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Asymptotic-Möbius maps.

Georg Grützner

Abstract

Roughly speaking, a map between metric spaces is an asymptotic-Möbius map if it almost preserves cross-ratios of points being a large distance apart from each other. We show that under such maps, some large-scale notions of dimension increase: asymptotic dimension for finitely generated nilpotent groups, telescopic dimension for $CAT(0)$ spaces.

1 Introduction

In 1999, Oded Schramm investigated the scaling limits of loop-erased random walks (LERW) [18]. Schramm observed that the conformal invariance of the limit process implies its encoding in corresponding conformal mappings f_t satisfying Loewner's differential equation

$$\frac{\partial f_t}{\partial t} = z \frac{\zeta(t) + z}{\zeta(t) - z} \frac{\partial f_t}{\partial z}.$$

This observation revolutionized two-dimensional statistical physics (see, e.g., a review of SLE by G. F. Lawler [11]).

There are generalizations for dimensions higher than 2[20]. However, the strength of conformal techniques in higher dimensions is severely limited. This is particularly clear in Liouville's 1850 theorem.¹ While all simply connected domains in \mathbb{R}^2 are conformally equivalent by the Riemann mapping theorem, any smooth conformal mapping on a domain of \mathbb{R}^n , with $n > 2$, is a composition of translations, similarities, orthogonal transformations and inversions.

Suitable classes of functions on metric spaces that capture conformal behavior are the classes of quasisymmetric (QS) maps, quasi-Möbius (QM) maps, and quasiconformal (QC) maps. Quasisymmetric maps transform balls into "quasiballs." The definition is based on the fact that conformal maps transform infinitesimal balls into infinitesimal ellipsoids. Quasi-Möbius maps preserve almost the cross ratio of points, and quasiconformal maps are an infinitesimal version of quasisymmetric maps. We always have $QS \Rightarrow QM \Rightarrow QC$ [19].

We are therefore interested in classes of functions that converge under scaling limits to the classes QS, QM or QC. In the eyes of a geometric

¹Monge 1850, pp. 609 - 616, a note contributed by Liouville as editor (Note VI: Extension au cas des trois dimensions de la question du tracé géographique.)

group theorist, we are looking for classes of maps between finitely generated groups Γ , with the word metric, which remain invariant under small changes in topology and which become conformal in the above sense when viewed from a distance. Specifically, we require that the class of maps is invariant under composition by quasi-isometries.

This chapter presents and examines a class of maps that mimics quasi-Möbius maps in the context of geometric group theory.

In [16] Pierre Pansu introduces a notion of large-scale conformal maps that mimics the infinitesimal behavior of conformal maps. In short, large-scale conformal maps map families of disjoint balls onto families of weakly overlapping quasiballs. It is a very flexible notion that includes, for example, coarse embeddings. However, this flexibility makes the asymptotic behavior of such maps less predictable.

Instead, we introduce a more restrictive and semantically simpler notion that mimics quasi-Möbius behavior. Roughly speaking, a map between metric spaces is an asymptotic-Möbius map if it nearly preserves the cross ratio of points that are a large distance apart. As a result, the results presented in this paper are much stronger than those in [16].

We postpone the precise definition of asymptotic-Möbius (AM) maps until later. Sources of examples of AM -maps are:

1. Quasi-isometric embeddings,
2. Sublinear-bi-Lipschitz equivalences (i.e., maps inducing Lipschitz equivalences on asymptotic cones [5]),
3. Snowflaking (i.e., replacing a metric by a power of it),
4. Assouad maps from doubling metric spaces to \mathbb{R}^N .

For example, any nilpotent Lie group or finitely generated group can be AM -embedded in a Euclidean space of sufficiently high dimension [1]. Moreover, we will encounter infinite dimensional examples.

1.1 Results

Our main result is that under AM -mappings a large-scale term of dimension increases. The relevant term depends on the class of the groups considered.

Theorem 3. *Let G and G' be finitely generated nilpotent groups and $f : G \rightarrow G'$ an AM -map, then $\text{asdim}(G) \leq \text{asdim}(G')$. Furthermore, if $\text{asdim}(G) = \text{asdim}(G')$, then the asymptotic cones of G and G' are isomorphic. Conversely, given nilpotent groups with isomorphic asymptotic cones, there exists an asymptotic-Möbius map between them.*

In the world of $CAT(0)$ -spaces, the analogous theorem takes the following form.

Theorem 4. *Let X and Y be $CAT(0)$ spaces. If there exists an AM -map $f : X \rightarrow Y$, then the telescopic dimension increases e.g. $\text{tele-dim}(X) \leq \text{tele-dim}(Y)$.*

2 Asymptotic-Möbius maps

2.1 Quasi-Möbius maps

Quasi-Möbius maps were introduced by Jussi Väisälä, among others, as a means of studying quasisymmetric maps and quasiconformal maps. Unlike quasisymmetric maps, quasi-Möbius maps do not have a point fixed at infinity.

Let X be a metric space, and $x, y, z, w \in X$ a quadruple of distinct points. Their *cross-ratio* is

$$[x, y, z, w] = \frac{d(x, z)d(y, w)}{d(x, w)d(y, z)}.$$

If X is unbounded, then the cross ratio extends to the one-point compactification of X [19]. A map $f : X \rightarrow Y$ is *quasi-Möbius* if there is a homeomorphism $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for all quadruples of distinct points $x, y, z, w \in X$,

$$[f(x), f(y), f(z), f(w)] \leq \eta([x, y, z, w]).$$

If f is a homeomorphism, then f is quasi-Möbius iff its inverse is quasi-Möbius.

Examples of quasi-Möbius maps are

1. the stereographic projection $\mathbb{R}^n \rightarrow S^n$,
2. the Cayley transformations (the complex, quaternionic and octonionic analogues of the stereographic projection),
3. the inversions $x \mapsto \frac{x}{|x|^2}$ in Banach spaces.

2.2 Asymptotic-Möbius maps

We need a criterion to tell when two points in a space are far apart. One way to do this is to separate them by sublinear growing functions. This idea was used by Y. Cornulier to define sublinear-bi-Lipschitz equivalences [5].

Given $u : \mathbb{R}_+ \rightarrow \mathbb{R}$, we say that the points x and y in X are separated by u from a fixed origin o if $d(x, y) > u(d(x, o) + d(o, y))$. For simplicity, we often write $x, y > u$ and $|x| = d(o, x)$ for the distance from the origin. We can call u a *gauge*, i.e., a function that sets a scale of "sight" as a function of "location". To fix a gauge means to decide what is "near" and "far away".

Definition 1. A function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is an *admissible gauge* if

1. it is non-decreasing, and
2. u grows sublinearly, i.e. $\limsup_{r \rightarrow \infty} \frac{u(r)}{r} = 0$.

Morally, asymptotic-Möbius maps at widely spaced points are quasi-Möbius.

Definition 2. A map $f : X \rightarrow Y$ is an asymptotic-Möbius map or AM-map if there exists an admissible gauge u and a homeomorphism $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for every x, y, z, w in X such that all 6 distances between $x, y, z, w > u$, then

$$[f(x), f(y), f(z), f(w)] < \eta([x, y, z, w]).$$

At some point we will need a coarse version of path connectivity.

Definition 3. A metric space X is asymptotically chained, if for every origin $o \in X$ there exists an admissible gauge v such that for all $x, y \in X$ there exists a chain $x_1 = x, \dots, x_{k+1} = y$ satisfying

$$\max_{i \in 1 \dots k} \{d(x_i, x_{i+1})\} < v(|x| + |y|).$$

Our most important technical step will be the following theorem, the proof of which will occupy section 2.4 after some introductory definitions.

Theorem 1. Let f be an AM-map from an asymptotically chained space X to a space Y that preserves diverging and bounded sequences, then f induces a continuous, injective, quasisymmetric map g between some asymptotic cones of X and Y .

2.3 Asymptotic-cones

The *asymptotic cone* of a metric space (X, d) , captures the geometry on the large scale of X . Roughly speaking, it formalizes the idea of snapshots of the space X taken by an observer moving farther and farther away from X . This sequence of snapshots can stabilize and the observer has the impression of seeing a single object. We call this object the asymptotic cone of X .

If R -balls $B(o, R)$ with metric $R^{-1}d$ are uniformly pre-compact, then the asymptotic cones can be constructed concretely by pointed Gromov-Hausdorff convergence [6], [8] In general, however, we have to resort to constructions via ultrafilters.

Definition 4. A (nonprincipal) ultrafilter ω over \mathbb{N} is a set of subsets of \mathbb{N} satisfying the following conditions:

1. If $A, B \in \omega$ then $A \cap B \in \omega$.
2. If $A \in \omega, A \subset B \subset \mathbb{N}$, then $B \in \omega$.
3. For every $A \subset \mathbb{N}$, either $A \in \omega$ or $\mathbb{N} \setminus A \in \omega$.
4. No finite subset of \mathbb{N} is in ω .

Equivalently, ω is a finitely additive measure on \mathbb{N} such that every subset has measure either 0 or 1 and every finite subset has measure 0.

If a statement $A(n)$ holds for all $n \in B$, where B belongs to an ultrafilter ω , then $A(n)$ is said to hold ω -almost surely.

Definition 5. Let ω be a (nonprincipal) ultrafilter over \mathbb{N} . An ω -limit of a sequence of points $\{x_n\}$ in a topological space X is a point x in X such that for any neighborhood U of x the relation $x_n \in U$ holds ω -almost surely.

If X is a Hausdorff space, then the ω -limit of a sequence is unique. We denote this point by $\lim_{\omega} x_n$.

Definition 6. *The ultrapower of a set X with respect to an ultrafilter ω , denoted by X^{ω} , consists of equivalence classes of sequences $\{x_n\}$, $x_n \in X$, where two sequences $\{x_n\}$ and $\{y_n\}$ are identical if and only if $x_n = y_n$ ω -almost surely.*

We adopt the notation $\{x_n\}_{\omega}$ for the equivalence class of sequences.

Let (X, d) be a metric space, ω an ultrafilter over \mathbb{N} and o an origin in X . Furthermore, let a sequence of numbers $\{d_n\}$ be given, with $\lim_{\omega} d_n = \infty$. Then there is a pseudo-metric on the ultrapower X^{ω} by setting the distance between $x = \{x_n\}_{\omega}$ and $y = \{y_n\}_{\omega}$ to

$$d_{\omega}(x, y) = \lim_{\omega} \frac{d(x_n, y_n)}{d_n}.$$

Let X_o^{ω} be the set of equivalence classes of elements $x = \{x_n\}_{\omega}$ in X^{ω} satisfying $\lim_{\omega} \frac{d(x_n, o)}{d_n} < \infty$ and two elements $x = \{x_n\}_{\omega}$ and $y = \{y_n\}_{\omega}$ of X^{ω} are identical if and only if $d_{\omega}(x, y) = 0$. The metric space $(X_o^{\omega}, d_{\omega})$ is called an *asymptotic cone* of (X, d) .

For points in the asymptotic cone $(X_o^{\omega}, d_{\omega})$ of (X, d) we use the notation x_{ω} .

Lemma 7. *If $x_{\omega} \neq y_{\omega}$, then for every admissible gauge u and all sequences $\{x_n\}$, $\{y_n\}$ representing x_{ω} and y_{ω} , respectively,*

$$d(x_n, y_n) > u(|x_n| + |y_n|), \quad \omega - a.s.$$

Proof: If there is an admissible function u and sequences $\{x_n\}$, $\{y_n\}$ such that

$$d(x_n, y_n) \leq u(|x_n| + |y_n|), \quad \omega - a.s.$$

then $\omega - a.s.$,

$$\frac{d(x_n, y_n)}{d_n} \leq \frac{u(|x_n| + |y_n|)}{d_n}.$$

So

$$\lim_{\omega} \frac{d(x_n, y_n)}{d_n} = 0.$$

■

Lemma 8. *Let X be an asymptotically chained metric space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X representing the same point x_{ω} different from the origin o_{ω} such that $x_n \neq y_n$ $\omega - a.s.$ Then, for any admissible gauge u , either*

$$d(x_n, y_n) > u, \quad \omega - a.s.$$

or there exists a sequence $\{w_n\}$ representing the same point s.t

$$d(x_n, w_n) > u \quad \omega - a.s.$$

and

$$d(y_n, w_n) > u \quad \omega - a.s.$$

Proof: Suppose $d(x_n, y_n) \leq u \omega - a.s.$ and assuming $|x_n|, |y_n| \rightarrow +\infty$. Since X is asymptotically chained, there is an admissible gauge v and a chain $x_1^n = o, \dots, x_{k+1}^n = y_n$ such that

$$\max_{i \in 1, \dots, k} \{d(x_i^n, x_{i+1}^n)\} < v(|y_n|).$$

Assuming n is large enough so that $2u(4|y_n|) < |y_n|$, then at least one point $x_{i_n}^n$ in the chain satisfies

$$2u(4|y_n|) < d(y_n, x_{i_n}^n) \leq 2u(4|y_n|) + v(|y_n|).$$

Define $w_n = x_{i_n}^n$.

For n large enough,

$$\begin{aligned} 2d(x_n, y_n) &\leq 2u(|x_n| + |y_n|) \\ &\leq |x_n| + |y_n| \\ &\leq 2|y_n| + d(x_n, y_n). \end{aligned}$$

It follows,

$$d(x_n, y_n) \leq u(4|y_n|).$$

and therefore

$$u(4|y_n|) < d(x_n, w_n) \leq 3u(4|y_n|) + v(|y_n|).$$

Choose N s.t. $\forall n \geq N$

$$3u(4|y_n|) + v(|y_n|) \leq 2|y_n|.$$

Then

$$\begin{aligned} u(|x_n| + |w_n|) &\leq u(|x_n| + |y_n| + d(y_n, w_n)) \\ &\leq u(2|y_n| + d(x_n, y_n) + 2u(4|y_n|) + v(|y_n|)) \\ &\leq u(2|y_n| + 3u(4|y_n|) + v(|y_n|)) \\ &\leq u(4|y_n|) \\ &< d(x_n, w_n). \end{aligned}$$

This means $x_n, w_n > u$ and

$$\frac{d(x_n, w_n)}{d_n} \leq \frac{3u(4|y_n|) + v(|y_n|)}{d_n} \rightarrow 0.$$

Related,

$$\begin{aligned} u(|y_n| + |w_n|) &\leq u(2|y_n| + d(y_n, w_n)) \\ &\leq u(2|y_n| + 2u(4|y_n|) + v(|y_n|)) \\ &\leq u(4|y_n|) \\ &< d(y_n, w_n). \end{aligned}$$

■

2.4 Proof of Theorem 1.

Proof: Let u be an admissible gauge as given by the definition of f . Fix an origin $o \in X$. Let B_n be the ball of radius n around o . For each n , choose a point z_n in B_n such that

$$d(f(o), f(z_n))$$

is 'maximal'. More precisely, let $\epsilon > 0$ and choose z_n such that

$$d(f(o), f(z_n)) > \sup_{x \in B_n} d(f(o), f(x)) - \epsilon.$$

The supremum on the right hand side exists. Otherwise, one could construct a bounded sequence in B_n mapped by f to an unbounded sequence. This contradicts the assumptions.

Moreover, the four-point condition

$$[f(x), f(w), f(y), f(z)] \leq \eta([x, w, y, z]),$$

reduces to the three-point condition

$$\frac{d(f(x), f(y))}{d(f(x), f(z))} \leq \eta\left(\frac{d(x, y)}{d(x, z)}\right),$$

by bringing w to a point at ∞ .

In what follows, we often abbreviate the notation $f(x)$ to x' .

Define

$$d_n = d(o, z_n)$$

and

$$d'_n = d(o', z'_n).$$

By construction and assumption over f , the sequence d'_n is divergent. Consequently, the sequence d_n must also be divergent by the assumption over f .

(1) We show that if $\frac{d(o, x_n)}{d_n}$ is bounded, then $\frac{d(o', f(x_n))}{d'_n}$ is also bounded.

If $x_\omega = o_\omega$, then $d(o, x_n) \leq d_n$ for n large, therefore $d(o', x'_n) \leq d'_n$.

If $x_\omega \neq z_\omega$, $x_\omega \neq o_\omega$, then by lemma 7 $\{x_n\}, \{z_n\}, \{o\}$ are separated by u ω -a.s., and the Möbius- condition yields

$$\frac{d(o', x'_n)}{d'_n} = \frac{d(o', x'_n)}{d(o', z'_n)} < \eta\left(\frac{d(o, x_n)}{d(o, z_n)}\right) = \eta\left(\frac{d(o, x_n)}{d_n}\right) \quad \omega - a.s.$$

Thus $\frac{d(o', x'_n)}{d'_n}$ is bounded.

If $x_\omega = z_\omega$, then by lemma 8 there are $\{w_n\}$ representing the same point, s.t. $\{w_n\}, \{x_n\}$ are u separated and $\{w_n\}, \{z_n\}$ are u separated. In particular, $\{w_n\}, \{x_n\}, \{o\}$ and $\{w_n\}, \{z_n\}, \{o\}$ are u separated. Again

using the Möbius condition,

$$\begin{aligned}
\frac{d(o', x'_n)}{d'_n} &= \frac{d(o', x'_n)}{d(o', z'_n)} \\
&= \frac{d(o', x'_n)}{d(o', w'_n)} \frac{d(o', w'_n)}{d(o', z'_n)} \\
&< \eta\left(\frac{d(o, x_n)}{d(o, w_n)}\right) \eta\left(\frac{d(o, w_n)}{d(o, z_n)}\right) \\
&= \eta\left(\frac{d(o, x_n)}{d_n} \frac{d_n}{d(o, w_n)}\right) \eta\left(\frac{d(o, w_n)}{d_n}\right) \\
&\rightarrow \eta(1)^2
\end{aligned}$$

Thus $\frac{d(o', x'_n)}{d'_n}$ is bounded.

(2) The mapping $\{x_n\} \rightarrow \{f(x_n)\}$ is well-defined from $X_\omega(d_n)$ to $Y_\omega(d'_n)$.

If $\{x_n\}$ and $\{y_n\}$ are two sequences representing the same point different from the origin o_ω , then by Lemma 8 and Lemma 7 either $\{x_n\}, \{y_n\}, \{o\}$ are separated by u , or there exists a sequence $\{w_n\}$ representing the same point, s.t. $\{x_n\}, \{w_n\}, \{o\}$ and $\{y_n\}, \{w_n\}, \{o\}$ are separated by u .

In the first case,

$$\begin{aligned}
\frac{d(x'_n, y'_n)}{d'_n} &= \frac{d(o', x'_n)}{d'_n} \frac{d(x'_n, y'_n)}{d(x'_n, o')} \\
&\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, y_n)}{d(x_n, o)}\right) \\
&= \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, y_n)}{d_n} \frac{d_n}{d(x_n, o)}\right) \\
&\rightarrow 0
\end{aligned}$$

In the second case,

$$\begin{aligned}
\frac{d(x'_n, y'_n)}{d'_n} &\leq \frac{d(x'_n, w'_n)}{d'_n} + \frac{d(w'_n, y'_n)}{d'_n} \\
&\leq \frac{d(o', x'_n)}{d'_n} \frac{d(x'_n, w'_n)}{d(x'_n, o')} + \frac{d(o', y'_n)}{d'_n} \frac{d(y'_n, w'_n)}{d(y'_n, o')} \\
&\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, w_n)}{d(x_n, o)}\right) + \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(y_n, w_n)}{d(y_n, o)}\right) \\
&\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, w_n)}{d_n} \frac{d_n}{d(x_n, o)}\right) + \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(y_n, w_n)}{d_n} \frac{d_n}{d(y_n, o)}\right) \\
&\rightarrow 0
\end{aligned}$$

Therefore, in both cases $x'_\omega = y'_\omega$.

Let $\{x_n\}$ be a sequence representing the origin o_ω . If $\{x_n\}$ is a bounded sequence, then under the assumption over f $\{x'_n\}$ is also bounded and thus $x'_\omega = o'_\omega$.

If $d(o, x_n)$ diverges, then $\{x_n\}$ and $\{o\}$ are u -separated. We can choose

$y_\omega \neq o_\omega$, then by lemma 7, $\{x_n\}, \{y_n\}, \{o\}$ are u -separated. Thus,

$$\begin{aligned} \frac{d(o', x'_n)}{d'_n} &= \frac{d(o', y'_n)}{d'_n} \frac{d(o', x'_n)}{d(o', y'_n)} \\ &\leq \frac{d(o', y'_n)}{d'_n} \eta \left(\frac{d(o, x_n)}{d(o, y_n)} \right) \\ &= \frac{d(o', y'_n)}{d'_n} \eta \left(\frac{d(o, x_n)}{d_n} \frac{d_n}{d(o, y_n)} \right) \\ &\rightarrow 0. \end{aligned}$$

Therefore, $o'_\omega = x'_\omega$.

(3) The map $g : \{x_n\} \rightarrow \{f(x_n)\}$ is quasisymmetric and in particular continuous and injective.

Let $\{w_n\}, \{x_n\}, \{y_n\}$ be sequences representing three distinct points w_ω, x_ω and y_ω in $X_\omega(d_n)$. By lemma 7, $\{w_n\}, \{x_n\}, \{y_n\}$ are pairwise u separated. So

$$\frac{d(x'_n, y'_n)}{d(x'_n, w'_n)} < \eta \left(\frac{d(x_n, y_n)}{d(x_n, w_n)} \right),$$

equivalently

$$\frac{d(x'_n, y'_n)}{d'_n} \frac{d'_n}{d(x'_n, w'_n)} < \eta \left(\frac{d(x_n, y_n)}{d_n} \frac{d_n}{d(x_n, w_n)} \right).$$

Taking the ω limit, we get

$$\frac{d_\omega(x'_\omega, y'_\omega)}{d_\omega(x'_\omega, w'_\omega)} \leq \eta \left(\frac{d_\omega(x_\omega, y_\omega)}{d_\omega(x_\omega, w_\omega)} \right).$$

So, if $g : \{x_n\} \mapsto \{f(x_n)\}$ is not constant, then it is continuous and injective. Note that $\{o'\}$ and $\{f(z_n)\}$ are not the same points, so g is indeed not constant. ■

3 Examples

3.1 Sublinear-Lipschitz equivalences

In [5], Y. Cornulier introduces *sublinear-Lipschitz maps*. A map $f : X \rightarrow Y$ between metric spaces is a sublinear-Lipschitz map if there is an admissible gauge $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$d(f(x), f(y)) \leq C d(x, y) + C' u(|x| + |y|), \quad \forall x, y \in X,$$

for some constants $C, C' > 0$.

Two sublinear-Lipschitz maps f, f' are equivalent if there is an admissible gauge v and a constant $C'' > 0$ such that

$$d(f(x), f'(x)) \leq C'' v(|x|)$$

for all $x \in X$.

Sublinear-Lipschitz maps between metric spaces form a category. Taking asymptotic cones, we obtain a functor from the sublinear-Lipschitz category to the Lipschitz category. The sublinear-Lipschitz category is in a sense the maximal category with such a property.

The isomorphisms in the sublinear-Lipschitz category are called sublinear-Lipschitz equivalences or SBE maps.

Proposition 9. *Every SBE-mapping $f : X \rightarrow Y$ is an AM-mapping with linear η .*

Proof: If f is SBE, then f is bi-Lipschitz except at scales below an admissible gauge u . Indeed, let $x, y \in X$ s.t.

$$d(x, y) \geq u(|x| + |y|).$$

Then

$$\begin{aligned} d(f(x), f(y)) &\leq cd(x, y) + Cu(|x| + |y|) \\ &\leq (c + C)d(x, y), \end{aligned}$$

and

$$\begin{aligned} d(f(x), f(y)) &\geq c'd(x, y) - C'u(|x| + |y|) \\ &\geq (c' - C')d(x, y) \end{aligned}$$

So there exists $D > 0$, s.t. for all $x, y > u$,

$$\frac{1}{D}d(x, y) \leq d(f(x), f(y)) \leq Dd(x, y).$$

In particular, f is an AM-mapping with linear η . ■

3.2 Assouad-type maps

How can one recognize when a metric space is bi-Lipschitz equivalent to an Euclidean space? (S. Semmes, On the nonexistence of bilipschitz parametrizations, 1996)

Simple as it sounds, this question is not obvious. If a metric space admits a bi-Lipschitz embedding in \mathbb{R}^n , then it is clearly doubling. The converse is not true, however; the 3-dimensional Heisenberg group with the Carnot-Carathéodory metric is doubling, but does not admit a bi-Lipschitz embedding in \mathbb{R}^n for any n .

Assouad's embedding theorem [1] [13] states that any snowflake $X^\alpha = (X, d^\alpha)$, $0 < \alpha < 1$, of a doubling metric space admits a bi-Lipschitz embedding in a Euclidean space.

It is clear that the Assouad embedding of a doubling metric space into Euclidean space is an asymptotic-Möbius map. In what follows, we construct an example of an Assouad mapping from an infinite dimensional Heisenberg group into a Hilbert space. This construction follows straightforwardly from a construction of Lee and Naor for the finite dimensional case [12].

Let H be an infinite dimensional complex Hilbert space. H carries the symplectic form $\Omega(a, b) = \text{Im}(\langle a, b \rangle)$.

The *infinite-dimensional Heisenberg group* \mathcal{H}_Ω , is the set of tuples (a, t) with $a \in H$, $t \in \mathbb{R}$ and the group law

$$(a, t)(a', t') = (a + a', t + t' + 2\Omega(a, a')).$$

Let G be a group with identity element e . A group seminorm on G is a function $G \rightarrow [0, \infty)$ satisfying $N(g^{-1}) = N(g)$ for all $g \in G$, $N(gh) \leq N(g) + N(h)$ for all $g, h \in G$ and $N(e) = 0$. Moreover, if $N(g) = 0$ if and only if $g = e$, then N is a group norm on G .

The function

$$N(a, t) = \sqrt{\sqrt{\|a\|^4 + t^2} + \|a\|^2},$$

is a group norm on \mathcal{H}_Ω . Namely, if N_1 and N_2 are group seminorms, then $\sqrt{N_1^2 + N_2^2}$ is a group seminorm. In this case, $N_1 = (\|a\|^4 + t^2)^{\frac{1}{4}}$ is the Koranyi norm. Given N ,

$$d_N((a, t), (a', t')) = N((a, t)(a', t')^{-1})$$

is a right invariant metric on \mathcal{H}_Ω .

Let G be a group, then a Hermitian kernel on G is a complex-valued function $K : G \times G \rightarrow \mathbb{C}$ satisfying $K(g, h) = \overline{K(h, g)}$ for all $g, h \in G$.

A Hermitian kernel on G is positive definite if

$$\sum_{i,j}^n K(g_i, g_j) c_i \overline{c_j} \geq 0$$

for all $g_1, \dots, g_n \in G$ and for all complex numbers $c_1, \dots, c_n \in \mathbb{C}$, with equality if and only if the c_i vanish. A Hermitian kernel on G is conditionally negative definite if

$$\sum_{i,j}^n K(g_i, g_j) c_i \overline{c_j} \leq 0$$

for all $g_1, \dots, g_n \in G$ and for all complex numbers $c_1, \dots, c_n \in \mathbb{C}$ satisfying $\sum_i^n c_i = 0$.

Theorem 2 (Schönberg [17]). *Let G be a group, $K : G \times G \rightarrow \mathbb{R}$ a real-valued kernel on G satisfying $K(g, g) = 0 \forall g \in G$. Then K is negative definite if and only if there exists a Hilbert space H and a function $T : G \rightarrow H$ such that $\forall g, h \in G : K(g, h) = \|T(g) - T(h)\|^2$.*

Thus, if d_N is negative definite, then there is a Hilbert space H and an isometry $T : (\mathcal{H}_\Omega, \sqrt{d_N}) \rightarrow H$. In particular, the mapping $T : (\mathcal{H}_\Omega, d_N) \rightarrow H$ is an AM mapping.

A complex-valued function F on a group G satisfying $F(g^{-1}) = \overline{F(g)}$ for all $g \in G$ induces a Hermitian kernel on G by defining $K_F(g, h) := F(gh^{-1})$. The function F is called positive definite if K_F is a positive definite Hermitian kernel on G .

Lemma 10. For any $\lambda \in \mathbb{R}$, the function

$$\Phi_\lambda(a, t) = e^{-|\lambda|\|a\|^2 + i\lambda t}$$

is positive definite on \mathcal{H}_Ω .

Proof: Indeed, the function Φ_λ satisfies the condition $\Phi_\lambda((a, t)^{-1}) = \overline{\Phi_\lambda((a, t))}$ and induces the kernel

$$K((a, s), (b, t)) := \exp(-|\lambda|\|a - b\|^2 + i\lambda(s - t - 2\Omega(a, b))).$$

The kernel can be rewritten as a product of three exponentials

$$\exp(-|\lambda|(\|a\|^2 + \|b\|^2)) \exp(i\lambda(s - t)) \exp(2|\lambda|(\operatorname{Re}(\langle a, b \rangle) - i \operatorname{sign}(\lambda) \Omega(a, b))).$$

The product of positive definite kernels is positive definite. The first two factors are positive definite because for all $c_1, \dots, c_n \in \mathbb{C}$ the matrix $(c_i \overline{c_j})_{i,j}$ is positive semidefinite. In particular, also the matrix $(\exp(i\lambda(s_i - t_j)) c_i \overline{c_j})_{i,j}$. The exponential of any positive definite kernel is again a positive definite kernel ([3] Proposition 8.2.). Thus the kernel K is positive definite if

$$\operatorname{Re}(\langle a, b \rangle) - i \operatorname{sign}(\lambda) \Omega(a, b)$$

is positive definite. This is $\langle a, b \rangle$ if $\lambda < 0$ and $\overline{\langle a, b \rangle}$ if $\lambda \geq 0$ and thus clearly positive definite. \blacksquare

Proposition 11. $d_N : \mathcal{H}_\Omega \times \mathcal{H}_\Omega \rightarrow H$ is conditionally negative definite.

Proof: The existence of $\frac{1}{2}$ -stable distributions implies that for all $\epsilon > 0$ there exists a non-negative integrable function $\varphi_\epsilon : \mathbb{R} \rightarrow [0, \infty)$ s.t. its Fourier transform $\hat{\varphi}_\epsilon(t) = e^{-\epsilon\sqrt{|t|}}$. Note that $\frac{1}{2\pi} \hat{\varphi}_\epsilon(x) = \varphi_\epsilon(x)$.

By the above lemma,

$$F_\epsilon(a, t) = \int_{\mathbb{R}} e^{-|\lambda|\|a\|^2 + i\lambda t} \varphi_\epsilon(\lambda) d\lambda$$

is positive definite on \mathcal{H}_Ω . Let

$$h_k(x) = \frac{k}{\pi} \frac{1}{k^2 + x^2}$$

denote the Cauchy distribution with scale parameter $k > 0$. We write around

$$\begin{aligned} F_\epsilon(a, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda t} \hat{h}_{\|a\|^2}(\lambda) \hat{\varphi}_\epsilon(\lambda) d\lambda \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda t} \widehat{h_{\|a\|^2} * \varphi_\epsilon}(\lambda) d\lambda \\ &= (h_{\|a\|^2} * \hat{\varphi}_\epsilon)(t). \end{aligned}$$

Since F_ϵ is positive definite, $\frac{1 - F_\epsilon}{\epsilon}$ is conditionally negative definite and

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1 - F_\epsilon(a, t)}{\epsilon} &= \lim_{\epsilon \rightarrow 0} \left[h_{\|a\|^2} * \frac{1 - \hat{\varphi}_\epsilon}{\epsilon} \right] \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \frac{1 - e^{-\epsilon\sqrt{|x|}}}{\epsilon} h_{\|a\|^2}(t - x) dx \\ &= \frac{\|a\|^2}{\pi} \int_{\mathbb{R}} \frac{\sqrt{|x|}}{\|a\|^4 + (t - x)^2} dx. \end{aligned}$$

We now show that for all $r, t \in \mathbb{R}$

$$r^2 \int_{\mathbb{R}} \frac{\sqrt{|x|}}{r^4 + (t-x)^2} dx = \pi \sqrt{\sqrt{r^4 + t^2} + r^2}.$$

By changing variables $x = r^2 y$ and $s = t/r^2$, the left hand side can be written as

$$\int_0^\infty \left(\frac{1}{1+(s-y)^2} + \frac{1}{1+(s+y)^2} \right) \sqrt{y} dy$$

This integral is equal to

$$\lim_{r \rightarrow 0} \lim_{R \rightarrow \infty} \frac{1}{2} \int_{C_{r,R}} \left(\frac{1}{1+(s-z)^2} + \frac{1}{1+(s+z)^2} \right) \sqrt{z} dz,$$

where $C_{r,R}$ is the keyhole contour with a branch cut along the positive real axis. The integrand has simple poles at $i \pm s$ and $-i \pm s$. By $\text{Res}\left(\frac{\sqrt{z}}{1+(s-z)^2}, i+s\right) = \frac{\sqrt{i+s}}{2i}$, $\text{Res}\left(\frac{\sqrt{z}}{1+(s-z)^2}, -i+s\right) = \frac{\sqrt{-i+s}}{-2i}$, $\text{Res}\left(\frac{\sqrt{z}}{1+(s+z)^2}, i-s\right) = \frac{\sqrt{i-s}}{2i}$ and $\text{Res}\left(\frac{\sqrt{z}}{1+(s+z)^2}, -i-s\right) = \frac{\sqrt{-i-s}}{-2i}$. Now adding all the residues and multiplying by $2\pi i$, we obtain the above equation as

$$\frac{\pi}{2} (\sqrt{i+s} - \sqrt{-i+s} + \sqrt{i-s} - \sqrt{-i-s}).$$

This further simplifies to

$$\begin{aligned} \pi \text{Re}(\sqrt{i+s} + \sqrt{i-s}) &= \pi (\text{Re}(\sqrt{i+s}) + \text{Im}(\sqrt{i+s})) \\ &= \pi \left(\sqrt{\frac{\sqrt{1+s^2}+s}{2}} + \sqrt{\frac{\sqrt{1+s^2}-s}{2}} \right) \\ &= \pi \sqrt{\sqrt{1+s^2}+1} \end{aligned}$$

So we have seen that the N -norm is a limit of conditionally negative definite functions on \mathcal{H}_Ω , so it is conditionally negative definite. \blacksquare

4 Applications to dimension theory

4.1 Dimension theory of finitely-generated groups

In 1993 M. Gromov introduced the notion of asymptotic dimension as a large scale analogue of Lebesgue's covering dimension [7]. The asymptotic dimension of a finitely generated group is a quasi-isometric invariant. Its most prominent application goes back to Guoliang Yu, who showed that any finitely generated group with finite homotopy type and finite asymptotic dimension satisfies the Novikov conjecture [21].

Definition 12. *Let X be a metric space. We say that the asymptotic dimension of X does not exceed n if for every uniformly bounded open cover \mathcal{V} of X there exists a uniformly bounded open cover \mathcal{U} of X of multiplicity $\leq n+1$ such that \mathcal{V} refines \mathcal{U} . We write $\text{asdim}X = n$ if it is true that $\text{asdim}X \leq n$ and $\text{asdim}X \not\leq n-1$*

The asymptotic dimension of a finitely generated group Γ has several interesting implications. For example, $\text{asdim } \Gamma = 0$ if and only if Γ is finite (Proposition 65. in [2]) and $\text{asdim } \Gamma = 1$ if and only if Γ is virtually free (Theorem 66. in [2]).

If G is a finitely generated nilpotent group, then its asymptotic cone G_ω is a Carnot group [14] and the asymptotic dimension of G coincides with the topological dimension of its asymptotic cone [2].

Theorem 3. *Let G and G' be a finitely generated nilpotent groups and $f : G \rightarrow G'$ an AM-map, then $\text{asdim}(G) \leq \text{asdim}(G')$. If G and G' are nilpotent Lie groups, then $\dim(G) \leq \dim(G')$. Furthermore, if $\text{asdim}(G) = \text{asdim}(G')$, then the asymptotic cones of G and G' are isomorphic graded Lie groups. Conversely, given nilpotent groups with isomorphic asymptotic cones, there exists an asymptotic-Möbius map between them.*

Proof: The mapping f induces a quasimetric mapping g between the asymptotic cones G_ω and G'_ω of G and G' , respectively.

In particular, g is an injective mapping from G_ω to G'_ω and hence the topological dimensions increase. Since G and G' are nilpotent, $\text{asdim}(G) = \text{top-dim}(G_\omega) \leq \text{top-dim}(G'_\omega) = \text{asdim}(G')$.

The case for nilpotent Lie groups follows from the fact that for nilpotent Lie groups $\text{top-dim}(G) = \text{top-dim}(G_\omega)$.

If $\text{asdim}(G) = \text{asdim}(G')$, the invariance of domain theorem implies that g is a homeomorphism. Since both g and g^{-1} are quasimetric, they are differentiable by Pansu's theorem a.e., and the differential of g is a graded isomorphism of groups at almost every point [15]. \blacksquare

4.2 Dimension theory of CAT(0)-spaces

Heinrich Jung established in the early 1900s that the diameter of a set in Euclidean space \mathbb{R}^n is related to the radius of the smallest enclosing ball by an inequality whose constant depends only on the dimension n [9]. More precisely, for any set $A \subset \mathbb{R}^n$

$$r \leq d(A) \sqrt{\frac{n}{2(n+1)}}, \quad (1)$$

where r is the radius of the smallest enclosing ball of A and $d(A)$ is the diameter of A .

The above inequality can be used to define a geometric dimension of CAT(0)-spaces. A CAT(0)-space X has *geometric dimension* less than n if and only if the Jung's inequality (1) is true in X [4].

Remark 13. *It turns out that the geometric dimension of a CAT(0)-space X is equal to the maximal topological dimension of all compact subsets in X [10].*

The geometric dimension can be transformed into a large-scale term in the following way. A CAT(0)-space X has *telescopic dimension* less than n if and only if for every $\delta > 0$ there is a $D > 0$ such that for every set A

in X with diameter greater than D ,

$$r \leq \left(\delta + \sqrt{\frac{n}{2(n+1)}} \right) d(A). \quad (2)$$

Somewhat more general,

Definition 14. *A space X whose asymptotic cones are CAT(0) has telescopic dimension $\leq n$ if every asymptotic cone has geometric dimension $\leq n$.*

We adopt the notation $\text{tele-dim}(X)$ for the telescopic dimension of X .

Theorem 4. *Let X and Y be spaces, whose asymptotic cones are CAT(0). If there exists an AM-map $f : X \rightarrow Y$, then the telescopic dimension increases e.g. $\text{tele-dim}(X) \leq \text{tele-dim}(Y)$.*

The theorem follows straight from theorem 1, remark 13 and the fact that the topological dimension increases under injective continuous maps.

It follows that there can be no AM-mapping from Euclidean buildings of finite rank r and hence telescopic dimension r to a Euclidean building of lower rank.

If there is a AM-mapping from a CAT(0)-space to a hyperbolic metric space, then X is also a hyperbolic metric space.

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