Teaching beyond the mathematics of the moment

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In this paper, we showcase a method of identifying opportunities during the lesson where the communicational patterns go beyond the mathematics of the moment, namely beyond the boundaries of the curriculum. Drawing on the literature around Horizon Content Knowledge and building on the Theory of Commognition, we look at a lesson from one teacher and study instances where her communication with the students goes beyond the mathematics of the moment. Our analysis identifies three types of opportunities (manifested, potentials of the discussion and potentials of the task). Further analysis of manifested opportunities indicates that intersubjectivity is a characteristic of the discourse of the teacher at the mathematical horizon.

Keywords: Discourse, mathematical horizon, teaching practices, mathematics.

Introduction

This paper reports on preliminary results from the first author's doctoral study that aims to explore in-service mathematics teachers' discourse that goes beyond the boundaries of the curriculum, what is described by Ball and Bass (2009) as "the larger significance of what may be only partially revealed in the *mathematics of the moment"* (para. 17, our emphasis). The importance of this idea has been described in the literature in connection to teacher actions in response to students' contributions and towards addressing students' learning needs (Ball & Bass, 2009). However, the boundaries of the curriculum, even more going beyond them, are not always identifiable. Initially, this study, in its effort to describe such boundaries, was inspired by Ball and colleagues' idea of the Horizon Content Knowledge (HCK), one of the mathematical domains of the Mathematical Knowledge for Teaching (MKT) model for teacher professional knowledge:

> an awareness – more as an experienced and appreciative tourist than as a tour guide – of the large mathematical landscape in which the present experience and instruction is situated. (Ball & Bass, 2009, para. 17)

HCK is theorised to influence, among others, teaching practices related to noticing and evaluating mathematical significance in what the students are saying, foreseeing and making connection across educational levels and disciplines and evaluating opportunities (Ball & Bass, 2009). So far, several studies drew on the idea of the HCK with variations in the use and the narratives about the 'horizon' metaphor (see also Papadaki, 2019). Evidence of the role of the HCK in the quality of teaching relys on these different uses and narratives of the 'horizon' metaphor. In our work, we argue that bridging the gap between an all-encompassing mathematical knowledge for teaching and teaching practices aiming to enrich the learners' experience of mathematics as a subject, could lessen the reliance on the metaphor. To this aim we propose a discursive approach to the "horizon" embedded in the communicational patterns of the classroom. Specifically we draw on the commognitive theory (Sfard, 2008) to address the theoretical and methodological question: *How can we identify opportunities during the lesson where the communicational patterns go beyond the mathematics of the moment?*

Theoretical background

To study the communicational patterns in the mathematics classroom, we use the *Theory of Commognition* (Sfard, 2008). According to this theory, a discourse is signified by four elements: *word use*, *endorsed narratives*, *visual mediators* and performed *routines*. Thus, mathematical discourse is the type of communication signified by specific word use (e.g., 'circles'), visual mediators (e.g., diagrams, symbols), endorsed narratives (e.g., theorems, definitions) and routines (e.g., calculating angles). Learning of mathematics is conceptualised as change in learners' discourse (Sfard, 2008). Tabach and Nachlieli (2016) define teaching as "the communicational activity the motive of which is to bring the learners' discourse closer to a canonic discourse" (p. 303). In this study, the canonic discourse is the Mathematical Discourse[1] of the class which is confined by the curriculum. The role of the teacher in mediating changes is vital. The *Mathematical Discourse for Teaching* (*MDT*) model (Cooper, 2016; Mosvold, 2015), an adaptation of the MKT, provides a basis to study teachers' discourses shifting the attention to word use, narratives, visual mediators, routines and "the kinds of mathematical activities that are valued" (Cooper, 2016, p. 21).

MDT consist of two main Discourses that could be viewed as a way to partially distinguish between mathematical narratives situated in the social context of the classroom (Mathematical Discourse) and pedagogical narratives situated in the context of teaching mathematics (Pedagogical Content Discourse). The two Discourses are distinguished into six sub-Discourses. One of them is the *Discourse at the Mathematical Horizon* (*DMH*), which is described as "patterns of mathematical communication that are appropriate in a higher grade level" (Cooper & Karsenty, 2018, p. 242). In our work DMH spans across the Mathematical and Pedagogical Discourses and regards the recognition of mathematical as well as pedagogical significance in students' work beyond the mathematics of the moment by addressing, for example, issues of content (e.g., 'what should be learnt') and access (e.g., 'who should learn').

So far, the MDT has been used in the context of teachers' professional development (e.g., Cooper, 2016; Mosvold, 2015). However, in our work we aim to identify and analyse aspects of DMH in lesson observations, where Mathematical and Pedagogical Discourses are intertwined in teaching actions. To this purpose, we encompass recent developments of the Theory of Commognition. We are interested in identifying teaching practices, seen as routines performed by the teacher in the mathematics classroom, that go beyond the expected Mathematical Discourse of that classroom. Attending to the mathematical elements of the communication between teacher and students could provide evidence about mathematical aspects of DMH. To account for pedagogical aspects, we adopt Nachlieli and Elbaum-Cohen's (2021) view that teaching practices could be considered as objects of Pedagogical Discourse. Furthermore, to investigate situations in which the communication between the teacher and her students makes or does not make sense at both ends, we use the notion of intersubjectivity, namely "an action that makes sense from the perspective of two discourses – the learner's and the expert's–which may be incommensurable" (Cooper & Lavie, 2021, pp. 8–9).

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[1] Here, the term 'Discourse' denotes community established patterns of communication while 'discourse' is used to refer to the individualised version of that communication between the interlocutors.

Data and methods

The study is conducted in England and participants are secondary school mathematics teachers, their students and teacher educators. In England, schools develop a program of study following the guidelines of the curriculum for Key Stages 3 and 4 (KS3, ages 11 to 14 and KS4, ages 15 to 16) and the specifications of the qualification provider the school collaborates with. Thus, what students are expected to learn each year varies across schools. Mathematics teachers prepare their lessons using resources available on the internet, textbooks or their school's repository. Data of lesson observations consist of an audio recording of the lesson and notes produced by the first author during the observation and revised shortly after. The notes include recreations of the writings on the board, the slides used by teachers and accounts of their actions during the lesson. The lesson was transcribed, we also produced a factual account summarising the lesson. The data collection process complies with the code of ethics of the University of East Anglia.

The analysis of the data consists of four phases. First, we created a template to study teacher's contributions by dividing each lesson into sections according to the actions of the teacher (e.g., talking to groups of students, introducing a task, initiating a whole class discussion etc.). Second, all sections were coded in relation to the mathematical objects (e.g., angles) and/or practices (e.g., measurement of angles) accounting for the discursive elements (word use, visual mediators, narratives, and routines) identified in each section. By comparing the codes with the topic of the lesson, as identified by the teacher and the specifications of the curriculum, we eliminated the sections that were directly related to what the students are expected to learn during the academic year. We regarded the remaining sections as opportunities to engage in communication beyond the mathematics of the moment and proceeded analysing them further. Thirdly, we categorised the identified opportunities in three groups: *potential of the task* (the mathematical object or practice is not made explicit during the lesson, it is only attributed by the analysis), *potential of the discussion* (the mathematical object or practice is made explicit in the lesson but is not addressed by the teacher), and *manifested* (the mathematical object or practice is made explicit and addressed by the teacher during the lesson). Finally, we revisited the opportunities proposed in the third phase to identify deviations or alignments in teacher-students communications from the intersubjectivity perspective.

Exemplification of the analysis: The lesson of the 'nine-point circle'

Here, we draw on one lesson of a newly qualified mathematics teacher, Liz (alias), and her 11-year-old students. The students are in Year 7, the first year of lower secondary education. Liz designed the lesson by combining resources available on the internet and the school's repository. For her lessons, Liz shares slides on the interactive whiteboard. At the beginning of the lesson we discuss here, Liz invited the students to calculate the value of an angle in a composite shape. Figure 1, depicts the main task of the lesson that consists of four parts (1, 2, 3 and 4). The common characteristic of the parts is a circle with nine equally spaced points on its circumference. Liz referred to this shape throughout the lesson as "the nine-point circle". In part 1, the students were asked to identify all the different triangles with one vertex on the centre of the circle and the others on two of the nine points and calculate their angles. In part 2, they were asked to find a way to calculate the angles in a triangle with all the vertices on the circle. In part 3, the students were asked to find their own triangles by

joining three points on the circle and work on their angles. Finally in part 4, Liz asked the students to calculate the angles ACB and ADB and tell her if they notice anything. By the end, Liz pointed out that the observation made in part 4 was related to one of the 'circle theorems' included in the highest band of their qualification requirements (GCSE, acquired at age 16).

In this lesson, we identified four opportunities to engage in communication beyond the mathematics of the moment. Two of them were categorised as *manifested* ('auxiliary lines' and the 'Star-Trek lemma'), one *potential of the discussion* and one *potential of the task.* In the following sections, we present the opportunities alongside the curriculum expectations.

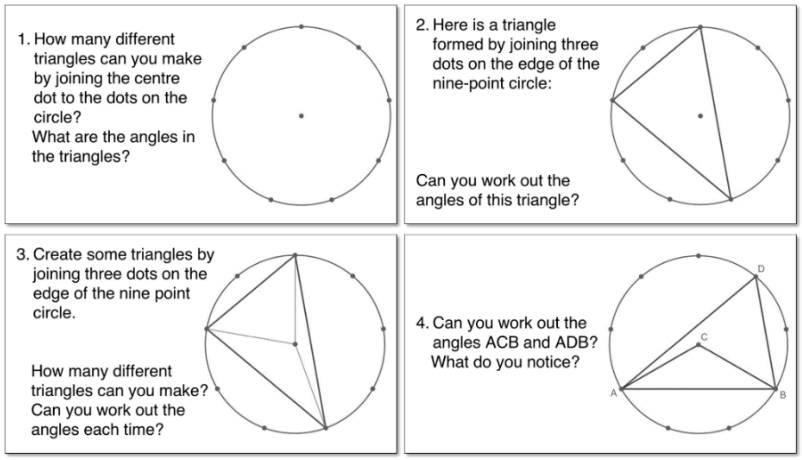


Figure 1: A recreation of the main task of the lesson

Manifested opportunities to go beyond the mathematics of the moment

Auxiliary lines: The completion of the main task, requires drawing auxiliary lines (i.e., additional lines) in order to divide the inscribed triangle in part 2 appropriately and calculate its angles. The process of drawing auxiliary lines was used by Liz as an approach to tackle an open problem.

Liz: Yeah. Right. [slide of part 2 on the board] So, I've got a slightly trickier question for you now. So here is a triangle formed by joining three dots on the edge of a nine-point circle, nine-point circle [points towards the diagram]. However, this time it doesn't go through the centre. Can you work out the angles of this triangle?

The process of drawing additional lines is not new to students at this age. As part of their primary and secondary school education, the students are expected to use auxiliary lines for the purposes of calculating the area of composite shapes or identifying a line of symmetry. Highly attaining students are also expected to learn to use auxiliary lines when proving certain theorems, in known situations, at the end of KS4, but not at Year 7. Also, this task requires the students to explore unfamiliar to them situations where auxiliary lines are necessary. Similar uses of auxiliary lines are observed in more of Liz's lessons suggesting that engagement with this process was not an one-off event.

Initially, Liz prompts the students to explore possible ways of tackling the problem. The students try to measure or estimate the angles of the given triangle. Five minutes later, she addresses the class:

Liz: […] it's taken him [student A] quite a long time to ask me a really, really good question. But, I think I should share it with everybody. He said, can I split it up into other triangles.

Student B: Oooh.

From a commognitive perspective, drawing auxiliary lines to modify the triangle in an appropriate way is a routine. For student A, the initiation of this routine is prompted when other routines failed. However, he requested Liz's approval before moving on. For other students the initiation was prompted by the teacher:

Liz:	Yeah, I'm not going to tell you the answer. I'm going to show you the clue. So, this is the clue. [draws three line segments from the centre to the edges of the triangle]
Student C:	Oh.
Liz:	[…] if you can split it into other triangles and using the triangles where we can where we can start. We know some facts, don't we? An isosceles triangle is nice. I'm not trying to find three unknown angles now, am I? I know that two are the same. […] plus we did all that work beforehand.

Here the procedure is presented to the students as a "clue". The use of the word could signify a step to a direction that was previously hidden from the students who are now invited to follow this clue to complete the task. The use of the phrase "split up into other triangles" and the absence of words like 'draw', 'line segment', that are usually present in formal geometric proofs, indicates that the discussion is mediated by the specific drawings of the task (Figure 1). We interpret the use of these words as evidence of intersubjectivity. For the students, this task is a new situation where they need to draw additional lines to solve a problem - similar to what they are doing when they are asked to calculate the area of a composite shape. While for Liz, this is a routine towards the modification of a shape in order to use properties established in earlier stages of a task. This routine is not part of the canonic discourse (Tabach & Nachlieli, 2016). Liz presents the students with a broad spectrum of situations that promote the routine of auxiliary lines beyond the requirements of the curriculum of Year 11. The instance presented here illustrates her teaching practice of providing gradual support to her students, of expanding the applicability of the routine and of establishing this process as conventional in geometric reasoning. Evidence of this practice were observed across Liz's lessons.

The Star-Trek lemma: Liz uses the 'nine-point circle' to introduce her students to a theorem about angles in circles also known as the Star-Trek lemma:

The angle subtended by an arc at the centre is twice the angle subtended at the circumference.

Liz presents the last part of the main task (part 4, Figure 1) as a challenge to the students:

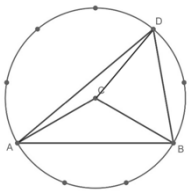
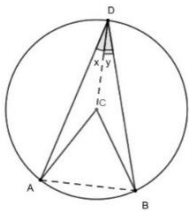
Liz:	So, I'm going to give you one last challenge, which is to take a look at this. [Liz puts up the slide with part 4 of the task]. And you're going to need to start calculating the angles and see if you notice anything. So, I'd like you, actually we can start from, we've got a starting point, haven't we? We know this angle here.

According to the curriculum, the Star Trek Lemma, named after the resemblance of the logo of the popular series, is one of the theorems that only highly attaining students are expected to learn to prove and apply during KS4. According to the school's program of study, Liz's students might come across circle theorems in Year 11 (age 16), subject to their grades until then, but not at Year 7.

The analysis shows that with the help of the nine-point circle and building upon the previous parts of the task, Liz and the students have an opportunity to go through the main ideas of the proof of the Star-Trek lemma despite not having yet engaged with algebraic routines and relevant terminology (e.g., arc, subtended, etc). Throughout the lesson, Liz and the students communicate using the words 'angle', 'triangle', 'circle' and 'point' or 'dot'. The new narratives about angles can be negotiated

through routines and other narratives endorsed in previous lessons or earlier parts of the task. In the lesson students engage with simple algebraic routines based on the visual mediator of 'the nine-point circle' in comparison to the proof of the lemma seen later in KS4. Table 1 illustrates our reconstruction of the narratives observed during the lesson (left column) alongside a canonic proof of the lemma seen in KS4 (right column). To produce the reconstructed narrative, we collated parts of the recording and the notes in a way that corresponds with the steps of the proof.

Table 1: Reconstruction of narratives observed in lesson (left) and a canonic proof seen in KS4 (right)

The nine-point Circle task	**A proof of the Star-Trek Lemma**
 [Spliting the shape into the triangles ACD, ACB and CDB.] **Liz:** Two sides when I'm going from the centre point out to the edge of my circle, that line there is exactly the same distance as that line there. DCA=160º [task 1/angles around a point] ADC=10º because the triangle is isosceles. ACB=120º BCD=80º [task 1] CDB=50º because the triangle is isosceles. ADB=60º **Student**: Mmm, Double!	 Draw AB and the radius CD. On the diagram angle ADC=x and CDB=y. Therefore, ADB=x+y Angle CAD=x because the triangle ACD is isosceles, CA=CD radius of the circle. And DCA=180º-2x (i) Also, angle DBC=y because the triangle DCB is isosceles, CB=CD radius of the circle. And BCD=180º-2y (ii) ACB+DCA+DCB=360º because angles around a point sum up to 360º. From (i) and (ii): ACB+180º-2x+180º -2y=360º⇔ ⇔ACB=2x+2y ⇔ ⇔ACB=2(x+y) ⇔ ⇔ACB=2ADB

The students are not yet familiar with advanced algebraic routines, such as rearranging algebraic equivalences with more than one variable. However, with the mediation of 'the nine-point circle' Liz and the students negotiate the Star-Trek lemma and its proof, using an example where the angles can be calculated. Students are familiar with routines such as using angle facts to substantiate their actions or naming unknown angles towards their calculating. Additionally, drawing auxiliary lines was introduced in part 2. The action above led Liz to conclude:

Liz: […] this angle here [centre] will always be double the size of that one there [circumference].

Phrases like “this angle here” and “that angle there” act as placeholders for missing words that the students might come across in the future. Liz is aware of the words but does not name them to her students. Yet, the constructed narrative make sense both from the perspective of Liz, as an application of the Star-Trek lemma, and to the students, as an observation that can be confirmed following the steps of the task. Also, she uses the word “always” to signpost the generality of the argument. The task and Liz’s actions gear students towards constructing a narrative about the two angles by the end of the lesson. Actions beyond the mathematics of the moment are seen here as she engages students with mathematical content that students might see in the future in ways that are accessible to them.

Potentials of the discussion and the task to go beyond the mathematics of the moment

We also identified potential opportunities where the discussion could go beyond the mathematics of the moment but were not addressed by Liz. In part 3, the students are asked to create as many different triangles as they can by joining three points on the ‘nine-point circle’, during a specified time. An opportunity that emerged regards the total number of the different triangles:

Student: Miss, do you have us drawing thousands of triangles?
Liz: Are there thousands of triangles to find?
Student: Yeah… [Liz moves on]

Later, Liz makes the following comment during the whole class discussion:

Liz: There are, I believe there are about thirty-six different triangles that you can make out of this. There are lots. But we're running out of time, so I can't show you and I can't load up the the document I wanted to show you.

Time limitations and technical difficulties seem to be the reason why this opportunity did not materialise despite Liz’s intentions. Determining the exact number of different triangles relies on observations about congruent shapes which is part of the students’ compulsory education, as well as elements of problem solving and combinatorics that go beyond the mathematics of the moment.

Finally, one opportunity we observed emerging from the task is to use the same visual mediator, the ‘nine-point circle’, to introduce more circle theorems e.g., exploring the angles of cyclic polygons. As mentioned earlier, circle theorems, including cyclic quadrilaterals, could be mentioned in KS4.

Discussion and Conclusion

The analysis yielded opportunities to engage in mathematical conversation beyond the mathematics of the moment. Some of these opportunities were taken (manifested) and others were missed (potentials of the discussion and the task). Using intersubjectivity, we discuss opportunities that materialised during the lesson, to identify the elements of the teacher communicational patterns at the Mathematical Horizon. The teaching actions exemplified here, regard Liz inviting her students to endorse intersubjective narratives (the Star-Trek lemma with word use and routines which are accessible by the students) or to perform intersubjective routines (auxiliary lines prompted by teacher’s ‘clues’). We observed Liz and her students using words, endorsing narratives and perform routines that make sense from both teacher and student perspectives through the mediation of the ‘nine-point circle’. Liz’s expectation is that students, with her support would utilise known routines to justify their arguments mathematically. Our observations are in line with nested routines and ritual-enabling opportunities to learn as stepping stones for the students to enter a new discourse (Nachlieli

& Tabach, 2019). The missed opportunities identified here indicate that the communication might be constrained by factors such availability of time and resources or teacher's priorities.

The data reported here, illustrate a theoretical frame and a method of analysis to account for teaching situations where the communication goes beyond the boundaries of the curriculum. We consider the opportunities discussed here as evidence of DMH and, thus, contributes to the work on how DMH is conceptualised and investigated in research. The identification of opportunities (manifested and potential) can be utilised to determine what facilitates or hinders the opportunities to go beyond the mathematics of the moment. What, for example, does stop teachers to act upon students' contributions that have potential for a discussion that goes beyond the mathematics of the moments? How could these discussions be beneficial to students? These are the next steps of our work on the analysis of data collected from teachers (interviews and class observations) and teacher educators (interviews). In our work, our attention is on teachers: we study their narratives and how they react to student contributions. We note, however, that this study does not explore connections between DMH and students' learning. Finally, what we discuss here is connected to our own horizons as mathematicians and mathematics education researchers. Therefore, further research and collective work will give a nuanced view of DMH and its implications in teaching and teacher professional development.

Acknowledgment

This work is part of the first author's doctoral project funded by the Faculty of Social Science of the University of East Anglia.

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