



HAL
open science

Mathematical knowledge for teaching: challenges and potential in the case of geometrical patterns

James Gray, Bodil Kleve

► **To cite this version:**

James Gray, Bodil Kleve. Mathematical knowledge for teaching: challenges and potential in the case of geometrical patterns. Twelfth Congress of the European Society for Research in Mathematics Education (CERME12), Feb 2022, Bozen-Bolzano, Italy. hal-03745388

HAL Id: hal-03745388

<https://hal.science/hal-03745388v1>

Submitted on 4 Aug 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Mathematical knowledge for teaching: challenges and potential in the case of geometrical patterns

James Gray¹ and Bodil Kleve²

¹Oslo Metropolitan University, Oslo, Norway; james.gray@oslomet.no; bodil.kleve@oslomet.no

We report on an observation of a lesson where students were working on geometrical pattern tasks in pairs. The teacher's intentions for the paired setting and his reflections upon seeing a video of the lesson are presented, and through the lens of a TRU analysis and the Knowledge Quartet, we identify potential for exploiting the task and the environment to afford students' mathematical engagement. We focus on the interplay between the Contingency and Foundation dimensions in the Knowledge Quartet, and the potential of developing retrospective awareness into in-the-moment awareness.

Keywords: Paired learning, geometrical patterns, knowledge quartet, awareness, teacher insight.

Introduction and theoretical background

The new mathematics curriculum in Norway (LK20) has a specified learning goal that 9th grade students should be able to explore in the sense of describing, explaining, and presenting the structure and development of geometrical patterns and number patterns. Although geometrical patterns were not outside the remit of the previous curriculum, they were not mentioned specifically and the requirement to teach them is new. Within mathematics education literature, geometrical patterns have been promoted as a rich entry point to algebraic and functional thinking (Bednarz et al., 1996; Carraher et al., 2008). However, working with geometrical patterns in this way presents both possibilities and challenges for students and teachers (Orton & Orton, 1999). Moss and McNab (2011) argue that the challenges are due to pedagogical choices and not due to an inherent difficulty in subject material. Thus, mathematical knowledge for and in teaching plays a central role. In this article, we will study one teacher, here called Erik, while working with geometrical patterns as a means for developing students' ability to express relationships algebraically in a 9th grade class. Specifically, we will address the research questions:

What demands were placed on the teacher's mathematical knowledge for teaching in exploiting the task and the environment to afford students' engagement with mathematics?

We explore how the teacher's intentions in carrying out exploratory activities were enacted in a setting where working in pairs was supposed to encourage collaboration. In that respect, we pay attention to how the potential in paired learning and the mathematical potential of the task were exploited and what demands it put on the teacher's mathematical knowledge for teaching, MKT. Several frameworks for investigating MKT have been developed, and much research on the theme has been reported (Skott et al., 2018). The Knowledge Quartet, KQ, (Rowland et al., 2005) is based on the work of Shulman and his categories of knowledge. KQ was developed through a grounded approach to video data focusing on how teachers' mathematical knowledge surfaced in mathematics lessons. 18 codes from the categorisation were grouped into 4 broad dimensions; Foundation (teacher's mathematical knowledge possessed), Transformation (teacher's capacity to transform his/her foundational knowledge to accessible knowledge for others), Connection (connections between concepts which bind different parts of the mathematics together), and Contingency.

Contingency is informed by the three other dimensions, and concerns situations in the classroom which are not planned for. In later studies, this dimension has been further explored. A new code, “Teacher’s Insight” which is demonstrated when a teacher is aware that students construct mathematical ideas and something which sounds “half baked”, was included in the Contingency dimension (Rowland, 2012). Rowland and Zazkis (2013) related the contingency dimension to Mason’s (1998) description of “knowing to act in the moment” (p.139). This in-the-moment-pedagogy is “the teacher’s capacity to engage flexibly and productively with their students” (Mason & Davis, 2013, p. 184). Knowing to act-in-the-moment, puts a demand on the teacher’s mathematical knowledge as well as on how he or she is aware of this knowledge and how it is used and exemplified. “A teacher who is aware [...] is in a position to direct student attention to what really matters” (ibid, p. 189). Contrary to teacher’s insight or awareness is what is known as funnelling of which the effects are described by Wood (1998): “students’ thinking is focused on trying to figure out the response the teacher wants instead of thinking mathematically himself” (p.172). A challenge for teachers will often be to balance between funnelling and scaffolding. A pitfall may be “fostering dependency on teachers and cap opportunities for more independent learning” (Mazenod et al., 2019, p. 2).

Methodology

We observed five mathematics lessons with one teacher (Erik) in two different mixed ability ninth grade classes. Here we report from one of these lessons. The students were seated in pairs and worked on growth pattern tasks. Prior to our first classroom observation, we had a semi-structured pre-interview with the teacher. We asked him to describe a typical mathematics lesson, his intentions in teaching mathematics and how he dealt with mixed abilities, adapted education, and grouping. Additionally, we interviewed some of the students in the classes. We also conducted a semi-structured post-interview with Erik one month after having observed in his classes. The interview included a session in which we showed him several episodes from the five observed lessons, so he could share his reflections with us. The lessons were videotaped, and the teacher wore a microphone. The camera followed the teacher as he moved from one pair to another. After the observations, we studied the videos and selected episodes which were transcribed and analysed.

During the classroom observation, we used the Teaching for Robust Understanding (TRU) Framework (Schoenfeld & Floden, 2014) as an observation scheme. Schoenfeld (2016) emphasizes that there is “no one ‘right way’ to teach. The key idea is that TRU specifies the attributes of learning environments in which students flourish” (p.2). In advance the researchers had prepared for the application of the framework to ensure reliability. The Framework offers ways to reflect along five dimensions: 1-The richness of the mathematical content (How accurate, coherent, and well justified is the mathematical content?). 2-Cognitive demand (To what extent are students supported in grappling with and making sense of mathematical concepts?). 3-Access to mathematics (To what extent are all students supported in meaningful participation in (group) discussions?). 4-Agency, Ownership, and Identity (To what extent did teacher support and/or group dynamics provide access to "voice" for students?) and 5-Uses of Assessment (To what extent does the teacher monitor and help students refine their thinking within small groups?). The Framework has different rubrics: Whole class activities; Small Group Work, Student Presentation, and Individual Work. Since students worked in pairs, we used the rubric for Small Group Work. TRU assigns scores of 1, 2, and 3 for each

dimension. Although we assigned approximate scores when observing, we find it more appropriate to report the analysis with aspects of the content within the levels rather than scores, and we relate to the dimensions of KQ with a focus on teacher’s insight and awareness in the Contingency dimension.

Findings and discussion

In the pre-interview, Erik explained his thinking in choosing the tasks that students were working on. We see that his intentions are inclusion based on differentiation:

To include everyone, even the ones who aren’t so good, I try to find self-differentiating tasks, that have a low entry threshold and that one can do a lot with them [e.g.] a figure that is growing.

In the post-interview, we asked Erik if he thought it made sense for the students to sit in pairs in mathematics lessons. He responded that he intended for them to problem-solve together:

After all, maths is a language... When they are going to solve a problem, they should talk together and come up with ideas on how to solve it. And then the other partner comes up with “maybe that way” and that they ...[in] a social way ... find some solution together.

Commenting in the post-interview he talked about the need for student agency:

[There are] many ways of seeing the figure. Then they make a formula that looks different at first, then they simplify it and then they get the same formula in the end. [You] can solve the problem with different approaches. What do they want to be left with? Well ... their agency that should be their own. [It] shouldn’t be that the subject has its agency and on the other side the students have their identity and their agency and [it] should be ... that they experience that they have a sense of determination over the subject...The subject should not define everything for them. They should also define what it means to have flexible solution strategies then, I think. For they should have their own strategies at the end of the day.

Erik used learning pairs actively in all his lessons. At the start of the lesson Erik encouraged the students to “help each other”. Based on what the students said in interviews, this seemed to be an established way of working: “if I cannot complete an exercise, I ask my partner for help”. This tells us that Erik’s intention that students should collaborate in pairs was exploited by the students.



Figure 1: Task: “Find a formula for the n -th figure”

In the following we explore to what extent Erik realised his aims. The episode below was typical of the interactions that we observed between Erik and different pairs. Here Finn, who was working at a desk with another boy, had his hand up. Erik had just finished helping another pair and went over to Finn. The task they were working on is shown in figure 1.

Finn: Is the exercise there ... is it ... F_n plus ... F_n plus two? Or?

Erik: erm ... $n[0]$.. [He takes the pen from the student] Add the bit that comes up here [He draws a dot to represent the figure F_0 on the task sheet.] This one will always be there. Then it's the arms that grow.

Finn: Yeah

Erik: It's always there. Plus one is always there. Right. That one, if you go back. F_0 , I mean, then you will get rid of those two, then you will just be left with ... with one, right. [He holds up one finger to Finn.]

Finn: Yeah

Erik: So there is plus one in each one there.

Finn: Hmm

Erik: Here is F_1 . What do you times one with to get two? What do you times two with to get four? Three to get six? Four to get eight? What do you times with there?

Finn: Four to get eight[?]

Erik: Yeah

Finn: Two

Erik: Two yeah. [Holds up two fingers.] Then it's always plus one and then it's n times by... [Knocks the table several times with his finger and then holds up two fingers.]

Finn: Two

Erik: $2n + 1$

Finn: Yeah ... yeah ... $2n + 1$ [?]

Erik: Yeah. Because there you have ... this one, if you look at this here, you will always have it ... that there, right ... that will always be there

Finn: Hmm

Erik: It becomes plus one.

Finn: Yeah

Erik: When you see... Here is F_1 , then it is times by two. Here is F_2 . Then it is times by ... here there are four. Here there are six. Here there are...?

Finn: Four

Erik: Eight

Finn: Ah [Sounds resigned and disappointed]

Erik: It's $2n$. Two times by one is two, plus one is three. Two times by two is four, plus one is five.

Finn: Hmm

Erik: Here there are eight arms [points to the next task on the sheet]. You may see it better there because it is a bit better version. [He stands up and walks away.]

Finn: Yeah

Initially there was discrepancy between the student's question and the teacher's response. In the task, the number of dots per figure were increasing by two. The student therefore asked: "is it $F_n + 2$?". The student was trying to express the solution as a recurrence relation and was grappling with "half baked" mathematical ideas. Here was a potential for the teacher together with the students to explore the relationship and difference between recurrence relations and closed formulas for geometrical patterns. As reported in the research literature, this is a challenge both for students and teachers (Moss & McNab, 2011) which leads to mathematical and pedagogical demands. The Foundation dimension of the KQ, as well as teacher's insight and awareness are of crucial importance in order to exploit this potential. Finn's idea was not followed up by the teacher, who guided him to a full solution to the task based on the closed formula. During this episode Erik addressed only the student who asked for help while the other student paid attention without contributing nor being invited to contribute to the conversation. The potential in paired seating was not exploited. This differs from Erik's intention about the students working in pairs to find a solution together.

In terms of the mathematical content offered in this classroom environment - the first dimension of the TRU framework (Schoenfeld & Floden, 2014) - the teacher's focus was on the answer and how to get there. The question "What do you times one with to get two?", is an example of funnelling (Wood, 1998). The task in hand has been reduced to a simple question about multiplication with answer "two". In the TRU framework, this would indicate a low score on cognitive demand as teacher intervention constrained students to activities such as applying straightforward or memorized procedures, and was explaining how he would solve the task (Schoenfeld & Floden, 2014). It appears that Finn did not see the relevance of the answer "two", since when Erik tried to get him to relate it to the closed formula, he needed to hold up two fingers as a prompt. Later Finn answered wrongly ("four") on a related line of questioning, another indication that he was being funnelled into giving correct responses. As in Wood's (1998) description of the effects of funnelling, Finn appeared to be focused on answering the teacher and abandoned his own recurrence-relation based ideas.

In the conversation Erik asked closed-ended questions, and Finn's contributions after the initial question were just single word responses. Thus, the two boys were not given the opportunity to discuss, explain or reflect on their mathematical ideas, processes that would indicate a fostering of the students' sense of agency. Much of the episode had the feel of a teacher monologue because Finn interjected the "yeah" responses rather than them being invited by pauses from Erik. The monologue with closed questions indicated the teacher's ownership of the mathematics.

In the post interview the teacher realised that he had been too eager to give the solution to the task, remarking on his impatience both before and after seeing himself in the video. In response to the video of the episode, Erik said:

I remember this one here, yes, the problem here is that he really struggled to understand - to crack the code here [What we] see immediately here [in the video] is that here I probably explain a bit more than...am probably, in a way, a little too quick to give him an answer or solution to the problem. I think. That he himself should have had a little longer to ponder the problem. I see that now, yeah. I should have done that. Eh, but it is clear that, on the other hand, if I help him with one task, then maybe he understands; can maybe move on to the next task and understand it ...

We see that Erik's focus is on helping the student. However, he expressed an insight about the student's mathematical capacity - that he should have given him some more time to ponder himself. We consider this expressed insight and awareness as a potential for the teacher to develop his in-the-moment pedagogy so he can act flexibly and productively with his students so he instead of funnelling can direct students' attention to what really matters. However, in the actual moment of this lesson, Erik did not seem to try to analyse and understand Finn's thinking, what he has said and why he said it. In the post interview, we suggested to Erik that when Finn said " F_n plus two" he was trying to express how the number of dots increases, and we asked Erik how he might get from there to the closed formula. He said:

He was thinking of this [the recurrence relation] I would assume. Because we spent a lot of time last year finding patterns in how it grows. We have been working on that. Then clearly to go from there and find that it is $2n + 1$. It is clear that we have not [done that]... We have worked a lot

with the graphical representation. Because we should have had a ... a type of square number variant then.

Here he went on talking about certain shapes (square numbers, rectangular numbers, and triangles numbers) that he had given the closed formula for on a hand-out sheet. In Norwegian, use of “clearly” in the sentence “Then clearly to go from there and find that it is $2n + 1$ ” is expressing that it is clearly difficult to do. We see that Erik struggled to suggest a way to support this transition. This quote indicates the heart of the matter: Erik struggled to understand the students’ utterances and he had to rely on his own solution and understanding due to not being able to transform the connection between the different parts of mathematics to something accessible for students.

The fourth dimension of the TRU framework considers the “extent to which students have opportunities to conjecture, explain, make mathematical arguments, and build on one another’s ideas, in ways that contribute to students’ development of agency, authority, and identities as doers of mathematics” (Shoenfeld & Floden, 2014, p.1). In the episode above the environment offered the chance for Finn to make a conjecture “ F_n plus two”. Finn’s subsequent contributions were constrained by the teacher’s questions. The episode above was picked out because it was typical of the interaction between Erik and the students. Thus, we argue that there was little opportunity for developing a sense of agency, authority, and an identity as a doer of mathematics in this classroom environment.

There was one interaction between Erik and another pair of students (Anna and Berit) which was atypical due to the students explicitly saying that they wanted to retain authority over their work. As Erik approached the desk Berit said “No, we don’t want any help from you” and waved her hand as if to indicate that Erik should go. Erik persisted to ask questions about Anna and Berit’s progress, but, before they would show anything, Anna demanded that Erik promise not to “give any hints at all”. Anna underlined this point by saying it three times. Apart from confirming how the classroom environment constrained the students’ ownership of the mathematics, the interaction between Anna, Berit and Erik indicates potential within this environment. We noted earlier that the mathematical authority was retained by the teacher in his interaction with Finn. In the interaction with Anna and Berit, we see a willingness from Erik to give that authority to the students, and a relaxed classroom environment that allows the students to challenge established patterns of social interaction.

The ensuing conversation with Anna and Berit returns to a pattern that resembles the other interactions with the students. Erik asked Anna about her solutions to the questions on the worksheet, and sometimes told her the solution before she had time to answer. Erik commented on one of the expressions that Anna had found by saying “No, this one I would multiply with this, to make it look prettier”. Anna objected but eventually said: “Ah, but OK” in a resigned way. Schoenfeld’s introduction to TRU refers to Engle when defining agency:

Learners have intellectual agency when they ... share what they actually think about the problem in focus rather than feeling the need to come up with a response that they may or may not believe in, but that matches what some other authority like a teacher or textbook would say is correct. (Engle, 2011, in Schoenfeld (2016), p.9).

In the exchange between Anna and Erik, the response from Erik constrained Anna to writing expressions that conformed to his idea of pretty.

Discussion and Conclusions

We would like to highlight that there are many positive aspects that we observed in this classroom: the teacher's intentions were for the students to collaborate and explore mathematics; he was using low threshold high ceiling tasks to promote inclusion and collaborative learning; there was a safe working environment as demonstrated by the girls' frank comments to the teacher; and the teacher was aware of the ideas of agency and ownership in the context of mathematics. The analysis of the mathematics revealed that students were directed to the answers and their form, not included as a pair, and the mathematics was reduced to a step-by-step procedure. Thus, the mathematics being offered by the interactions with the teacher is only limited in nature and scope. From the pre interview, this seemed not to be his intention. However, when reflecting on this episode in the post interview, rather than having shown in-the-moment awareness, which is an important aspect of the contingency dimension of the KQ (Mason & Davis, 2013; Rowland & Zazkis, 2013) he displayed a retrospective awareness. It is unclear if the teacher was aware that the way he spoke mathematics with the students reduced their access to it. Rather than the potential of students' half-baked ideas being exploited, learning was capped by over-nurturing, fostering teacher dependency. In particular, there was little awareness of and planning to address the difficulties of the transition from recursive relations to closed formula. For teachers to understand students' starting points, and how these can be built on to develop genuine understanding and agency, this study has demonstrated the importance of developing awareness of the interplay between the Foundation and Contingency dimensions of the KQ, especially the aspects of teacher insight and awareness in the Contingency dimension. There is potential here in the teacher's retrospective awareness, and further research could look at how retrospective awareness can develop to an in the moment awareness.

Furthermore, we saw that it is not sufficient to place students in pairs to promote collaboration. There is a possible mismatch between the teacher's intention (problem solving together) and the teacher's instruction to the students (to help each other) - the latter does not necessarily imply working on a joint product. The teacher's intention also contrasts with the manner of the help (a conversation with an individual not a pair) and the nature (demonstration through funnelling). Exploiting the environment with paired learning thus places demand on teachers' skills to initiate a mathematically rich discussion. We have identified the need for Teacher Insight to be able to build on the students' utterances. Thus, the interplay between the Contingency and Foundation dimensions of the teacher's MKT plays a crucial role here. Initiating a mathematically rich discussion in paired learning activities may place other demands on a teacher's MKT than orchestrating a whole class discussion does. We see this as an avenue for further research. In our experience, paired learning has become widespread in Norwegian schools in the last decade and, thus, the demands of paired learning for teacher knowledge have general implications beyond the teacher in the present study.

Acknowledgment

This research was funded by the Norwegian Research Council, grant number 287132.

References

Bednarz, N., Kieran, C., & Lee, L. (1996). *Approaches to Algebra, Perspectives for Research and Teaching*. Springer

- Carraher, D., Martinez, M., & Schlieman, A. (2008). Early algebra and mathematical generalization. *ZDM Mathematics Education*, 40, 3–22.
- Engle, R. A. (2011). The productive disciplinary engagement framework: Origins, key concepts and continuing developments In D. Y. Dai (Ed.), *Design research on learning and thinking in educational settings: Enhancing intellectual growth and functioning*. Taylor & Francis.
- Mason, J. (1998). Enabling Teachers to be Teachers: Necessary levels of Awareness and Structure of Attention. *Journal of Mathematics Teacher Education*, 1(3), 243–267.
- Mason, J., & Davis, B. (2013). The Importance of Teachers' Mathematical Awareness for In-the-Moment Pedagogy. *Canadian Journal of Science, Mathematics and Technology Education*, 13(2), 182–197.
- Mazenod, A., Francis, B., Archer, L., Hodgen, J., Taylor, B., Tereshchenko, A., & Pepper, D. (2019). Nurturing learning or encouraging dependency? Teacher constructions of students in lower attainment groups in English secondary schools. *Cambridge Journal of Education*, 49(1), 53–68. <https://doi.org/10.1080/0305764X.2018.1441372>
- Moss, J., & McNab, S. (2011). *Early algebraization, 2011 - Springer*. Springer.
- Orton, A., & Orton, J. (1999). Pattern and the approach to algebra. In A. Orton (Ed.), *Pattern in the Teaching and Learning of Mathematics* (pp. 104–120). Cassell
- Rowland, T. (2012). *The Knowledge Quartet*. <http://www.knowledgequartet.org/>
- Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, 8(3), 255–281.
- Rowland, T., & Zazkis, R. (2013). Contingency in Mathematics Classroom: Opportunities Taken and Opportunities Missed. *Canadian Journal of Science, Mathematics and Technology Education*, 13(2), 137–153.
- Schoenfeld, A. H. (2016). *An Introduction to the Teaching for Robust Understanding (TRU) Framework*. <http://truframework.org>
- Schoenfeld, A. H., & Floden, R. E. (2014). *The TRU Math Scoring Rubric*. <http://ats.berkeley.edu/tools.html>
- Skott, J., Mosvold, R., & Sakonidis, H. (2018). Classroom practice and teachers' knowledge, beliefs and identity In T. Dreyfus, M. Artigue, D. Potari, S. Prediger, & K. Ruthven (Eds.), *Developing Research in Mathematics Education* (pp. 162–180). Routledge.
- Wood, T. (1998). Alternative patterns of communication in mathematics classes: Funneling or focusing? In H. Steinbring, M. G. B. Bussi, & A. Sierpiska (Eds.), *Language and communication in the mathematics classroom* (pp. 167–178). National Council of Teachers of Mathematics.