Basic Mental Models of Equations -Theoretical Conception and Practical Meaning

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A Basic Mental Model (BMM) of a mathematical concept is a content-related interpretation that gives meaning to this concept. There exist many suggestions for BMMs in mathematics education, e.g., for natural numbers, functions, derivatives or integrals. In this article we present four BMMs of an equation and discuss the relation between these BMMs and the solving of equations. Particularly, we are interested in the interrelationship between these BMMs and the use of digital technologies in this solving process. These theoretical considerations bring up research questions which should be answered to achieve a better understanding of the concept of equation and the solving processes.

Keywords: Equation, solving of equations, Basic Mental Models, digital technologies.

Basic Mental Models (BMM)

The concept of Basic Mental Model (in German: Grundvorstellungen) has been well established in German-speaking didactics of mathematics for many years (Hofe v. & Blum, 2016). *A BMM of a mathematical concept is a content-related interpretation that gives meaning to this concept, providing relations to meaningful contexts* (see Greefrath et al., 2016). The *meaning* of a concept is constituted by its interpretation in scientific mathematics, in school mathematics and in the (historical) development of the concept (Kilpatrick et al., 2005). BMMs can be conceived as prerequisites for students to deal with mathematical concepts in an insightful way.

The concept of BMM can be used both in a *normative* (prescriptive) and an *individual* (descriptive) sense (see Hofe et al., 2005).

- *Normative* BMMs are the answer to the question: How should students generally and *ideally* think of a given mathematical concept? They are identified by didactical analyses of the mathematical concept. They can be used as educational guidelines and to specify learning objectives for mathematics lessons. The determination of BMMs is a didactical challenge for researchers and requires a subject-oriented classification of mathematical and real-life situations of the concept.
- *Individual* BMMs are individual mental models or concepts students *actually* develop in learning processes and problem-solving situations. They can vary from or represent only part of normative BMMs, they even can be based on misconceptions. Individual BMMs are the result of the personal development of meaning and the integration of the concept into an individual's personal worldview.

BMMs of mathematical concepts can be considered within the theoretical framework of "Concept Image – Concept Definition" (Tall & Vinner, 1981). They are parts or subsets of the "Concept Image" of a mathematical concept. While "Concept Image" refers to all individual mental images identified with the concept, BMMs are the *core* or *central components* of these images.

The development of BMMs is a creative task for mathematics educator. Based on the concept definition, a critical investigation of the use of this concept and the involved perceptions in problem solving situations in scientific mathematics, in university mathematics lessons and in the school curriculum give hypotheses of the existence of different BMMs. These have to be empirically verified, like Greefrath et al. (2021) have done this for the concepts of derivative and integral.

A mathematical spotlight: equations and equivalence of equations

Equations with only numbers or quantities are statements which can be true or false. E.g., the equation $3 + 5 = 8$ is a true, $3 + 5 = 7$ is a false statement. Equations with unquantified variables, e.g., $T_1(x) = T_2(x)$, T_1, T_2 algebraic terms, are predicates, which will turn into statement if the variables are quantified, e.g., for $x = 4$, $3 \cdot x + 1 = 7$ is a false statement, for $x = 2$ it is a true statement. Two equations are *equivalent*, if

- their sets of solutions are the same or
- there is an equivalence transformation which transforms one equation into the other.

Each injective function applied to both expressions of an equation is an equivalence transformation. This applies especially for the elementary transformations like addition and multiplication with real numbers (for more details see Arcavi et al., 2017).

BMMs of an equation

BMMs of an equation form a content basis for what students normatively need to work with equations. They are central and important in developing perceptions and (mental) representation about basic activities while solving equations. BMMs are based on mathematical aspects of equations: the definition of an equation, the use of the equals sign and the relationship between the equation concept to the concepts of algebraic expressions and functions.

- **Operational BMM**: An equation is understood as a calculation or transformation. The equal sign is seen as an operational sign, which indicates a reading direction of the equation in the sense of a "resulting-in" sign.
  - Arithmetic calculations: The equation $3 + 4 = 7$ can be expressed as "$3 + 4 \rightarrow 7$" or verbally by "3 plus 4 results in 7". It is the predominant BMM of an equation in primary school.
  - Transformations of algebraic expressions: Equations like $(a+b)^2 = a^2 + 2ab + b^2$ can be expressed by $(a+b)^2 \rightarrow a^2 + 2ab + b^2$ or verbally by "expanding $(a+b)^2$ results in $a^2 + 2ab + b^2$". "Results in" can also be read—with "factorizing" instead of "expanding"—in the other direction.
- **Relational BMM**: An equation is understood as a task to determine numbers or quantities for the expressions on both sides of the equation to get the same value or quantity on both sides. The equal

sign is seen as a relation sign. The variable here is understood as an unknown which has to be determined.[1]

The visualization for or a representation of this BMM is the model of an equal arm balance. In this model, each expression is represented as a combination of weights. Numerical values are represented by unit weights and each variable value is assigned by a fixed—initially unknown to the spectator—weight. The equality of values of both terms is expressed in the balance of the scales. Like in Figure 1, the balance model is suitable especially for linear equations. These are carried out as actions of removing the same weights from or adding the same weights on both scales.

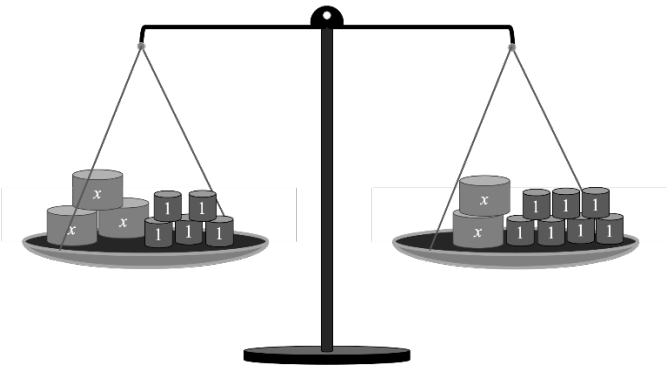

Figure 1. An equal arm balance

- **Functional BMM**: An equation $T_1(x) = T_2(x)$ is a comparison of two expressions which are understood as functions with $y = T_1(x)$ and $y = T_2(x)$. Here, too, the equals sign is understood relationally, but the BMM of the variable is that of a changing number or quantity.

In this functional interpretation, the variable x "passes through" the values of the definition range of the two terms $T_1(x)$ and $T_2(x)$. This results in the equation becoming a false or true statement, depending on the values of $T_1(x)$ and $T_2(x)$. However, we want to determine the value(s) x_o with the same function values $T_1(x_o) = T_2(x_o)$. In a graphical representation of the two functions with $y = T_1(x)$ and $y = T_2(x)$, solving an equation means determine the intersection points of the two function graphs. This is possible for nearly all types of functions. Figure 2 shows an example of a quadratic equation.

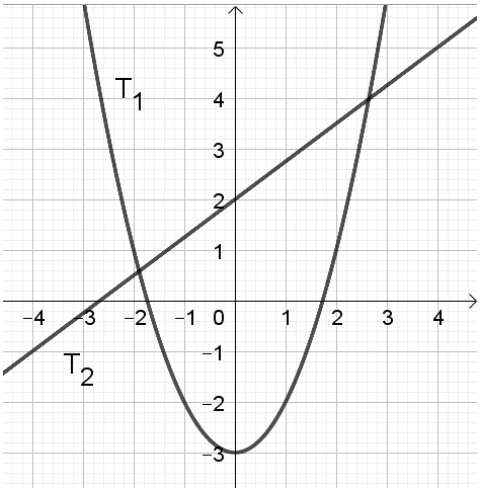

Figure 2. The equation $x^2 - 3 = 0.75x + 2$ leads to the two functions with $T_1(x) = x^2 - 3$ and $T_2(x) = 0.75x + 2$

This BMM can be transferred to a system of equations with 2 variables if a function is defined by each of the two given equations whose common function values are to be determined.

- **Object-BMM**: An equation is regarded as a mathematical object that has characteristic

[1] We refer to the three central BMMs for variables without explaining their background (see e.g. MacGregor and Stacey (1997) or Oldenburg (2019) for details): The variable as a general number, the variable as an unknown number and the variable as changing number or quantity.

properties, such as the number of possible solutions, the definition range or special solution algorithms. E. g.

- The quadratic equation $x^2 + x - 3 = 0$ has exactly two solutions. There exist several methods to calculate these solutions.
- The equation $x^2 + y^2 = z^2$, $x, y, z \in \mathbb{N}$ is a Diophantine equation whose solutions are called Pythagorean number triples.
- The equation $x^2 + y^2 = r^2$ with $x, y \in \mathbb{R}$, $r \in \mathbb{R}^+$ is an equation of a circle with the coordinate origin as the center and the radius r.

The first two BMMs are based on the operational and relational meaning of the equal sign and thus mark the transition from the arithmetic-dominated mathematics of the primary level to the algebra of the secondary level. The third and fourth BMM represent an understanding of equations on (lower and upper) secondary level. They will be especially important while working with digital technologies. There is a hierarchy in the appearance of these BMMs from primary to upper secondary school: Operational BMM → Relational BMM → Functional BMM → Object-BMM .

Central activities and working with equations

The meaningful working with and the flexible use of concepts require a wider view of the concept, especially seeing the concept in relation to different applications and to different representations (see e.g., Freudenthal (1983)). At least since the works of Piaget and Aebli, and further those of Vygotsky and Leont'ev, real hands-on activities are seen as central and fundamental for the development of mental structures or (thinking) operations. The development of BMMs must therefore be seen in close relation to activities and related tasks with this concept.

Referring to the quite elementary "Input-Operation-Output"-model concerning the working with tasks (see e.g., Günster and Weigand (2020)), this model can be transferred to three *central activities* while working with equations (in mathematics lessons). These *central activities* are:

- **Input or setting up an equation**: Setting up an equation in such a way that the central relations presented in a problem are expressed appropriately in the equation. It is therefore a matter of translating a problem situation into the language of mathematics.
- **Operate on or anticipating and applying transformations**: Developing strategies which can be flexibly used to solve equations in a problem-adequate way.
- **Output or interpreting**: Gaining insights into an issue by means of an equation, appropriately transformed equations or the solution(s) of an equation.

These central activities require *syntactic* knowledge concerning the structure of expressions that determine the equation and knowledge of solution strategies concerning different equation types. Furthermore, flexibility and symbol sense are required, and the ability to transfer equation to other (equivalent) equations (see Kieran (1992), Oleksik (2019)).

Semantic knowledge is required with regard to the meaning of equations in relation to environmental situations, inner-mathematical models and representations, and the interpretation of the solution(s) in relation to these situations.

Digital technologies—nowadays and in the future

For solving equations, there is already a wider range of digital technologies that can be roughly divided—not non-overlapping—into two categories: on the one hand, *tools* for solving equations and, on the other hand, *learning environments*. *Tools* such as spreadsheets or computer algebra systems allow the user alternative ways of using them. *Learning environments* are structured learning arrangements designed with a teaching objective, using tools and visual aids. Figure 3 shows an interactive digital simulation of a beam balance and the solving of a linear equation. Moreover, there are digital assessment systems such as STACK (Sangwin, 2013) or learning apps like *Photomath*[2], *Chegg Math Solver*[3], *Cymath*[4] or *Mathway*[5]. A didactic discussion of such programs can be found in Arcavi et al. (2017, p. 106). These systems will certainly be further developed in the coming years.

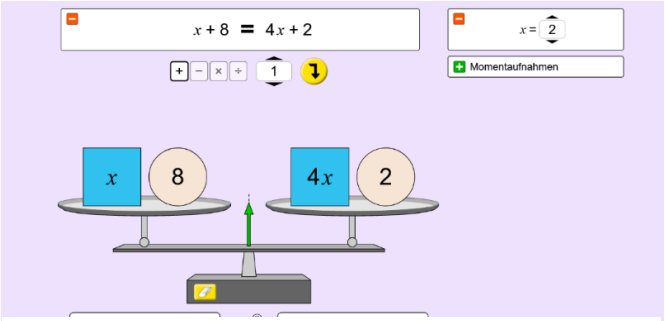

Figure 3. Simulation of a balance (University of Colorado Boulder)

BMMs and the solving of equations with digital technologies (DT)

The four BMMs allow meaningful explanations for different algebraic methods of solving equations. Since the main focus here is on the significance of the BMMs and the use of DT, the operational BMM, which essentially plays the significant role in primary school and lower secondary school, will not be discussed (cf. Arcavi et al. (2017)).

The Object-BMM

A CAS solves equations on the numeric and symbolic levels, and it is a formulary that offers formulas for linear, quadratic, like in Figure 4, and special types of polynomial functions of higher order than 2.

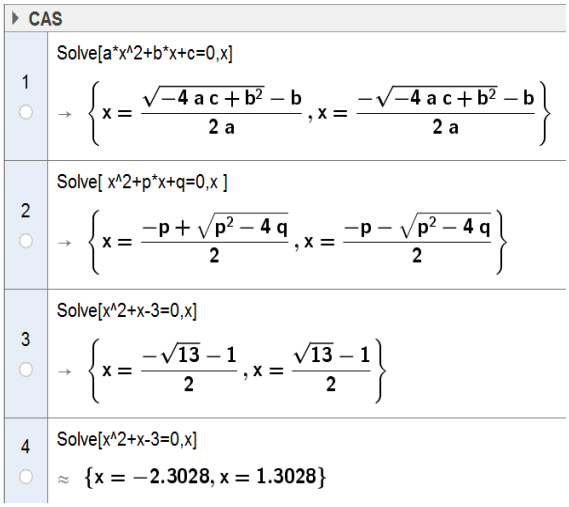

Figure 4. Solving quadratic equations with a CAS

2 https://photomath.app/
3 https://www.chegg.com/math-solver
4 www.cymath.com/
5 www.mathway.com

The Object-BMM can be integrated into an Input-Operation-Output model:

"Operation" may be seen as a "black box", if the learner does not know the algorithm used in this "box". But it might also be a "white box", if the CAS is "only" used as a tool to get the result quicker than with paper and pencil. Furthermore, to turn the "black box" into a "white box" it might be helpful to explain "Operation" and the Object-BMM—in dependence of the type of the equation—by referring to the relational and the functional BMM.

The functional BMM

A CAS allows the solving of equations on different representation levels. It calculates the zeros of (special types of) functions on the symbolic and numeric level by only pressing one button and it visualizes the solutions on the graphic level. The solutions are the zeros of the function. Figure 5 shows an equation with a parameter, the solutions are given dynamically by varying the parameter b with a slider.

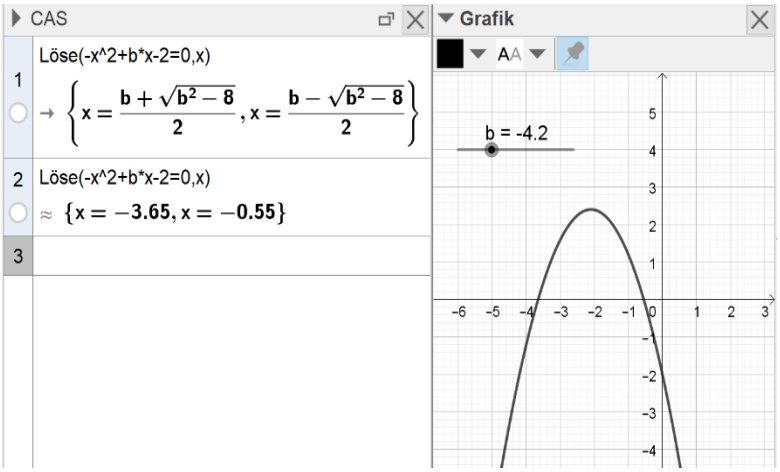

Figure 5. Solving quadratic equations (with a parameter b) and the functional BMM

Special equations of higher degree, particularly equations of degree 3 and e.g., trigonometric and exponential equations can be solved—quite often—numerically, sometimes symbolically, and nearly always graphically.

Solving (more complex) equations with a CAS is not only a "pressing a button"-activity. Basic knowledge of solving different types of equations and different strategies for solving of equations are necessary, especially if an approach that had been used did not lead to a successful solution. Figure 6 shows an example of the solving of the equation $1 + sin(x) = 2^x$. In this case, trying to get a symbolic or numeric solution with a CAS is not possible.

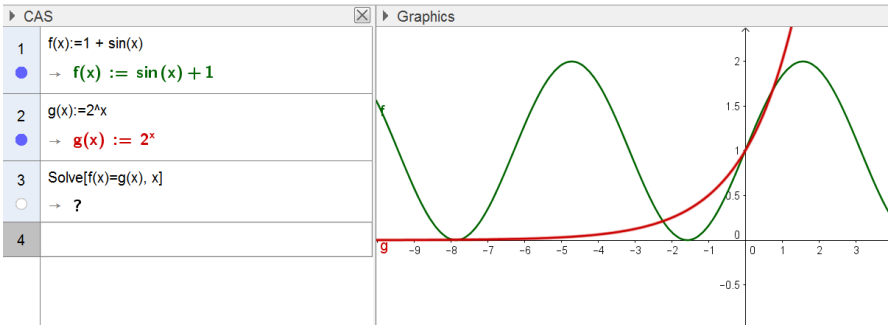

Figure 6. Graphical solution of the equation $1 + sin(x) = 2^x$

The relational BMM

Concerning the functional BMM, a CAS is a *tool* for solving equations. However, a CAS can also be used as a *teaching-learning system* and integrated into a *learning environment*. An example is the step-by-step execution of (equivalence) transformations for an equation. For this, arithmetic

operations are applied to an equation as a whole while the algebraic transformations can be visualized graphically. The rather confusing static Figure 7 has to be seen in a dynamic step-by-step presentation to show more clearly the invariance of the x-coordinate of the intersection point of the graphs and hence the consistency of the solution of each transformed equation.

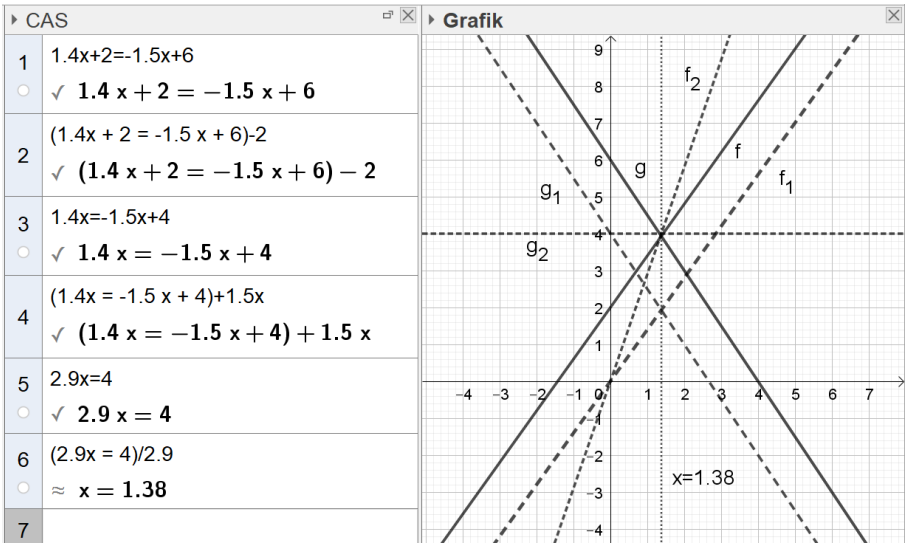

Figure 7. Step-by-step transformations of an equation

Conclusion and research questions

Solving of equations—in this article—is seen in relation to the BMMs of equations (operational, relational, functional BMM and Object BMM) and the use of DT, mainly CAS, but with different kinds of representations. The first research question is the substantial question about the existence and the kind of individual BMMs:

1. *Are the four BMMs represented in students' thinking if they solve equations and how is the relationship between BMM and the way of solving an equation?*

We suspect that the BMMs have to be distinguished concerning different types of equation, especially linear, quadratic and "other" equations, and different methods of solving equations (especially symbolic, graphic, numeric).

The second research question is about the three *central activities* while working with equations and their relation to the BMMs?

2. *How do the BMMs of equation interact with the central activities while solving equations?*

Concerning the solving of equations with DT, the interrelationship between three dimensions—BMMs, DT and central activities—seems to be a crucial point. The third research question aims to get criteria for the development of tasks for constructively generating BMMs of equations.

3. *Which tasks and activities—especially with the use of DT—support the development of BMMs for equations (depending on the different types of equations)?*

The analyses in this article are the theoretical basis for answering these research questions empirically. This will be the next step in this research project.

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