

Towards implementing Black Boxes in Mathematical Modeling an epistemological Approach

Adapting models from science and science education

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This article explores the concept of black boxes within mathematical modeling. First, we perform a literature-based clarification of the term black box, the use of a black box, and explore the use of this construct in science, and workplace. A black box is characterized by the following issues: (a) being believed to be distinct, (b) having observable (and relatable) inputs and outputs, (c) being black (that is, opaque to the observer). Afterwards, we demonstrate the uses of black boxes within mathematical modeling. We generalize these uses to a model of how black boxes can be implemented in mathematics education. Based on this model, we derive educationally relevant activities for using and opening such a black box. The application of these activities is illustrated in more details using an example of physical chemistry: the Arrhenius Equation.

Keywords: Black box, mathematical modeling, STEM

Introduction

Mathematical modeling is an important part of mathematics education (Kaiser, 2020). English (2016) described modeling as a powerful tool to implement 21st century skills in mathematic classrooms. This is because real-world problems connect mathematics with aspects of the world, and are often linked with mathematical modeling and mathematical models (s.f. Artigue et al., 2007; Kaiser & Stender, 2013; Lingefjärd, 2006).

In practice, many models use black boxes. For example, they serve as simplifications of phenomena or approximations in modeling tasks (e.g. Bissell, 2004; Buchberger, 1990; Hoiijnen et al., 1992; Straesser, 2007; Williams & Wake, 2007). Black boxes are often used, especially in work places and science models (Straesser, 2007; Williams & Wake, 2007). In order to understand how the term black box is used and whether or how it can be utilized for mathematics, we have used an epistemological approach and analyzed tasks of science. Because of these findings, we argue that we should include teaching about black boxes in mathematics education. However, there is no consensus (yet) on how exactly we should do that. Hence, the aim of this paper is to give an overview of possible approaches, to present our own concept, and to deepen existing approaches in the process.

Theoretical background: Black Boxes

Kaiser and Stender (2013) proposed two different directions for mathematical modeling: “*mathematics for applications, models and modeling*” (p. 278), and “*applications, models and modeling for the learning of mathematics*” (p. 278). The second headline emphasizes the goal of learning mathematics. In this paper, we want to implement black boxes into mathematical modeling in the context of STEM. Therefore, we will refer to the first headline. The realistic or applied approach

of mathematics puts the solution of the real-world problem in the center. To describe the processes while working out the solution different modeling cycles have been developed (e.g. Blum & Leiß, 2007; Kaiser & Stender, 2013). In the following, we will refer to the modeling cycle proposed by Kaiser and Stender (2013). This corresponds to the mathematical modeling cycle in physics education of Redish and Bing (2009) excluding the real-world situations and fits the later presented problem.

Modeling with Black Boxes

As described above assumptions are obligatory during modeling. These can be interpreted as black boxes. In addition, well-accepted mathematical models can work as a starting point of further real-world problems. Therefore the problem solver must not have fully understood the origin model or its emergence, but its input and output behavior. Accordingly, this mathematical model is used as a black box. In our contribution, we give an overview of black boxes in general and in particular in the context of mathematical modeling. We will present a way to open black boxes, but without giving an answer to the question if and when one should open a black box.

The term black box is usually used without a precise definition, but in a kind of intuitive understanding (e.g. Bissell, 2004; Buchberger, 1990; Fojo et al., 2017; Hoiijnen et al., 1992; Straesser, 2007; Williams & Wake, 2007). We use the definition of black box provided by Glanville (1982). Following a philosophical approach he characterizes a black box as: “[...] (a) being believed to be distinct, (b) having observable (and relatable) inputs and outputs, (c) being black (that is, opaque to the observer)” (Glanville, 1982, p. 1). We want to emphasize how we understand this definition and give an example. One aspect that we want to enhance is that even though the definition claims observable inputs and outputs, it does not mean that the researcher is already fully aware of its input-output behaviour. Notably, the fact that a black box is both observable and relatable does not necessarily imply that it is also deterministic. Indeed, statistical systems can also be understood as a black box.

State of the art: Black Boxes in Mathematics Education

In mathematics education, black boxes are associated with Computer-Algebra-Systems or other technical systems (Buchberger, 1990; Peschek, 1999; Williams & Wake, 2007). For this usage of black boxes, Peschek (1999) defines a black box as something “[...] in denen operatives Wissen so materialisiert ist, dass es als Ganzes aufrufbar und einsetzbar wird, ohne dass ihre innere Funktionsweise verstanden werden oder auch nur bekannt sein muss” [in which operational knowledge is materialized in such a way that it can be called up and used as a whole without its internal functioning being understood or even having to be known] (p. 1). This mathematical perspective on black boxes is not sufficient. On this account, the focus in the following is not on trivialized mathematics that is taken over by computer systems, but on the usage of black boxes in modeling and STEM education (e.g. Krell & Hergert, 2019; Krell & Krüger, 2017; Upmeier zu Belzen, Krüger, & van Driel, 2019).

Modeling with Black Boxes

In later approaches in science education, a model-based approach is often chosen and the terms *model of* and *model for* are mentioned in this discussion (Gouvea & Passmore, 2017). The aspect *model for* something means using the model to gain more information about the real world or to make

predictions. Therefore they turn into epistemic tools (Gouvea & Passmore, 2017). In this application, it can be useful or even necessary to use a model as a black Box. The idea of black box might emphasize the fact, that a model is a useful thinking tool, but not a precise description of reality. This misconception can otherwise lead to difficulties understanding more complex systems that cannot be explained with the origin model or even prevent scientists to gain new epistemological knowledge (Taber, 2017). But a black box in science education can not only be used to emphasize the model as a tool but is also an approach for modeling itself, by using a black box activity (e.g. Ruebush et al., 2009; Warren, 2001).

Opening a Black Box

The metaphor of opening a black box represents the precise exploration and questioning the inner functioning for which the black box stands for. There are two reasons for opening a black box. The first reason is to gain scientific knowledge about the black box itself. The motivation to gain scientific knowledge about the black box can be research or education. In the context of research, the scientist wants to gain knowledge about the unknown. In science or mathematic education, the incentive to open a black box can be versatile. By opening a black box students can learn something about the model itself, the modeling process and lead to a deeper understanding about the black box, for example, a model that serves as a tool. This can and should prevent application errors. The second reason is that the black box cannot be distinguished.

Black Box Concept

In this section, we explain how we should include black boxes into mathematics education. Therefore, we want to distinguish between a black box that evolved from a white box and a black box that should only represent the unknown. The first, for example, involves mathematical models whose modeling is not apparent to the user, but the user knows about their validity. The second one implies for example phenomena in nature that have not yet been explored in detail. The later criterion of Glanville's definition mentions the user's point of view, and brings both kinds of black boxes together. This view of using a black box is illustrated in Figure 1.

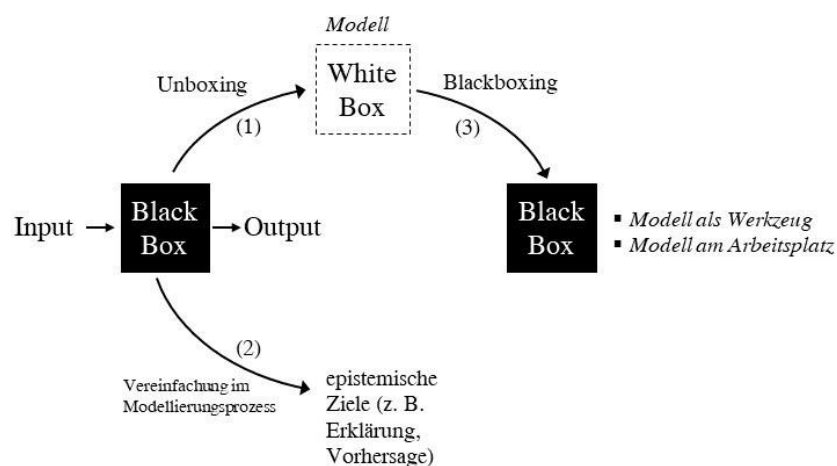


Figure 1: Extended Approach: illustration of the black box, refined from Glanville (1982, p. 1)

We justify this distinction due to their vocational use. By studying unknown phenomena, researchers are trying to unbox the unknown (1) (Glanville, 1982). After unboxing the unknown, it might be

useful to turn the white box into a black box (3). Williams and Wake (2007) point out two reasons for black boxes: time and space. However, for mathematics, only time is relevant.

Time represents the black boxes that changed models and/or mathematical work into instruments, and tools (Williams & Wake, 2007). This leads to efficient use of those models but goes along with the leakage of knowledge. For example, the model designer knows the simplifications they made, but the user does not. This is a typical situation at work (Straesser, 2007). CAS-Systems are self-constructed by the designer who is fully aware of the underlying algorithm, the technic, and mathematical knowledge, but it is opaque to its observer/user. The command for differentiation or integration serves as an example. In this application, the software is taking over the mathematical work (Williams & Wake, 2007).

In contrast to opening the black box, it might be useful, not to unbox the black box (2). This may seem to contravene the aim to study the unknown, but “[...] our ability to overcome and cope with ignorance and thus is a primitive of learning and, hence, of science” (Glanville, 1982, p. 1). This can be for example a simplification and is therefore essential in the work of modeling (Bissell, 2004; e.g. Hoijnen et al., 1992). Hoijnen et al. (1992) depict an application for the black box as an approximation in mathematical modeling in a scientific context. He describes the use of a black box to estimate the energy that is gained out of biomass. This enables to calculate auto-and heterotrophic biomass yield the without studying the microorganisms that are not relevant for the modeling (Hoijnen et al., 1992).

One reason for those simplifications is the fact, that exact scientific research is determined by the limits of the measurable aspects (Ortlieb, 2008). Another reason is the intricacy of parts of phenomena that turn out irrelevant compared to the phenomena. Bissell (2004) gives an example in the context of communication and control engineering and points out that:

In this approach, a complex linear electrical network is represented by its input-output behavior; at this level of abstraction, the precise nature of the interconnections of components inside the black box becomes irrelevant (p. 312).

This leads to the conclusion, that in the scientific context it is necessary to leave some phenomena opaque as a black box to achieve epistemic aims. Therefore, this way of working with models and modeling has to be part of the scientific and mathematic curriculum.

Activities to open a Black Box in Education

The result of an opening process for example by modeling is a more specific model, which still leaves a part of the real word opaque. Latour (1994) describes this endless process as follows: “Each of the parts inside the black box is a black box full of parts” (p. 37). An endless process is incompatible to gain results, efficiency, and education. Therefore, the necessity of a *temporary ending* is essential. In the following, we want to picture relevant activities to work with black boxes, and an option to bring the endless process of opening with a temporary ending together. We try to identify boxes in boxes. This leads to a new modeling cycle, which is illustrated in Figure 2.

The first level represents the black box itself, for example, an equation that is used as a tool. The reasons for this usage have been explained in previous sections. The target itself defines the different underlying levels and can appear in different ways. By opening a black box students can learn

something about the model itself, this might be by the interpretation of the model or learning about the modeling process itself. While doing so, assumptions or underlying models appear in this process, which are represented by the smaller boxes below, and defined as new black boxes. Due to this definition, there is a point, where the opening process is finished due to its target, even though not everything is visible to the observer. The most difficult part of this is to know the point of whether to stop the process, or to go further to fulfill the target. This topic needs further research.

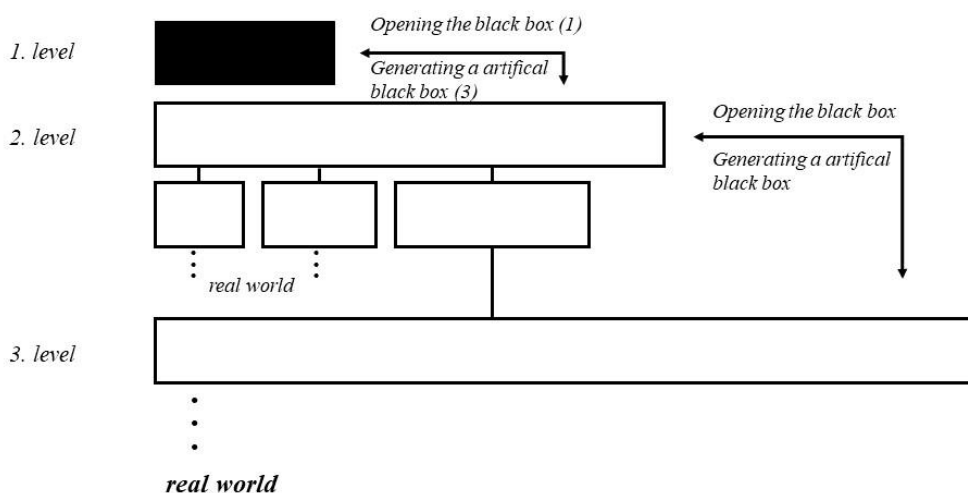


Figure 2: How to open and generate a Black Box.

Another activity to open the black box is that the students interfere it, and make assumptions from the input-output performance. This modeling access is called a black box activity in science education (Krell et al., 2019). This activity can be applied in a different way. It can be linked to a real-world model or applied on a black box for educational reason.¹ The usage of this black box is only to foster students' understanding of models and modeling in science showed positive effects (Krell & Hergert, 2019). Here, modeling serves as a tool for opening a black box.

The described modeling tasks retrieve in the atomistic approach of mathematical modeling, that is besides the holistic approach consensus on the teaching and learning of mathematical modeling (Kaiser, 2020). In the following section, we illustrate the benefit of our model by applying it theoretically to the Arrhenius equation.

Application: The Arrhenius Equation

The Arrhenius Equation (1) is one example in science (education), how a model is used as a black box. The equation is one of the most important equations in physical chemistry (Logan, 1982). For example, it is used in material research for battery technology (Breuer et al., 2015; Ren et al., 2015).

$$k = A \cdot e^{-\frac{E_A}{R \cdot T}} \xleftrightarrow{\text{Linearization}} \ln \frac{k}{A} = -\frac{E_A}{R} \cdot \frac{1}{T} \quad (1)$$

A common application in science education is the ascertainment of the activation energy of the sucrose inversion. To do so the students only need the scientific model as a tool. In this process, the

¹ e.g. Freie Universität Berlin. Modelle der Biologie. <https://tetfolio.fu-berlin.de/web/440484> 29.08.2021

model is embedded as a black box and will not be opened, which is an example for the first level of Figure 2. The modeling steps for the ascertainment are described in Figure 3 based on the modeling cycle of Kaiser and Stender (2013), in which the real situation is replaced with the black box. In this example the Arrhenius Equation is the black box.

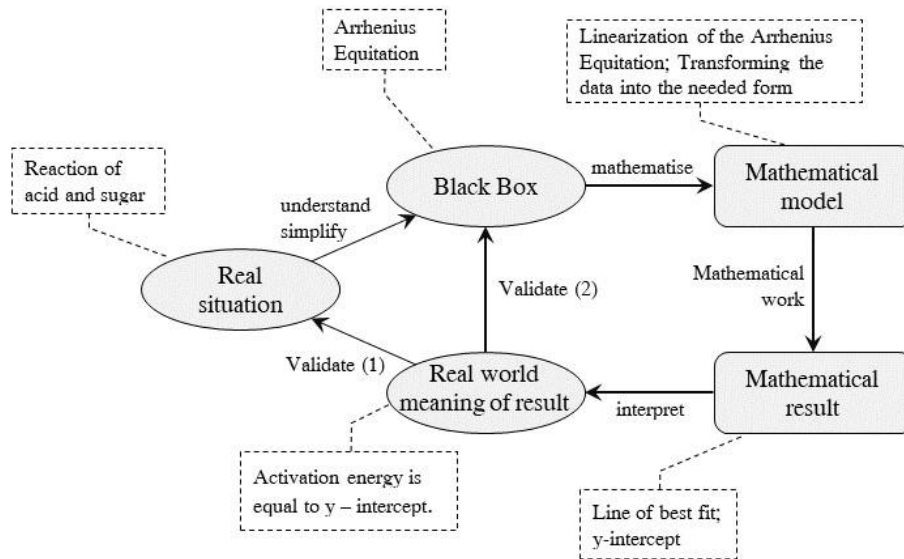


Figure 3: modeling cycle to ascertain the activation energy of the sucrose inversion according to Kaiser and Stender (2013, p. 279)

The modeling task is the ascertainment of the reaction of acid and sugar that cannot be measured directly. The students need to consider, how they can ascertain the activation energy only with measurable data. The Arrhenius Equation is a tool to determine the wanted quantity. To do so the reaction rate is necessary. This one is ascertainable with the changing turning angle of the liquid during the reaction. These results are the base of the following usage of the Arrhenius Equation to determine the activation energy. In the next step, the data is transferred to fit into the mathematical model. This data set is mapped to a coordinate system and the line of best fit is plotted. This leads to the so-called Arrhenius slope (Figure 4). The students have to read off the y-intercept of the slope that corresponds to the activation energy. After this, the results are validated (2) by approximating possible mistakes during the measurement and its analysis. Therefore, every step of the modeling is revised. Following the steps of the modeling cycle can help to structure the propagation of uncertainty. After that the results are compared with the literature value (validate (1)).

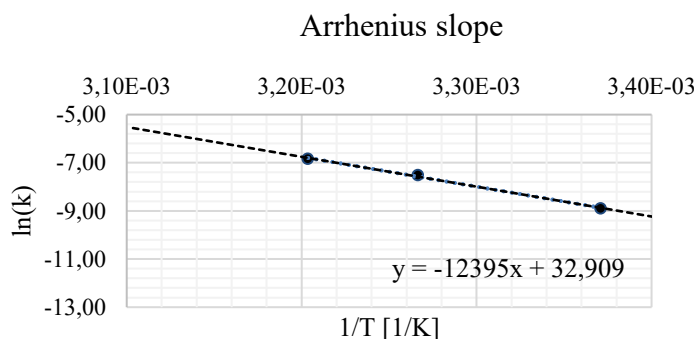


Figure 4: Arrhenius slope

In this described usage, the equation is used as a black box. In the context of education, a reason to unbox the equation is to gain knowledge about the model. This takes us from knowing 'how' to knowing 'why'. The interpretation reveals the connection between the different variables but does not explain the different parameters themselves. This leads to an underlying level. For example, the activation energy and its factors can be interpreted as another black box.

Summary

In this paper, we focused on how we can apply black box techniques in mathematical modeling. By using literature-based normative approach, we developed a model to implement black boxes in mathematical modeling, and argued for its benefits in mathematical education. These benefits show themselves in the necessity for application in science and workplaces, and within the modeling of real-world situations. However, the critical reflection of this process is an open question for future research. This, most importantly, includes deciding when and where to apply these techniques ("critical opening"), both in an educational setting and in the real-world.

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