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# The “Sense-Making-Algebra” project for Hungarian seventh graders

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*According to the Hungarian National Curriculum, the systematic study of algebra starts in the seventh grade. In the initial phase, traditional methods focus less on generalization as a cornerstone of algebra and more on algebraic operations. Matched with the traditional syllabus, the authors have incorporated lessons into the teaching process at this early stage of algebra learning that focus on generalization activities and the meaningful use of symbols. The authors investigated how generalization activities manifest themselves during these lessons.*

*Keywords: Algebraic reasoning, generalizations, using symbols, algebra in the early grades, lower secondary level education.*

## Introduction

The cognitive ability to abstract is given special attention in the Hungarian National Core Curriculum in grades 7-8 (Government Decree, 2012, rev. 2020). The generalization of experiences and the need to justify relationships are also expected. The curriculum also requires the learner to formalize the mathematical content of everyday problems and use letters to denote unknown quantities. To begin teaching using symbols, a procedural approach based on “letter arithmetic,” i.e., practicing operations with algebraic expressions based on definitions and rules, is a traditional teaching strategy also presented in Hungarian textbooks. However, less emphasis is on generalization and the meaningful use of symbols in that early stage. Discrepancies between development goals and the practice in teaching early algebra were the impetus of our research. Improving this situation requires recognizing that algebraic thinking, which involves the meaningful use of symbols, is a form of mathematical sense-making related to symbolization (Schoenfeld, 2007). Many authors explain the sense-making process of a mathematical concept by connecting it to prior knowledge (Scheiner, 2016). Palatnik & Koichu (2017) found that the criterion for successful implementation of the sense-making process is that students are adequately engaged in the task and that the learning environment, which in their scenario was project work, is supportive. Inspired by these ideas, the authors of this paper added mathematical problem situations to the traditional “letter arithmetic” to support the generalization process and the meaningful use of variables. This paradigm is named the “Sense-Making-Algebra” teaching strategy by the authors of this paper. A rationale behind this teaching strategy is to allow students to see the power of symbolization as early as possible in the algebra learning process. Furthermore, with a problem-oriented teaching approach (Kónya & Kovács, 2021), the authors aimed to create a supportive learning environment.

The authors’ research questions are, (1) how do seventh-grade students’ generalization activities manifest themselves during the “Sense-Making-Algebra” intervention, and (2) what resources do they

employ to create their generalization? We hypothesize that most learners can generalize from experience and argue for generalization using natural language or symbols after the teaching unit.

This paper describes a part of the broader experiment: the content and analysis of the first experimental lesson and the related follow-up test. In this lesson, we used the so-called “calendar problem,” which has been discussed in various places in the literature, as an effective initiative to encourage generalization (Friedlander & Hershkowitz, 1997; Huang et al., 2014).

The first author performed the action research reported in this paper in her three seventh-grade classes. The second author acted as a critical friend during the teaching experiment. The study provides an example of how to put into practice the sense-making use of variables. Although there are some well-prepared collections on teaching strategies and recommendations on algebra learning (e.g., Friedlander & Arcavi, 2017), these works are focused on equations. The authors found fewer compendiums of a similar style at the pre-equations stage of algebra learning, with generalization and reasoning in focus, which also motivated this research.

### **Theoretical underpinning**

Several scholars attempted to investigate the nature and content of algebraic thinking. There are several perspectives on what defines algebraic thinking, but many agree that generalization, or the ability to discern the general in the specific, is a crucial element (Pittalis & Zacharias, 2019). The present article focuses mainly on generalization, which may be accomplished using natural language and symbols. While outlining the theoretical basis, we emphasize this perspective, and in this regard, the work of Kaput (2007) serves as the theoretical framework for our research.

Freudenthal (1977) made an early attempt to conceptualize school algebra and included algebraic thinking in the subject. Freudenthal highlighted that the ability to describe relations and solve procedures in a general way is part of algebra. In this novel perspective, algebra is seen as a human cognitive activity. Thus, those who conceive algebra as reasoning prefer to examine how students think and speak about it. Arcavi (1994) considers the dichotomy of generalization and symbol usage in school algebra and claims that many students who master algebraic methods often fail to perceive algebra as a tool for comprehending, expressing, conveying generalizations, and constructing mathematical arguments. According to Kieran’s (2004) model for conceptualizing algebraic activity, algebra is a multidimensional activity that includes numerous ways of thinking. Algebraic thinking approaches quantitative situations that aim to find relationships and structure using not strictly letters-symbolic approaches. Kaput (2007) emphasizes two core aspects of algebraic reasoning. One of them is generalization, and expressing generalizations in increasingly coherent symbol usage (Core Aspect A). “Increasingly” means that initially, the students use their resources, typically natural language, but later switch to conventional representational forms. Core Aspect B means the syntactically guided action on symbols. Kaput argues that the “Core Aspect B” should come later than the “Core Aspect A” since rule-based actions on symbols depend on knowing the allowed combinations of symbols, particularly which combinations are equivalent to others.

## Method

The action research took place from December 2020 till March 2021 in a Hungarian practicing school for teacher training. A total of 68 seventh-grade students aged 13-14 years (hereafter S01-S68) from three classes (hereafter classes b, d, z) participated in the experiment. The groups have three maths lessons a week, each 40 minutes long. The teacher was the same in all three classes. Based on the textbook used in the school, the teacher's syllabus allocated 28 lessons to algebra instruction. The intervention took place in the first teaching unit (Algebraic expressions and operations with algebraic expressions), for which the curriculum allocated 11 lessons. The teacher conducted four experimental lessons in this teaching unit. The authors only report on the first experimental lesson in this paper and the follow-up test relating to it. The authors used the following tools for data collection and analysis: (1) classroom observation through a visiting teacher's field notes; (2) the teacher's research journal; (3) copies of students' written outputs.

The first lesson was inspired by Huang et al. (2014), where the authors report a lesson on investigating patterns in calendars, focusing on improving empirical reasoning towards deductive proof. The planning of lessons followed a problem-oriented teaching approach (Kónya & Kovács, 2021). It means that students: (1) analyzed mathematical problems; (2) were allowed to reflect on their own and their classmates' thinking critically in classroom discussions; (3) were encouraged to explain and justify their thinking.

The plan of the first lesson is as follows. Part 1: pattern finding. Students looked for patterns in the December 2020 calendar during the preparatory phase of the lesson. First, the teacher chose the diagonal 1-9-17-25 (Figure 1, left). Then, after determining that the numbers in the sequence increased by 8, students worked in pairs to explore similar patterns (Figure 1, right). The purpose of the pattern-finding task was to make students aware of the mathematical structure of the calendar, which is already familiar from everyday life.






**Figure 1. The December 2020 calendar with diagonal patterns**

Part 2: magician's trick. The principal part of the lesson was based on a magician's trick. "Choose a number with a top right and a bottom left neighbor. Then, you tell me the sum of the three numbers, and I will guess the chosen number!" The processing of the magician's trick had three stages: (1) Numerical experience. The number said by the student and the number thought of was recorded on the board. (2) Formulating the rule: how did the magician do the calculations? (3) Explanation of the rule. The theoretical consideration behind placing the problem in a playful context was to engage as many students as possible in the task and maintain motivation.

In the follow-up test, two compulsory and one optional assignment were given to the students (Table 1). The first and second problem was close to the calendar task in the lesson. These two tasks were chosen to require generalization and make it easy to argue without using symbols. On the other hand, the third task was assumed to be much more difficult to explain without symbols.

**Table 1: The follow-up test (for March calendar)**

<p><b>Task 1.</b> (Compulsory) Nándi selected a number in the calendar and added its left and right neighbors. When he halved the sum, he got exactly the number he had chosen. Do you think this is always true if you choose a number with a left and a right neighbor? Justify!</p>	
<p><b>Task 2.</b> (Compulsory) Csenge chose a number from the calendar with both a lower and an upper neighbor. She added the two neighbors together and then halved the sum. What has she experienced? Will this always be true if you choose a number that has a lower and an upper neighbor? Justify!</p>	
<p><b>Task 3.</b> (Optional) Laci had a magician trick. If someone told him the sum of three numbers in the calendar in the shape of a V, he would guess the middle number. (For example, if someone told him that the sum is 22, he would know that the middle number is 12.) However, unfortunately, he forgot how the trick worked. Help him!</p>	

To assess the follow-up test, the authors examined the students' solutions, for which they created the following code system (Table 2).

**Table 2: The coding system to evaluate students' outcomes**

Code	Description	Definition
NA	No Answer	One of the following elements: The student has not submitted a solution. No explanation. The student did not understand the text. The student tries to give reasons but does not give a relevant explanation.
AR	Arithmetic level	The student makes relevant calculations only on some numbers.
ALG	Algebraic level, without symbol	The student argues generally and correctly, using natural language without using symbols.
ALG+	Algebraic level, using a symbol	Three criteria must be fulfilled. (1) A symbol represents a number in the configuration. (2) Student correctly expresses the relationships between the numbers, and (3) student performs algebraic operations.

## Findings and analysis

During the pattern-finding task (part 1 of the lesson), students have skillfully formulated the rules in the natural language during classroom discussions. Moreover, the generalization process emerged naturally:

- S45: Is what we are talking about a feature unique to December or not?
- S32: [Enthusiastically] There are seven numbers next to each other, that is, they are arranged by week, so for each month, it is [true].

While working on the magician's trick (part 2 of the lesson), the experience was written on the whiteboard (Table 3), and the one who figured out the trick could be the next magician.

**Table 3: Numerical experience in class *d***

The magician	Teacher	Teacher	Teacher	S31	S31
Sum	33	66	69	73	75
Number thought	11	22	23	“You made a mistake.”	25

Finally, students formulated the rule in their own words; they had to divide the sum of three numbers by three to get the number they had chosen.

When questioned why the trick worked, a faulty analogy emerged. Several students suggested that it should be divided by three because three numbers were added. The teacher took advantage of this situation and involved the class in discussing the mistake. The following dialogue is from class *b*, although almost the same happened in class *z*.

- Teacher: [Addressing the faulty analogy.] Look at the diagonal patterns in the calendar! [She points to the corresponding pattern, see Figure 1 (right) that she deliberately left on the board at the beginning of the lesson.]
- S03: [Uncertainly] The [number] six is the key.
- S14: One number is six less and the other six more [than the number one thinks].
- T: Let us write what he said! Let, for example, the number thought be 11. Thus  

$$(11 - 6) + 11 + (11 + 6) = 3 \cdot 11 = 33.$$
- T: What if the thought number is not 11?
- S: [Unrecognized student from the class] It can be anything
- S14: Let it be  $x$ !

First, S14 tried to explain the rule with his natural resource, which was the basis of the teacher's general example. They concluded with the deductive proof  $(x - 6) + x + (x + 6) = 3 \cdot x$ . It is worth noting that the students explained the rule of merging starting from the problem situation; this is a point where the advantage of the “Sense-Making Algebra” approach is very apparent.

In class *d*, after the general examples, S46 proposed denoting the numbers with  $a, b, c, d$ , i.e.,  $a + b + c = d$ . Although the need to use letters emerged in the student, she did not express the relationships between the symbols. The student’s typical failure provided an appropriate opportunity for the teacher to draw attention to the principle that symbols must express the mathematical relations extracted from the situation.

In two classes, the students wanted to create their own magic tricks. The need for generalization was raised naturally in all three classes: do the series apply in other months, does the magic trick work in other directions, or other months? The use of letters and symbols occurs spontaneously in two of the three classes, with minimal teacher guidance. However, more teacher guidance and questions were needed in the third class.

The result of the coding of the follow-up test is shown in Table 4.

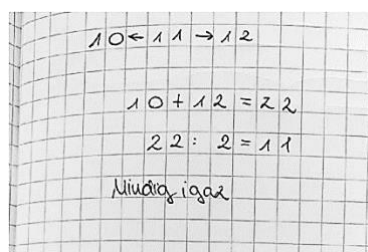
**Table 4: Result of the follow-up test**

	Task 1	Task 2	Task 3
NA	23	28	61
AR	6	5	0
ALG	25	23	0
ALG+	14	12	7
SUM	68	68	68

39 students gave algebraic level answers (ALG and ALG+) in the first task, i.e., 57% of all students. The result is similar in the second task, 52%, 35 out of 68 students. 21% of all students used symbols to express their reasoning (ALG+) in the first task, while 18% in the second task. The following examples are representative of each category.

NA (Task 1., S12) “Yes, that is true. Because if a number has a left and a right neighbor, both will be odd or even, and if you add two odd or even numbers together and divide by two, you always get an integer.” What student writes is true in itself, but it does not explain the experience.

AR (Task 1, S22) See Figure 2. The student refers to only one particular case and concludes that the rule is always valid. (Translation of the text in the figure: “always true”).



**Figure 2: AR solution of Task 1 by S22**

ALG (Task 1. S1) “The rule is valid because if you add a number one greater than the number you have chosen and a number one less than the number you have chosen, you will always end up with

twice the chosen number.” The student correctly expresses the relationship between numbers in natural language; his reasoning is general, but he does not use symbols.

ALG+ (Task 3. S4) See Figure 3. The student represented the central element of the V-shape by  $z$ , correctly determined the other two elements ( $z - 8, z - 6$ ), and performed the merge correctly ( $3z - 14$ ). The number needed is then determined by thinking backward from the sum. Translation of the text: “Add fourteen to the total and divide by three.”

③  $y = y$   
 $z - 8 + z - 6 + z = 3z - 14$   
شركه كوروا لله اذني تيم ناهي اس ابله ابله  
donarrnal -

**Figure 3: ALG+ solution for Task 3 by S4**

The follow-up test also showed an example of an unexpected generalization: S15 wrote: “Conclusion: if you add the neighbors of a number (it can be second neighbors, third neighbors, fourth neighbors), you always get twice the number. It is because both neighbors differ by the same difference from the number.”

## Discussion and pedagogical implications

The authors’ first research question was: how do seventh-grade students’ generalization activities manifest themselves during the “Sense-Making-Algebra” intervention? Consistent with Friedlander & Hershkowitz (1997), our research confirmed that students generalize and reason willingly and satisfactorily at the early algebra learning stage. A possible good choice for this process is the “calendar problem,” as presented by Huang et al., (2014). The mathematical structure is simple, based on addition, so it can be used to pose problems accessible to a wide range of learners. Furthermore, we found that making and justifying conjectures occurs naturally using this type of exploration task. Moreover, this study found that the “problem-oriented” teaching approach supported generalization and reasoning activities, providing a good space for classroom discourse (Kónya & Kovács, 2021). Generalization, in many cases, did not occur as an individual activity but as a consequence of group discourse. The starting point for generalization could be the students’ questioning in addition to the teacher’s initiative.

The second research question was: what resources do students employ to create their generalization? Students expressed their ideas verbally in natural language resources in classroom discourse, which facilitated generalization. The predominance of textual formulation remained in written work until the end of the learning cycle. However, about 20% of the students were already confident using symbols in problem situations in our experiment. Although we did not use project work as a learning environment, we saw an identical process described by Palatnik & Koichu (2017), who reported that students developed and justified claims, made generalizations, addressed why-questions, and established coherence among the explored objects.



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