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# Recognition of an equation as an algebraic description of contextualised situations by fifth graders

Esperanza López Centella<sup>1</sup>, Jana Slezáková<sup>2</sup> and Darina Jirotková<sup>2</sup>

<sup>1</sup>University of Granada, Faculty of Educational Sciences, Granada, Spain; [esperanza@ugr.es](mailto:esperanza@ugr.es)

<sup>2</sup>Charles University, Faculty of Education, Prague, Czech Republic; [jana.slezakova@pedf.cuni.cz](mailto:jana.slezakova@pedf.cuni.cz),  
[darina.jirotkova@pedf.cuni.cz](mailto:darina.jirotkova@pedf.cuni.cz)

*In this study we qualitatively explore the conceptual and semantic understanding of equations of elementary school pupils who were introduced to solving basic linear equations. We analyse the individual answers of 38 fifth grade students in a Czech public primary school to a paper-based task consisting of justifying whether the equation  $5+2x=13$  algebraically describes each one of 5 contextualised situations presented. Under a grounded theory approach, we provide a system of categories of the students' strategies to address this question. Our findings show the students' abilities to deal with the task and to infer true mathematical facts about equations, the wide variety and nature of students' strategies and the apparent dependency of the strategy on the system of representation of the situation.*

*Keywords: Equations, contextualised situations, elementary students, strategies.*

## Introduction

Gradually, algebraic activity is finding its place in primary school classrooms, following the guidelines of international organisations of Mathematics Education (National Council of Teachers of Mathematics [NCTM], 2000) and the evidence and conclusions from multiple research works on early algebra (e.g., Kaput, 1999). Getting acquainted with the notions of unknown amounts and variables, establishing dependency relationships between variables, developing functional thinking, representing information in different systems and transferring it from one to another, symbolising and using meaningfully algebraic notation are part of these algebraic activities. In this scenario, equations bring a great chance to deal with several of them at once. In the context of a task involving a linear equation and five contextualised situations verbally and pictorially presented (Figure 1) to elementary students of fifth grade, the aims of this research study are:

- (a) Explore the students' abilities to identify and justify the given equation as an algebraic transcription of a particular situation,
- (b) Describe the strategies used by students to address this question.

## Background and theoretical framework

A branch of research studies in algebraic thinking have focused on epistemological analysis of scholar work with equations and the exploration of different aspects of children's performances when working with them, discerning in such a way a demarcation between arithmetic and algebra. Filloy and Rojano (1989) suggest the terms *arithmetic equations* to refer to equations with an unknown on only one side of the equal sign (so that undoing the operation at hand is enough to know its value), and *algebraic equations* to those where the unknown appears in both sides and therefore must be operated on. Balacheff (2001) understands that the shift from an arithmetic to an

algebraic interpretation of equality corresponds to a shift of emphasis in the validation of the problem solution: from a *pragmatic control*, where the solution is validated arithmetically with reference to the initial context of the problem; to a *theoretical control*, where the solution is validated with reference to mathematical principles. Concerning the ways to formulate equations from verbal data, Herscovics (1989) recognises *syntactic* and *semantic translations* as different procedures, referring respectively to direct translation of key words to symbols, and the attempt to express the meaning of the problem. According to Hoch and Dreyfus (2004), “any algebraic expression represents an algebraic structure. The external appearance or shape reveals, or if necessary can be transformed to reveal, an internal order. The internal order is determined by the relationships between the quantities and operations that are the component parts of the structure” (p. 50). In this paper we refer to these notions as *external structure* and *internal structure* of an equation. We say that an equation *algebraically describes* a contextualised situation if there is a correspondence relationship between the equation terms and a set of elements of the context in such a way that the equality established between the relationships of the equation terms is also fulfilled between the homologous relationships of the corresponding elements of the context.

Other works explore the processes of mathematical reasoning of equations in primary mathematics lessons through operations with structures of computing-terms (Nührenbörger & Schwarzkopf, 2016) and how students can come to understand and use the syntactic rules of algebra based on their understanding about relations between quantities (Brizuela & Schliemann, 2014). These studies show that dealing with equations is not beyond ten years old students’ mathematical understanding and that much more could be achieved if algebraic activities became part of the daily mathematics classes offered to elementary school children. In this sense, Figueira-Sampaio et al. (2009) propose a constructivist computational tool to assist in learning equations of first degree in primary school, to illustrate the idea of equilibrium and properties of the equality. Otten et al. (2019) report that the algebraic strategies such as restructuring, isolation and substitution that primary school students used when working with a hanging-mobile during their teaching experiment were later used by these students for solving linear equations in new contexts.

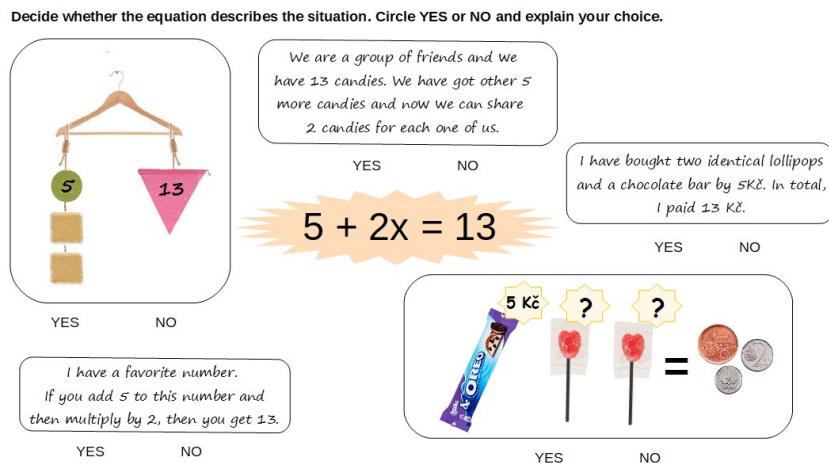
## **Research methodology**

This study is part of a larger qualitative and descriptive research project, based on classroom interventions to explore the understanding of elementary students in algebraic activities.

### **Participants and design of the task**

Thirty-eight fifth grade elementary students (10-11 years old) of mixed abilities in mathematics from the same public primary school in the city of Prague (Czech Republic) participated in the study. They were two class groups, both taught by the same teacher. Although equations are included in the second cycle (6-9 grades) of elementary school in the Czech curriculum (VÚP, 2017, pp. 31-34), the participant students had been briefly introduced by their teacher to symbolic letters and solving simple linear equations in the classroom. They had been taught by a method partially inspired in Hejný Method (Hejný, 2012). They had played with numbers and arithmetic operations subject to certain conditions and worked with unknown amounts. They had no prior experience with activities as the involved in our present study.

According to our research aims, we presented to the students the task shown in Figure 1.



**Figure 1: English version of the worksheet presented to the students**

The worksheets for them were drafted in Czech language, the students’ mother language in which they also wrote down their answers. Table 1 specifies the task variables considered.

**Table 1: Task variables and features of the worksheet situations**

	Friends	Lollipops	Labels	Number	Hanger
Representation	Verbal	Verbal	Pictorial	Verbal	Pictorial
Unknown	Number of friends	Price of lollipop	Price of lollipop	Favorite number	Weight of square
Order of appear.	13, 5, 2	2, 5, 13	5, 1, 1, 13	5, 2, 13	5, 1, 1, 13
Possible equa.	$(13+5)/x=2$	$2x+5=13$	$5+x+x=13$	$(x+5)2=13$	$5+x+x=13$
Described?	No	Yes	Yes	No	Yes

**Collection of data and data analysis**

The data collection was performed in one session at the beginning of the first semester of the Czech school year 2019-2020, during normal mathematics class time of the participant groups. The worksheets were given and administered by the teacher to the students with the firm instruction of doing the task individually and writing down their explanations with pen on the sheet (and its back if they needed it). During their work, the students did not receive any feedback about the correction of their answers or suitability of their strategies. Upon completion, students’ worksheets were delivered to the research team and constitute the data for this qualitative research.

We considered as units of analysis the students’ written answers to the different items of the task: selection of yes/no and corresponding explanation for each situation. We transcribed and translated the answers from Czech to English. After thorough reviews, we categorised the data following the principles of Grounded Theory (Corbin & Strauss, 1990). In agreement with our research goals, we establish categories of the focus of their performances and of their strategies, as shown below. Students’ anonymity was ensured by assigning each a label: Si, i=1, ...,38.

## Focus of performance

- Resolution. Ignoring the given equation, the student tries to find a solution for each situation. If the student can solve it, she/he circles “Yes”. Otherwise, she/he circles “No”.
- Discrimination. Considering the given equation, the student decides—based on one or more strategies—whether this algebraically describes each situation. If the student thinks so, she/he circles “Yes”. Otherwise, she/he circles “No”.

## Strategies

- Identification. The student establishes a one-to-one correspondence between the contextual elements of the situation and the terms of the given equation.
- Equation solving. ▸ Operational (O). The student follows the algebraic rules and standard solving equation process. ▸ Trial and error (T&E). The student assigns a value to the unknown and checks whether the equality of the equation holds, repeating the process until finding a solution or stopping after some trials. ▸ Memory and mental calculation (MC). The student does not annotate anything about the solving process, using own record of numerical facts and mental calculation.
- Divisibility. The student discusses the existence of solution relying on divisibility arguments.
- Solution checking. The student arithmetically checks whether the solution of the given equation is also a solution for the situation at hand.
- Equations comparison. The student formulates an equation for the situation at hand and compares it—without solving it—with the given equation.
- Solutions comparison. The student formulates an equation for the situation at hand and solves it, numerically comparing its solution with the solution obtained by solving the given equation.
- Analogy. The student recognises a situation as an analogy of another, copying the answer provided in it.
- Numbers comparison. The student compares the numerical data appearing in the statement/picture of the situation at hand with those of the given equation.

Note that, in contrast to the solutions comparison strategy, in the solution checking the student does not formulate any equation of the situation at hand.

## Results

### Focus of performance

As shown in Table 2, students’ participation in the task was notably high: 186 out of potential 190 answers were provided. More than the half of the students (20 of 38) focused their performances on discrimination—as requested by the task—, while the remaining students (18) did on resolution. Restricting ourselves to the first ones, it is remarkable the high rate of right answers. Interestingly enough, there were those who deliberately circled both “Yes” and “No” for the same situation. This is the case of  $S_{11}$  for the Hanger situation:

$S_{11}$ : Yes, since  $5+4=9+4=13$ . No, since  $5+?+?=15$  [he writes down two 5’s and two arrows from each one of them to each question mark].

He argues similarly in the Labels situation:

S<sub>11</sub>: Yes,  $5+2x=13|-5, 2x=8|:2$  [vertical bars are part of his symbolism],  $1x=4$  [he writes 4 next to each question mark of the lollipop price labels]. No, because the price of lollipop could be 4.50.

After formulating an equation for it, getting its solution and applying the solutions comparison strategy, S<sub>11</sub> also thinks in a different way: he considers another value for the unknown and checks that the equality of the equation does not hold for that value, what he offers as evidence that the equation might not describe the situation. The answer of S<sub>37</sub> (focused on resolution) to Friends situation also illustrates this: “Yes: How many people were there? 9 people. No: 1 to 8 people”.

**Table 2: Total number of students’ answers of each type**

		Friends	Lollipops	Labels	Number	Hanger
Resolution (18)	Yes	12	13	14	4	10
	No	4	4	3	14	6
	Yes & No	2	1	1	0	1
	No answer	0	0	0	0	1
Discrimination (20)	Yes	0	19	18	1	17
	No	19	1	0	17	2
	Yes & No	0	0	1	1	1
	No answer	1	0	1	1	0

The lack of an explicit question in the statements prevented some students to identify an unknown susceptible of being involved in an equation. This is the case of S<sub>13</sub> in Friends situation:

S<sub>13</sub>: We do not have any unknowns. It is not a task, because there is no question to answer (it is all information).

Order of operations potentially matters. It turns interesting to analyse the only “Yes” answer in Number situation (restricting to discrimination as focus):

S<sub>17</sub>:  $5+2x=13, 5+2x=13|-5, 2x8=13|:2, x=4$  [each step is reproduced as originally written by the student, typos included]. If you add 5 and multiply 2x, you get 13.  $13-5=8, 8:2=4$ .

On one hand, she solves the given equation (getting  $x=4$ ). On the other hand, she formulates the Number situation in her own words. Then, apparently in order to find the unknown favorite number, she tries to undo the operations that, applied on this number, give 13. She replaces addition by subtraction and multiplication by division, but she does not reverse the application order of the operations (she calculate  $(13-5):2=4$  instead of  $13:2-5$ ). In this way she gets 4 as outcome and, comparing the solutions, she circles “Yes”.

### Strategies

Table 3 shows the frequencies of use of each strategy by the students who focused their performances on discrimination (20), as requested by the task.

**Table 3: Frequencies of the strategies used by students focused on discrimination**

	Friends	Lollipops	Labels	Number	Hanger
Identification	0	2	2	0	3
(O, T&E, MC)	(6, 0, 0)	(3, 0, 3)	(6, 0, 0)	(3, 1, 0)	(4, 0, 0)
Divisibility	0	0	0	7	0
Solution checking	0	0	6	1	6
Equations comparison	6	6	1	6	3
Solutions comparison	6	7	3	5	3
Analogy	2	2	9	0	4
Numbers comparison	2	0	0	0	0
Total	22	23	27	23	23

As evidenced by the total number of strategies used in each situation, students often involved more than one strategy in their answers. The frequency of each strategy seems to be influenced by the representation system of each situation. In verbal situations, equations comparison together with solutions comparison and equation solving were the most used strategies, while in pictorial situations the most frequent were analogy and solution checking. Verbal statements awakened the students' need to formulate an equation describing the situation, while pictures let them more easily perceive analogies with other situations and they offered a scenario (the "blanks" of the pieces of cloth and of the lollipops price labels) to implement solution checking. In Friends situation (where the unknown is the number of friends), some students misunderstood the role of the number of candies, as they compared or related it to the value (4) of the unknown for the given equation:

S<sub>18</sub>:  $5 + \text{two } x = \text{thirteen} \rightarrow x=4, 13-5=8, 8:2=4$  [she solves the given equation]. No. Everybody would have to get 4 c.

S<sub>8</sub>: No. This equation shows that one man would get 4 candies [referring to the solution of the given equation].

In the same situation, some students' answers reveal how important the external structure and numerical data were for them, obviating the internal relations:

S<sub>5</sub>: No. It will not work because there should be 13 of anything.

S<sub>33</sub>: No.  $13+5=18$ . This task does not work because it comes out of 18 candies, but we need 13 candies in total.

S<sub>23</sub>: Yes, because 1,2 [referring to Friends and Lollipops situations] are the same because the numbers are the same.

In contrast, other students referred to mathematical properties when comparing equations. In Lollipops situation, S<sub>13</sub> alludes to commutativity (involving symbolic letters and numbers):

S<sub>13</sub>: Yes. This equation would look  $2x+5=13$ , but we can switch 5 and  $2x$  [comparing it with the given equation  $2x+5=13$ ].

Both disambiguation and interpretation of the statements also become crucial. For the same situation, S<sub>10</sub> justifies her negative answer as follows:

S<sub>10</sub>: I say no, because it says “two identical lollipops and chocolate for 5 Kč” so this all together written “for 5 Kč” and then total costs 13 Kč. It is an illogical task. “My choice”.

Some students used the same letter to symbolise different quantities. This is the case of S<sub>30</sub> in his attempt of syntactic translation of the Number statement:

S<sub>30</sub>: No because it would be:  $x+5= x$ ,  $x \cdot 2 = 13$  but it is not:  $5+2x=13$  and also does not happen  $x \cdot 2=13$  since 13 is not a divisor of two. [He verbally exchanges multiple by divisor.]

In Hanger situation, this same student does not interpret the need for the numbers in the squares to be equal, but to add up to the difference (8) between the quantities in both sides (13 and 5):

S<sub>30</sub>: Yes, because the two numbers under 5 can be any way, so there could be 2 times 4, or 5 and 3 etc.

## Discussion and conclusions

In this work we show elementary students’ abilities to recognise whether a linear equation describes different contextualised situations. We shed light on the individual strategies that fifth grade students newly initiated in solving equations used to discern this. We highlight the wide variety and nature of their strategies as well as the different frequencies of their use depending on the system of representation of the situations. Faced with verbally presented situations, students tended to formulate an equation based on syntactic and semantic translation of the statement, to compare it with the given equation. Faced with pictorially presented situations, their tendency was to allude to an analogy between the situation at hand and another one of the tasks, concluding then the same as they did in the analogous situation. Alternatively, in this case, they also preferred to check whether the solution of the given equation was a solution for the situation at hand. This finding can be used with teaching purposes. It is also remarkable that students used more than one strategy to justify their decisions and that they did not systematise their strategies but adapting them at each situation. This is illustrated by the fact that seven used divisibility arguments to discuss the Number situation.

The students’ performances indicate that they were able to infer mathematical truths not explicitly taught to them when briefly introduced to equations in the classroom. Behind their use of solution checking and solutions comparison strategies are the following facts, connected to the internal structure of equations (related to the equivalence of linear equations and uniqueness of solution): (a) If the solution of a linear equation is a solution for a situation, then this equation can algebraically describe that situation; (b) if two linear equations have the same solution, then they can algebraically describe the same situations. In contrast, most of the students who used the equations comparison strategy based their strategy on the external structure of the equations. Although this did not prevent them to get right answers in this task, it is not a safe practice, since two equivalent equations can have different external structures. Another practice that stands out in the students’ performances is the frequent mathematical checking of their answers (e.g., S<sub>18</sub>: 1 lollipop costs 4Kc,  $13-5=8$ ,  $8:2=4$ ,  $4+4+5$ ,  $8+5=13$ ). This validation was mostly pragmatic (Balacheff, 2001), although also a few students referred to mathematical rules in their justifications (see S<sub>13</sub> above). Based on our findings, as many other authors (e.g., Brizuela & Schliemann, 2004), we conclude that algebraic activities and tasks in which the notion of linear equations is implicitly or explicitly involved could be worked at this educational level, serving to enrich students’ mathematical thinking.



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