



HAL
open science

Asymptotic-Möbius maps

Georg Alexander Gruetzner

► **To cite this version:**

| Georg Alexander Gruetzner. Asymptotic-Möbius maps. 2023. ⟨hal-03746707v5⟩

HAL Id: hal-03746707

<https://hal.science/hal-03746707v5>

Preprint submitted on 21 Feb 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Asymptotic-Möbius maps.

Georg Grützner

Abstract

Roughly speaking, a map between metric spaces is an asymptotic-Möbius map if it almost preserves cross-ratios of points being a large distance apart from each other. We show that under such maps, some large-scale notions of dimension increases: asymptotic dimension for finitely generated nilpotent groups, telescopic dimension for $CAT(0)$ spaces.

1 Introduction

In 1999, Oded Schramm investigated the scaling limits of loop-erased random walks (LERW) [18]. Schramm observed that the conformal invariance of the limit process implies its encoding in corresponding conformal mappings f_t satisfying Loewner's differential equation

$$\frac{\partial f_t}{\partial t} = z \frac{\zeta(t) + z}{\zeta(t) - z} \frac{\partial f_t}{\partial z}.$$

This observation revolutionized two-dimensional statistical physics (see, e.g., a review of SLE by G. F. Lawler [11]).

There are generalizations for dimensions higher than 2[20]. However, the strength of conformal techniques in higher dimensions is severely limited. This is particularly clear in Liouville's 1850 theorem.¹ While all simply connected domains in \mathbb{R}^2 are conformally equivalent by the Riemann mapping theorem, any smooth conformal mapping on a domain of \mathbb{R}^n , with $n > 2$, is a composition of translations, similarities, orthogonal transformations and inversions.

Suitable classes of functions on metric spaces that capture conformal behavior are the classes of quasisymmetric (QS) maps, quasi-Möbius (QM) maps, and quasiconformal (QC) maps. Quasisymmetric maps transform balls into "quasiballs." The definition is based on the fact that conformal maps transform infinitesimal balls into infinitesimal ellipsoids. Quasi-Möbius maps preserve almost the cross ratio of points, and quasiconformal maps are an infinitesimal version of quasisymmetric maps. We always have $QS \Rightarrow QM \Rightarrow QC$ [19].

We are therefore interested in classes of functions that converge under scaling limits to the classes QS, QM or QC. In the eyes of a geometric

¹Monge 1850, pp. 609 - 616, a note contributed by Liouville as editor (Note VI: Extension au cas des trois dimensions de la question du tracé géographique.)

group theorist, we are looking for classes of maps between finitely generated groups Γ , with the word metric, which remain invariant under small changes in topology and which become conformal in the above sense when viewed from a distance. Specifically, we require that the class of maps is invariant under composition by quasi-isometries.

This chapter presents and examines a class of maps that mimics quasi-Möbius maps in the context of geometric group theory.

In [16] Pierre Pansu introduces a notion of large-scale conformal maps that mimics the infinitesimal behavior of conformal maps. In short, large-scale conformal maps map families of disjoint balls onto families of weakly overlapping quasiballs. It is a very flexible notion that includes, for example, coarse embeddings. However, this flexibility makes the asymptotic behavior of such maps less predictable.

Instead, we introduce a more restrictive and semantically simpler notion that mimics quasi-Möbius behavior. Roughly speaking, a map between metric spaces is an asymptotic-Möbius map if it nearly preserves the cross ratio of points that are a large distance apart. As a result, the results presented in this paper are much stronger than those in [16].

We postpone the precise definition of asymptotic-Möbius (AM) maps until later. Sources of examples of AM -maps are:

1. Quasi-isometric embeddings,
2. Sublinear-bi-Lipschitz equivalences (i.e., maps inducing Lipschitz equivalences on asymptotic cones [5]),
3. Snowflaking (i.e., replacing a metric by a power of it),
4. Assouad maps from doubling metric spaces to \mathbb{R}^N .

For example, any nilpotent Lie group or finitely generated group can be AM -embedded in a Euclidean space of sufficiently high dimension [1]. Moreover, we will encounter infinite dimensional examples.

1.1 Results

Our main result is that under AM -mappings a large-scale term of dimension increases. The relevant term depends on the class of the groups considered.

Theorem 3. *Let G and G' be finitely generated nilpotent groups and $f : G \rightarrow G'$ an AM -map, then $\text{asdim}(G) \leq \text{asdim}(G')$. Furthermore, if $\text{asdim}(G) = \text{asdim}(G')$, then the asymptotic cones of G and G' are isomorphic. Conversely, given nilpotent groups with isomorphic asymptotic cones, there exists an asymptotic-Möbius map between them.*

In the world of $\text{CAT}(0)$ -spaces, the analogous theorem takes the following form.

Theorem 4. *Let X and Y be $\text{CAT}(0)$ spaces. If there exists an AM -map $f : X \rightarrow Y$, then the telescopic dimension increases e.g. $\text{tele-dim}(X) \leq \text{tele-dim}(Y)$.*

2 Asymptotic-Möbius maps

2.1 Quasi-Möbius maps

Quasi-Möbius maps were introduced by Jussi Väisälä, among others, as a means of studying quasisymmetric maps and quasiconformal maps. Unlike quasisymmetric maps, quasi-Möbius maps do not have a point fixed at infinity.

Let X be a metric space, and $x, y, z, w \in X$ a quadruple of distinct points. Their *cross-ratio* is

$$[x, y, z, w] = \frac{d(x, z)d(y, w)}{d(x, w)d(y, z)}.$$

If X is unbounded, then the cross ratio extends to the one-point compactification of X [19]. A map $f : X \rightarrow Y$ is *quasi-Möbius* if there is a homeomorphism $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for all quadruples of distinct points $x, y, z, w \in X$,

$$[f(x), f(y), f(z), f(w)] \leq \eta([x, y, z, w]).$$

If f is a homeomorphism, then f is quasi-Möbius iff its inverse is quasi-Möbius.

Examples of quasi-Möbius maps are

1. the stereographic projection $\mathbb{R}^n \rightarrow S^n$,
2. the Cayley transformations (the complex, quaternionic and octonionic analogues of the stereographic projection),
3. the inversions $x \mapsto \frac{x}{|x|^2}$ in Banach spaces.

2.2 Asymptotic-Möbius maps

We need a criterion to tell when two points in a space are far apart. One way to do this is to separate them by sublinear growing functions. This idea was used by Y. Cornulier to define sublinear-bi-Lipschitz equivalences [5].

Given $u : \mathbb{R}_+ \rightarrow \mathbb{R}$, we say that the points x and y in X are separated by u from a fixed origin o if $d(x, y) > u(d(x, o) + d(o, y))$. For simplicity, we often write $x, y > u$ and $|x| = d(o, x)$ for the distance from the origin. We can call u a *gauge*, i.e., a function that sets a scale of "sight" as a function of "location". To fix a gauge means to decide what is "near" and "far away".

Definition 1. A function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is an *admissible gauge* if

1. it is non-decreasing, and
2. u grows sublinearly, i.e. $\limsup_{r \rightarrow \infty} \frac{u(r)}{r} = 0$.

Morally, asymptotic-Möbius maps at widely spaced points are quasi-Möbius.

Definition 2. A map $f : X \rightarrow Y$ is an asymptotic-Möbius map or AM-map if there exists an admissible gauge u and a homeomorphism $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for every x, y, z, w in X such that all 6 distances between $x, y, z, w > u$, then

$$[f(x), f(y), f(z), f(w)] < \eta([x, y, z, w]).$$

At some point we will need a coarse version of path connectivity.

Definition 3. A metric space X is asymptotically chained, if for every origin $o \in X$ there exists an admissible gauge v such that for all $x, y \in X$ there exists a chain $x_1 = x, \dots, x_{k+1} = y$ satisfying

$$\max_{i \in 1 \dots k} \{d(x_i, x_{i+1})\} < v(|x| + |y|).$$

Our most important technical step will be the following theorem, the proof of which will occupy section 2.4 after some introductory definitions.

Theorem 1. Let f be an AM-map from an asymptotically chained space X to a space Y that preserves diverging and bounded sequences, then f induces a continuous, injective, quasisymmetric map g between some asymptotic cones of X and Y .

2.3 Asymptotic-cones

The *asymptotic cone* of a metric space (X, d) , captures the geometry on the large scale of X . Roughly speaking, it formalizes the idea of snapshots of the space X taken by an observer moving farther and farther away from X . This sequence of snapshots can stabilize and the observer has the impression of seeing a single object. We call this object the asymptotic cone of X .

If R -balls $B(o, R)$ with metric $R^{-1}d$ are uniformly pre-compact, then the asymptotic cones can be constructed concretely by pointed Gromov-Hausdorff convergence [6], [8] In general, however, we have to resort to constructions via ultrafilters.

Definition 4. A (nonprincipal) ultrafilter ω over \mathbb{N} is a set of subsets of \mathbb{N} satisfying the following conditions:

1. If $A, B \in \omega$ then $A \cap B \in \omega$.
2. If $A \in \omega, A \subset B \subset \mathbb{N}$, then $B \in \omega$.
3. For every $A \subset \mathbb{N}$, either $A \in \omega$ or $\mathbb{N} \setminus A \in \omega$.
4. No finite subset of \mathbb{N} is in ω .

Equivalently, ω is a finitely additive measure on \mathbb{N} such that every subset has measure either 0 or 1 and every finite subset has measure 0.

If a statement $A(n)$ holds for all $n \in B$, where B belongs to an ultrafilter ω , then $A(n)$ is said to hold ω -almost surely.

Definition 5. Let ω be a (nonprincipal) ultrafilter over \mathbb{N} . An ω -limit of a sequence of points $\{x_n\}$ in a topological space X is a point x in X such that for any neighborhood U of x the relation $x_n \in U$ holds ω -almost surely.

If X is a Hausdorff space, then the ω -limit of a sequence is unique. We denote this point by $\lim_{\omega} x_n$.

Definition 6. *The ultrapower of a set X with respect to an ultrafilter ω , denoted by X^{ω} , consists of equivalence classes of sequences $\{x_n\}$, $x_n \in X$, where two sequences $\{x_n\}$ and $\{y_n\}$ are identical if and only if $x_n = y_n$ ω -almost surely.*

We adopt the notation $\{x_n\}_{\omega}$ for the equivalence class of sequences.

Let (X, d) be a metric space, ω an ultrafilter over \mathbb{N} and o an origin in X . Furthermore, let a sequence of numbers $\{d_n\}$ be given, with $\lim_{\omega} d_n = \infty$. Then there is a pseudo-metric on the ultrapower X^{ω} by setting the distance between $x = \{x_n\}_{\omega}$ and $y = \{y_n\}_{\omega}$ to

$$d_{\omega}(x, y) = \lim_{\omega} \frac{d(x_n, y_n)}{d_n}.$$

Let X_o^{ω} be the set of equivalence classes of elements $x = \{x_n\}_{\omega}$ in X^{ω} satisfying $\lim_{\omega} \frac{d(x_n, o)}{d_n} < \infty$ and two elements $x = \{x_n\}_{\omega}$ and $y = \{y_n\}_{\omega}$ of X^{ω} are identical if and only if $d_{\omega}(x, y) = 0$. The metric space $(X_o^{\omega}, d_{\omega})$ is called an *asymptotic cone* of (X, d) .

For points in the asymptotic cone $(X_o^{\omega}, d_{\omega})$ of (X, d) we use the notation x_{ω} .

Lemma 7. *If $x_{\omega} \neq y_{\omega}$, then for every admissible gauge u and all sequences $\{x_n\}$, $\{y_n\}$ representing x_{ω} and y_{ω} , respectively,*

$$d(x_n, y_n) > u(|x_n| + |y_n|), \quad \omega - a.s.$$

Proof: If there is an admissible function u and sequences $\{x_n\}$, $\{y_n\}$ such that

$$d(x_n, y_n) \leq u(|x_n| + |y_n|), \quad \omega - a.s.$$

then $\omega - a.s.$,

$$\frac{d(x_n, y_n)}{d_n} \leq \frac{u(|x_n| + |y_n|)}{d_n}.$$

So

$$\lim_{\omega} \frac{d(x_n, y_n)}{d_n} = 0.$$

■

Lemma 8. *Let X be an asymptotically chained metric space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X representing the same point x_{ω} different from the origin o_{ω} such that $x_n \neq y_n$ $\omega - a.s.$ Then, for any admissible gauge u , either*

$$d(x_n, y_n) > u, \quad \omega - a.s.$$

or there exists a sequence $\{w_n\}$ representing the same point s.t

$$d(x_n, w_n) > u \quad \omega - a.s.$$

and

$$d(y_n, w_n) > u \quad \omega - a.s.$$

Proof: Suppose $d(x_n, y_n) \leq u \omega - a.s.$ and assuming $|x_n|, |y_n| \rightarrow +\infty$. Since X is asymptotically chained, there is an admissible gauge v and a chain $x_1^n = o, \dots, x_{k+1}^n = y_n$ such that

$$\max_{i \in 1, \dots, k} \{d(x_i^n, x_{i+1}^n)\} < v(|y_n|).$$

Assuming n is large enough so that $2u(4|y_n|) < |y_n|$, then at least one point $x_{i_n}^n$ in the chain satisfies

$$2u(4|y_n|) < d(y_n, x_{i_n}^n) \leq 2u(4|y_n|) + v(|y_n|).$$

Define $w_n = x_{i_n}^n$.

For n large enough,

$$\begin{aligned} 2d(x_n, y_n) &\leq 2u(|x_n| + |y_n|) \\ &\leq |x_n| + |y_n| \\ &\leq 2|y_n| + d(x_n, y_n). \end{aligned}$$

It follows,

$$d(x_n, y_n) \leq u(4|y_n|).$$

and therefore

$$u(4|y_n|) < d(x_n, w_n) \leq 3u(4|y_n|) + v(|y_n|).$$

Choose N s.t. $\forall n \geq N$

$$3u(4|y_n|) + v(|y_n|) \leq 2|y_n|.$$

Then

$$\begin{aligned} u(|x_n| + |w_n|) &\leq u(|x_n| + |y_n| + d(y_n, w_n)) \\ &\leq u(2|y_n| + d(x_n, y_n) + 2u(4|y_n|) + v(|y_n|)) \\ &\leq u(2|y_n| + 3u(4|y_n|) + v(|y_n|)) \\ &\leq u(4|y_n|) \\ &< d(x_n, w_n). \end{aligned}$$

This means $x_n, w_n > u$ and

$$\frac{d(x_n, w_n)}{d_n} \leq \frac{3u(4|y_n|) + v(|y_n|)}{d_n} \rightarrow 0.$$

Related,

$$\begin{aligned} u(|y_n| + |w_n|) &\leq u(2|y_n| + d(y_n, w_n)) \\ &\leq u(2|y_n| + 2u(4|y_n|) + v(|y_n|)) \\ &\leq u(4|y_n|) \\ &< d(y_n, w_n). \end{aligned}$$

■

2.4 Proof of Theorem 1.

Proof: Let u be an admissible gauge as given by the definition of f . Fix an origin $o \in X$. Let B_n be the ball of radius n around o . For each n , choose a point z_n in B_n such that

$$d(f(o), f(z_n))$$

is 'maximal'. More precisely, let $\epsilon > 0$ and choose z_n such that

$$d(f(o), f(z_n)) > \sup_{x \in B_n} d(f(o), f(x)) - \epsilon.$$

The supremum on the right hand side exists. Otherwise, one could construct a bounded sequence in B_n mapped by f to an unbounded sequence. This contradicts the assumptions.

Moreover, the four-point condition

$$[f(x), f(w), f(y), f(z)] \leq \eta([x, w, y, z]),$$

reduces to the three-point condition

$$\frac{d(f(x), f(y))}{d(f(x), f(z))} \leq \eta\left(\frac{d(x, y)}{d(x, z)}\right),$$

by bringing w to a point at ∞ .

In what follows, we often abbreviate the notation $f(x)$ to x' .

Define

$$d_n = d(o, z_n)$$

and

$$d'_n = d(o', z'_n).$$

By construction and assumption over f , the sequence d'_n is divergent. Consequently, the sequence d_n must also be divergent by the assumption over f .

(1) We show that if $\frac{d(o, x_n)}{d_n}$ is bounded, then $\frac{d(o', f(x_n))}{d'_n}$ is also bounded.

If $x_\omega = o_\omega$, then $d(o, x_n) \leq d_n$ for n large, therefore $d(o', x'_n) \leq d'_n$.

If $x_\omega \neq z_\omega$, $x_\omega \neq o_\omega$, then by lemma 7 $\{x_n\}, \{z_n\}, \{o\}$ are separated by u ω -a.s., and the Möbius- condition yields

$$\frac{d(o', x'_n)}{d'_n} = \frac{d(o', x'_n)}{d(o', z'_n)} < \eta\left(\frac{d(o, x_n)}{d(o, z_n)}\right) = \eta\left(\frac{d(o, x_n)}{d_n}\right) \quad \omega - a.s.$$

Thus $\frac{d(o', x'_n)}{d'_n}$ is bounded.

If $x_\omega = z_\omega$, then by lemma 8 there are $\{w_n\}$ representing the same point, s.t. $\{w_n\}, \{x_n\}$ are u separated and $\{w_n\}, \{z_n\}$ are u separated. In particular, $\{w_n\}, \{x_n\}, \{o\}$ and $\{w_n\}, \{z_n\}, \{o\}$ are u separated. Again

using the Möbius condition,

$$\begin{aligned}
\frac{d(o', x'_n)}{d'_n} &= \frac{d(o', x'_n)}{d(o', z'_n)} \\
&= \frac{d(o', x'_n)}{d(o', w'_n)} \frac{d(o', w'_n)}{d(o', z'_n)} \\
&< \eta\left(\frac{d(o, x_n)}{d(o, w_n)}\right) \eta\left(\frac{d(o, w_n)}{d(o, z_n)}\right) \\
&= \eta\left(\frac{d(o, x_n)}{d_n} \frac{d_n}{d(o, w_n)}\right) \eta\left(\frac{d(o, w_n)}{d_n}\right) \\
&\rightarrow \eta(1)^2
\end{aligned}$$

Thus $\frac{d(o', x'_n)}{d'_n}$ is bounded.

(2) The mapping $\{x_n\} \rightarrow \{f(x_n)\}$ is well-defined from $X_\omega(d_n)$ to $Y_\omega(d'_n)$.

If $\{x_n\}$ and $\{y_n\}$ are two sequences representing the same point different from the origin o_ω , then by Lemma 8 and Lemma 7 either $\{x_n\}, \{y_n\}, \{o\}$ are separated by u , or there exists a sequence $\{w_n\}$ representing the same point, s.t. $\{x_n\}, \{w_n\}, \{o\}$ and $\{y_n\}, \{w_n\}, \{o\}$ are separated by u .

In the first case,

$$\begin{aligned}
\frac{d(x'_n, y'_n)}{d'_n} &= \frac{d(o', x'_n)}{d'_n} \frac{d(x'_n, y'_n)}{d(x'_n, o')} \\
&\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, y_n)}{d(x_n, o)}\right) \\
&= \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, y_n)}{d_n} \frac{d_n}{d(x_n, o)}\right) \\
&\rightarrow 0
\end{aligned}$$

In the second case,

$$\begin{aligned}
\frac{d(x'_n, y'_n)}{d'_n} &\leq \frac{d(x'_n, w'_n)}{d'_n} + \frac{d(w'_n, y'_n)}{d'_n} \\
&\leq \frac{d(o', x'_n)}{d'_n} \frac{d(x'_n, w'_n)}{d(x'_n, o')} + \frac{d(o', y'_n)}{d'_n} \frac{d(y'_n, w'_n)}{d(y'_n, o')} \\
&\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, w_n)}{d(x_n, o)}\right) + \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(y_n, w_n)}{d(y_n, o)}\right) \\
&\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, w_n)}{d_n} \frac{d_n}{d(x_n, o)}\right) + \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(y_n, w_n)}{d_n} \frac{d_n}{d(y_n, o)}\right) \\
&\rightarrow 0
\end{aligned}$$

Therefore, in both cases $x'_\omega = y'_\omega$.

Let $\{x_n\}$ be a sequence representing the origin o_ω . If $\{x_n\}$ is a bounded sequence, then under the assumption over f $\{x'_n\}$ is also bounded and thus $x'_\omega = o'_\omega$.

If $d(o, x_n)$ diverges, then $\{x_n\}$ and $\{o\}$ are u -separated. We can choose

$y_\omega \neq o_\omega$, then by lemma 7, $\{x_n\}, \{y_n\}, \{o\}$ are u -separated. Thus,

$$\begin{aligned} \frac{d(o', x'_n)}{d'_n} &= \frac{d(o', y'_n)}{d'_n} \frac{d(o', x'_n)}{d(o', y'_n)} \\ &\leq \frac{d(o', y'_n)}{d'_n} \eta \left(\frac{d(o, x_n)}{d(o, y_n)} \right) \\ &= \frac{d(o', y'_n)}{d'_n} \eta \left(\frac{d(o, x_n)}{d_n} \frac{d_n}{d(o, y_n)} \right) \\ &\rightarrow 0. \end{aligned}$$

Therefore, $o'_\omega = x'_\omega$.

(3) The map $g : \{x_n\} \rightarrow \{f(x_n)\}$ is quasisymmetric and in particular continuous and injective.

Let $\{w_n\}, \{x_n\}, \{y_n\}$ be sequences representing three distinct points w_ω, x_ω and y_ω in $X_\omega(d_n)$. By lemma 7, $\{w_n\}, \{x_n\}, \{y_n\}$ are pairwise u separated. So

$$\frac{d(x'_n, y'_n)}{d(x'_n, w'_n)} < \eta \left(\frac{d(x_n, y_n)}{d(x_n, w_n)} \right),$$

equivalently

$$\frac{d(x'_n, y'_n)}{d'_n} \frac{d'_n}{d(x'_n, w'_n)} < \eta \left(\frac{d(x_n, y_n)}{d_n} \frac{d_n}{d(x_n, w_n)} \right).$$

Taking the ω limit, we get

$$\frac{d_\omega(x'_\omega, y'_\omega)}{d_\omega(x'_\omega, w'_\omega)} \leq \eta \left(\frac{d_\omega(x_\omega, y_\omega)}{d_\omega(x_\omega, w_\omega)} \right).$$

So, if $g : \{x_n\} \mapsto \{f(x_n)\}$ is not constant, then it is continuous and injective. Note that $\{o'\}$ and $\{f(z_n)\}$ are not the same points, so g is indeed not constant. ■

3 Examples

3.1 Sublinear-Lipschitz equivalences

In [5], Y. Cornulier introduces *sublinear-Lipschitz maps*. A map $f : X \rightarrow Y$ between metric spaces is a sublinear-Lipschitz map if there is an admissible gauge $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$d(f(x), f(y)) \leq C d(x, y) + C' u(|x| + |y|), \quad \forall x, y \in X,$$

for some constants $C, C' > 0$.

Two sublinear-Lipschitz maps f, f' are equivalent if there is an admissible gauge v and a constant $C'' > 0$ such that

$$d(f(x), f'(x)) \leq C'' v(|x|)$$

for all $x \in X$.

Sublinear-Lipschitz maps between metric spaces form a category. Taking asymptotic cones, we obtain a functor from the sublinear-Lipschitz category to the Lipschitz category. The sublinear-Lipschitz category is in a sense the maximal category with such a property.

The isomorphisms in the sublinear-Lipschitz category are called sublinear-Lipschitz equivalences or SBE maps.

Proposition 9. *Every SBE-mapping $f : X \rightarrow Y$ is an AM-mapping with linear η .*

Proof: If f is SBE, then f is bi-Lipschitz except at scales below an admissible gauge u . Indeed, let $x, y \in X$ s.t.

$$d(x, y) \geq u(|x| + |y|).$$

Then

$$\begin{aligned} d(f(x), f(y)) &\leq cd(x, y) + Cu(|x| + |y|) \\ &\leq (c + C)d(x, y), \end{aligned}$$

and

$$\begin{aligned} d(f(x), f(y)) &\geq c'd(x, y) - C'u(|x| + |y|) \\ &\geq (c' - C')d(x, y) \end{aligned}$$

So there exists $D > 0$, s.t. for all $x, y > u$,

$$\frac{1}{D}d(x, y) \leq d(f(x), f(y)) \leq Dd(x, y).$$

In particular, f is an AM-mapping with linear η . ■

3.2 Assouad-type maps

How can one recognize when a metric space is bi-Lipschitz equivalent to an Euclidean space? (S. Semmes, On the nonexistence of bilipschitz parametrizations, 1996)

Simple as it sounds, this question is not obvious. If a metric space admits a bi-Lipschitz embedding in \mathbb{R}^n , then it is clearly doubling. The converse is not true, however; the 3-dimensional Heisenberg group with the Carnot-Carathéodory metric is doubling, but does not admit a bi-Lipschitz embedding in \mathbb{R}^n for any n .

Assouad's embedding theorem [1] [13] states that any snowflake $X^\alpha = (X, d^\alpha)$, $0 < \alpha < 1$, of a doubling metric space admits a bi-Lipschitz embedding in a Euclidean space.

It is clear that the Assouad embedding of a doubling metric space into Euclidean space is an asymptotic-Möbius map. In what follows, we construct an example of an Assouad mapping from an infinite dimensional Heisenberg group into a Hilbert space. This construction follows straightforwardly from a construction of Lee and Naor for the finite dimensional case [12].

Let H be an infinite dimensional complex Hilbert space. H carries the symplectic form $\Omega(a, b) = \text{Im}(\langle a, b \rangle)$.

The *infinite-dimensional Heisenberg group* \mathcal{H}_Ω , is the set of tuples (a, t) with $a \in H$, $t \in \mathbb{R}$ and the group law

$$(a, t)(a', t') = (a + a', t + t' + 2\Omega(a, a')).$$

Let G be a group with identity element e . A group seminorm on G is a function $G \rightarrow [0, \infty)$ satisfying $N(g^{-1}) = N(g)$ for all $g \in G$, $N(gh) \leq N(g) + N(h)$ for all $g, h \in G$ and $N(e) = 0$. Moreover, if $N(g) = 0$ if and only if $g = e$, then N is a group norm on G .

The function

$$N(a, t) = \sqrt{\sqrt{\|a\|^4 + t^2} + \|a\|^2},$$

is a group norm on \mathcal{H}_Ω . Namely, if N_1 and N_2 are group seminorms, then $\sqrt{N_1^2 + N_2^2}$ is a group seminorm. In this case, $N_1 = (\|a\|^4 + t^2)^{\frac{1}{4}}$ is the Koranyi norm. Given N ,

$$d_N((a, t), (a', t')) = N((a, t)(a', t')^{-1})$$

is a right invariant metric on \mathcal{H}_Ω .

Let G be a group, then a Hermitian kernel on G is a complex-valued function $K : G \times G \rightarrow \mathbb{C}$ satisfying $K(g, h) = \overline{K(h, g)}$ for all $g, h \in G$.

A Hermitian kernel on G is positive definite if

$$\sum_{i,j}^n K(g_i, g_j) c_i \overline{c_j} \geq 0$$

for all $g_1, \dots, g_n \in G$ and for all complex numbers $c_1, \dots, c_n \in \mathbb{C}$, with equality if and only if the c_i vanish. A Hermitian kernel on G is conditionally negative definite if

$$\sum_{i,j}^n K(g_i, g_j) c_i \overline{c_j} \leq 0$$

for all $g_1, \dots, g_n \in G$ and for all complex numbers $c_1, \dots, c_n \in \mathbb{C}$ satisfying $\sum_i^n c_i = 0$.

Theorem 2 (Schönberg [17]). *Let G be a group, $K : G \times G \rightarrow \mathbb{R}$ a real-valued kernel on G satisfying $K(g, g) = 0 \ \forall g \in G$. Then K is negative definite if and only if there exists a Hilbert space H and a function $T : G \rightarrow H$ such that $\forall g, h \in G : K(g, h) = \|T(g) - T(h)\|^2$.*

Thus, if d_N is negative definite, then there is a Hilbert space H and an isometry $T : (\mathcal{H}_\Omega, \sqrt{d_N}) \rightarrow H$. In particular, the mapping $T : (\mathcal{H}_\Omega, d_N) \rightarrow H$ is an AM mapping.

A complex-valued function F on a group G satisfying $F(g^{-1}) = \overline{F(g)}$ for all $g \in G$ induces a Hermitian kernel on G by defining $K_F(g, h) := F(gh^{-1})$. The function F is called positive definite if K_F is a positive definite Hermitian kernel on G .

Lemma 10. For any $\lambda \in \mathbb{R}$, the function

$$\Phi_\lambda(a, t) = e^{-|\lambda|\|a\|^2 + i\lambda t}$$

is positive definite on \mathcal{H}_Ω .

Proof: Indeed, the function Φ_λ satisfies the condition $\Phi_\lambda((a, t)^{-1}) = \overline{\Phi_\lambda((a, t))}$ and induces the kernel

$$K((a, s), (b, t)) := \exp(-|\lambda|\|a - b\|^2 + i\lambda(s - t - 2\Omega(a, b))).$$

The kernel can be rewritten as a product of three exponentials

$$\exp(-|\lambda|(\|a\|^2 + \|b\|^2)) \exp(i\lambda(s - t)) \exp(2|\lambda|(\operatorname{Re}(\langle a, b \rangle) - i \operatorname{sign}(\lambda) \Omega(a, b))).$$

The product of positive definite kernels is positive definite. The first two factors are positive definite because for all $c_1, \dots, c_n \in \mathbb{C}$ the matrix $(c_i \overline{c_j})_{i,j}$ is positive semidefinite. In particular, also the matrix $(\exp(i\lambda(s_i - t_j)) c_i \overline{c_j})_{i,j}$. The exponential of any positive definite kernel is again a positive definite kernel ([3] Proposition 8.2.). Thus the kernel K is positive definite if

$$\operatorname{Re}(\langle a, b \rangle) - i \operatorname{sign}(\lambda) \Omega(a, b)$$

is positive definite. This is $\langle a, b \rangle$ if $\lambda < 0$ and $\overline{\langle a, b \rangle}$ if $\lambda \geq 0$ and thus clearly positive definite. \blacksquare

Proposition 11. $d_N : \mathcal{H}_\Omega \times \mathcal{H}_\Omega \rightarrow H$ is conditionally negative definite.

Proof: The existence of $\frac{1}{2}$ -stable distributions implies that for all $\epsilon > 0$ there exists a non-negative integrable function $\varphi_\epsilon : \mathbb{R} \rightarrow [0, \infty)$ s.t. its Fourier transform $\hat{\varphi}_\epsilon(t) = e^{-\epsilon\sqrt{|t|}}$. Note that $\frac{1}{2\pi} \hat{\varphi}_\epsilon(x) = \varphi_\epsilon(x)$.

By the above lemma,

$$F_\epsilon(a, t) = \int_{\mathbb{R}} e^{-|\lambda|\|a\|^2 + i\lambda t} \varphi_\epsilon(\lambda) d\lambda$$

is positive definite on \mathcal{H}_Ω . Let

$$h_k(x) = \frac{k}{\pi} \frac{1}{k^2 + x^2}$$

denote the Cauchy distribution with scale parameter $k > 0$. We write around

$$\begin{aligned} F_\epsilon(a, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda t} \hat{h}_{\|a\|^2}(\lambda) \hat{\varphi}_\epsilon(\lambda) d\lambda \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda t} \widehat{h_{\|a\|^2} * \varphi_\epsilon}(\lambda) d\lambda \\ &= (h_{\|a\|^2} * \hat{\varphi}_\epsilon)(t). \end{aligned}$$

Since F_ϵ is positive definite, $\frac{1 - F_\epsilon}{\epsilon}$ is conditionally negative definite and

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1 - F_\epsilon(a, t)}{\epsilon} &= \lim_{\epsilon \rightarrow 0} \left[h_{\|a\|^2} * \frac{1 - \hat{\varphi}_\epsilon}{\epsilon} \right] \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \frac{1 - e^{-\epsilon\sqrt{|x|}}}{\epsilon} h_{\|a\|^2}(t - x) dx \\ &= \frac{\|a\|^2}{\pi} \int_{\mathbb{R}} \frac{\sqrt{|x|}}{\|a\|^4 + (t - x)^2} dx. \end{aligned}$$

We now show that for all $r, t \in \mathbb{R}$

$$r^2 \int_{\mathbb{R}} \frac{\sqrt{|x|}}{r^4 + (t-x)^2} dx = \pi \sqrt{\sqrt{r^4 + t^2} + r^2}.$$

By changing variables $x = r^2 y$ and $s = t/r^2$, the left hand side can be written as

$$\int_0^\infty \left(\frac{1}{1+(s-y)^2} + \frac{1}{1+(s+y)^2} \right) \sqrt{y} dy$$

This integral is equal to

$$\lim_{r \rightarrow 0} \lim_{R \rightarrow \infty} \frac{1}{2} \int_{C_{r,R}} \left(\frac{1}{1+(s-z)^2} + \frac{1}{1+(s+z)^2} \right) \sqrt{z} dz,$$

where $C_{r,R}$ is the keyhole contour with a branch cut along the positive real axis. The integrand has simple poles at $i \pm s$ and $-i \pm s$. By $\text{Res}\left(\frac{\sqrt{z}}{1+(s-z)^2}, i+s\right) = \frac{\sqrt{i+s}}{2i}$, $\text{Res}\left(\frac{\sqrt{z}}{1+(s-z)^2}, -i+s\right) = \frac{\sqrt{-i+s}}{-2i}$, $\text{Res}\left(\frac{\sqrt{z}}{1+(s+z)^2}, i-s\right) = \frac{\sqrt{i-s}}{2i}$ and $\text{Res}\left(\frac{\sqrt{z}}{1+(s+z)^2}, -i-s\right) = \frac{\sqrt{-i-s}}{-2i}$. Now adding all the residues and multiplying by $2\pi i$, we obtain the above equation as

$$\frac{\pi}{2} (\sqrt{i+s} - \sqrt{-i+s} + \sqrt{i-s} - \sqrt{-i-s}).$$

This further simplifies to

$$\begin{aligned} \pi \text{Re}(\sqrt{i+s} + \sqrt{i-s}) &= \pi (\text{Re}(\sqrt{i+s}) + \text{Im}(\sqrt{i+s})) \\ &= \pi \left(\sqrt{\frac{\sqrt{1+s^2}+s}{2}} + \sqrt{\frac{\sqrt{1+s^2}-s}{2}} \right) \\ &= \pi \sqrt{\sqrt{1+s^2}+1} \end{aligned}$$

So we have seen that the N -norm is a limit of conditionally negative definite functions on \mathcal{H}_Ω , so it is conditionally negative definite. \blacksquare

4 Applications to dimension theory

4.1 Dimension theory of finitely-generated groups

In 1993 M. Gromov introduced the notion of asymptotic dimension as a large scale analogue of Lebesgue's covering dimension [7]. The asymptotic dimension of a finitely generated group is a quasi-isometric invariant. Its most prominent application goes back to Guoliang Yu, who showed that any finitely generated group with finite homotopy type and finite asymptotic dimension satisfies the Novikov conjecture [21].

Definition 12. *Let X be a metric space. We say that the asymptotic dimension of X does not exceed n if for every uniformly bounded open cover \mathcal{V} of X there exists a uniformly bounded open cover \mathcal{U} of X of multiplicity $\leq n+1$ such that \mathcal{V} refines \mathcal{U} . We write $\text{asdim}X = n$ if it is true that $\text{asdim}X \leq n$ and $\text{asdim}X \not\leq n-1$*

The asymptotic dimension of a finitely generated group Γ has several interesting implications. For example, $\text{asdim } \Gamma = 0$ if and only if Γ is finite (Proposition 65. in [2]) and $\text{asdim } \Gamma = 1$ if and only if Γ is virtually free (Theorem 66. in [2]).

If G is a finitely generated nilpotent group, then its asymptotic cone G_ω is a Carnot group [14] and the asymptotic dimension of G coincides with the topological dimension of its asymptotic cone [2].

Theorem 3. *Let G and G' be a finitely generated nilpotent groups and $f : G \rightarrow G'$ an AM-map, then $\text{asdim}(G) \leq \text{asdim}(G')$. If G and G' are nilpotent Lie groups, then $\dim(G) \leq \dim(G')$. Furthermore, if $\text{asdim}(G) = \text{asdim}(G')$, then the asymptotic cones of G and G' are isomorphic graded Lie groups. Conversely, given nilpotent groups with isomorphic asymptotic cones, there exists an asymptotic-Möbius map between them.*

Proof: The mapping f induces a quasimetric mapping g between the asymptotic cones G_ω and G'_ω of G and G' , respectively.

In particular, g is an injective mapping from G_ω to G'_ω and hence the topological dimensions increase. Since G and G' are nilpotent, $\text{asdim}(G) = \text{top-dim}(G_\omega) \leq \text{top-dim}(G'_\omega) = \text{asdim}(G')$.

The case for nilpotent Lie groups follows from the fact that for nilpotent Lie groups $\text{top-dim}(G) = \text{top-dim}(G_\omega)$.

If $\text{asdim}(G) = \text{asdim}(G')$, the invariance of domain theorem implies that g is a homeomorphism. Since both g and g^{-1} are quasimetric, they are differentiable by Pansu's theorem a.e., and the differential of g is a graded isomorphism of groups at almost every point [15]. ■

4.2 Dimension theory of CAT(0)-spaces

Heinrich Jung established in the early 1900s that the diameter of a set in Euclidean space \mathbb{R}^n is related to the radius of the smallest enclosing ball by an inequality whose constant depends only on the dimension n [9]. More precisely, for any set $A \subset \mathbb{R}^n$

$$r \leq d(A) \sqrt{\frac{n}{2(n+1)}}, \quad (1)$$

where r is the radius of the smallest enclosing ball of A and $d(A)$ is the diameter of A .

The above inequality can be used to define a geometric dimension of CAT(0)-spaces. A CAT(0)-space X has *geometric dimension* less than n if and only if the Jung's inequality (1) is true in X [4].

Remark 13. *It turns out that the geometric dimension of a CAT(0)-space X is equal to the maximal topological dimension of all compact subsets in X [10].*

The geometric dimension can be transformed into a large-scale term in the following way. A CAT(0)-space X has *telescopic dimension* less than n if and only if for every $\delta > 0$ there is a $D > 0$ such that for every set A

in X with diameter greater than D ,

$$r \leq \left(\delta + \sqrt{\frac{n}{2(n+1)}} \right) d(A). \quad (2)$$

Somewhat more general,

Definition 14. *A space X whose asymptotic cones are CAT(0) has telescopic dimension $\leq n$ if every asymptotic cone has geometric dimension $\leq n$.*

We adopt the notation $\text{tele-dim}(X)$ for the telescopic dimension of X .

Theorem 4. *Let X and Y be spaces, whose asymptotic cones are CAT(0). If there exists an AM-map $f : X \rightarrow Y$, then the telescopic dimension increases e.g. $\text{tele-dim}(X) \leq \text{tele-dim}(Y)$.*

The theorem follows straight from theorem 1, remark 13 and the fact that the topological dimension increases under injective continuous maps.

It follows that there can be no AM-mapping from Euclidean buildings of finite rank r and hence telescopic dimension r to a Euclidean building of lower rank.

If there is a AM-mapping from a CAT(0)-space to a hyperbolic metric space, then X is also a hyperbolic metric space.

References

- [1] ASSOUAD, P. Plongements lipschitziens dans \mathbf{R}^n . *Bull. Soc. Math. France* 111, 4 (1983), 429–448.
- [2] BELL, G., AND DRANISHNIKOV, A. Asymptotic dimension. *Topology Appl.* 155, 12 (2008), 1265–1296.
- [3] BENYAMINI, Y., AND LINDENSTRAUSS, J. *Geometric nonlinear functional analysis. Vol. 1*, vol. 48 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2000.
- [4] CAPRACE, P.-E., AND LYTCHAK, A. At infinity of finite-dimensional CAT(0) spaces. *Math. Ann.* 346, 1 (2010), 1–21.
- [5] DE CORNULIER, Y. On sublinear bilipschitz equivalence of groups. *Ann. Sci. Éc. Norm. Supér. (4)* 52, 5 (2019), 1201–1242.
- [6] GROMOV, M. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, 53 (1981), 53–73.
- [7] GROMOV, M. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, vol. 182 of *London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, Cambridge, 1993, pp. 1–295.
- [8] GROMOV, M. *Metric structures for Riemannian and non-Riemannian spaces*, vol. 152 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1999. Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.
- [9] JUNG, H. Über die kleinste Kugel, die eine räumliche Figur einschliesst. *Journal für die reine und angewandte Mathematik (Crelles Journal)* 1901 (1 1901), 241–257.
- [10] KLEINER, B. The local structure of length spaces with curvature bounded above. *Math. Z.* 231, 3 (1999), 409–456.
- [11] LAWLER, G. F. Conformal invariance and 2D statistical physics. *Bull. Amer. Math. Soc. (N.S.)* 46, 1 (2009), 35–54.
- [12] LEE, J. R., AND NAOR, A. L^p metrics on the Heisenberg group and the Goemans-Linial conjecture.
- [13] NAOR, A., AND NEIMAN, O. Assouad’s theorem with dimension independent of the snowflaking. *Rev. Mat. Iberoam.* 28, 4 (2012), 1123–1142.
- [14] PANSU, P. Croissance des boules et des géodésiques fermées dans les nilvariétés. *Ergodic Theory Dynam. Systems* 3, 3 (1983), 415–445.
- [15] PANSU, P. Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math. (2)* 129, 1 (1989), 1–60.
- [16] PANSU, P. Large scale conformal maps. *Ann. Sci. Éc. Norm. Supér. (4)* 54, 4 (2021), 831–887.

- [17] SCHOENBERG, I. J. Metric spaces and positive definite functions. *Trans. Amer. Math. Soc.* *44*, 3 (1938), 522–536.
- [18] SCHRAMM, O. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.* *118* (2000), 221–288.
- [19] VÄISÄLÄ, J. Quasi-Möbius maps. *J. Analyse Math.* *44* (1984/85), 218–234.
- [20] WERNER, W., AND POWELL, E. *Lecture notes on the Gaussian free field*, vol. 28 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 2021.
- [21] YU, G. The Novikov conjecture for groups with finite asymptotic dimension. *Ann. of Math. (2)* *147*, 2 (1998), 325–355.

GEORG GRÜTZNER
 UNIVERSITÉ PARIS-SACLAY, CNRS
 LABORATOIRE DE MATHÉMATIQUES D'ORSAY
 91405 ORSAY CÉDEX, FRANCE
 DATASHAPE, CENTRE INRIA SACLAY
 91120 PALAISEAU, FRANCE
georg.gruetzner@universite-paris-saclay.fr

Orsay, September 30, 2021.
 2020 *Mathematical Subject Classification*.
Keywords and phrases. Möbius geometry, large scale geometry.