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► **To cite this version:**

Jorunn Reinhardtsen, Martin Carlsen. Computational and analytical perspectives in introductory algebra: a theoretical contribution. Twelfth Congress of the European Society for Research in Mathematics Education (CERME12), Feb 2022, Bozen-Bolzano, Italy. hal-03745191

HAL Id: hal-03745191

<https://hal.science/hal-03745191>

Submitted on 3 Aug 2022

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Calculational and analytical perspectives in introductory algebra: a theoretical contribution

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In this theoretical paper we discuss what a calculational perspective on mathematical activity entails and discuss its relevance in terms of related speech genres in the introductory algebra classroom. Learning in the introductory algebra classroom is multifaceted because major shifts in form (the algebraic syntax versus arithmetic syntax) and function (analytic rather than calculational) are expected to take place. This theoretical argument is exemplified through student utterances from a Californian 6th grade classroom discussion, in which students' calculational perspective on mathematical activity is apparent. The data reveal traits and intentions from both traditional and alternative instructional genres. Our main argument is that the students' approaches to generalize provided patterns were limited to their perceptual field and thus fail to include a deductive argument, the key feature of algebraic reasoning.

Keywords: Algebraic reasoning, analytic perspective, calculational perspective, introductory algebra, speech genre.

Introduction

This theoretical paper discusses two perspectives on mathematical activity, *calculational* and *analytic*, that are discernible in introductory algebra classrooms. Different frames of reference, or speech genres (Bakhtin, 1986), can present communicational challenges and limit possibilities for algebra learning. We take these two perspectives as points of departure and discuss these in terms of previous research. A calculational perspective is exemplified through discussions of patterning tasks in a US 6th grade classroom.

Hewitt (2019), in a theoretical contribution, suggested that students in school ought to focus on “structure within complex examples” (p. 558) to develop algebraic thinking. Moreover, the students should be encouraged to make their reasoning explicit through expressing what they see rather than performing any mathematical calculations. Montenegro et al. (2018) argued in a similar manner from an empirical point of view. They found that students succeeded in obtaining a functional relationship in a figural pattern when the students turned away from the patterns in numbers and turned towards structure in the figural pattern engaged with. Both Hewitt (2019) and Montenegro et al. (2018) thus argued in favour of an analytic approach over a calculational one.

Although the goal of algebra teaching is for students to engage in analytical thinking, the calculational discourse persists in many classrooms. We argue that research has to acknowledge and investigate the role that this discourse plays in students' participation in algebraic activity. Learning in the introductory algebra classroom is multifaceted because major shifts in form (the algebraic syntax versus arithmetic syntax) and function (analytic rather than calculational) are expected to take place. Sfard (2007) addresses the bewilderment involved in the shift in function of mathematical activity

algebra teaching may cause for students through her heading “When the rules of discourse change, but nobody tells you”. We would like to contribute theoretical insights into the characteristics of students’ calculational perspective in introductory algebra classrooms

A sociocultural approach to mathematical thinking

To address students’ perspective on mathematical activity in classroom interactions we draw on Radford (2014) who defines thinking as cultural, embodied and material. Students mathematical thinking is not a separated and ideal internal process but occur as they use cultural artifacts (function table, algebraic syntax, etc.) and other semiotic means in social and goal-oriented activity. The semiotic resources the students use to solve patterning problems, such as gestures, spoken and written words, numbers, function tables and algebraic symbols, shape the form and generality of their mathematical thinking (Radford, 2018). Furthermore, the patterning activity itself and the cultural artifacts employed gain meaning from cultural practices such as algebraic reasoning and schooling, of which language is a prominent aspect (Sfard, 2007).

We draw on Bakhtin’s (1986) notion of *speech genres* when considering the nature of different perspectives on mathematical activity and their role in shaping the classroom discourse. According to Bakhtin, speech genres are extremely heterogeneous, and examples varies from everyday small talk and narratives to sports commentary and different kinds of scientific statements. As an inclusive concept, speech genre is not generally defined. Instead, we conceptualize it in terms of the specific speech genres discussed in this study. For Bakhtin (1986), the utterance, i.e. the unit of language in use and in context, was the basic unit of analysis and he argued that *addressivity* is the key to understanding it. Addressivity refers to the dialogic nature of the utterance, as it is shaped to address a particular listener (real or imagined) foreseeing his response. Furthermore, Bakhtin found our utterances relatively stable in forms of construction and argued that no utterance is given that does not belong to a speech genre, emphasizing that we are not always conscious of it: “[we] speak in diverse genres without suspecting that they exist.” (Bakhtin, 1986, p. 78).

Related speech genres in the introductory algebra classroom

In the introductory algebra classroom, we argue that there are several related speech genres with deep cultural roots at play. A genre form may include linguistically coded intentions and rhetoric that a current speaker may not be aware of (Gerofsky, 1999). Gerofsky argued for the importance of intertextuality between related genres as “a genre which, in its form and addressivity, recalls other familiar genres may bring to consciousness the hidden ground and intentions embedded within [these]” (p. 38). Genres speak to students telling them what to expect and what is expected of them. Radford (2001) argued that algebra as a way of reasoning involves dealing with indeterminate quantities in an analytic and deductive manner with the intensions of explaining and arguing about general relationships and methods for solving problems. Thus, algebra is a specific genre form.

Another speech genre at play is instructional discourse. Mehan (1979) showed that utterances in the classroom (mathematics as well as other subjects) followed a distinct pattern of initiation-reply-evaluation (I-R-E). Thus, it can be seen as a rather standard and rigid speech genre, also found to dominate mathematics teaching in American classrooms (Stigler & Hiebert, 1999).

Experimental studies have developed alternative programs to open up classroom discourse, in which students explore new problems, pose conjectures and argue (Lampert, 1990). Reform documents, such as Standards 2000, aim to bring these ideas into the ordinary classroom (Hiebert, 1999). The speech genre involved shares some ground and intentions with algebraic reasoning since algebra grew out of a sociocultural context of explaining and arguing (Radford, 2001).

Hiebert (1999) explained that the implementation of alternative programs in school districts have not been effective for a simple but unappreciated reason, since it “is hard to change the way we teach” (p. 15). Furthermore, Hiebert argued that there is overwhelming evidence that what students learn in traditional teaching is unsatisfactory and consists of the following: “In most classrooms, students have more opportunities to learn simple calculation procedures, terms, and definitions than to learn more complex procedures and why they work or to engage in mathematical processes other than calculation and memorization” (p. 12). This supports the notion of students developing a calculational perspective on mathematical activity in mathematics classrooms. Furthermore, this genre can be seen to emerge as students’ uptake of teachers’ intentions in traditional instruction. A genre analysis of classroom interactions can shed new light on the communicational processes that shape the introductory algebra classroom discourse. In this paper we limit the further discussion to explore the nature of a calculational perspective and contrast it to an analytic perspective.

Calculational and analytical perspectives in introductory algebra

We propose a new theoretical stance on student activity in the introductory algebra classroom, i.e. a calculational perspective. Based on previous research on algebra learning, we single out as key demarcations between algebraic and calculational perspectives: (1) ways of interpreting signs and operations; (2) strategies for solving problems; and (3) ways of justifying solutions. This is contrasted to an analytic perspective, which is a central feature of algebraic reasoning (Radford, 2012).

As regards the first demarcation, Booth (1984) and Kieran (1981) found that students working with numbers and their operations tended to interpret the operational signs and the equal sign as signaling types of actions rather than structure. Additionally, Kieran showed that students developed a limited view of what constitutes a mathematical solution and were likely to search for numerical answers (lack of closure). Radford (2018), upon repeatedly observing that 4th, 5th, and 6th grade students included an equal sign in their generalizations (i.e. $n + n = x$), argued that the conceptual challenge for students was not necessarily the alphanumeric symbols themselves, but a need for reconceptualizing numerical operations.

Students holding a calculational perspective are limited to a processual view of signs and operations. In contrast, an analytic perspective includes a structural view of operations: (1) signaling mathematical structures that are relevant and can be used in transformations of expressions (Kieran, 2018); and (2) seeing expressions as objects in themselves that can be operated upon, substituted, or classified (Sfard, 2007). These are interdependent of a developing awareness of mathematical generalities (Kieran, 2018) and an acceptance of indeterminacy, both in operations (operating with indeterminacy as if it were a number) and solutions (Radford, 2018).

Concerning students’ problem-solving strategies, the second demarcation, students often use a guess-and-check strategy to solve new algebraic problems, including patterning tasks (Bednarz & Janvier,

1996; Lannin, 2005; Radford, 2018). Bednarz and Janvier (1996) found that students using a guess-and-check strategy were not able to accept indeterminacy and remained dependent on performing calculations on numbers. In contrast, students that used a strategy of determining the underlying structure of the problem (which included the linking of different mathematical structures such as for example additive and multiplicative to determine how many equal parts) were able to explain an algebraic equation when shown such a solution.

Addressing the third demarcation, students' justifications, Lannin (2005) showed that 6th grade students mainly used empirical results to determine and justify their pattern generalizations when left to themselves. They tended to focus on particular values to determine and verify generalizations, rather than a general relation. Radford (2012) argued that algebra is an analytic art. Formulas must be deduced and not guessed for students to engage in algebraic reasoning. They have to come up with a deductive argument. A deductive argument, for example in a patterning activity, includes three components: (1) noticing a commonality between the terms $p_1, p_2, p_3, \dots, p_k$; (2) generalizing this to all subsequent terms $p_{k+1}, p_{k+2}, p_{k+3}, \dots$; and (3) using the commonality to formulate an expression for any term in the sequence, i.e. an explicit strategy to generalize a sequence (Radford, 2008). However, students more often pursue a recursive strategy where each term (p_k) is determined from the previous one (p_{k-1}) (Lannin et al., 2006). Radford (2012) argued that such a generalization is of an arithmetic nature, as opposed to algebraic, as it is limited to include terms within the perceptual field. It does not include the third component of a deductive argument. Nevertheless, guess-and-check and recursive strategies might be helpful in generalizing a pattern. Particularly, a recursive strategy can give insight into the pattern's rate of change, which again can be helpful in developing an explicit expression. However, research shows that students struggle to see the connection between the two ways of generalizing a pattern (Lannin et al., 2006).

Ellis (2007) showed that the processes of generalization and justification were interwoven in students' activity. 7th grade students' generalizing acts were directly linked to the justifications they gave. On the other hand, their view of what counts as an acceptable justification influenced their generalization process and students following a deductive justification scheme developed more sophisticated generalizations from initially limited or unhelpful generalizations.

In sum, a calculational perspective on mathematical activity includes (but is not limited to): 1) viewing numbers as the main objects of activity and other signs and operations as actions to perform; 2) employing strategies rooted in calculational processes in problem solving, and 3) relying on empirical results for justification. These aspects may confirm and reinforce each other as Ellis (2007) pointed out, and they may function as barriers for students' engagement in algebraic, analytic reasoning. The intention of the calculational genre, as opposed to the algebraic genre, is to perform calculations following known procedures to produce correct numerical solutions.

Students' calculational perspective in a 6th grade classroom

To empirically exemplify our theoretical contribution, we draw on data from a Californian 6th grade classroom where students were working with patterning tasks. The background for this paper is an international algebra project called VIDEOMAT (see Kilhamn & Säljö, 2019). An in-depth analysis of students' discourse in classroom patterning activity showed that students across countries pursued

similar solving strategies. Reinhardtson and Givvin (2019) found that students focused on processes of calculations rather than structure and quantitative relationships in their work, i.e. they drew on operations on numbers rather than introduced algebraic ideas and symbols.

The teacher's approach to working with every patterning task involved three main phases of activity, classroom exploration of a pattern, making a function table and developing an explicit generalization from a recursive one when possible, and extending the generalization to all subsequent terms and creating an algebraic expression. The teacher first asked for students' observations and suggestions before proceeding, allowing students to pose conjectures and ideas.

We structure the following section according to the three main characteristics of the calculational perspective outlined above. The first two episodes exemplify students' view of numbers as the main objects of activity and other signs and operations as actions to perform. The third episode exemplify students use of strategies rooted in calculational processes, thus relying on empirical justification.

Explicit and recursive generalizations: "Multiplying by 4 or they're adding 4"

In the first lesson the teacher introduced the metaphor of function machine, emphasizing an explicit relationship between sets of numbers in the second phase of activity. In the first phase of activity the class discussed the numerical sequence 4, 8, 12, 16, __, __, __. Students suggested both recursive generalizations: "Like it's plus 4, 4 plus 4 equals 8, and 8 plus 4 equals 12, and so on", and explicit ones: "I did multiples of 4, 20, 24, 28".

In phase two, the teacher drew a function chart and a function machine and then asked the students to discuss shortly with their seat-partner: "And if I put a 2 in, what's going on in here, so that I get an 8 out. What's happening in this machine that represents this number sequence?". One student, Mara, made a link to the previous discussion and said to her partner: "Multiplying by 4 or they're adding 4". Another student, Ivy, offers a more literal explanation: "And then it came out in a different number, so it's a function machine".

The ideas presented by students in the whole class discussion in the first phase are mainly concerned with how to calculate the terms in the sequence. Mara's uptake of the calculational ideas as presented in phase two shows that these are accessible to other students. However, making sense of the difference between the two generalizations, recursive ($p_k = p_{k-1} + 4$) and explicit ($p_k = 4k$) and the relationship between them, requires looking at the numbers and operations analytically.

Neither Ivy nor Mara picked up on the significant changes in the perspective that the teacher was emphasizing. Ivy offered a literal interpretation and referred to a transformation of a number, as if by magic. Mara linked the teacher's question to the previous discussion but did not notice that the operation of "adding 4" no longer was appropriate, i.e. the horizontal relationship between numbers in the two columns only corresponds to an explicit generalization of the sequence.

Using letters: "Because you can do $4n$... n can be any number, so multiply it"

In the third phase of working with the sequence above after having determined the 10th and 20th terms of the sequence together, the teacher asked the students to discuss the general expression with their seat-partner "How do I show 4 times any number. We're just calling it n for right now".

Trace explained his expression to his partner: “ $4n$. Because you can do $4n$, yea, $4n$. n can be any number, so multiply it, $4 \text{ dot } n$, $4n$, stuff like that”. Trace made sense of his expression on the terms of being able to perform a calculation when replacing n with a number: “Because you can do $4n \dots n$ can be any number, so multiply it”. Thus, Trace was able to respond to the teacher’s question of how to show “4 times any number”. However, as Küchemann (1978) pointed out, there is a leap between being able to evaluate an expression and using a letter as a variable to express mathematical structures and relationships.

In the third lesson the class worked with a figural pattern. In phase one the whole class discussion centered around rate of change. However, in the second phase of creating a function table guided by the teacher, the students no longer made references to the figures but quickly noticed a familiar (quadratic) relationship between the numbers in the two columns. Luna set up multiple numerical expressions such as $1 \cdot 1 = 1$, $2 \cdot 2 = 4$ and $3 \cdot 3 = 9$, and in the third phase she suggested: “It’s called n times n equals x ”. Her generalization was developed and verified through calculating numerical expressions, and her algebraic expression also resembled these.

Guess-an-check strategy: “I tried the next number which is 6”

The teacher emphasized the relationship between a recursive and an explicit generalization as a strategy to determine the explicit expression throughout the classroom work. However, the students mainly used a guess-and-check strategy when working independently with function charts.

In the third lesson the teacher gave the students a function chart in which the left column included the numbers 1-8, with x as the last entry, and the right column included the four first entries: 5, 11, 17, 23. The instructions were to fill in missing numbers, describe patterns and write an expression.

The teacher, in a whole class setting, asked Liam to explain what he noticed. He explained that he first looked at “1 times something gets to 5”, deciding this being 5. He then tried the next pair of numbers in the chart (2, 11), but found that $2 \cdot 6 \neq 11$: “So, I realized that you could do, I tried the next number which is 6”. He then explained he did minus one to get the number he needed; five. He then tried the sequence of operations regarding the next pair of numbers in the chart (2,11): $6 \cdot 2 = 12$, $12 - 1 = 11$, and found that it worked. Liam demonstrates how the students approached the number sequences through guess-and-check. Also when working with figural sequences, the students used this strategy. Lisa, in the fourth lesson, explained her approach: “Um, I tried it, I kind of did, um. I don’t know how to explain it, but I started with doing multiplying it by 1 and adding 2, but it didn’t work. So, I tried multiplying by 2 and subtracting 1”. The students working with function charts increasingly made explicit generalizations, but these were mainly based on numerical schemes rather than analytical ones. This finding coincides with Ellis’ (2007) argument.

Synthesizes and concluding remarks

In this paper we set out to come up with theoretical insights into the characteristics of students’ calculational perspective in introductory algebra classrooms. We have argued that developing algebraic reasoning is a challenging process as previously met calculational perspectives interfere when trying to reason analytically about number patterns and figural patterns. The empirical examples demonstrate that students approach problems in introductory algebra through a calculational

perspective. They face difficulties making sense of the difference between a recursive and an explicit generalization (cf. Lannin, 2005; Lannin et al., 2006), they evaluate letters but struggle using them to express structure and generalities (cf. Küchemann, 1978; Radford, 2018), and they use a guess-and-check strategy (cf. Bednarz & Janvier, 1996; Ellis, 2007).

The classroom data reveal traits and intentions from both traditional and alternative instructional genres. Elements such as letting students explore sequences, notice patterns, and suggest ideas invited students to engage in generalization processes. However, the use of a function chart and algebraic notation as a step-by-step process heavily supported by the teacher share traits with the traditional instructional discourse and may have informed students' participation in the activity. Deeply rooted norms (cf. Stigler & Hiebert, 1999) of emphasizing procedures and products (solutions, facts, etc.) rather than mathematical processes (problem solving, generalizing, etc.) came to the fore in the classroom, despite efforts by the teacher to focus on the latter in discussions.

We argue in accordance with Radford (2012) that the students' calculational perspective is of an arithmetic nature, as opposed to an algebraic nature. The students' approaches to generalize the patterns are limited to their perceptual field, and thus fail to include a deductive argument, the key feature of algebraic reasoning (Ellis, 2007; Radford, 2012). Despite the teacher's efforts, the students do not make sense of the shifts in form (the algebraic syntax versus arithmetic syntax) and function (analytic rather than calculational) in this classroom. These expected shifts unfortunately do not take place. More research is needed to investigate the role that the calculational perspective play in students' participation in algebraic activity.

Radford (2018) found that symbolic thinking, in which letters are used to develop a generalization, took a long time for students to develop. Furthermore, it developed in line with an increasingly analytic approach to patterning activity. A genre analysis of the introductory algebra classroom supports this finding. A genre form includes many aspects that reinforce each other, as we have pointed out concerning calculational and analytical genres. Our discussion suggests the importance of addressing the elements of the algebraic genre as a consorted effort.

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