



HAL
open science

Mental calculations with rational numbers: strategies' range and strategy selection

Ioannis Papadopoulos, Michail Karakostas

► To cite this version:

Ioannis Papadopoulos, Michail Karakostas. Mental calculations with rational numbers: strategies' range and strategy selection. Twelfth Congress of the European Society for Research in Mathematics Education (CERME12), Feb 2022, Bozen-Bolzano, Italy. hal-03746631v2

HAL Id: hal-03746631

<https://hal.science/hal-03746631v2>

Submitted on 17 Oct 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Mental calculations with rational numbers: strategies' range and strategy selection

Ioannis Papadopoulos¹ and Michail Karakostas²

¹Aristotle University of Thessaloniki, Faculty of Education, Greece, ypapadop@eled.auth.gr

²University of Western Macedonia, Greece, milakis123@gmail.com

In this paper primary and secondary school students are followed while they try to perform operations with rational numbers first mentally and then in paper-and-pencil. The same tasks were given to the participants of both groups. The findings give evidence that no matter the context (mental or written calculations) the dominance of the mental form of the written algorithm is unquestionable for both groups. Moreover, most of the student were not able to explain their decision to use the algorithm in their mental calculations. The students who were able to justify their choice claimed that their criterion was the accuracy, speed and easiness provided by the algorithm which creates a paradox since for all the items on the study the algorithm is the most time-consuming strategy.

Keywords: Mental calculations, rational numbers, written algorithm.

Introduction.

Mental calculations with rational numbers have been the focus of several studies (Caney & Watson, 2003; Rezat, 2011; Carvalho & Ponte, 2019). The ability to perform mentally such calculations is considered significant. According to the Principles and Standards of the NCTM (2000), “students should develop and adapt procedures for mental calculations and computational estimation with fractions, decimals, and integers” (p. 220). In the last CERME, Papadopoulos et al. (2019) highlighted the dominance of the use of mental form of the written algorithm as a way to calculate mentally the outcome of certain operations with rational numbers. The discussion that followed the presentation was fruitful and raised questions concerning the possible reason the participants choose this strategy, or whether they know alternative strategies and if yes why they do not use them. Moreover, the audience in the conference was wondering whether the results would be the same in case the students were asked to solve the same tasks in paper-and-pencil.

In this setting the current study attempts to give some answers to the above-mentioned concerns. So, the research questions are as follows:

- (i) What is the range of the strategies employed by primary and secondary education students when they execute operations (first mentally and then in a written form) that involve rational numbers?
- (ii) On what criterion do they choose their mental calculation strategy?

Theoretical background

Most of the studies on mental calculation focus on whole numbers and their four operations (Rezat, 2011). However, in recent years there is an ongoing interest on mental calculations with rational numbers examining issues such as the comparison of rational numbers (Yang et al., 2009) and the operations with them (Caney & Watson, 2003; Papadopoulos et al., 2019). Caney and Watson (2003) recorded the strategies the participants used while trying to calculate mentally. They ended with a

series of different strategies: *'changing operation'* (e.g., subtraction to addition), *'changing representation'* (e.g., fractions to decimals and vice versa), *'use of equivalents'*, *'use of known facts'* (e.g., times table), *'repeated addition/multiplication'* (e.g., by doubling, halving), *'use of bridging'* (bridged to one/whole), *'working with parts of a second number'* (split the second number by place value or by parts), *'working from the left/right'*, *'using mental picture'*, *'using mental form of written algorithm'* and *'using memorized rules'*. They group these strategies in two main categories: instrumental and conceptual strategies. The first refer to strategies based on the use of procedural paths learned by rote. The second occurs when students make use of their knowledge on the specific set of numbers and operations in order to calculate mentally. Callingham and Watson (2004) found that in the realm of rational numbers it is easier to work mentally with fractions rather than with decimals or percentages and that division is more demanding compared to addition and subtraction (which holds equally for the set of whole numbers). Moreover, it seems that in the context of the addition and subtraction of decimal numbers the students preferred strategies they already used for mental calculations with whole numbers (Rezat, 2011). In their study on mental calculations with rational numbers by primary school students Carvalho and Ponte (2019) highlight two main findings: In their effort to work mentally the students (i) tend to use the strategy of *'changing representation'* (from fractions to decimals), and (ii) *'use numerical relationships strategies supported by propositional representations'* (p. 393). Another related issue is the ability of flexibility in mental calculations that has been raised by scholars such as Rathgeb-Schnierer and Green (2015) and the lack of which has been recorded not only for students but for teachers also (Lemonidis et al., 2018). Finally, Papadopoulos et al. (2019) who tried to record and compare strategies used by participants across all educational levels for the same tasks found that the mental form of the written algorithm dominated in the participants choices no matter the educational level (almost 43% for primary school students, 31% for secondary school students, 46% for University students from the Department of Primary Education and 59% for University students from the Mathematics and Engineering Departments). All the other strategies' percentages were close to 1%-2%. This raises the issue of how the solvers choose strategies in mental calculations. Indeed, the choice of the proper strategy is of critical importance. Threlfall (2000) described the process of choosing strategies as a series of certain steps: (i) prior analysis of the problem to recognize features that are associated to different possible strategies, (ii) decide which ones of these strategies are viable in terms of knowledge and skills, (iii) decide between the viable strategies the one that is (possibly) the easiest, and (iv) carry through the decision in practice. But as he admits *'neither children nor adults actually calculate in that way'* (p. 84). The idea of teaching criteria for deciding in advance which strategy to use is rather not feasible and perhaps this explains why there are no suggestions in the relevant research literature for direct teaching on how to decide strategically the proper way of mental calculation (Threlfall, 2002). So, there is no proof on the way students choose to calculate mentally and this is why in this paper we chose to examine this issue. Threlfall (2002) claims that mental calculation strategies are not purposefully selected by the students, but they occur since the students are interested in finding the solution rather than the method. More precisely, he claims that *"They are ways of thinking about mental calculations that do not describe the whole sequence to the solution, but concern just some of the steps, for example ways of beginning, ways of thinking about the numbers, and ways of relating the numbers to other knowledge"* (p. 42).

Setting of the study

The total number of the participants was 127 students from primary and secondary education. More precisely there were 65 students from grades 5 and 6, and 62 students from grades 10, 11 and 12 in a rural area of Northern Greece.

Five tasks in total were given to the students (Fig. 1)

$\frac{3}{6} + \frac{4}{8}$	$4 \times \frac{6}{12}$	$1\frac{1}{2} - \frac{1}{4}$	$2\frac{5}{10} \times 2$	$8 : \frac{4}{8}$
Item 1	Item 2	Item 3	Item 4	Item 5

Figure 1: Tasks posed to the students

The tasks' design followed two principles. First, there are more than one ways to calculate each item. Second, all the items can be calculated in a fast and easy way if the solvers notice the quantities involved in each operation. Therefore, item 1 includes the sum of two halves and therefore the answer is 1. Item 2 is about four halves, therefore 2. In item 3 if the solver sees $\frac{1}{2}$ as $\frac{1}{4} + \frac{1}{4}$ it is easy then to get the result $1\frac{1}{4}$. For item 4 the result should be the double of 2.5, therefore 5. Finally, for item 5 the solver must think how many halves are needed to get 8. Therefore, the answer is 16. This small collection of items involves all the four operations and a variety of combination of numbers (fractions, whole numbers, mixed numbers).

The whole study consisted of a two-step process. Initially all the participants were interviewed individually. They were asked to solve the tasks one-by-one mentally vocalizing their thoughts while solving them. They were not allowed to make written calculations or to keep some notes. There was no time restriction, and they could skip tasks in case they felt they couldn't solve them. No feedback was provided to them during their effort. All interviews were recorded and transcribed.

For the second part, the students were invited to work individually on paper-and-pencil. Now they had to solve each item with as many different ways as they could. After completing the solutions for each item, the students had also to write down their answer for the question: Can you explain why from all these different ways of solving this item, did you choose this specific strategy for your mental calculation?

The transcribed protocols (first part) and the students' worksheets (second part) constituted our data. For the first part of the study the data were analyzed in two levels. First, the answers were categorized according to whether they were correct, incorrect, unanswered or not codable. Second, the correct answers were distributed to the different strategies of Caney and Watson (2003). It must be said however that not all the strategies appeared in the students' answers and that some new strategies emerged. For the second part, the data analysis took place at a qualitative level on the basis of content analysis following a more deductive sort of thematic analysis (Mayring, 2014). The data were coded independently by the authors and validity and reliability were established by comparing sets of independent results, clarifying codes and re-coding data until agreement.

Results and Discussion

Table 1 summarizes the distribution of the total number of answers (635 answers for the first part of mental calculation and 721 for the second part of written calculations) across the different strategies.

Table 1: Distribution of answers across different strategies for both parts of the study

	1. Written algorithm	2. Equivalent	3. Parts of a number	4. Change representation	5. Reduction
Mental calculation	229/635 (36.06%)	6/635 (0.94%)	5/635 (0.78%)	8/635 (1.25%)	4/635 (0.62%)
Written calculation	296/721 (41.05%)	26/721 (3.6%)	7/721 (0.97%)	20/721 (2.77%)	10/721 (1.38%)
	6. Algebraic	7. Combine	8. Incorrect	9. Unanswered	10. No-codable
Mental calculation	0/635 (0%)	7/635 (1.1%)	263/635 (41.41%)	112/635 (17.64%)	1/635 (0.15%)
Written calculation	2/721 (0.27%)	37/721 (5.13%)	252/721 (34.95%)	68/721 (9.43%)	3/721 (0.41%)

Examining the different strategies, it seems to be an overlapping between the use of *Equivalent* and use of *Reduction* since from the mathematical point of view it is the same. Reduction results always to equivalent fractions. We consider them different based on the wording of the participating students who treated them as such. It was the solver's aim as this was expressed verbally that made us to decide if the solution is associated with one strategy or another. Another new strategy is the 'algebraic' one (despite its limited presence) that was not included in the list of Caney and Watson (2003). An example of this approach collected from answers in item 1 is:

$$\frac{3}{6} + \frac{4}{8} = x \Rightarrow 48 \cdot \frac{3}{6} + 48 \cdot \frac{4}{8} = 48x \Rightarrow 8 \cdot 3 + 6 \cdot 4 = 48x \Rightarrow 48x = 48 \Rightarrow x = 1.$$

The category of 'combination' has been also added to include answers that combine more than one strategy at the same calculation. For example, in the following answer for item 5 the student combined the *Reduction* and *Written algorithm* strategies: $8 \div \frac{4}{8} = 8 \div \frac{4 \div 4}{8 \div 4} = 8 \div \frac{1}{2} = 8 \cdot \frac{2}{1} = 16$. For the same item, another response combined the *Equivalent* and *Change representation* strategies: $8 \div \frac{4}{8} = 8 \div \frac{1}{2} = \frac{8}{0.5} = 16$.

Two interesting observations can be made based on the arithmetical data of Table 1. The first is related with the range of the strategies employed by the participants in their mental and written calculations. It can be said that the results are more or less the same in both cases. Papadopoulos et al. (2019) highlighted the dominance of the mental form of the written algorithm in mental calculations with rational numbers. But it seems now that the situation is the same no matter the way of calculation (mental or written). Therefore, it is not the context of the calculation that promotes the use of the algorithm. The second observation is that almost the total number of the collected answers is around the triplet algorithm-incorrect-unanswered (almost 94% and 86% for the mental and written

calculation respectively). This means that the main option for a correct answer is to use the algorithm. Otherwise, the most possible is to get an incorrect answer or skip the task.

The situation remains the same when the data from Table 1 are distributed across the two samples in Table 2. The algorithm percentages for primary education students for the mental and written calculations are 29.84% and 37.73% respectively. Interestingly no other strategies are employed by the primary school students except 2 answers using the ‘parts of numbers’ in the mental part of the study, and 2 cases in the written part (‘parts of numbers’ and ‘change representation’). All the students (almost 100%) are gathered around the same triplet mentioned earlier. The secondary education students employed several strategies in their mental and written calculations. However, the frequency of these strategies is small compared with the use of the algorithm. So, in the mental part of the study 42,58% of the correct answers were based on the use of algorithm while only 28 out of 310 answers (9,03%) employed other strategies. The situation is improved in the written part of the study since the percentage of the correct answers that use other strategies (except from algorithm) is increased to 25,31% (100 out of 395 answers). But again the dominance of the algorithm (43.79%) is unquestioned.

Table 2: Distribution of strategies across the two samples for both parts of the study

	Mental calculation		Written calculation	
	Primary Education	Secondary Education	Primary Education	Secondary Education
Written Algorithm	97 (29.84%)	132 (42.58%)	123 (37.73%)	173 (43.79%)
Equivalent		6		26
Parts of number	2	3	1	6
Change represent		8	1	19
Reduction		4		10
Algebraically				2
Combination		7		37
Incorrect	138 (42.46%)	125 (40.32%)	148 (45.39%)	104 (26.32%)
Unanswered	88 (27.07%)	24 (7.74%)	53 (16.25%)	15 (3.79%)
Non-codable		1		3
TOTAL	325	310	326	395

An alternative way to organize the data for the written phase of the study is according to the number of different strategies per item employed by the participants (Table 3). From the 325 answers given by the primary school students (65 students × 5 items) 123 employed one strategy (i.e., the mental form of the written algorithm) and the rest of them but one did not employ any strategy. Only one

student was able to use 2 different strategies for one item. The situation is slightly better for the secondary education students ($62 \times 5 = 310$ answers). There was a small number of answers that employed 2, 3, and 4 strategies (49, 13, and 3 answers respectively). But again, the main characteristic is that the majority of the answers are around 0 and 1 strategies. So, our claim that it is not the context that imposed the use of the algorithmic approach is rather strengthened.

Table 3: Number of strategies for the same task

Number of different strategies	Primary Education	Secondary Education	TOTAL
0 strategies	201 (61.84%)	122 (39.35%)	323
1 Strategy	123 (37.84%)	123 (39.67%)	246
2 strategies	1	49	50
3 strategies		13	13
4 strategies		3	3
TOTAL	325	310	635

Therefore, almost half of the attempts represent lack of any strategy while in the case of the successful attempts almost 8 out of ten were based on the use of just one strategy, that is the use of the mental form of the written algorithm. So, the research question about the range of the strategies employed in these two different contexts can be answered in a rather clear manner: No matter the context, the students can merely use the algorithm in their calculations, or they are unable to respond successfully.

Table 4: Reason for choosing a strategy

Reason of choosing a strategy	Primary Education	Secondary Education	TOTAL
Easy / fast / efficient	7 (2.15%)	80 (25.8%)	87 (13.7%)
Teaching practice	1	2	3
Certainty	1	28 (9.03%)	29
Lack of another knowledge	8	5	13
I don't know	308 (94.76%)	195 (62.90%)	503 (79.21%)
TOTAL	325	310	635

The second research question aims to reveal the criterion the students use to choose their mental calculation strategy. Our hope was to contribute to the issue raised by Threlfall (2002) that there is no proof on the way students choose to calculate mentally. The analysis of the collected answers resulted in five categories (Table 4). The students explained that they chose the specific approach in their mental calculation because (i) they think that this was the easier, faster, and more efficient way for the calculation, (ii) this is the way they were taught to calculate with rational numbers, (iii) they

felt secure with the specific strategy, (iv) this was the only strategy they knew, and (v) they were unable to provide any explanation for their decision. For the primary school students what is impressive is the very small number of responses. In (only) 17 (out of 325) cases the students were able to explain the way they worked. This means that for almost 95 out of 100 cases they were not able either to justify their choice or to find any strategy that would serve their purpose (unanswered items). For those who were able to provide an explanation this was mainly the easiness/fastness/efficiency of the algorithmic approach. For the secondary education students, for almost 63 (out of 100) cases the students were unable to justify their choices. From those who gave explanations, again most of them justified their choice on the basis of the easiness/fastness/efficiency of the algorithmic approach. In comparison to the primary school students the only difference here was the increased number of answers (28 vs 1) referring to the issue of the certainty the students felt with using the specific strategy for mental calculations. Actually, focusing on the total sample what is evident is the students' inability to justify their choices. But what is especially interesting is a paradox that seems to appear after analysing all the responses. From Tables 1-3 it can be seen that the students exhibited an almost exclusive preference to the mental form of the written algorithm. This choice was later justified by them as the most easy, fast, and efficient way of calculating. Its efficiency is unquestionable. Indeed, the correct application of the algorithm guarantees the correct result. But it is interesting that they consider it easy and fast. For example, for item 1, the sum $\frac{3}{6} + \frac{4}{8}$ can be immediately (in an easy and fast way) be seen as the sum of two halves which is equal to 1. On the contrary, the participants preferred to make the fractions having the same denominator ($\frac{3}{6} + \frac{4}{8} = \frac{12}{24} + \frac{12}{24}$), to add them ($\frac{3}{6} + \frac{4}{8} = \frac{12}{24} + \frac{12}{24} = \frac{24}{24}$), to reduce the sum, to find 1 ($\frac{3}{6} + \frac{4}{8} = \frac{12}{24} + \frac{12}{24} = \frac{24}{24} = 1$) and they considered this process as the most easy and fast.

Conclusions

The ability for mental calculations and the selection of the most suitable strategy for mental calculations are considered especially significant by the research community. In this paper we followed primary and secondary school students in their attempt to calculate (mentally and in paper-and-pencil) the same collection of tasks. It seems that the algorithm was the only option for primary school students no matter whether they calculated mentally or in paper-and-pencil. The secondary school students exhibited an ability to use alternative strategies, but the presence of these strategies was very small compared to the use of the algorithm. Most of the participants used just one strategy for their calculations in paper-and-pencil, which is indicative of limited flexibility (Heinze et al., 2009). Finally, most of the students justified their choice of the algorithmic approach considering it as being easy, fast, and efficient. From the mathematics point of view this creates a paradox since the use of algorithm provokes an increased cognitive load compared to many other mental approaches in calculating. The findings reveal some interesting aspects about mental calculations, but these findings cannot be generalized due to the relatively small number of participants. However, they deepen our understanding of the topic and challenge us for a future study to strengthen our arguments made here.

References

Callingham, R., & Watson, A. (2004). A developmental scale of mental computation with part-whole numbers. *Mathematics Education Research Journal*, 16(2), 69–86.

- Caney, A., & Watson, J. M., (2003). Mental computation strategies for part-whole Numbers. *AARE 2003 International Education Research Conference*, Auckland, University of Tasmania. Retrieved from <https://www.aare.edu.au/data/publications/2003/can03399.pdf>
- Carvalho, R., & da Ponte, J. (2019). Mental computation: an opportunity to develop students' strategies in rational number division. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of CERME11* (pp. 387-394). Utrecht, The Netherlands: ERME. hal-02400904
- Heinze, A., Star, J. R., & Verschaffel, L. (2009). Flexible and adaptive use of strategies and representations in mathematics education. *ZDM Mathematics Education*, 41(5), 535-540.
- Lemonidis, C., Tsakiridou, E., & Meliopoulou, I. (2018). In-service teachers' content and pedagogical content knowledge in mental calculations with rational numbers. *International Journal of Science and Mathematics Education*, 16(6), 1127-1145.
- Mayring, P. (2014). *Qualitative content analysis: Theoretical foundation, basic procedures and software solution*. Klagenfurt: Beltz
- Papadopoulos, I., Panagiotopoulou, S., & Karakostas, M. (2019). Mental calculations with rational numbers across educational levels. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of CERME11* (pp. 469-476). Utrecht, The Netherlands: ERME. hal-02401039.
- Rathgeb-Schnierer, E., & Green, M. (2015). Cognitive flexibility and reasoning patterns in American and German elementary students when sorting addition and subtraction problems. In K. Krainer & N. Vondrova (Eds.), *Proceedings of CERME9* (pp. 339-345). Prague, Czech Republic: ERME. hal-01281858
- Rezat, S. (2011). Mental calculation strategies for addition and subtraction in the set of rational numbers. In M. Pytlack, T. Rowland, & E. Swodoba (Eds.), *Proceedings of CERME7* (pp. 396-405). Rzeszow, Poland: CERME.
- Threlfall, J. (2000). Mental calculation strategies. *Research in Mathematics Education*, 2(1), 77-90.
- Threlfall, J. (2002). Flexible mental calculation. *Educational Studies in Mathematics*, 50(1), 29-47.
- Yang, D. C., Reys, R. E., & Reys, B. J. (2009). Number sense strategies used by pre-service teachers in Taiwan. *International Journal of Science and Mathematics Education*, 7(2), 383-403