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# Generalizing mathematically through changes in referents

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*In this pilot, opportunities to engage in mathematical generalization were identified in a section of a textbook from year 6. From a social-semiotic perspective, we explored how these opportunities were constructed linguistically. While passive verb forms and nominalizations constructed an independent character of mathematics, logic expansions constructed limitations for the generalizations. Changes in referents constructed opportunities for generalizing actions.*

*Keywords: Mathematical generalization, social semiotics, transduction, textbooks, primary school.*

## Background

Mathematical generalization (henceforth MG) has been raised as a big idea in mathematics, for example through scaffolding algebraic reasoning, functional thinking and many other mathematical activities (e.g., Dörfler, 1991). It is also known that students often have difficulties with, among other things, expressing generality and using generalized language (Mason, 1996). Generalized language may here be interpreted as a part of disciplinary literacy (Shanahan & Shanahan, 2008), as specialized texts and literacy skills are expected of students at more advanced levels of studies. However, big ideas are also a part of primary school mathematics (Madej, 2021). Therefore, the expected literacy skills and specialization are important to scrutinize also in texts for primary school.

Despite its importance MG is not in focus in the 2011 Swedish steering documents. The course plans hardly mention aspects of generalization (Hemmi, Lepik & Viholainen, 2013), and it has been shown that Swedish primary school textbooks after 2011 contain very low proportions of generalized arithmetic (Bråting, Madej & Hemmi, 2019). Still, textbooks largely seem to organize the teaching in Swedish primary school classrooms (Koljonen, 2020). This makes Swedish textbooks an interesting starting point for understanding opportunities to engage in MG.

Different aspects of language have been compared for various school years in Swedish textbooks in various subjects (e.g., Österholm & Bergqvist, 2013). Aspects of language have also been studied in textbooks in other countries (e.g., Alshwaikh, 2016). While the degree of abstraction and generalization has been studied in Swedish educational texts for social science, natural science and literature (Edling, 2006), to our knowledge so far, the ways in which MG is presented have not been investigated from a linguistic point of view. Therefore, this pilot study aims to explore and describe linguistic aspects of MG in a section of a textbook. The research questions are:

- What opportunities for mathematical generalization can be identified in the textbook section?
- In what ways are linguistic resources used to construct the mathematical generalization?

## **Mathematical generalization and a social semiotic perspective on mathematics**

MG entails both generalization as an object or conclusion, and generalizing as an act or process (Ellis, 2007; Harel & Tall, 1991; Mason, 1996). It has been regarded as e.g., mental processes or social interaction across agents and within specific contexts, and may be expressed through gestures, images and other semiotic resources, as well as formal symbols or words (Dörfler, 1991; Harel & Tall, 1991; Radford, 2018). Ellis (2007) takes an actor-oriented perspective and empirically identifies different ways that a learner may engage in MG. In her taxonomy, these ways are described as the *generalizing actions* of relating, searching and extending, and the *reflection generalizations* of identification or statement, definition and influence. Generalizing actions are inferred through activity and talk, while reflection generalizations are inferred through statements or the use of a result of a generalization. Since textbooks are always oriented towards the students using them, Ellis' actor-oriented perspective seems useful for the present study. However, it is not what students actually do, say or write when using textbooks, or students' mental processes, which are in focus here. Adapting Ellis (2007), we look at explicit opportunities in the textbooks to engage in generalizing actions, to read or state reflection generalizations or to read or use a result of such a generalization.

To understand how MG is constructed through written language and other semiotic resources in textbooks, we build on a social semiotic perspective and Systemic Functional Linguistics, SFL (e.g. O'Halloran, 2005). Central in this perspective is that in any act of communication we make choices of language in order to construct a certain meaning (e.g. Halliday & Matthiessen, 2004). This is also the case for semiotic resources such as mathematical notation and images (O'Halloran, 2005). Further, a change in the semiotic mode, the process of *transduction*, includes ontological shifts (Kress, 2010). Through the "re-articulation of meaning from the entities from one mode into the entities of the new mode" (Kress, 2010, p. 125), we believe such shifts may be important for the ontological construction of MG. In this paper, various semiotic resources are analysed with SFL to understand opportunities to read and write MG in a primary school textbook section.

### **Methods**

A textbook for year 6, the final year of Swedish primary school, was considered suitable since MG is often considered rather difficult for students to master (Mason, 1996), and therefore, it can be expected that a book for older students will contain the largest proportion of opportunities for MG. The text analysed in this study is taken from a year 6 book translated from Finnish into Swedish. This particular book seemed to include a large variety of opportunities for MG, compared to others.

In this pilot, we test the methods of analysis. We do not identify all textbook sections where MG is offered. Therefore, we selected an initial section on patterns, which was spontaneously deemed to offer MG. The selection is small, but we do not seek to generalize the findings in this pilot to a larger body of texts. To answer the research questions, two analyses were conducted. Both analyses were conducted by one of the researchers, checked by the other, and then discussed until agreed.

### **The analysis of mathematical generalization**

To identify the opportunities to engage in MG, we used a taxonomy of mathematical generalization (Ellis, 2007). It is presented in our version adapted to textbooks in Table 1.

**Table 1: Questions for identifying opportunities to mathematical generalization**

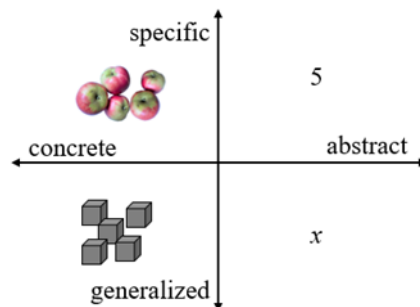
MG (Ellis, 2007)	Questions posed to the text
<i>Generalizing action</i>	<i>Do the students have the opportunity to ...</i>
Relating	<ul style="list-style-type: none"> <li>• relate situations through the formation of an association between two or more problems or situations?</li> <li>• relate objects through the formation of a similarity between two or more present objects?</li> </ul>
Searching	<ul style="list-style-type: none"> <li>• detect a stable relationship between two or more objects.</li> <li>• test if a procedure remains valid for all cases?</li> <li>• check whether a detected pattern remains stable across all cases?</li> <li>• determine if the outcome of the action is identical every time?</li> </ul>
Extending	<ul style="list-style-type: none"> <li>• expand to a larger range of cases than that from which the phenomenon originated?</li> <li>• remove particulars to develop a global case?</li> <li>• operate on an object to generate new cases?</li> <li>• repeat an existing pattern to generate new cases?</li> </ul>
<i>Reflection generalization</i>	<i>Do the students have the opportunity to write (in tasks) or read (in introductory text) ...</i>
Identification or statement	<ul style="list-style-type: none"> <li>• the identification of a property?</li> <li>• a statement of commonality or similarity, or of a general phenomenon?</li> </ul>
Definition	<ul style="list-style-type: none"> <li>• a definition of a class of objects all satisfying a given relationship, pattern or other phenomena?</li> </ul>
Influence	<ul style="list-style-type: none"> <li>• an implementation of a previously developed generalization or an adaption of an existing generalization to apply to a new problem or situation?</li> </ul>

### **The SFL-analysis**

Linguistic generalizations may be constructed through choices of referents and nominalizations (Halliday & Matthiessen, 2004). These are the main features explored in this analysis.

*Referents* are experiential elements in the text. Different types of noun phrases, and how they change, indicate different ways of using language (Halliday & Matthiessen, 2004). Referents can have an everyday character or be technical, and they may be presented in a variety of semiotic modes (Kress & Van Leeuwen, 2006). They may also be placed on scales between physical and abstract, specific and general (Edling, 2006). A move in the text from a specific to a general referent may realize a linguistic generalization in the text, whereas a move from a concrete to an abstract referent instead

may realize a linguistic abstraction in the text (Figure 1). When a referent is changed into a different semiotic resource it is called a transduction which at the same time re-articulates the ontological meaning of the referent (Kress, 2010). For instance, a picture of 5 apples which then is followed by the number 5 entails an ontological shift from concrete to abstract meaning. Linguistic changes of referents including transduction may thus indicate opportunities in the textbook for MG. This will be explored in the analysis.



**Figure 1: Abstraction and generalization in different referents, adapted from Edling (2006)**

A *nominalization* is an incongruent expression, or grammatical metaphor (Halliday & Matthiessen, 2004), where a noun (e.g. *subtraction*) is used instead of the corresponding verb or adjective form (e.g. *subtract*). Nominalization is a means of constructing generalization because an operational process, expressed by a verb, is reified into a general concept by the nominal form. Operational processes are mostly tied to specific situations where calculations are needed, whilst general concepts describe mathematical relations without necessarily linking to specific situations. Nominalization also hides human participation, since there is no explicit human agent carrying out the action. This downplaying of human agents further accentuates generalization. Therefore, generalization may also be indicated through a *passive verb form*, e.g. in the phrase “can be used” instead of “you can use”. Our analysis explores in what respect nominalizations and passive verb forms contribute to MG. To avoid overlooking other important linguistic features, the instances of MG in the textbook section were read, reread and discussed with respect to the semiotic resources used in them. For instance, expansions of clause complexes show in what ways a text may be developed (Halliday & Matthiessen, 2004); this might contribute to MG. However, the analysed text mostly consists of main clauses.

Finally: In a certain sense, textbook tasks comprise one half of a dialogue where the textbook author asks questions and the students answer. We acknowledge that in such a dialogue, any feasible reflection generalization will be constructed in the answer and not the question. Since this study does not look at actual answers, opportunities for reflection generalization are therefore not linguistically analysed in the tasks, only in the introductory text.

## Results

Opportunities to engage in mathematical generalization, MG, are given in the form of generalizing actions and to some extent in the form of reflection generalization. To answer the research questions, the results are structured by what is identified in introductory text and tasks respectively. All translations from the textbook section are ours.

## Nominalizations and passive verb forms constructing an independent character

The introductory text provides opportunities to engage in *generalizing actions* and read *reflection generalizations*. The top of the page question and the two following sentences (Fig.2) constitute a prompt to search for relationships, and thus models *searching* in the form of detecting a stable relationship between objects. A similar statement is made for decreasing number sequences using subtraction and division. Finally, “You can also find out the rule ...” models *extending* to other ways of investigating number sequences. All three statements “If the number sequence [increases/decreases] ...” and “You can also ...” are descriptions of general strategies for investigating patterns and therefore examples of *reflection generalization as identification or statement*. Hence, in the introductory text, the generalizing actions and reflection generalizations seem to coincide

### 4.4 Finding patterns in number sequences

Seeing and using patterns in sequences of numbers can be used in problem solving and programming.

How do you find the next number in the sequence of numbers?

First you find out if the numbers in the sequence become larger or smaller.

- If the number sequence increases, you can find out the following number using addition or multiplication.

#### 2, 5, 8, 11, 14

The rule is +3 (add three).  
The following numbers are thus 17 and 20.

- If the number sequence decreases, you can find out the following number using subtraction or division.

#### 30, 26, 22, 18, 14

The rule is -4 (subtract 4).  
The following numbers are thus 10 and 6.

There are many different ways to form patterns in sequences of numbers. You can also find out the rule by comparing how the differences between the numbers increase.

+3, +6, +12, +24,  
2, 5, 11, 23, 47,

The rule here is that the difference of two adjacent numbers are doubled.

- Look at the series of images
  - Write a sequence of numbers that describes how the number of sticks in the figures changes.
  - Write a rule that describes how the number of sticks in the figures changes.
  - How many sticks does the seventh figure consist of?



figure 1



figure 2



figure 3

- How do you know your answer is correct?

- How does the sequence of numbers continue?

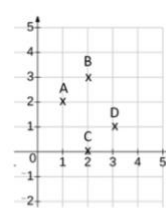
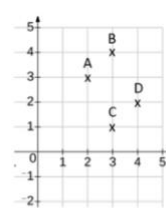
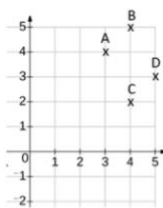
- 1, 5, 9, 13, 17,
- 37, 34, 31, 28, 25,
- 1, 2, 4, 7, 11,
- 3, 6, 12, 24, 48,
- 160, 80, 40, 20, 10,

Do like this:

Rule:
Next number:

- Write the coordinates of the points A, B, C and D in the fourth image.

- Draw the coordinate plane. Draw the points and write the coordinates of the fourth image.
- How are the new x- and y-coordinates formed?



Do like this:

a.	A = ( , )	B = ( , )
	C = ( , )	D = ( , )
b.	Answer:	



Illustration of a magpie with a red pencil in its beak

Figure 2: The textbook section Find patterns in number sequences

The written referents in the three statements in the introductory text mentioned above, e.g. “patterns in number sequences” and “the following number” change to the specific numbers e.g. “2, 5, 8, 11, 14”. The ontological meaning in this transduction shifts from generalized to specific. The specific number sequences in the introduction thus function to explain and unpack the *reflection generalizations*. The nominalizations in the four arithmetic operations in these statements construct the number sequences as structures: it does not matter which specific numbers are added, multiplied, subtracted or divided, because it is the operations in general which are relevant, not their results. The passive verb form “are doubled” constructs *searching* as something independent from human beings: the activity does not render different results (e.g. as tripling instead of doubling), depending on who performed the investigation. This independent character is a quite fundamental aspect of MG. Finally, the introductory text is expanded logically in the reflection generalizations “If the number sequence

[increases/decreases] ...” constructing a condition for when the procedure described in the main clause is valid. These expansions thus limit the range of the generalizations.

### **Generalizing actions in tasks constructed through changes in referents**

Task 1 prompts a re-articulation of the number of sticks in a given pattern into a number sequence. Through forming a similarity between the two objects, an opportunity to *relating* is constructed. The ontological meaning is re-articulated through a transduction from the generalized-concrete sticks to specific-abstract numbers. An opportunity to engage in *searching* is given through writing the rule for how the pattern changes. Here, the ontological meaning shifts from the specific-abstract numbers to the generalized-abstract rule, formulated as “+3 (add three)”. 1c prompts to continue the number sequence and generate a new case, the “seventh figure”. Here, the transduction re-articulates the generalized-abstract “a rule”, to the specific-abstract number of sticks which is supposed to be calculated. In this way, the ontological shift supports a control of the identified rule and the test for new cases, thus *extending* by continuing the pattern. The last part of task 1 gives the opportunity to write an *identification or statement*. However, this opportunity could not be analysed linguistically.

Task 2 is fairly similar to task 1. It prompts to re-articulate specific-abstract number sequences as generalized-abstract rules for the sequences and thus gives an opportunity to *searching*. The prompt in the last part is to re-articulate new cases of specific-abstract numbers, which gives an opportunity to extending. Since no concrete figures are given, this task does not comprise *relating*. The picture of the checkered piece of paper in the task and what is written on it, explicitly models how to engage in searching and extending the number sequences. In task 1 and 2, *searching* is then constructed through transduction from specific-abstract to generalized-abstract, whereas *extending* is constructed through the ontological shift back from generalized-abstract to specific-abstract meaning.

Task 3 gives the opportunity to a *reflection generalization* as an *influence* since the previously developed generalization of how to work with patterns is adapted to a new situation, consisting of coordinates which constitute successive sets of ordered pairs. The first step prompts to “[w]rite the coordinates ... in the fourth image”, so focus is on extending through continuing the pattern, to generate a new case. *Relating* and *searching* are thus not supported ahead of extending, as in the previous tasks. 3a prompts “draw” and “write”. It could be argued that these requests include relating the points A, B, C and D in the sequence of graphs to their more abstract re-articulations as sets of ordered pairs of numbers. However, it is only the points and “coordinates of the fourth image” which are explicitly asked for. In 3b, the question “How are the new x- and y-coordinate formed?” could include searching for a stable relationship between the points A, B, C and D in the sequence of graphs. Just as in 3a, only the “new” coordinates are asked for. The picture of the checkered piece of paper in task 3 is different from in task 2. It supports writing down the coordinates for one set of points A, B, C and D, and to write the “Answer”.

To solve task 3, learners thus need to recognize the steps of relating and searching the pattern in the first three graphs without a prompt to do so. *Extending* to the fourth picture is the only explicit request. The steps include transductions from the graphical mode to numbers (3a), and from the numbers to formulating the rule of the pattern (3b), respectively. As in task 1 and 2, *searching* seems to be constructed through an ontological shift from the specific-abstract to the generalized-abstract. The

graphs are interpreted as concrete referents in this study since they have a spatial extension. *Relating* in task 3 is thus constructed through an ontological shift from concrete to generalized meaning.

## Concluding remarks

This pilot analysis of a section of a mathematics textbook has revealed opportunities to engage in *generalizing actions*, as well as reading and writing *reflection generalizations*. However, no opportunities to read or write *definitions* are given, nor to read a reflection generalization as *influence*. Swedish textbooks have low proportions of generalized arithmetic (Bråting et al, 2019) and mathematical generalization, MG, include many aspects (Ellis, 2007). We therefore believe that a forthcoming comparative study of textbooks would reveal differences in opportunities for MG. Further, different proportions of *searching* might be a distinguishing feature since searching has been prominent in the textbook example analysed in this study. A study of students' solutions or work with the textbook section may contribute to the understanding of what MG actually takes place.

In the introductory text, opportunities for reading *identification or statement* are constructed through nominalizations of the four arithmetic operations. The range of two *statements* about patterns are constructed through logical expansions in the text. Moreover, passive verb forms and nominalizations construct patterns as structures, independent of human agents. In this sense, the *identification or statement* does not only model reflection generalization, but also expresses the general character of mathematics. These linguistic features may be interpreted as a part of the disciplinary literacy (Shanahan & Shanahan, 2008) which is expected of students, in order to understand MG. Finally, the reflection generalizations in the introductory text are unpacked and explained through the ontological shift which occurs in the transduction from written text to number sequences. In this way we can see that various semiotic resources are used to express MG.

Opportunities for generalizing actions in the analysed tasks are mainly constructed through different transductions, i.e. changes in the referents' semiotic mode. When *relating*, the ontological meaning shifts from concrete to more abstract; when *searching*, it shifts from specific-abstract to general-abstract. When *extending*, the ontological meaning shifts back from generalized-abstract to specific-abstract referents. Therefore, for a textbook to support opportunities for learning *generalizing action*, we believe that transduction may be a key feature. Further, to enhance opportunities for MG attention should be paid not only to the change between concrete and abstract referents, but also between referents which are specific and generalized, and referents presented in different semiotic modes.

## References

- Alshwaikh, J. (2016). Investigating geometry curriculum in Palestinian textbooks: Towards multimodal analysis of the Arabic mathematics discourse. *Research in Mathematics Education*, 18(2), 165–181. <https://doi.org/10.1080/14794802.2016.1177580>
- Bråting, K., Madej, L. & Hemmi, K. 2019. Development of algebraic thinking: opportunities offered by the Swedish curriculum and elementary mathematics textbooks. *Nordic Studies in Mathematics Education*, 24(1), 27–49. <http://uu.diva-portal.org/smash/get/diva2:1290139/FULLTEXT01.pdf>
- Dörfler, W. (1991). Forms and means of generalization in mathematics. In A. Bishop (Ed.), *Mathematical knowledge: Its growth through teaching* (pp. 63–85). Erlbaum.



- Edling, A. (2006). *Abstraction and authority in textbooks. The textual paths towards specialized language*. (Doctoral dissertation, Uppsala University, Sweden) <http://www.diva-portal.org/smash/get/diva2:168583/FULLTEXT01.pdf>
- Ellis, A. (2007). A Taxonomy for Categorizing Generalizations: Generalizing Actions and Reflection Generalizations. *The Journal of the Learning Sciences*, 16(2), 221–262. <http://dx.doi.org/10.1080/10508400701193705>
- Hemmi, K. Lepik, M. & Viholainen, A. (2013). Analysing proof-related competences in Estonian, Finnish and Swedish mathematics curricula – towards a framework of developmental proof. *Journal of Curriculum Studies*, 45(3), 354–378. <https://doi.org/10.1080/00220272.2012.754055>
- Halliday, M. A. K., & Matthiessen, Christian M. I. M. (2004). *An introduction to functional grammar* (3rd ed.). Routledge.
- Harel, G., & Tall, D. O. (1991). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics*, 11(1), 38–42. <https://www.jstor.org/stable/40248005>
- Koljonen, T. (2020). *Finnish mathematics curriculum materials and teachers' interaction with them in two cultural-educational contexts*. (Doctoral dissertation, Åbo Akademi University). <https://www.doria.fi/handle/10024/178511>
- Kress, G. R. (2010). *Multimodality: a social semiotic approach to contemporary communication*. Routledge.
- Kress, G. R., & Van Leeuwen, T. (2006). *Reading images: The grammar of visual design* (2nd ed.). Routledge.
- Madej, L. (2021). *X – men sen då? Algebrans stora idéer från första klass till högre matematik. Med fokus på tidig algebra i Sverige*. [X – then what? Big ideas in algebra from primary education to higher mathematics. With a focus on early algebra in Sweden]. (Doctoral dissertation, Uppsala University). <http://uu.diva-portal.org/smash/get/diva2:1503354/FULLTEXT01.pdf>
- Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra* (pp. 65–86). Kluwer.
- Radford, L. (2018). The emergence of symbolic algebraic thinking in primary school. In C. Kieran (Ed.) *Teaching and learning algebraic thinking with 5-12-year-olds, ICME-13 monographs* (pp. 3–25). Springer.
- Shanahan, T. & Shanahan, C. (2008). Teaching disciplinary literacy to adolescents: rethinking content-area literacy. *Harvard Education Review*, 78(1), 40–59. <https://doi.org/10.17763/haer.78.1.v62444321p602101>
- Österholm, M. & Bergqvist, E. (2013). What is so special about mathematical texts? Analyses of common claims in research literature and of properties of textbooks. *ZDM Mathematics Education*, 45(5), 751–763. <https://doi.org/10.1007/s11858-013-0522-6>