y_n| \rightarrow +\infty$ . Since  $u$  is a homeomorphism of  $\mathbb{R}^+$ ,  $u$  is increasing.

Since  $(X, o)$  is uniformly  $\mathcal{O}(\text{id})$ -separated, there exists  $v \in \mathcal{O}(\text{id})$  s.t. for all  $n$  large enough, there exists  $w_n$  that satisfies

$$2u(4|y_n|) < d(y_n, w_n) \leq 2u(4|y_n|) + v(|y_n|) \quad (1)$$

For  $n$  large enough,

$$\begin{aligned} 2d(x_n, y_n) &\leq 2u(|x_n| + |y_n|) \\ &\leq |x_n| + |y_n| \\ &\leq 2|y_n| + d(x_n, y_n). \end{aligned} \quad (2)$$

Therefore,

$$d(x_n, y_n) \leq u(4|y_n|).$$

and hence

$$u(4|y_n|) < d(x_n, w_n) \leq 3u(4|y_n|) + v(|y_n|). \quad (3)$$

Pick  $N$  s.t.  $\forall n \geq N$

$$3u(4|y_n|) + v(|y_n|) \leq 2|y_n|.$$

Then

$$\begin{aligned} u(|x_n| + |w_n|) &\leq u(|x_n| + |y_n| + d(y_n, w_n)) \\ &\leq u(2|y_n| + d(x_n, y_n) + 2u(4|y_n|) + v(|y_n|)) \\ &\leq u(2|y_n| + 3u(4|y_n|) + v(|y_n|)) \\ &\leq u(4|y_n|) \\ &< d(x_n, w_n). \end{aligned} \quad (4)$$

That means  $x_n, w_n > u$  and

$$\frac{d(x_n, w_n)}{d_n} \leq \frac{3u(4|y_n|) + v(|y_n|)}{d_n} \rightarrow 0.$$

Similarly,

$$\begin{aligned} u(|y_n| + |w_n|) &\leq u(2|y_n| + d(y_n, w_n)) \\ &\leq u(2|y_n| + 2u(4|y_n|) + v(|y_n|)) \\ &\leq u(4|y_n|) \\ &< d(y_n, w_n). \end{aligned} \quad (5)$$

■

**Theorem 3.** *Let  $f$  be an AM-map from a uniformly  $\mathcal{O}(\text{id})$ -separated space  $X$  to a space  $Y$  that preserves diverging and bounded sequences, then  $f$  induces a continuous, injective, quasi-symmetric map  $g$  between some asymptotic cones of  $X$  and  $Y$ .*

**Proof:** Let  $u \in \mathcal{O}(\text{id})$  as given by the definition of  $f$ . Fix an origin  $o \in X$ . Let  $B_n$  be the ball of radius  $n$  around  $o$ . For every  $n$ , pick a point  $z_n$  in  $B_n$  such that

$$d(f(o), f(z_n)) \quad (6)$$

is 'maximal'. More precisely, let  $\epsilon > 0$  and pick  $z_n$  such that

$$d(f(o), f(z_n)) > \sup_{x \in B_n} d(f(o), f(x)) - \epsilon. \quad (7)$$

The supremum on the right-hand-side exists. If not we could construct a bounded sequence in  $B_n$  that is mapped by  $f$  to an unbounded sequence. This contradicts the assumptions.

Define

$$d_n = d(o, z_n) \quad (8)$$

and

$$d'_n = d(o', z'_n). \quad (9)$$

By construction and assumption on  $f$ , the sequence  $d'_n$  diverges. Hence, also the sequence  $d_n$  must diverge by assumption on  $f$ .

(1) We show that if  $\frac{d(o, x_n)}{d_n}$  is bounded, then  $\frac{d(o', f(x_n))}{d'_n}$  is bounded as well.

If  $x_\omega = o_\omega$ , then  $d(o, x_n) \leq d_n$  for  $n$  large, therefore  $d(o', x'_n) \leq d'_n$ .

If  $x_\omega \neq z_\omega$ ,  $x_\omega \neq o_\omega$ , then by lemma 7  $\{x_n\}, \{z_n\}, \{o\}$  are  $\mathcal{O}(\text{id})$ -separated and the Möbius-condition gives

$$\frac{d(o', x'_n)}{d'_n} = \frac{d(o', x'_n)}{d(o', z'_n)} < \eta \left( \frac{d(o, x_n)}{d(o, z_n)} \right) = \eta \left( \frac{d(o, x_n)}{d_n} \right).$$

Thus  $\frac{d(o', x'_n)}{d'_n}$  is bounded.

If  $x_\omega = z_\omega$ , by lemma 8, there exists  $\{w_n\}$  s.t.  $\{w_n\}, \{x_n\}$  are  $u$  separated and  $\{w_n\}, \{z_n\}$  are  $u$  separated. In particular,  $\{w_n\}, \{x_n\}, \{o\}$  and  $\{w_n\}, \{z_n\}, \{o\}$  are  $u$  separated. Using the Möbius-condition again,

$$\begin{aligned} \frac{d(o', x'_n)}{d'_n} &= \frac{d(o', x'_n)}{d(o', z'_n)} \\ &= \frac{d(o', x'_n)}{d(o', w'_n)} \frac{d(o', w'_n)}{d(o', z'_n)} \\ &< \eta \left( \frac{d(o, x_n)}{d(o, w_n)} \right) \eta \left( \frac{d(o, w_n)}{d(o, z_n)} \right) \\ &= \eta \left( \frac{d(o, x_n)}{d_n} \frac{d_n}{d(o, w_n)} \right) \eta \left( \frac{d(o, w_n)}{d_n} \right) \\ &\rightarrow \eta(1)^2 \end{aligned} \quad (10)$$

Thus  $\frac{d(o', x'_n)}{d'_n}$  is bounded.

(2) The map  $\{x_n\} \rightarrow \{f(x_n)\}$  is well-defined from  $X_\omega(d_n)$  to  $Y_\omega(d'_n)$ .

If  $\{x_n\}$  and  $\{y_n\}$  are two sequences representing the same point different from the origin  $o_\omega$ , then by lemma 8 and lemma 7, either  $\{x_n\}, \{y_n\}, \{o\}$  are separated by  $u$ , or there exists a sequence  $\{w_n\}$  representing the same point s.t.  $\{x_n\}, \{w_n\}, \{o\}$  and  $\{y_n\}, \{w_n\}, \{o\}$  are separated by  $u$ .

In the first case,

$$\begin{aligned} \frac{d(x'_n, y'_n)}{d'_n} &= \frac{d(o', x'_n)}{d'_n} \frac{d(x'_n, y'_n)}{d(x'_n, o')} \\ &\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, y_n)}{d(x_n, o)}\right) \\ &= \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, y_n)}{d_n} \frac{d_n}{d(x_n, o)}\right) \\ &\rightarrow 0 \end{aligned}$$

In the second case,

$$\begin{aligned} \frac{d(x'_n, y'_n)}{d'_n} &\leq \frac{d(x'_n, w'_n)}{d'_n} + \frac{d(w'_n, y'_n)}{d'_n} \\ &\leq \frac{d(o', x'_n)}{d'_n} \frac{d(x'_n, w'_n)}{d(x'_n, o')} + \frac{d(o', y'_n)}{d'_n} \frac{d(y'_n, w'_n)}{d(y'_n, o')} \\ &\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, w_n)}{d(x_n, o)}\right) + \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(y_n, w_n)}{d(y_n, o)}\right) \\ &\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, w_n)}{d_n} \frac{d_n}{d(x_n, o)}\right) + \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(y_n, w_n)}{d_n} \frac{d_n}{d(y_n, o)}\right) \\ &\rightarrow 0 \end{aligned}$$

Hence, in both cases  $x'_\omega = y'_\omega$ .

Let  $\{x_n\}$  be a sequence representing the origin  $o_\omega$ . If  $\{x_n\}$  is a bounded sequence, then by assumption on  $f$  also  $\{x'_n\}$  is bounded and therefore  $x'_\omega = o'_\omega$ .

If  $d(o, x_n)$  diverges, then  $\{x_n\}$  and  $\{o\}$  are  $u$ -separated. We may pick  $y_\omega \neq o_\omega$ , then by lemma 7,  $\{x_n\}, \{y_n\}, \{o\}$  are  $u$ -separated. Thus,

$$\begin{aligned} \frac{d(o', x'_n)}{d'_n} &= \frac{d(o', y'_n)}{d'_n} \frac{d(o', x'_n)}{d(o', y'_n)} \\ &\leq \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(o, x_n)}{d(o, y_n)}\right) \\ &= \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(o, x_n)}{d_n} \frac{d_n}{d(o, y_n)}\right) \\ &\rightarrow 0. \end{aligned} \tag{11}$$

Therefore  $o'_\omega = x'_\omega$ .

(3) The map  $\{x_n\} \rightarrow \{f(x_n)\}$  is quasi-symmetric and in particular continuous and injective.

Let  $\{w_n\}, \{x_n\}, \{y_n\}$  be sequences representing three distinct points  $w_\omega, x_\omega$  and  $y_\omega$  in  $X_\omega(d_n)$ . By lemma 7,  $\{w_n\}, \{x_n\}, \{y_n\}$  are pairwise  $u$  separated.



Thus

$$\frac{d(x'_n, y'_n)}{d(x'_n, w'_n)} < \eta \left( \frac{d(x_n, y_n)}{d(x_n, w_n)} \right),$$

equivalently

$$\frac{d(x'_n, y'_n)}{d'_n} \frac{d'_n}{d(x'_n, w'_n)} < \eta \left( \frac{d(x_n, y_n)}{d_n} \frac{d_n}{d(x_n, w_n)} \right).$$

Taking the  $\omega$ -limit gives

$$\frac{d_\omega(x'_\omega, y'_\omega)}{d_\omega(x'_\omega, w'_\omega)} \leq \eta \left( \frac{d_\omega(x_\omega, y_\omega)}{d_\omega(x_\omega, w_\omega)} \right).$$

Hence, if  $\{x_n\} \mapsto \{f(x_n)\}$  is non-constant, then it is continuous and injective. Notice that  $\{o'\}$  and  $\{f(z_n)\}$  do not represent the same points. ■

## 3 Examples

### 3.1 Sublinearly Lipschitz Equivalences

In [?] Y. Cornulier introduces *sublinearly Lipschitz maps*. A map  $f : X \rightarrow Y$  between metric spaces is a sublinearly Lipschitz map, if there exists an admissible function  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  such that

$$d(f(x), f(y)) \leq Cd(x, y) + C'u(|x| + |y|), \quad \forall x, y \in X,$$

for some constants  $C, C' > 0$ .

Two sublinearly Lipschitz maps  $f, f'$  are equivalent, if there exists an admissible function  $v$  and a constant  $C'' > 0$ , such that

$$d(f(x), f'(x)) \leq C''v(|x|)$$

for all  $x \in X$ .

Sublinearly Lipschitz maps between metric spaces form a category. Taking asymptotic cones yields a functor from the sublinearly Lipschitz category to the Lipschitz category. The sublinearly Lipschitz category is in some sense the maximal category with such a property.

The isomorphisms in the sublinearly Lipschitz category are called sublinearly Lipschitz equivalences, or SBE maps.

**Proposition 9.** *Every SBE-map  $f : X \rightarrow Y$  is an AM-map with linear  $\eta$ .*

**Proof:** If  $f$  is SBE, then  $f$  is bi-Lipschitz except at scales below some admissible function  $u$ . Indeed, let  $x, y \in X$  s.t.

$$d(x, y) \geq u(|x| + |y|).$$

Then

$$\begin{aligned} d(f(x), f(y)) &\leq cd(x, y) + Cu(|x| + |y|) \\ &\leq (c + C)d(x, y), \end{aligned} \tag{12}$$

and

$$\begin{aligned} d(f(x), f(y)) &\geq c'd(x, y) - C'u(|x| + |y|) \\ &\geq (c' - C')d(x, y) \end{aligned} \tag{13}$$

Hence, there exists  $D > 0$ , s.t. for all  $x, y > u$ ,

$$\frac{1}{D}d(x, y) \leq d(f(x), f(y)) \leq Dd(x, y).$$

In particular,  $f$  is an *AM*-map with linear  $\eta$ . ■

### 3.2 Assuad-type maps

Assuad's embedding theorem [A, NN] says that every snowflake  $X^\alpha = (X, d^\alpha)$ ,  $0 < \alpha < 1$ , of a doubling metric space admits a bi-Lipschitz embedding into some Euclidean space.

Clearly, the Assuad embedding of a doubling metric space into Euclidean space is an asymptotic-Möbius map.

In the following we construct an example of a Assuad type map from an infinite dimensional Heisenberg group into a Hilbert space.

Let  $H$  be an infinite dimensional complex Hilbert space.  $H$  carries the symplectic form  $\omega(a, b) = \text{Im}(\langle a, b \rangle)$ .

The *infinite dimensional Heisenberg group*  $\mathcal{H}_\omega$ , is the set of tuples  $(a, t)$  with  $a \in H$ ,  $t \in \mathbb{R}$  and group law

$$(a, t)(a', t') = (a + a', t + t' + \frac{1}{2}\omega(a, a')).$$

The function

$$N(a, t) = \sqrt{\sqrt{\|a\|^4 + t^2} + \|a\|^2},$$

is a group norm on  $\mathcal{H}_\omega$ . Indeed, if  $N_1$  and  $N_2$  are group norms, then  $\sqrt{N_1^2 + N_2^2}$  is a group norm. In this case  $N_1 = (\|a\|^4 + t^2)^{\frac{1}{4}}$  is the Koranyi-norm. Given  $N$ ,

$$d_N((a, t), (a', t')) = N((a, t)(a', t')^{-1})$$

is a right-invariant metric on  $\mathcal{H}_\omega$ .

**Theorem 4.** *Let  $G$  be a group,  $K : G \times G \rightarrow \mathbb{R}$  a real valued kernel on  $G$  satisfying  $K(g, g) = 0 \ \forall g \in G$ . Then  $K$  is negative definite if and only if there exists a Hilbertspace  $H$  and a function  $T : G \rightarrow H$  such that  $\forall g, h \in G : K(g, h) = \|T(g) - T(h)\|^2$ .*

Thus, if  $d_N$  is negative definite, then there exists a Hilbertspace  $H$  and an isometry  $T : (\mathcal{H}_\omega, \sqrt{d_N}) \rightarrow H$ . In particular, the map  $T : (\mathcal{H}_\omega, d_N) \rightarrow H$  is an *AM*-map.

**Proposition 10.**  $d_N : \mathcal{H}_\omega \times \mathcal{H}_\omega \rightarrow H$  is an *AM*-map.

## 4 Dimension theory

### 4.1 Finitely-generated groups

**Definition 11.** Let  $X$  be a metric space. We say that the asymptotic dimension of  $X$  does not exceed  $n$  provided for every uniformly bounded open cover  $\mathcal{V}$  of  $X$  there is a uniformly bounded open cover  $\mathcal{U}$  of  $X$  of multiplicity  $\leq n + 1$  so that  $\mathcal{V}$  refines  $\mathcal{U}$ . We write  $\text{asdim}X = n$  if it is true that  $\text{asdim}X \leq n$  and  $\text{asdim}X \not\leq n - 1$ .

Let  $X$  be a finitely generated nilpotent group, then its asymptotic cone is a Carnot group and the asymptotic dimension of  $X$  agrees with the homogeneous dimension of its asymptotic cone.

**Theorem 5.** Let  $G$  and  $G'$  be a finitely generated nilpotent groups and  $f : G \rightarrow G'$  an AM-map, then  $\text{asdim}(G) \leq \text{asdim}(G')$ . Furthermore, if  $\text{asdim}(G) = \text{asdim}(G')$ , then the cones of  $G$  and  $G'$  are diffeomorphic.

**Proof:** The map  $f$  induces a quasi-symmetric map  $g$  between asymptotic cones  $G_\omega$  and  $G'_\omega$  of  $G$  and  $G'$  respectively.

In particular  $g$  is an injective map from  $G_\omega$  to  $G'_\omega$  and therefore the topological dimensions increase. Since  $G$  and  $G'$  are nilpotent,  $\text{asdim}(G) = \text{top-dim}(G_\omega) \leq \text{top-dim}(G'_\omega) = \text{asdim}(G')$ .

If  $\text{asdim}(G) = \text{asdim}(G')$ , the invariance of domain theorem implies that  $g$  is a homeomorphism. Since both  $g$  and  $g^{-1}$  are quasi-symmetric, they are a.e. differentiable by Pansu's theorem and  $g$  is a diffeomorphism. ■

### 4.2 $CAT(0)$ -spaces

**Definition 12.** A space  $X$  whose asymptotic cone is  $CAT(0)$  has telescopic dimension  $\leq n$ , if every asymptotic cone has geometric dimension  $\leq n$ . (Caprace, Lythak)

We adopt the notation  $\text{teledim}(X)$  for the telescopic dimension of  $X$ .

The geometric dimension is inductively defined. A discrete space has geometric dimension 0 and the dimension of  $X$  is the dimension of the space of directions at a point +1 (Kleiner, 1999).

**Theorem 6.** Let  $X$  and  $Y$  be  $CAT(0)$  spaces. If there exists an AM-map  $f : X \rightarrow Y$ , then the telescopic dimension increases e.g.  $\text{tele-dim}(X) \leq \text{tele-dim}(Y)$ .

As a consequence, there cannot exist an AM-map from Euclidean buildings of finite rank  $r$  and thus telescopic dimension  $r$ , to a Euclidean building of lower rank.

If there exists an AM-map from a  $CAT(0)$  space or finitely generated group  $X$  to a hyperbolic metric space, then  $X$  is a hyperbolic metric space as well.

Analogously,

**Theorem 7.** Let  $G$  and  $G'$  be nilpotent Lie groups, and  $f : G \rightarrow G'$  an AM-map, then  $\text{dim}(G) \leq \text{dim}(G')$ .

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