



HAL
open science

Irrational numbers in the teaching of the Pythagorean theorem: practical argumentation in a Lesson Study and Didactic Suitability training course

Viviane Hummes, Telesforo Sol, Alicia Sánchez, Adriana Breda, Vicenç Font

► To cite this version:

Viviane Hummes, Telesforo Sol, Alicia Sánchez, Adriana Breda, Vicenç Font. Irrational numbers in the teaching of the Pythagorean theorem: practical argumentation in a Lesson Study and Didactic Suitability training course. Twelfth Congress of the European Society for Research in Mathematics Education (CERME12), Feb 2022, Bozen-Bolzano, Italy. <hal-03745383>

HAL Id: hal-03745383

<https://hal.science/hal-03745383v1>

Submitted on 4 Aug 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Irrational numbers in the teaching of the Pythagorean theorem: practical argumentation in a Lesson Study and Didactic Suitability training course

Viviane Hummes, Telesforo Sol, Alicia Sánchez, Adriana Breda and Vicenç Font

University of Barcelona, Spain;

vhummes@ub.edu; telesforo.sol@ub.edu; asanchezb@ub.edu; adriana.breda@ub.edu;
vfont@ub.edu

This work investigates the role of the Didactic Suitability Criteria in the argumentation oriented towards action when introducing irrational numbers in the teaching of the Pythagorean Theorem in a teacher training course that combines Lesson Study and Didactic Suitability. We analysed the dialogues of the lesson planning stage of the Lesson Study cycle by means of an argumentative trajectory and the Didactic Suitability Criteria, also considering practical argumentation. We evidenced the conflicts between the cognitive and the epistemic suitability criteria in which teachers gave more importance to one criterion than another depending on the context. Epistemic suitability gives rise to considering a representative sample of problems for a partial meaning of the Pythagorean Theorem, and cognitive suitability promotes argumentation about which intended meanings can be achieved.

Keywords: Irrational numbers, pythagorean theorem, practical argumentation, lesson study, didactic suitability.

Introduction

Research in Mathematics Education highlights the need to consider students' previous knowledge in teaching and learning processes. This entails the teacher should know and comprehend students' previous knowledge and then decide what to do in the instructional process. Among some trends that regard the importance of previous knowledge, we focus on Lesson Study (Huang et al., 2019), which originated in Japan and later spread to other countries, and the Didactic Suitability Criteria (Breda et al., 2018), proposed by the Onto-Semiotic Approach (Godino et al., 2019).

In Lesson Study, implicit agreements between participants on the aspects that are positively valued emerge. These aspects can be reinterpreted in terms of the components and indicators of the Didactic Suitability Criteria. In Lesson Study, some of these components and indicators may emerge as consensuses from the reflection of the group of teachers. This justifies the inclusion of the Didactic Suitability Criteria in a training course involving Lesson Study. This phenomenon is the origin of proposals for teacher training which focus on the development of reflection on teaching practice by combining Lesson Study and Didactic Suitability Criteria (Hummes et al., 2019). In addition, analysis of teachers' reflections indicates that teachers engage in argumentation about the actions that they agree to undertake, which some authors call practical argumentation (Gómez, 2017). This argumentation occurs when the participant teachers try to find a consensus and explain their reasons as equals. The structure of this type of argument is of interest in our research.

This work is part of broader research whose objectives are designing and implementing training experiences aimed at developing reflection and teachers' argumentative competence in order to study: i) How they argue in their lessons; ii) What is their practical reasoning about the actions they decide to carry out in lessons; and iii) How Didactic Suitability Criteria relate to (teachers') arguments about their practice during the teaching and learning processes. In particular, this study focuses on an episode from a training course for mathematics teachers that combines the Lesson Study and the Didactic Suitability Criteria. It aims to analyse the argumentation oriented towards action when introducing irrational numbers in the teaching of the Pythagorean Theorem and the role of the Didactic Suitability Criteria in this argumentation.

Theoretical Framework

Lesson Study: This is the collaborative and detailed design of a lesson, its implementation and direct observation in the classroom, and its joint analysis after its implementation (Huang et al., 2019). A group of teachers and experts with a common concern about their students' learning gather, plan a lesson, and finally analyse and discuss what they observed in the implementation. Multiple iterations of this process bring the teachers many opportunities to discuss the students' learning and how the teaching influences this learning. A Lesson Study cycle should follow these stages (Lim-Ratnam, 2013): 1) study of the curriculum and goals, when participants choose content to teach and set the learning goals; 2) lesson planning, when participants set the objectives of the lesson and describe its development; 3) implementation and observation of the lesson, when they record the impact of the planning on the students' learning and collect data generated from the observation; 4) joint reflection on the collected data, when participants use the data from the observation to reflect on the implemented lesson, the students' learning and the previous planning.

Didactic Suitability: In the Onto-Semiotic Approach, the didactic suitability of an instructional process is the degree to which it meets certain characteristics that allow it to be described as suitable (optimal or adequate) to achieve an adaptation between the personal meanings developed by students (learning) and the institutional meanings intended or implemented (teaching), taking into account the circumstances and available resources (environment) (Godino et al., 2019). This is a multidimensional construct that consists of six suitability criteria: 1) epistemic criterion, to assess whether the mathematics that is taught is 'good mathematics'; 2) cognitive criterion, to assess, before starting the instructional process, whether what is intended to be taught is at a reasonable distance from what students know; 3) interactional criterion, to assess whether the interaction solves students' doubts and difficulties; 4) mediational criterion, to assess the adequacy of time and material resources used in the instructional process; 5) emotional criterion, to assess the students' involvement in the classroom; and 6) ecological criterion, to assess the adequacy of the instructional process to the educational project of the school, the curricular guidelines, and the conditions of the social and professional environment, among others factors (Godino et al., 2019).

Argumentation oriented towards action: In the Theory of Communicative Action, argumentation is defined as a "type of speech in which participants state the validity claims that have turned dubious and try to accept or decline them using arguments" (Habermas, 1987, p. 37). We focus on argumentation oriented towards action (or practical argumentation) and on the consensus that

emerges from the reflection of a group of teachers on their own practice. Gómez (2017) defines practical argumentation as a “reasoning in social contexts directed to choose an action to solve a practical problem” (p. 215). The argumentative speech may be considered as a process, a procedure, and a product (Habermas, 1987). As a process, one may exclude any coercion and focus on the search for truth as an action directed to understanding. As a procedure, the discursive process of understanding is regulated as a cooperative work division between proponents (P) and opponents (O), where a validity claim (VC) that has become a problem is stated, examining with arguments whether this claim is recognized or not. In the case addressed here, some VC are discussed and the teachers can be distinguished, based on their reasons -supporting and providing argumentative strength (AS) to the VC, or trying to invalidate it (InvVC)- to recognise or not its validity. Finally, as a product, the argumentative speech is producing appropriate arguments that are convincing by their intrinsic properties and can be used to accept or refuse the VC. Arguments are the ways to obtain a common acknowledgment of the VC that the proponent states hypothetically, and through which an opinion can become a knowledge or an action. The argument trajectory (Ramos, 2006) is a tool that allows the elements of the argumentative speech mentioned above to be identified. In this study, we used an argumentative trajectory to analyse the participants’ practical argumentation.

Methodology

This is a qualitative-interpretive study. Eight mathematics teachers working in schools in the south of Brazil (students aged 11-18) participated in this research. They were graduated in mathematics and had three-to-fifteen-years of teaching experience. In addition, three of them had a master’s degree in Mathematics Education. The training program was planned as a face-to-face implementation, but it had to be restructured as a virtual implementation due to the COVID-19 pandemic. The sessions were conducted using Skype and were recorded with the participants’ permission. The training course followed these phases: 1) implementation of two complete Lesson Study cycles (two groups of four teachers, each group developed a cycle); 2) introduction of the Didactic Suitability Criteria as a tool to guide the teacher reflections; 3) new analysis of the implemented lesson and its redesign using these criteria. The first author led the training course and acted as a participant observer.

We identified different episodes of practical argumentation between the participants along the phases of the course. We also identified the role of the Didactic Suitability Criteria within the arguments given in each session, with different relevance. The analysis presented in this work can be reproduced for each of these episodes. However, due to a lack of space, we present the analysis of only one episode from the lesson-planning stage of the Lesson Study conducted by one of the groups. The participants chose the Pythagorean Theorem as the topic to be taught as it allowed them to implement the original lesson and its redesign for two different student groups (aged 14-15) at two different moments of the semester.

With the argumentative trajectory, we considered the three aspects of the argumentation in the episode: ‘process’, aiming to achieve ideal conditions for communication between participants; ‘procedure’, considering the teachers’ argumentations in the form of cooperative work division between proponents and opponents; and ‘product’, obtaining appropriate arguments to accept or not the VC about including irrational numbers in the teaching of the Pythagorean Theorem. In addition,

the argumentative trajectory allowed the argumentation to be related to the components and indicators of the Didactic Suitability Criteria, in order to identify the role of these criteria in the argumentation oriented towards action about introducing irrational numbers in the teaching of the Pythagorean Theorem. The analysis followed some phases similar to those used by Ramos (2006): i) We reviewed the sessions to identify the episodes of practical argumentation between the teachers, meaning discussions between them about the decisions made to plan the lesson. ii) Once we identified the episodes of argumentation in the videos, we recognised the teachers' roles, the achieved consensus, the invalidated claims, and the reasons to invalidate them, using the constructs described in the theoretical section (P, O, VC, InvVC, AS) plus RMC (rationally motivated consensus), and CO (consensus by omission). iii) We made an argumentative trajectory to visualise the relation between VC, InvVC, and AS in the episode, and thus conclude with the RMC. We distinguished the participants as P or O, and identify who participated in the CO. Regarding the latter, some elements of the argumentation are implicitly expressed in the sessions. These elements were inferred in the argumentative trajectory from the general context where the episodes of argumentation occur. To do this, we reviewed the videos recorded during the first and second stages of the Lesson Study cycle several times. iv) We analysed the argumentative trajectory using the components and descriptors of the Didactic Suitability Criteria to identify the role of these criteria in the participants' argumentation. v) We used triangulation by experts to validate the obtained results.

Analysis

In this section, we present the analysis of an argumentation episode that occurred in the planning phase of the Lesson Study cycle implemented by four participant teachers (P1, P2, P3, P4). In the first phase of the Lesson Study cycle, where learning goals were established, P1 suggested a problem about finding the length of the diagonal of a square of side one using the Pythagorean Theorem, aiming to include examples of triangles with irrational lengths, in addition to examples with natural lengths. However, at that moment, the teachers did not discuss this proposal. Nevertheless, in the phase of planning the lesson, the teachers considered presenting examples of right-angled triangles with irrational lengths. They suggested using the Pythagorean Theorem to identify the length of the diagonal of a square of side one. In the following lines, we present the argumentation episode developed by the teachers:

- P1: I think that this will be light for them [he refers to the Pythagorean Theorem as a relation between natural lengths in a right-angled triangle]. They will have problems when we talk about the square root of two, since [...] students have a great difficulty understanding the irrational numbers and, when we make the square of side one, I think that it will generate certain difficulty. Don't you think so? (VC1).
- P2: But the idea was only showing that it also works for the square root, was it not? (InvVC1).
- P1: This is the idea. In this case, we should remind that the square of the square root of two is two. (AS1 to VC1).
- P4: They didn't learn that. (AS2 to VC1).
- P1: Then, as they have not seen that, there should be an easy way to verify... Did they not learn the notion of the square root of two in the number line last year? (AS3 to VC1)

- P4: We worked on the irrational numbers quite well. But what you said is true, they have difficulties accepting that the square root of two is a number. I doubt that they remember it. (AS4 to VC1)
- P1: [...] we could propose how a square of area two is. For instance, you have a square of side one, then the area is one. In order to obtain the area two, what must happen? If the square side is two, then the area will be four. Thus, they will have a notion that, in order to obtain an area two, the length of the square side should be a number between one and two [...]. (AS5 to VC1)
- P4: I think that this is a wonderful idea when the lesson is face-to-face. (InvAS5 to VC1)
- P3: Because many times they see a root and get blocked. They do not even wait to see what you want to do. I think that we should do a previous class to delve into the content of roots. (VC2)
- P4: I think that, after learning the Pythagorean Theorem and really showing that it works, we could make a triangle with two sides one and try to find the third side. Then, I can mention that the square root of two is an irrational number and that length that we have just found is an irrational number. (VC3)
- P1: What about using the ruler and the compass to make, for example, a square of side one and using the compass to show the length of the diagonal on the number line? What do you think? (VC4)
- P4: I like it. (AS1 to VC4)

We identified proponents (P1, P3, P4) and opponents (P2, P4). The theses are: i) introducing irrational numbers with the Pythagorean Theorem applied to the right-angled triangle of cathetuses one; ii) the sample of problems cannot be extended due to a lack of previous knowledge and the classroom conditions.

Argumentative trajectory

i) P1 posed the problem of working on the Pythagorean Theorem with irrational lengths. (VC1) ii) P2 tried to invalidate VC1 returning to the initial proposal of just showing some examples of right-angled triangles with irrational lengths. (InvVC1) iii) P1 considered that it is important to bear in mind the properties of irrational numbers, specifically, the operations with roots. (AS1 to VC1) iv) P4 indicated that the students did not learn operations with irrational numbers. (AS2 to VC1) v) P1 tried to discover what the students know about irrational numbers. He said that they should find an easy way to verify the Pythagorean Theorem with irrational numbers. (AS3 to VC1) vi) P4 explained that she has worked on that, but she was not sure whether the students remember it. She confirms that students also had difficulties understanding that the square root of two is a number. (AS4 to VC1) vii) P1 proposed to remind the students that the square root of two is an irrational number, doing an approximation of the decimal expansion. (AS5 to VC1) viii) P4 argued that this would be a good idea if this were a face-to-face implementation. (InvAS5 to VC1) ix) P3 highlighted the need of doing a previous lesson about operations with roots to address the students' difficulties. (VC2) x) P4 only mentioned that the number found is an irrational number. (VC3) xi) P1 proposed that they could use the ruler and the compass to make a square of side one and show the irrational number that corresponds to the length of the diagonal, representing it on the number line. (VC4) xii) P4 said that she liked that idea. (AS1 to VC4) xiii) Participants achieved a rationally motivated consensus (RMC). P2 and P3 did it by omission (CO).

Participants continued discussing about the resources for the lesson. The following questions arose: How could they use the ruler and the compass in a virtual lesson? How will they record the lesson? Which should be the position of P4 during the recording with the resources?

Analysis of the argumentative trajectory from the perspective of the Didactic Suitability

i) P1 highlighted the students' difficulties during the learning of irrational numbers. He questioned whether "the intended meanings can be achieved", an indicator of the component "prior knowledge" of the cognitive criterion. ii) P2 proposed to present a situation with irrational numbers as the lengths of the sides of a right-angled triangle, without delving into the concept of irrational numbers and operations with them. P2 tried to maintain the idea of having a representative sample of problems within the same partial meaning of the Pythagorean Theorem. This is related to the indicator "for one or more partial meanings, a representative sample of problems is provided" of the component "representativeness of the complexity of the mathematical object" of the epistemic criterion. iii) P1 considered whether "the students have the previous knowledge necessary to learn the topic", particularly the knowledge on operations with irrational numbers. It is related to the component "prior knowledge" of the cognitive criterion. iv) P4, who implemented the lesson and knew the students, confirmed the students' lack of knowledge on operations with irrational numbers. The component "prior knowledge" of the cognitive criterion emerged again. v) When P1 asked if the students placed irrational numbers on the number line, he looked into the students' previous knowledge. Thus, the component "prior knowledge" of the cognitive criterion was present again. vi) P4 confirmed the students' difficulty understanding the square root of two as a number. This assertion also corresponds to the component "prior knowledge" of the cognitive criterion. vii) P1 proposed an activity to work on the idea of irrational number as a relation between the area of a square and its sides (particularly, he proposed the square of area two), searching for an approximation of the decimal expansion, as a way to review irrational numbers for the students. He tried to propose a task that included a relevant mathematical process. This is an indicator of the component "richness of processes" of the epistemic criterion. viii) P4 explained that the idea of P1 would be wonderful for a face-to-face lesson. Although she did not mention it, P4 implicitly referred to the "classroom conditions" and "teacher-student interaction". These are components of the mediational and the interactional criteria respectively. ix) P3 mentioned a possible aversion to the lesson, when the students see the roots of non-perfect square numbers. This idea is related to the component "emotions" of the emotional criterion. In addition, she proposed to do a pre-lesson activity to address the prior knowledge necessary to comprehend the mathematical object that would be taught. This proposal is related to the indicator "adaptation of the intended meanings" of the cognitive criterion. x) P4 proposed only to show that using the Pythagorean Theorem they could obtain an irrational number (in this case, the square root of two), without doing a mathematically rich process (a component of the epistemic criterion) or considering the students' lack of previous knowledge on irrational numbers (a component of the cognitive criterion). xi) P1 proposed the use of material resources (ruler and compass), a component of the mediational criterion. At the same time, he proposed an activity rich in mathematical processes (using the ruler to translate the length of the diagonal of the square on the number line). It is related to the component "richness of processes" of the epistemic criterion. This episode ends with the discussion about the means and resources for the lesson. Thus, both the mediational and interactional criteria were highlighted.

In this episode, the practical argument took place when the teachers considered an indicator of the epistemic criterion: a representative sample of problems must be provided. The group tried to find a certain balance between the epistemic and the cognitive criteria and, in this search, they resorted to other Didactic Suitability Criteria (mediational and interactional criteria). The group agreed to briefly introduce the square root of two as an irrational number after applying the Pythagorean Theorem to a right triangle with length one. This was, in a certain way, a reformulation of the initial proposal that triggered the episode. The consensus obtained depended on the relevance of the different didactic suitability criteria in the argumentative trajectory.

Conclusions

The episode of argumentation that we analysed occurred in the lesson-planning phase. When the teachers were planning to work on the Pythagorean Theorem as a relation between the lengths of the sides of a right-angled triangle, they proposed to extend the sample of problems with tasks that included right-angled triangles with irrational lengths, in addition to tasks with natural numbers (Pythagorean triples). This generated an episode of practical argumentation about which actions to do, that evidences how the teachers gave importance to the students' knowledge of irrational numbers to teach the Pythagorean Theorem. Throughout the episode of argumentation, the teachers became aware of the students' knowledge through the validity claims that they made.

In order to answer our study aims, first, we could identify moments of practical argumentation in the course of Lesson Study and Didactic Suitability Criteria. As explained in the last section, the analysis of the argumentative trajectory from the perspective of the didactic suitability made evident the important role of the Didactic Suitability Criteria to argue either for or against a certain action. Didactic Suitability Criteria were essential to provide argumentative strength to the proponent. In terms of Toulmin's (Molina et al., 2019) model, the Didactic Suitability Criteria took the role of the warrant. An area of interest is why Didactic Suitability Criteria provide argumentative strength and are present even when they were not taught. Our interpretation is that this happens because the Didactic Suitability Criteria emerge from a wide consensus among the educational community.

Moreover, it is worth noting that some criteria that are recommendable a priori, such as working on the Pythagorean Theorem in a mathematically rich way or considering the students' previous knowledge, can be in conflict when they are applied to a certain context. Regarding the epistemic suitability, the sample of problems for a partial meaning of the Pythagorean Theorem (the relation between the lengths of the sides of a right-angled triangle) should be representative. The cognitive suitability fosters arguments about the intended partial meanings that are achievable and the students' previous knowledge. In this case, in the practical argumentation about introducing irrational numbers in the teaching of the Pythagorean Theorem, conflict arose between the cognitive suitability criterion and the epistemic suitability criterion. Then, the teachers focused on one criterion or the other, considering the specific context. We observed that the teachers who are in favor of extending the sample of problems, focus on epistemic suitability. While the teachers in favor of limiting the extension of the sample focus on cognitive aspects.

In the broader project, where this study is located, the main objective is to develop teacher reflection. In that sense, it is already known that teaching the Didactic Suitability Criteria to the participants

improves reflection. In addition, with the design and implementation of the course that combines the Lesson Study and Didactic Suitability Criteria, it is expected to obtain data to analyse the practical argumentation in different stages of the course and, in this way, verify the influence of the combination of Lesson Study and Didactic Suitability Criteria in the promotion of teacher reflection as a professional competence. From the analysis carried out, our conclusion is that we could identify several moments of practical argumentation in which the Didactic Suitability Criteria had a relevant role. In our opinion, this occurred because the training course implemented facilitates collective argumentation, among other reasons.

Acknowledgment

Work developed as part of the project PGC2018-098603-B-I00 (MINECO/FEDER, EU) and with the support of the Full Doctorate Abroad, process number 88881.173616/2018-01 (CAPES/Brasil).

References

- Breda, A., Font, V. & Pino-Fan, L. (2018). Criterios valorativos y normativos en la Didáctica de las Matemáticas: el caso del constructo idoneidad didáctica. *Bolema*, 32(60), 255–278.
- Godino, J. D., Batanero, C. & Font, V. (2019). The onto-semiotic approach: Implications for the prescriptive character of didactics. *For the Learning of Mathematics*, 39(1), 37–42.
- Gómez, J. (2017). ¿Qué es la argumentación práctica? *Revista Co-herencia*, 14(27), 215–243.
- Habermas, J. (1987). *Teoría de la acción comunicativa I: Racionalidad de la acción y racionalización social*. Taurus.
- Huang, R., Takahashi, A., & da Ponte, J. P. (2019). *Theory and Practice of Lesson Study in Mathematics*. Springer.
- Hummes, V. B., Breda, A., Seckel, M. J., & Font, V. (2020). Criterios de Idoneidad Didáctica en una clase basada en el Lesson Study. *Praxis & Saber*, 11(26), Article e10667.
- Hummes, V. B., Font, V., & Breda, A., (2019). Combined use of the Lesson Study and the Criteria of Didactical Suitability for the development of the reflection on the own practice in the training of mathematics teachers. *Acta Scientiae*, 21(1), 64–82.
- Lim-Ratnam, C. (2013). Lesson Study step by step: How teacher learning communities improve instruction. *International Journal for Lesson and Learning Studies*, 2(3), 304–306.
- Molina, Ó., Font, V., & Pino-Fan, L. R. (2019). Estructura y dinámica de argumentos analógicos, abductivos y deductivos: Un curso de geometría del espacio como contexto de reflexión. *Enseñanza de las Ciencias*, 37(1), 93–116.
- Ramos, A. B. (2006). *Objetos personales matemáticos y didácticos del profesorado y cambios institucionales. El caso de la contextualización de las funciones en una facultad de ciencias económicas y sociales* [Doctoral dissertation, Universitat de Barcelona]. Dipòsit Digital de la Universitat de Barcelona. <http://diposit.ub.edu/dspace/handle/2445/41428>