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# Asymptotic-Möbius maps.

Georg Grützner

## Abstract

Roughly speaking, a map between metric spaces is asymptotically Möbius if it induces quasi-Möbius maps on asymptotic cones. We show that under such maps, some large-scale notions of dimension increase: asymptotic dimension for finitely generated nilpotent groups, telescopic dimension for  $CAT(0)$  spaces.

## 1 Introduction

### 1.1 Microscopic conformality

Examples of conformal maps appeared very early in history. Already in 1569 discovered Mercator a map to project the sphere onto an infinite cylinder, today known as the Mercator planisphere. Distances under conformal maps can change dramatically. The metric distortion of the Mercator map clearly blows up near the poles. Nevertheless, the metric distortion can often be estimated. This led to metric spaces analogous of conformal maps such as quasi-symmetric maps or quasi-Möbius maps [2], [3], [4].

### 1.2 Mesoscopic conformality

Much later it was discovered that conformality may manifest itself as well in discrete settings. The beginning marks Paul Koebe's circle packing theorem in 1931. A circle packing of the 2-sphere is a collection of interior-disjoint disks. The incidence graph of a circle-packing has a vertex for every circle and an edge between two vertices whenever the corresponding circles touch. Koebe discovered, that every triangulation of the 2-sphere is the incidence graph of a circle-packing which is unique up to Möbius-transformations of the sphere.

### 1.3 A new class of maps

In this paper, we go one step further and define a class of asymptotic-Möbius maps. Roughly speaking, asymptotic means that our definitions are unaffected by local changes in metric or topology. Technically, it means that pre- or post-composition of asymptotic-Möbius maps with quasi-isometries are again asymptotic-Möbius.

## 1.4 Examples

We postpone the precise definition until later. Sources of examples of  $AM$ -maps are:

1. Quasi-isometric embeddings are  $AM$ -maps.
2. SBE-maps are  $AM$ -maps.
3. Snowflaking (i.e. replacing a metric by a power of it) gives rise to an  $AM$ -map.
4. Assuad maps from doubling metric spaces of  $\mathbb{R}^N$ .

Every nilpotent Lie group or finitely generated group can be  $AM$ -embedded in Euclidean space of sufficiently high dimension [1].

## 1.5 Results

Our main results is that under  $AM$ -maps some large-scale notion of dimension increases. The relevant notion depends on the class of groups considered.

**Theorem 1.** *Let  $G$  and  $G'$  be finitely generated nilpotent groups and  $f : G \rightarrow G'$  an  $AM$ -map, then  $asdim(G) \leq asdim(G')$ . Furthermore, if  $asdim(G) = asdim(G')$ , then the cones of  $G$  and  $G'$  are diffeomorphic.*

In the world of  $CAT(0)$ -spaces, the analogous theorem takes the following form.

**Theorem 2.** *Let  $X$  and  $Y$  be  $CAT(0)$  spaces. If there exists an  $AM$ -map  $f : X \rightarrow Y$ , then the telescopic dimension increases e.g.  $tele-dim(X) \leq tele-dim(Y)$ .*

## Asymptotic-Möbius maps

Let  $o$  be a fixed origin in  $X$ . Let  $u$  be a homeomorphism of  $\mathbb{R}^+$  such that  $u(x) = o(x)$ . We say that the distance between  $x, y > u$ , if  $d(x, y) > u(d(x, o) + d(o, y))$ . For simplicity we often write  $|x| = d(o, x)$ .

**Definition 1.** *A map  $f : X \rightarrow Y$  is an asymptotic-Möbius map or  $AM$ -map, if there exists a map  $u \in o(id)$  and a homeomorphism  $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that for every  $x, y, z, w$  in  $X$  such that all 6 distances between  $x, y, z, w > u$ , then*

$$[f(x), f(y), f(z), f(w)] < \eta([x, y, z, w]).$$

**Proposition 2.** *Every SBE-map  $f : X \rightarrow Y$  is an  $AM$ -map with linear  $\eta$ .*

**Proof:** If  $f$  is SBE, then  $f$  is bi-Lipschitz except at scales below some function  $u \in o(id)$ . Indeed, let  $x, y \in X$  s.t.

$$d(x, y) \geq u(|x| + |y|).$$

Then

$$\begin{aligned} d(f(x), f(y)) &\leq cd(x, y) + Cu(|x| + |y|) \\ &\leq (c + C)d(x, y), \end{aligned} \quad (1)$$

and

$$\begin{aligned} d(f(x), f(y)) &\geq c'd(x, y) - C'u(|x| + |y|) \\ &\geq (c' - C')d(x, y) \end{aligned} \quad (2)$$

Hence, there exists  $D > 0$ , s.t. for all  $x, y > u$ ,

$$\frac{1}{D}d(x, y) \leq d(f(x), f(y)) \leq Dd(x, y).$$

In particular,  $f$  is an  $AM$ -map with linear  $\eta$ . ■

**Lemma 3.** *If  $x_\omega \neq y_\omega$ , then for all  $u \in \mathcal{O}(\text{id})$  and all sequences  $\{x_n\}, \{y_n\}$  representing  $x_\omega$  and  $y_\omega$  respectively,*

$$d(x_n, y_n) > u(|x_n| + |y_n|), \quad \omega - a.e.$$

**Proof:** If there exists  $u \in \mathcal{O}(\text{id})$  and sequences  $\{x_n\}, \{y_n\}$  such that

$$d(x_n, y_n) \leq u(|x_n| + |y_n|), \quad \text{for } \omega - a.e. n$$

then for  $\omega - a.e. n$ ,

$$\frac{d(x_n, y_n)}{d_n} \leq \frac{u(|x_n| + |y_n|)}{d_n}.$$

Thus

$$\lim_{\omega} \frac{d(x_n, y_n)}{d_n} = 0$$

.

**Lemma 4.** *Let  $X$  be a length-space. Let  $\{x_n\}$  and  $\{y_n\}$  be two sequences in  $X$  representing the same point  $x_\omega$  different from the origin  $o_\omega$ , and such that  $x_n \neq y_n$  for  $\omega - a.e. n$ . Then for every  $u \in \mathcal{O}(\text{id})$ , either*

$$d(x_n, y_n) > u,$$

*or there exists a sequence  $\{w_n\}$  representing the same point s.t.*

$$d(x_n, w_n) > u$$

*and*

$$d(y_n, w_n) > u$$

*for  $\omega - a.e. n$ .*

**Proof:** Let  $v \in \mathcal{O}(\text{id})$  arbitrary. Suppose  $d(x_n, y_n) \leq u$  for  $\omega - a.e. n$ . By assumption,  $|x_n|, |y_n| \rightarrow +\infty$ . Since  $u$  is a homeomorphism of  $\mathbb{R}^+$ ,  $u$  is increasing. Pick  $N$  s.t.  $\forall n \geq N$

$$3u(2(|x_n| + |y_n|)) + v(|y_n|) \leq |x_n| + |y_n|.$$

Since  $X$  is a length-space and has approximate mid-points, we may pick  $w_n$  s.t.

$$2u(2(|x_n| + |y_n|)) < d(y_n, w_n) \leq 2u(2(|x_n| + |y_n|)) + v(|y_n|) \quad (3)$$

By the triangle inequality

$$u(2(|x_n| + |y_n|)) < d(x_n, w_n) \leq 3u(2(|x_n| + |y_n|)) + v(|y_n|). \quad (4)$$

Then

$$\begin{aligned} u(|x_n| + |w_n|) &\leq u(|x_n| + |y_n| + d(y_n, w_n)) \\ &\leq u(|x_n| + |y_n| + 2u(2(|x_n| + |y_n|)) + v(|y_n|)) \\ &\leq u(2(|x_n| + |y_n|)) \\ &< d(x_n, w_n). \end{aligned} \quad (5)$$

That means  $x_n, w_n > u$  and

$$\frac{d(x_n, w_n)}{d_n} \leq \frac{3u(2(|x_n| + |y_n|)) + v(|y_n|)}{d_n} \rightarrow 0.$$

Similarly,

$$\begin{aligned} u(|y_n| + |w_n|) &\leq u(|x_n| + |y_n| + d(x_n, w_n)) \\ &\leq u(|x_n| + |y_n| + 3u(2(|x_n| + |y_n|)) + v(|y_n|)) \\ &\leq u(2(|x_n| + |y_n|)) \\ &< d(y_n, w_n). \end{aligned} \quad (6)$$

■

**Theorem 3.** *Let  $f$  be an AM-map from a length-space  $X$  to a space  $Y$  that preserves diverging and bounded sequences, then  $f$  induces a continuous, injective, quasi-symmetric map  $g$  between some asymptotic cones of  $X$  and  $Y$ .*

**Proof:** Let  $u \in \mathcal{O}(\text{id})$  as given by the definition of  $f$ . Fix an origin  $o \in X$ . Let  $B_n$  be the ball of radius  $n$  around  $o$ . For every  $n$ , pick a point  $z_n$  in  $B_n$  such that

$$d(f(o), f(z_n)) \quad (7)$$

is 'maximal'. More precisely, let  $\epsilon > 0$  and pick  $z_n$  such that

$$d(f(o), f(z_n)) > \sup_{x \in B_n} d(f(o), f(x)) - \epsilon. \quad (8)$$

The supremum on the right-hand-side exists. If not we could construct a bounded sequence in  $B_n$  that is mapped by  $f$  to an unbounded sequence. This contradicts the assumptions.

Define

$$d_n = d(o, z_n) \quad (9)$$

and

$$d'_n = d(o', z'_n). \quad (10)$$

By construction and assumption on  $f$ , the sequence  $d'_n$  diverges. Hence, also the sequence  $d_n$  must diverge by assumption on  $f$ .

(1) We show that if  $\frac{d(o, x_n)}{d_n}$  is bounded, then  $\frac{d(o', f(x_n))}{d'_n}$  is bounded as well.

If  $x_\omega = o_\omega$ , then  $d(o, x_n) \leq d_n$  for  $n$  large, therefore  $d(o', x'_n) \leq d'_n$ .

If  $x_\omega \neq z_\omega$ ,  $x_\omega \neq o_\omega$ , then by lemma 3  $\{x_n\}, \{z_n\}, \{o\}$  are  $\mathcal{O}(\text{id})$ -separated and the Möbius-condition gives

$$\frac{d(o', x'_n)}{d'_n} = \frac{d(o', x'_n)}{d(o', z'_n)} < \eta\left(\frac{d(o, x_n)}{d(o, z_n)}\right) = \eta\left(\frac{d(o, x_n)}{d_n}\right).$$

Thus  $\frac{d(o', x'_n)}{d'_n}$  is bounded.

If  $x_\omega = z_\omega$ , by lemma 4, there exists  $\{w_n\}$  s.t.  $\{w_n\}, \{x_n\}$  are  $u$  separated and  $\{w_n\}, \{z_n\}$  are  $u$  separated. In particular,  $\{w_n\}, \{x_n\}, \{o\}$  and  $\{w_n\}, \{z_n\}, \{o\}$  are  $u$  separated. Using the Möbius-condition again,

$$\begin{aligned} \frac{d(o', x'_n)}{d'_n} &= \frac{d(o', x'_n)}{d(o', z'_n)} \\ &= \frac{d(o', x'_n)}{d(o', w'_n)} \frac{d(o', w'_n)}{d(o', z'_n)} \\ &< \eta\left(\frac{d(o, x_n)}{d(o, w_n)}\right) \eta\left(\frac{d(o, w_n)}{d(o, z_n)}\right) \\ &= \eta\left(\frac{d(o, x_n)}{d_n} \frac{d_n}{d(o, w_n)}\right) \eta\left(\frac{d(o, w_n)}{d_n}\right) \\ &\rightarrow \eta(1)^2 \end{aligned} \tag{11}$$

Thus  $\frac{d(o', x'_n)}{d'_n}$  is bounded.

(2) The map  $\{x_n\} \rightarrow \{f(x_n)\}$  is well-defined from  $X_\omega(d_n)$  to  $Y_\omega(d'_n)$ .

If  $\{x_n\}$  and  $\{y_n\}$  are two sequences representing the same point different from the origin  $o_\omega$ , then by lemma 4 and lemma 3, either  $\{x_n\}, \{y_n\}, \{o\}$  are separated by  $u$ , or there exists a sequence  $\{w_n\}$  representing the same point s.t.  $\{x_n\}, \{w_n\}, \{o\}$  and  $\{y_n\}, \{w_n\}, \{o\}$  are separated by  $u$ .

In the first case,

$$\begin{aligned} \frac{d(x'_n, y'_n)}{d'_n} &= \frac{d(o', x'_n)}{d'_n} \frac{d(x'_n, y'_n)}{d(x'_n, o')} \\ &\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, y_n)}{d(x_n, o)}\right) \\ &= \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, y_n)}{d_n} \frac{d_n}{d(x_n, o)}\right) \\ &\rightarrow 0 \end{aligned}$$

In the second case,

$$\begin{aligned}
\frac{d(x'_n, y'_n)}{d'_n} &\leq \frac{d(x'_n, w'_n)}{d'_n} + \frac{d(w'_n, y'_n)}{d'_n} \\
&\leq \frac{d(o', x'_n)}{d'_n} \frac{d(x'_n, w'_n)}{d(x'_n, o')} + \frac{d(o', y'_n)}{d'_n} \frac{d(y'_n, w'_n)}{d(y'_n, o')} \\
&\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, w_n)}{d(x_n, o)}\right) + \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(y_n, w_n)}{d(y_n, o)}\right) \\
&\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, w_n)}{d_n} \frac{d_n}{d(x_n, o)}\right) + \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(y_n, w_n)}{d_n} \frac{d_n}{d(y_n, o)}\right) \\
&\rightarrow 0
\end{aligned}$$

Hence, in both cases  $x'_\omega = y'_\omega$ .

Let  $\{x_n\}$  be a sequence representing the origin  $o_\omega$ . If  $\{x_n\}$  is a bounded sequence, then by assumption on  $f$  also  $\{x'_n\}$  is bounded and therefore  $x'_\omega = o'_\omega$ .

If  $d(o, x_n)$  diverges, then  $\{x_n\}$  and  $\{o\}$  are  $u$ -separated. We may pick  $y_\omega \neq o_\omega$ , then by lemma 3,  $\{x_n\}, \{y_n\}, \{o\}$  are  $u$ -separated. Thus,

$$\begin{aligned}
\frac{d(o', x'_n)}{d'_n} &= \frac{d(o', y'_n)}{d'_n} \frac{d(o', x'_n)}{d(o', y'_n)} \\
&\leq \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(o, x_n)}{d(o, y_n)}\right) \\
&= \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(o, x_n)}{d_n} \frac{d_n}{d(o, y_n)}\right) \\
&\rightarrow 0.
\end{aligned} \tag{12}$$

Therefore  $o'_\omega = x'_\omega$ .

(3) The map  $\{x_n\} \rightarrow \{f(x_n)\}$  is quasi-symmetric and in particular continuous and injective.

Let  $\{w_n\}, \{x_n\}, \{y_n\}$  be sequences representing three distinct points  $w_\omega, x_\omega$  and  $y_\omega$  in  $X_\omega(d_n)$ . By lemma 3,  $\{w_n\}, \{x_n\}, \{y_n\}$  are pairwise  $u$  separated. Thus

$$\frac{d(x'_n, y'_n)}{d(x'_n, w'_n)} < \eta\left(\frac{d(x_n, y_n)}{d(x_n, w_n)}\right),$$

equivalently

$$\frac{d(x'_n, y'_n)}{d'_n} \frac{d'_n}{d(x'_n, w'_n)} < \eta\left(\frac{d(x_n, y_n)}{d_n} \frac{d_n}{d(x_n, w_n)}\right).$$

Taking the  $\omega$ -limit gives

$$\frac{d_\omega(x'_\omega, y'_\omega)}{d_\omega(x'_\omega, w'_\omega)} \leq \eta\left(\frac{d_\omega(x_\omega, y_\omega)}{d_\omega(x_\omega, w_\omega)}\right).$$

Hence, if  $\{x_n\} \mapsto \{f(x_n)\}$  is non-constant, then it is continuous and injective. Notice that  $\{o'\}$  and  $\{f(z_n)\}$  do not represent the same points. ■

## 2 Dimension theory

**Definition 5.** Let  $X$  be a metric space. We say that the asymptotic dimension of  $X$  does not exceed  $n$  provided for every uniformly bounded open cover  $\mathcal{V}$  of  $X$  there is a uniformly bounded open cover  $\mathcal{U}$  of  $X$  of multiplicity  $\leq n + 1$  so that  $\mathcal{V}$  refines  $\mathcal{U}$ . We write  $\text{asdim}X = n$  if it is true that  $\text{asdim}X \leq n$  and  $\text{asdim}X \not\leq n - 1$ .

Let  $X$  be a finitely generated nilpotent group, then its asymptotic cone is a Carnot group and the asymptotic dimension of  $X$  agrees with the homogeneous dimension of its asymptotic cone.

**Theorem 4.** Let  $G$  and  $G'$  be a finitely generated nilpotent groups and  $f : G \rightarrow G'$  an AM-map, then  $\text{asdim}(G) \leq \text{asdim}(G')$ . Furthermore, if  $\text{asdim}(G) = \text{asdim}(G')$ , then the cones of  $G$  and  $G'$  are diffeomorphic.

**Proof:** The map  $f$  induces a quasi-symmetric map  $g$  between asymptotic cones  $G_\omega$  and  $G'_\omega$  of  $G$  and  $G'$  respectively.

In particular  $g$  is an injective map from  $G_\omega$  to  $G'_\omega$  and therefore the topological dimensions increase. Since  $G$  and  $G'$  are nilpotent,  $\text{asdim}(G) = \text{top-dim}(G_\omega) \leq \text{top-dim}(G'_\omega) = \text{asdim}(G')$ .

If  $\text{asdim}(G) = \text{asdim}(G')$ , the invariance of domain theorem implies that  $g$  is a homeomorphism. Since both  $g$  and  $g^{-1}$  are quasi-symmetric, they are a.e. differentiable by Pansu's theorem and  $g$  is a diffeomorphism. ■

**Theorem 5.** Let  $X$  and  $Y$  be  $CAT(0)$  spaces. If there exists an AM-map  $f : X \rightarrow Y$ , then the telescopic dimension increases e.g.  $\text{tele-dim}(X) \leq \text{tele-dim}(Y)$ .

As a consequence, there cannot exist an AM-map from Euclidean buildings of finite rank  $r$  and thus telescopic dimension  $r$ , to a Euclidean building of lower rank.

If there exists an AM-map from a  $CAT(0)$  space or finitely generated group  $X$  to a hyperbolic metric space, then  $X$  is a hyperbolic metric space as well.

Analogously,

**Theorem 6.** Let  $G$  and  $G'$  be nilpotent Lie groups, and  $f : G \rightarrow G'$  an AM-map, then  $\dim(G) \leq \dim(G')$ .



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