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Grade 3 Students’ Algebraic Thinking Abilities: An empirically Validated Theoretical Framework

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The present study proposed and empirically validated a theoretical framework describing algebraic thinking abilities for Grade 3 students. Four algebraic abilities were incorporated in this framework: ‘relational manipulation of equalities/inequalities’, ‘compose and decompose number and arithmetical expressions’, ‘functional thinking’ and ‘representing-modeling’. The study involved 124 students. Analysis showed that the three first constructs compose a general thinking ability that could be considered as an index of Grade 3 students’ capacity to respond to a variety of algebraic thinking tasks. This general ability was a strong predictive factor of students’ representing-modeling ability.

Keywords: Early-algebra, functional thinking, generalized arithmetic, modeling.

Introduction

Initiatives worldwide have underlined the significance of early algebra in mathematics education and stressed that to meet the goal of developing a fundamental algebraic understanding, students in elementary school should be involved in activities that prepare them for algebra in later grades (National Council of Teachers of Mathematics, 2000; Stephens, et al., 2017). Early algebraic thinking can be coherently conceptualized as a synthesis of different content strands, concepts, processes or forms of reasoning that relate to the ideas of equivalence, properties of numbers and operations, variable, proportional reasoning, modelling, and functional thinking (Chimoni, et al., 2018; Kaput, 2008; Kieran, et al., 2016). Research studies suggest that elementary school children could engage in sophisticated ways of algebraic thinking, such as generalizing, representing, justifying, and reasoning with mathematical structure and relationships (Stephens et al., 2017). However, the existence of different approaches to the notion of algebraic thinking, in terms of content strands, concepts, processes and reasoning forms makes it challenging to provide a coherent description of the nature of algebraic thinking. The relation between young students’ algebraic thinking abilities, remains under-researched. In this study, as a first step, we explore Grade 3 students’ thinking algebraic abilities to identify their relations based on empirical data.

Theoretical Considerations

Several research efforts concentrated on the analysis of the nature and content of algebraic thinking and provided a list of characteristics of algebraic thinking in all grades (e.g., Blanton et al., 2011). An overarching definition suggested that algebraic thinking is a ‘habit of mind’ that enables students to identify and express mathematical structure and relationships (Blanton & Kaput, 2005). Researchers
claimed that a fundamental element of algebraic thinking is generalization, that is the ability to see the general in the particular (Kaput, 2008; Kieran, 2007). Chimoni et al. (2018) provided a synthesis of the literature, suggesting four basic dimensions in terms of (a) content strands, (b) concepts, (c) processes, and (d) reasoning forms: A number of theoretical frameworks conceptualized algebra in terms of content strands and concepts. For instance, Kaput’s (2008) model asserts that generalizing and symbolizing are tightly linked in that symbols allow generalizations to be expressed in a stable and compact form, throughout generalized arithmetic, functional thinking, and modeling languages. Generalized arithmetic entails noticing relationships between numbers, the manipulations of operations and their properties, and the transformation and solution of equations (Chimoni et al., 2018). Functional thinking has generally been defined as the process of building, describing, and reasoning with and about functions (Stephens, et al., 2017). Modeling languages involves the use of symbols for developing models, and re- translating between models and situations. Kaput’s framework was empirically validated by Pitta-Pantazi et al. (2020) for grades 8 and 9.

Kieran (2007) conceptualized algebra as a multidimensional activity that encompasses various types of tasks and ways of thinking. She suggested three types of activities: generational, transformational, and global, meta-level. The generational activities refer to the generation of equations and expressions in various situations and involve exploration of problem situations and numerical and geometrical patterns that lead to the formulation of generalizations, and exploring numerical relations. The transformational activities refer to the transformation of expressions by applying specific rules and involve conceptual understanding of algebraic objects. The global, meta-level activities are not strictly algebraic in nature, but algebraic tools are needed to be investigated and involve general mathematical processes, such as proving, studying functional relations, and identifying structure. Further, Kieran (2007) suggested that the smooth transition from arithmetic to algebra could be achieved by focusing on (a) the relations of numbers and not only on calculations; (b) relations between operations; (c) representation and solution of problems; (d) the use of numbers and letters; (e) the meaning of the equal sign. This list of activities provides a comprehensive lens to examine algebraic thinking in terms of content strands and specific algebraic concepts. In addition, Drijvers et al., (2011) distinguished four strands in algebra teaching: patterns and formulas; restrictions; functions; and language. Patterns and formulas involve searching for regularity, patterns and structures and embeds generalization. Restrictions entail manipulating equations or inequalities, such as finding what value of the unknown variable satisfies the required conditions. Functions reflect algebra as the study of relations and functions. Language concerns algebra as a symbolic system. Driscoll (1999) provided a description of algebraic thinking in terms of habits of mind that fits with the frameworks suggested by Kieran (2007) and Drijvers et al. (2011). He suggested ‘doing and undoing mathematical processes’, ‘identifying and representing functional rules’, and ‘thinking about computations independently of particular numbers’.

In terms of processes and reasoning forms, a number of research studies proposed noticing, representing, and justifying with mathematical structure and relations as core processes for searching similarities and differences (Blanton et al., 2011; Jeannotte & Kieran, 2017). Abductive, inductive and deductive reasoning were proposed as important types of reasoning forms in algebraic situations (Chimoni et al., 2018). For instance, abductive reasoning is necessary at the stage where individuals
develop a prediction about a plausible generalization (Rivera & Becker, 2007), while inductive reasoning is required to grasp a generality through noticing how a local commonality hold across all terms. The variety of the frameworks described above shows the complexity in defining early algebraic thinking by clarifying its differences from arithmetical thinking and defining well-accepted algebraic abilities for young students (Chimoni et al., 2018). Furthermore, to date, these frameworks have not been extensively validated based on empirical data for young students.

The Present Study

The main purpose of the study is to examine, based on a synthesis of well-accepted theoretical frameworks, Grade 3 students’ algebraic thinking abilities. We define algebraic thinking abilities as the capacity of the individuals to perform various early algebra tasks and include both relevant knowledge, reasoning skills, and algebraic processes, such as generalizing, representing, justifying and reasoning with mathematical relationships (Blanton et al., 2011). We propose algebraic thinking abilities in an attempt to describe the structure of algebraic thinking by amalgamating various types of algebraic tasks, processes and practices. The proposed framework is based on (a) Kaput’s (2008) algebra core content areas, (b) Blanton et al.’s, (2011) description of algebraic processes, (c) Kieran’s (2007) algebraic types of activities, (d) Drijvers et al.’s, (2011) algebra strands, and (e) Driscoll’s (1999) algebraic habits of mind. The framework (see Figure 1) involves four distinct but correlated factors and defines a measurement model of young student’s algebraic thinking abilities. The innovative aspect of the proposed framework is the fact that it integrates aspects of the fore-mentioned frameworks that meet Grade 3 students’ early-algebraic experiences and needs. It is grounded on embedding early algebraic processes and practices in specific content areas to define algebraic thinking abilities that Grade 3 students should develop based on well-accepted frameworks, contemporary curricula, and policy documents. The proposed thinking abilities can be used as measurement indicators of Grade 3 algebraic thinking.

The first factor, relational manipulation of equalities/inequalities, corresponds to students’ capacity to manipulate equations and equalities, find the value of the unknown in equations and inequalities that are represented in the form of an empty box to be filled or in balance scale equalities and inequalities that are suitable for Grade 3. It embeds Driscoll’s (1999) idea of doing and undoing mathematical procedures. The second factor, compose and decompose numbers and arithmetical expressions, includes students’ ability to conceptualize the relations and properties of numbers, relations between operations, to reflect and make predictions about computations and solve problems independently of particular numbers, based on number-property generalizations. In addition, it includes the ability to interpret equalities expressed in different forms (Blanton et al., 2011). Students need to decide as a result of noticing an arithmetic or computational structure, before proceeding to a second action. Conclusively, it conceptualizes students’ capacity to make numeric and arithmetical computations, solve numeric related problems, and interpret equivalence expressions by grasping mathematical structure and relations in these situations, abstracting from arithmetic properties and objectifying the generalized properties. The third factor, functional thinking, adopts the definition proposed by Pittalis et al. (2020) for young students, suggesting that it encompasses student’s capacity
to notice, generalize, and abstract relations between covarying quantities/variables, represent these relations through natural language, symbols, and appropriate representations and use the generalized representations in problem solving situations. Finally, representing-modeling languages, represents students’ capacity to represent word-problem situations that embed relations among quantities using a variety of representations, such as number sentences, literal symbols, and figural models (Kaput, 2008; Kieran et al., 2016). The research questions of the study were: Could different algebraic thinking tasks be categorized on the basis of the factors of the proposed theoretical framework? What is the structure and the relations between the proposed algebraic thinking abilities, as they are projected through Grade 3 students’ responses?

Measures

The test items were adopted or developed based on previous research studies (Blanton et al. 2011; Chimoni et al., 2018). The multiple-choice tasks were considered as correct or incorrect, while the open tasks were given partial marks. The first factor, relational manipulation of equalities/inequalities, was measured by three types of tasks: completing the missing value in an equity, finding the value of the unknown in an equation that was presented in a balance-scale form, and finding one possible value of an unknown variable in an inequality that was presented in a pictorial form (Drijvers et al., 2011). Variables in balance-scale equations and inequalities were represented by pictorial symbols (see Table 1). The second factor, compose and decompose numbers and arithmetical expressions, was measured by four types of tasks. The first type required students to solve number-property problems, such as conceptualizing odd and even numbers. The second type of tasks provided students the result of an addition or subtraction and asked students to find the sum or difference based on the derived facts (Driscoll, 1999). Generalization of the derived fact was necessary to conceptualize the structure of the provided and the given calculations and grasp the differences. The third type required students to predict whether ‘big calculations’ result to odd or even numbers, without making any calculations but noticing the odd/even property of the numbers and generalizing the result of adding/subtracting odd/even numbers. The fourth type provided students two equalities in the form of balance-scale situation. Student had to find the heaviest toy by interpreting the equalities and making assumptions regarding the relations of the involved objects. The representing-modeling languages factor was measured by three types of tasks that required translating a word-problem situation by noticing the relation between the involved quantities. In the first type students had to represent the problem in a number sentence form, in the second one in a model form (Kieran et al., 2016), and the third one in literal symbols. Functional thinking was measured by tasks that entail four modes of thinking (Pittalis et al., 2020): recursive patterning (14 tasks), covariational thinking (5 tasks), correspondence-particular (7 tasks), and correspondence-general (7 tasks). Due to space limitations, we do not present the functional thinking tasks.

Participants, Procedure and Data Analysis

Consent forms for students to participate in the study were distributed to two urban primary schools in Cyprus. The schools, teachers and students involved participated voluntarily, thus our sampling was a convenient one. Parents’ consent forms were returned for 95% of the students, resulting in a sample of 124 Grade 3, 9-year-old students (65 males and 59 females).
Table 1: Examples of tasks

<table>
<thead>
<tr>
<th>Factor</th>
<th>Type of task</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relational manipulation of equalities and inequalities</td>
<td>(a) Completing equalities</td>
<td>(a) Fill in the missing number: $12 + 4 = \square + 6$, $3 \times \square = 15$</td>
</tr>
<tr>
<td></td>
<td>(b) Solving equations in balance scale form</td>
<td>(b) Find the value of the symbol.</td>
</tr>
<tr>
<td></td>
<td>(c) Solving inequalities</td>
<td>(c) Find a possible value for the symbol.</td>
</tr>
<tr>
<td>Compose and decompose numbers and arithmetical expressions</td>
<td>(a) Solving number-property problems</td>
<td>(a) I had in my pocket more than 12, but less than 15 euro. I spent all my money to buy pencils that cost 2 euro each. How much money I had in my pocket?</td>
</tr>
<tr>
<td></td>
<td>(b) Make calculations based on derived facts</td>
<td>(b) Find the following if you know that $118+8=126$.</td>
</tr>
<tr>
<td></td>
<td>(c) Make predictions based on number-properties</td>
<td>128+8, 118+18, 118+9, 218+8, 119+18</td>
</tr>
<tr>
<td></td>
<td>(d) Interpret equities</td>
<td>(c) Does the following give even result (without making calculations)?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>122+18, 15+45, 478+222+444, 333-115</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(d) Which toy is the heaviest?</td>
</tr>
<tr>
<td>Representing-modeling languages</td>
<td>(a) Number sentence</td>
<td>(a) Which number sentence represents the problem?</td>
</tr>
<tr>
<td></td>
<td>(b) Model</td>
<td><em>I had 4 pencils. I gave 3 pencils to my brother. How many pencils have I got now?</em></td>
</tr>
<tr>
<td></td>
<td>(c) Literal Symbols</td>
<td>$3 + 3 + 4 = \square$ $3 \times 4 = \square$ $4 + 3 = \square$ $4 - 3 = \square$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(b) Which model represents the situation?</td>
</tr>
<tr>
<td></td>
<td></td>
<td><em>Nick has 5 stamps less than Sophie.</em></td>
</tr>
<tr>
<td></td>
<td></td>
<td><em>Katy (K) and Lia (L) have altogether 3 stamps less than Mary (M).</em></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$K + L - 3 = M$ $K + L + 3 = M$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$K + L = M + 3$ $K + L + M = 3$</td>
</tr>
</tbody>
</table>

The tasks of the study were split into two parts. Consideration was given to the number of tasks and time required to be approximately the same. Each part was administered in the form of a written test during one school period. The two parts were administered in two successive weeks. The instructions were provided in written and verbal form. Confirmatory factor analysis was used to examine the validity of our proposed, a priori model by using MPLUS 8.0 (Muthén & Muthén, 1998-2007). To evaluate model fit, three widely accepted fit indices were computed: $\chi^2$/df should be <2; the Comparative Fit Index should be >.9; and the root mean-square error of approximation (RMSEA) should be <.08. The Cronbach’s alpha index of internal consistency was very good ($\alpha=.81$).
Results

Confirmatory factor analysis (CFA) was used to evaluate the construct validity of the model. Based on the results, the a-priori model matched the data set of the present study and determined the “goodness of fit” of the hypothesized latent construct. Analysis showed that the fit-indices of the hypothesized model were excellent ($\chi^2$/df=1.27, CFI=.97, and RMSEA=.05), validating empirically the fit of the structure of the model to the empirical data. Figure 1 presents the observed variables (different types of tasks in Table 1), the corresponding factor loadings and the factor correlations.

CFA showed that the factor loadings of the tasks were statistically significant and most of them were rather large, ranging from .42 to .83, giving support to the assumption that all latent factors were adequately measured by the observed variables. In accordance with our theoretical assumption, all measures were clustered into four first-order factors in the expected factor-loading pattern. Thus, analysis showed that algebraic thinking consists of four interrelated abilities that is (a) relational manipulation of equalities/inequalities, (b) compose and decompose numbers and arithmetical expressions, (c) functional thinking, and (d) representing-modeling language. The correlations between the four abilities were significant (see Figure 1) and ranged from .64 to .88.

![Figure 1: The standardized solution of the proposed framework](image)

$p < .05$, F1: Relational Manipulation of Equalities/Inequalities, F2: Compose and Decompose Numbers and Arithmetical Expressions, F3: Functional thinking, F4: Representing-modeling Languages

To investigate the relations between the four algebraic thinking abilities, we examined the fit to the data of alternative structural models, hypothesizing a direct sequential path between the four factors or the existence of a higher-order thinking ability. The model that had the best fitting indices ($\chi^2$/df=1.25, CFI=.97, and RMSEA=.04) showed that F1, F2, and F3 compose a higher-order factor,
F100, reflecting students’ general algebraic ability to manipulate relationally equalities and inequalities, compose and decompose numbers and arithmetic structures, generalize and abstract functional relations (see Figure 2). Analysis showed that F100 could accurately explain students’ variances in F1, F2 and F3, suggesting that young students develop these abilities in parallel, their development is interrelated, and they share equally important contribution in building up a general functional thinking ability. F100 could be considered as an index of students’ readiness to engage in algebraic thinking situations by noticing, interpreting, abstracting, and effectively using structure in arithmetic situations, number and operations properties, and quantity relations. It is the consequence of the correlations between F1, F2 and F3 that underlies a synthesis of specific arithmetic and algebraic thinking abilities. The regression coefficient of F100 on F4 was 0.76 ($p<0.05$), indicating that F100 is a strong predictive factor of students’ modeling language ability using a variety of representational forms. It could be supported that F100, as a general arithmetic-algebraic ability predicts rather accurately ‘representing-modeling languages’ that has a salient algebraic-nature.

Figure 2: The relations between algebraic thinking abilities

**Discussion**

The contribution of the study lies on the empirical evaluation of a proposed model that unpacks the dimensions of Grade 3 students’ algebraic thinking. The results of the study showed that Grade 3 students’ variances in algebraic situations might be modelled by four distinct and interrelated algebraic thinking abilities; relational manipulation of equalities and inequalities, compose and decompose numbers and arithmetical expressions, functional thinking, and representing-modeling languages. Structural analysis showed that the three abilities, relational manipulation of equalities and inequalities, compose and decompose numbers and arithmetical expressions, and functional thinking compose a general algebraic thinking ability that can be considered as an index of Grade 3 students’ capacity to respond adequately in a variety of arithmetic-algebraic thinking tasks (Blanton et al., 2011, Driscoll, 1999). This general ability proved to be a strong predictive factor of students’ ability to represent word problems and situations using number sentence, models or symbols. It could be supported that compose and decompose numbers and arithmetical expressions facilitates grasping the numerical relations in a problem and conceiving how the quantities set an equivalence or an equation, functional thinking contributes in noticing the relations between the involved quantities in the word situation, and relational manipulation of equalities and inequalities to discern the known and the unknown in the formed expression. The results of the present study are important in terms of teaching implications. The framework helps teachers to get a better understanding of algebraic thinking and the specific type of abilities that Grade 3 students should develop. Furthermore, the description of algebraic thinking abilities may inform teachers about students’ potential difficulties and thinking requirements in a variety of situations.
References


