Using the graph when talking about functional relations in Grade 1: The importance of terminology

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This study investigates how the graph representation creates opportunities for young students to develop an understanding of functional relationships in pattern generalizations. The empirical data is from an educational teacher-focused classroom design research focusing on generalizations in arithmetical growing patterns in Grade 1. The results show that the students in Grade 1 are given an opportunity to reason mathematically in both recursive- and covariational thinking. The results also show how the teaching provided opportunities for the students to use multiple representations of functional thinking and how oral language is a common representation to describe relationships. However, using a well-thought-out terminology to exploit the potential of the graph representation when discussing functional relationships and generalizations appears to be important.

Keywords: Generalizations, functional relationships, graph representation, elementary school, terminology.

Introduction

The concept of *generalization* is used in various ways and in relation to multiple mathematical contents, for example, generalized arithmetic or generalized relationships between quantities. Generalizing the functional relationship between two variables in growing patterns has been used successfully, even with young students, and research indicates that young students can develop their functional thinking (e.g., Blanton & Kaput, 2004; Blanton et al., 2015; Stephens et al., 2017). However, teachers as well as students have difficulties in working with generalizations and students are often not given opportunities to develop mathematical generalizations (e.g., Stylianides & Silver, 2009). In the Algebraic Thinking Group at CERME, there is an ongoing discussion about figural patterns and how they can support young students' understanding of algebraic generalizations. Nevertheless, what is unsolved is how teaching can contribute to students' shifting their focus "from the visual structure to the algebraic structure" (Chimoni et al., 2019, p. 530) and how teaching can contribute to young students' overcoming the difficulties described by Stylianides and Silver (2009) as a way to support their understanding of functional relationships. In an attempt to contribute insights about teaching functional relationships, an educational teacher-focused classroom design research with three teachers was conducted (Sterner, 2019; 2021). This paper builds on the teaching of one Grade 1 teacher who used graphs to talk about functional relationships and generalizations and includes reflections from the three teachers in the design research (intervention). The research question in the paper is, "In what way does the graph representation support teaching and provide learning opportunities about functional relationships and generalizations in Grade 1?". What is interesting in relation to young students' learning about functional relationships is the teachers' reflections on their challenges.

Background and conceptual frames for the intervention

Functional thinking is a part of algebraic thinking that, focuses on the relationship between two (or more) quantities. The definition central for this study is from Wilkie (2016): "Functional thinking relates to understanding the notion of *change* and how varying quantities (or "variables") relate to each other" (p. 247). Research indicates that teaching functional thinking in elementary school is dominated by identifying recursive patterns, where the relationship between quantities is often missing (Blanton et al., 2015; Carraher et al., 2008). In addition, young students are often introduced to variable notation as a quantity with a "missing value", and thus the relationship between the independent and dependent variables and proportional reasoning is lacking in elementary school (Blanton et al., 2015; Carraher et al., 2008).

In Swedish primary schools (Grades 1–6), it is not common to teach functional relationships; the concepts of "double" and "a half" are used in lower grades when talking about proportional relations. In this study, design principles (DPs) are used as a theoretical frame to focus on generalizations and functional relationships. As can be seen from the Methodology section, the research project is intended as an educational design research. The content of the design principles (DPs) used in this study are formulated based on previous research on generalizations and functional relationships in teaching in elementary school. I interpret these principles as theoretical frames for the intervention as well as a guide for the content of the teachers' teaching, and the following design principles have been used. The second (DP2) is particularly central in this study.

DP1: The students should be given opportunities to identify a pattern, structure the pattern, and generalize the pattern (Mulligan et al., 2013; Stylianides & Silver, 2009).

DP2: The students should be given opportunities to work with algebraic reasoning, including functional thinking and proportional relationships and determining relations between two or more varying quantities (Beckman & Izák, 2015; Blanton et al., 2015; Blanton & Kaput, 2004).

Theoretical frames in the analysis

The analysis was done in three steps. In the first step, the definition from Wilkie (2016) was used in relation to the content of the design principles. The first analysis resulted in the empirical data, including students' and the teachers' discussions about the content of the design principles. In the second step, the second strand of Kaput's (2008) three algebra content strands was used, *Functions, Relations, and Joint Variation*. This strand provided an opportunity to highlight sequences in the empirical material, including functions, relations, and joint variation. In the third and last step of the analysis, I explored how the graph representation supports teaching and young students' learning of functional relationships and generalizations. Based on Kaput's (2008) algebraic thinking and the core aspects, Blanton et al., (2018) and Blanton et al, (2019) worked on generalization and functional thinking. Blanton et al., (2019) suggested four essential algebraic thinking practices: *generalizing, representing, justifying,* and *reasoning with* mathematical structure and relationships. In the last analysis, I used Blanton et al.'s (2019) essential algebraic thinking practices of generalizing as a theoretical frame. The second and the third practices, representing and justifying, became the most central practices to explore how teaching can be formulated to contribute to young students' learning of functional relationships and generalization in arithmetical growing patterns. The combination of representing generalizations and justifying generalizations made visible what opportunities the graph representation provides to students to justify generalizations.

Methodology

The study outlined in this paper is a part of an educational teacher-focused classroom design project inspired by *classroom design with teachers* (Stephan, 2015). One aspect of mathematical reasoning central in this paper is generalizing the functional relationship between two variables. In addition to the author, three mathematics teachers from different schools in Sweden, one from Grade 1 (Jonna) and two from Grade 6 (Clara and Irma), participated and collaborated in three recurring design cycles, an intervention, over a period of nine months (Sterner, 2019). The intervention is guided by design principles (Greeno, 2006; McKenney & Reeves, 2012), which are then used as the theoretical frame for the mathematical content of the intervention and as a guide for the content of the teaching (see "Background and conceptual frames for the intervention" section).

The empirical data in this paper include video recordings from various parts of the design process: five common meetings from the design– and refining phase and three lessons from Grade 1. Other data material includes field notes from the Grade 1 classroom, observations, copies of student work, and tape recordings of reflections with the teacher directly after teaching. Jonna, as well as the other teachers are experienced and development-oriented. The students are 6 or 7 years old, and the study is conducted with half the class (twelve students). The teacher (Jonna) describes functional relationships as a new and unknown way to learn pattern generalization in Grade 1. As mentioned, the design principles served as a theoretical frame for the intervention and teaching, and thus empirical data from both the teaching in the classroom and discussions from the design process were necessary. The empirical data was analyzed in three steps and three different frameworks were used (Blanton et al., 2018; Blanton et al., 2019; Kaput, 2008; Wilkie, 2016).

Ethical considerations

The intention was not to generalize the results but rather to exemplify the opportunities and challenges in teaching functional relationships in Grade 1. In the intervention, it was important to consider the teachers' and the students' participation and the chosen mathematical content.

The intervention in Grade 1

In the intervention, the three teachers (Clara, Irma, and Jonna) designed various tasks to support students in Grade 1 to *identify*, *structure*, and *generalize* different patterns (DP1). In the initial stage of the teaching in Grade 1, the students were given the opportunity to identify and structure different sequences of repeating patterns (e.g., AABAABAAB…) as well as conceptualize the idea of a unit of a repeating pattern.

One task concerning a certain number of dogs and the corresponding number of tails, ears, and legs was used to exemplify direct proportionality and to support functional relationships, as well as the importance of how quantities vary in relation to each other (DP2). This growing pattern task was used in teaching in both Grades 1 and 6 and was taken from an earlier study inspired by Blanton and Kaput (2004) and Mulligan et al. (2013). Students in Grade 1 counted and structured the tails, ears, and legs and represented the data and their functional thinking in different ways, for example, by drawing pictures, putting matches in piles, and making simple tables. The whole class talked about the patterns, for instance, those in the relationship between the number of dogs and the number of tails, ears, and legs (i.e., 1 dog has 2 ears…2 dogs have 4 ears…3 dogs have 6 ears… Each dog has 2 ears, and each dog has 4 legs). In the whole-class discussion, Jonna, the teacher in Grade 1, structured the data in function tables and made a graphical representation in a Cartesian coordinate system. They made graphical representations of the three relationships, where the number of dogs was illustrated as a function of the number of tails, ears, or legs respectively ($y = x,\ y = 2x,$ or $y = 4x$). Finally, Jonna and the students discussed the three different graphs. In the class, questions were asked like the following, "Which of these graphs represent the tails of the dogs and how do you know that?", "How many dogs do we need for 6 tails?", or "How many dogs do we need for 6 ears?" Some of the students counted, and some used the graph to recognize the patterns and the functional relationship.

Result and analysis

Despite that the teachers found it challenging to teach about functional relationships and pattern generalizations in the classroom, it turned out that a significant part of the classroom conversation worked well as an entry to discussing functional relationships. There was a learning potential to work with the graph representation and discuss the relation between quantities in Grade 1.

In the reflection directly after the teaching, Jonna talked about her perceived challenges in the teaching. Jonna talked both about the lack of words to describe the functional relationships and the challenges in explaining generalizations in Grade 1. Similar challenges arose in the discussions in the design process. The transcript below is from the teachers' discussion in the design process after the teaching had been completed, whereby the teachers refer to the previously mentioned tasks with dogs and the number of legs.

Clara: I don't have the words to talk about the functions the relation between dogs and legs. I don't know what questions I should ask the students to give them opportunities to develop generalizations.

Jonna: To be honest… I am not sure what a generalization includes. Is it enough when the students say, "Every time I add 4 more…"?

The transcript highlights two thoughts that often arose in the intervention when the teachers discussed generalizations in patterns and functional thinking. First, the teachers talk about challenges in how to talk about functional relationships. They talk about the lack of words to describe and express how quantities vary in relation to each other. In the intervention, the concept of a functional relationship is new and unknown in relation to pattern generalization the teachers used before ("double" and "a half") when working with relationships in elementary school. Now, the teachers are searching for words to describe and explain, for example, coordinates, independent- and dependent variables, proportional relationships, graphs, functions, and the x- and y-axes. The teachers are searching for the terminology that they have not previously used in their teaching.

Secondly, the teachers expressed uncertainty about when the students' arguments are enough to represent a generalization. In the dialogue, Jonna's reflections about whether it is enough when the students say, "Every time, I add 4 more." This can be interpreted as the student having identified a pattern and is somehow about to generalize the pattern. In the design process, the teachers discuss "When can a generalization be seen as a generalization?" in Grade 1.

At the end of the intervention, the teachers planned activities that would allow the students to identify functional relationships and patterns from what they called the "opposite way". They start from the graph representation, the function, and ask the students to find patterns. They begin an activity like the previously mentioned activity involving functional relationships with discrete numbers, that is the dogs and the relationships between the numbers of tails, ears, or legs ($y = x, y = 2x$ or $y = 4x$). The graph representation creates opportunities to talk about functional relationships in discrete values in Grade 1. However, in the intervention, the participants problematize the graph's visualization of continuous values.

Before the lesson in Grade 1, Jonna expressed worry about explaining and talking about functional relationships to the students. She mentions again the missing terminology for talking about how quantities vary in relation to each other. Jonna starts to give all the students a Cartesian coordinate system with two graphical representations and an empty table. The two graphs (the green and the red graph, see Fig. 1) illustrate the functions $y = x$ and $y = 3x$.

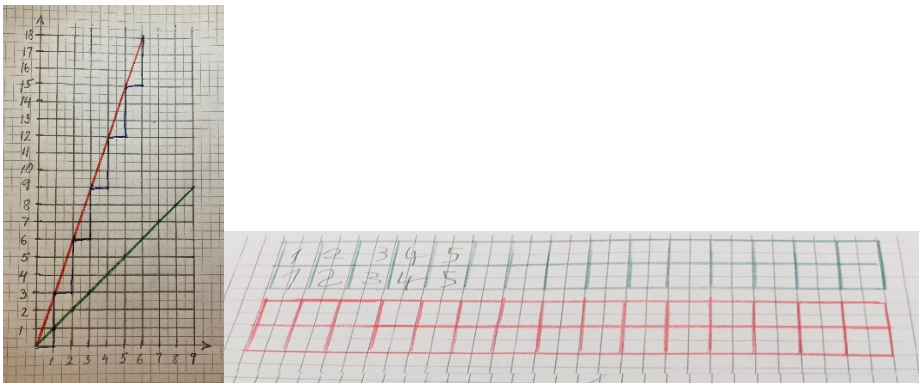

Figure 1: The Cartesian coordinate system and Karim's table,

illustrating the functions $y = x$ and $y = 3x$

The students used the graphs, wrote the independent and the dependent values in the tables, and then identified the pattern. Some of the students had to write many coordinates in the tables before they identified the pattern and tried justifying the pattern with oral language. Some students found the pattern quite fast and tried to express the pattern illustrating the function $y = 3x$ in the classroom. One of the students, Karim, talks to the teacher; he points to the camera stand with three legs and says, "Every time we have a new camera stand, we have three more legs."

In the class discussion, Jonna, the teacher, initiated talks about the identified patterns and used the tables and the graphs to talk about predicting near data. The results show that the teacher-led-class discussions included recursive- and covariational reasoning. The teacher and the students spoke about what emerges in a single sequence, namely, what happed to the dependent value? This short transcript is an example of a conversation in the discussion relating to one of the graphs in Figure 1, that of the “red” graph illustrating, $y = 3x$.

1	Teacher, Jonna:	Can someone describe the identified pattern?
2	Student, Sara:	The pattern increases by 3.
3	Teacher, Jonna:	How do you know that the pattern increases by 3?
4	Student, Ida:	In the table, it is “3–jumps”.
5	Teacher, Jonna:	Yes, what can the red graph illustrate?
6	Student, Alex:	A snowman with 3 snowballs.
7	Student, Tanesha:	A tricycle with 3 wheels.
8	Teacher, Jonna:	Yes, how many wheels do we need for 2 tricycles?
9	Student, Frank:	We need 6 wheels.
10	Teacher, Jonna	Tell me, how do you know that?
11	Student, Frank:	3 wheels are needed for each tricycle.

The observed teaching offers several opportunities to represent pattern generalizations in multiple ways, such as using pictures, tables, graphs, or oral language. The students used the Cartesian coordinate system and structured coordinates in a table. The transcript indicates that the students only have their oral language to describe the patterns, as they have no access to variable notation. However, the most common way to talk about generalization and functional relationships was in a recursive pattern, which means describing the variation and the relation in the dependent value, for example, line 2 (“… increases by 3”) and line 4 (“… 3-jumps”). It continues like this, and the teacher tries to initiate discussions, including those centered around the relationship between the independent and dependent variables. An initial reasoning, including the covariational relationship, is visible at the end of the transcript in lines 8-11: “3 wheels are needed for each tricycle.”

The class discussion continues and the teacher, Jonna, points to the Cartesian coordinate system and asks the students to say something about the green and red functions ($y = x$ and $y = 3x$). One student notes that “the red line goes straight up compared to the green one.” Jonna, the teacher, agrees and asks the students if anyone wants to justify why this may be so. One of the students points to the red graph and shows what he calls the “small tents” [the tent is much bigger in the red graph]. The teacher says, “Yes … that’s right … and how can we explain this big and small tent?”. In the class, the silence is noticeable before one of the students points to the red graph and says, “The red graph could be the tricycle, 1 tricycle has 3 wheels”. The student points to the “tent” in the y-axis and counts “1, 2, 3…”. The student then goes on to point to the x-axis and y-axis and the coordinate (2.6) and says, “2 tricycles have 6 wheels…”.

The discussion ends with the teacher asking what the green graph could illustrate. Several students recognize the graph and the table from the last lesson about the dogs, and their corresponding tails. Another student connects this to the tricycle and says, “…The green line shows a unicycle.”

Discussion

This study sheds light on the role of the graph representation and how it can develop students' functional thinking. The results provide insights about the opportunities the graph representation invites, in terms of both the teachers' and students' reasoning about functional relationships when working with pattern generalizations in Grade 1. Although the graph representation first and foremost invites discussions of recursive reasoning, both the teacher and the students in Grade 1 also succeeded in discussing covariational relationships.

Despite Jonna's worries before the lesson, it seems that functional thinking is visible in the class discussions. For example, Jonna's teaching visualizes one way to try to get closer and talk about how quantities vary in relation to each other, which could be compared to what Wilkie (2016) claims, "the notion of *change* and how varying quantities (or "variables") relate to each other" (p. 247). The graph representation supports the teacher and the students to justify the relationship between independent- and dependent variables in the oral language without variable notations. This suggests that my results show that the graph representation can be seen as a tool to use in elementary school when transforming what Chimoni et al. (2019) call shifting focus "from the visual structure to algebraic structure." However, Jonna and the other teachers in the intervention are still searching for the terminology to talk about functional relationships in elementary school.

The results also indicate what the teacher in the study expresses "I am not sure what a generalization includes. Is it enough when the students say, Every time I add 4 more…?". This phrase is a recursion and could be interpreted as a generalization in Grade 1. One may wonder what else could be done if there was some big number of wheels and one more tricycle was added and in what way would symbols for the unknown number of wheels and tricycles be introduced. Maybe the discussion about functional relationships with the support of the graph representation can be seen as the first step towards variable notations, which does not just support a missing value. This falls in line with both Carraher et al., (2008) and Stephens et al. (2017), who stress the importance of using coordinated changes in quantities when teaching generalizations. However, the study shows the importance of using a well-thought-out terminology to exploit the potential of the graph representation when discussing functional relationships and generalizations. What is interesting in this example is how the teacher's reflections enable the teacher to find new and unknown "ways" to teach generalization.

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