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# Relational thinking with indefinite quantities - case studies from elementary school children

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*Early algebra is becoming more and more important in research of mathematics education. Relational thinking and variables are emphasized as essential sub-areas of algebraic thinking. The article provides an insight into kindergarten and elementary school children's abilities to establish relationships between quantities. The children's approaches can be described in a continuum between number-oriented and structure-oriented approaches. The influence of indefinite quantities can be shown by comparing three case studies.*

*Keywords: Elementary school mathematics, early algebra, relational thinking, variable, task design.*

## Introduction

The present study is about early algebra, which importance is steadily increasing for mathematics teaching in elementary schools (e.g. Kieran, 2018; Cai & Knuth, 2011). Studies show that an algebraization of mathematics lessons in elementary school is beneficial for itself as well as for later algebra learning (e.g. Mason, 2008; Kaput, 2008). Compared to arithmetic thinking, algebraic thinking is characterized by a structural rather than an operational view of mathematical objects (Steinweg, 2013; Kieran, 2004). Kieran (2004) named the consideration of relational aspects of operations instead of their computation as an essential way of thinking in early algebra. Likewise, comparing mathematical expressions and a relational understanding of the equal sign plays an essential role in early algebraic thinking. The creation of relationships between mathematical expressions defines algebra and can be described in particular in the aspect of relational thinking, which forms the theoretical framework of the present study.

## Theoretical framework

Relational thinking is to be regarded as a sub-area of early algebra and can be defined as follows:

“Sentences have to be considered as wholes instead of as processes to do step by step. When students analyze expressions, they compare elements on one side of the equal sign to elements on the other side of the equal sign or they look for relations between elements on one side of the equation” (Molina & Ambrose, 2008; p.64).

Fostering relational thinking with the help of equations is a possibility and has already been studied (e.g. Carpenter et al., 2003; Molina et al., 2005; Molina et al., 2008). Nonetheless, non-formal representations can also be used to establish relationships between elements (Schliemann et al., 2007). They offer the advantage of granting access to younger children as well and thus capturing pre-school starting points for mathematical learning. Furthermore, problems of the formal representation are avoided. This includes the relational understanding of equal signs, which can be seen as a prerequisite for using relational thinking with regard to equations (cf. Molina et al., 2005).

In addition to relational thinking, dealing with variables is an essential aspect of algebraic thinking. In principle, three aspects of variables can be identified: variables as unknowns (e.g.  $5 + x = 8$ ), variables as indeterminate like in functional relationships or as a general number, for example to describe the laws of calculation (e.g. Freudenthal, 1983). The influence of various variables on relational thinking is illustrated by a study by Stephens and Wang (2008). They showed that the inclusion of multiple variables supports the use of relational thinking. Secondary school students had to put numbers in the boxes of the equation  $18 + \square = 20 + \square$ , establishing relationships between two indeterminate values. Compared to placeholder problems with only one unknown, these equations could stimulate students to think relationally rather than using operational solutions.

Relational thinking as the creation of relationships between mathematical expressions - be it equations, operations or relationships in factual relationships - is of great importance in early algebra. In particular, the connection between variable concepts and the use of the relationships between mathematical elements is challenging and requires further investigation, particularly with regard to young children of preschool and elementary school age. Therefore, the research questions examined in this article are the following: *How do elementary school children establish relationships between known, unknown and indefinite quantities that are represented by real material? What influence does the different use of known, unknown and indefinite quantities have on the use of relational ways of thinking?*

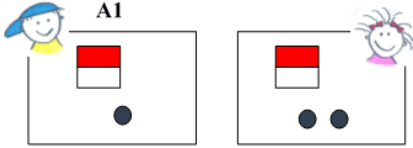
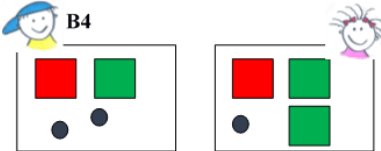
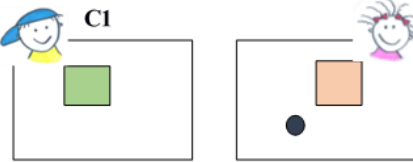
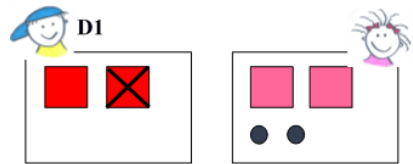
## Method

The aim of the study is an exploratory recording of the abilities of preschool and elementary school children to think relationally<sup>1</sup>. As Stephens and Wang's (2008) studies have shown, the use of multiple variables within equations is helpful in stimulating relational thinking. With regard to the age group of kindergarten and elementary school children, a design was chosen in which equations with known and unknown quantities were represented with the help of material. Based on Melzig (2013), tasks were created in which boxes and marbles represent known and unknown quantities. Melzig showed that boxes open up a first access to a sustainable understanding of variables. The non-formal task design with real material (boxes and marbles) should also enable preschool children to show their relational thinking and can serve as a starting point for developing sustainable ideas about variables. Various equations with one or more variables have been translated into an arrangement of different colored boxes and single lying marbles. A story was told: two children play with marbles, some of which they keep in colorful boxes. Within a task there are the same number of marbles in boxes of the same color. Boxes of the same color may contain a different number of marbles in a different task. There were 12 tasks in 4 different task types.

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<sup>1</sup> Other studies used the data to examine the conceptualization of the variables (e.g. Lenz, 2021).

**Table 1: Overview of the tasks.**

Task type	Example of the tasks
<p style="text-align: center;">Type A: “The same?”</p> <p>Two given sets of quantities have to be compared with each other. The content of the boxes is not known and does not need to be determined to answer the question. The boxes can be seen as a variable as an indefinite.</p>	<div style="text-align: center;">  <p><b>A1</b></p> </div> <p>Task A1: “Does both children have the same amount of marbles? How did you get that?”</p>
<p style="text-align: center;">Type B: “How many?”</p> <p>The contents of a box must be determined. The amount of marbles asked for can clearly be determined and so appears as an unknown. Exercises B1 and B4 also contain additional boxes (the red ones), which content is not known and does not necessarily have to be determined in order to find a solution.</p>	<div style="text-align: center;">  <p><b>B4</b></p> </div> <p>Task B4: How many marbles have to be in a green box, so that both children have the same amount of marbles? How did you get that?</p>
<p style="text-align: center;">Type C: “How many?”</p> <p>Both children have boxes of unknown content. In contrast to the previous type, no specific quantities can be given for them. In order to answer the question, children can state a relationship between the amount of marbles in the boxes. Since the contents of the boxes cannot be clearly determined in comparison to task type B, it can be seen as indefinite.</p>	<div style="text-align: center;">  <p><b>C1</b></p> </div> <p>Task C1: How many marbles have to be in the boy's box so that both children have the same amounts of marbles? How did you get that?</p>
<p style="text-align: center;">Type D: “Make them equal”</p> <p>Both children have the same amounts of marbles. The interviewer makes a transformation by removing or adding a box by one child in the task. Children have to decide what amounts of marbles they have to give to the other child in the task or take away from the child to the interviewer, to make the quantities equal again. Tasks D1 could be answered with a specific number of marbles, while in task D2 a relationship between two indefinite quantities had to be established.</p>	<div style="text-align: center;">  <p><b>D1</b></p> </div> <p>Task D1: Both children have the same amount of marbles. Now, the boy gives one of his boxes to the interviewer. How much does the girl have to give to the interviewer, so that both children have the same amount of marbles again?</p>

To get insight into children's ways of thinking and their use of relational thinking, a qualitative survey method using interviews was chosen. The study follows a diagnostic approach and is not to be understood as an intervention. The tasks were dealt with in video-recorded, semi-standardized individual interviews with 80 children in three age groups. 5-6 years old kindergarten children (N = 25), 7-8 years old second-graders (N = 29) and 9-10 years old fourth-graders (N = 26) took part. The children were asked to explain their approach. The transcribed interviews formed the data basis for

the subsequent qualitative content analysis (cf. Mayring, 2010) and the method for analyzing interviews according to Schmidt (2005). Categories were formed out of all of the interview-transcripts which described children's approaches to answer the tasks. They were created deductively based on preliminary theoretical considerations and inductively obtained from the data. The categories were recorded in a coding guide. After re-coding all the material, an overview of the categories of the entire data material was given in frequency tables. These were used for further analysis by pointing out possible relationships that need to be checked in individual cases. In a last step, in-depth case analyzes were made (Schmidt, 2005).

## **Findings**

After answering the tasks, the children were asked how they got their answer. Based on the analysis of the interviews, this article focuses on comparing the approaches described by the primary school children. First, the evaluation dimensions are presented and related to the theoretical background. Then insights into three case studies from primary school children are given.

### **Number-oriented and structure-oriented approaches**

The approaches described by the children for processing the tasks can be described across all tasks in a continuum between number-oriented and structure-oriented approaches.

#### **Number-oriented approach**

The number-oriented approach focuses on the specific amounts of the marbles. Because the calculation of sums instead of relating quantities predominates, it can be characterized as an arithmetic way of thinking.

#### **Structure-oriented approach**

In a structure-oriented approach, children make gestural or linguistically clear that equal subsets are related to each other. The focus is on the quantities themselves and not on their value ("they are the same" instead of naming the specific number). Children take a structural perspective on the task and make connections between sub-structures of the task. According to Molina and Ambrose (2008), this approach can be characterized as relational thinking.

### **Three case studies**

The distinction between number-oriented and structure-oriented approaches is compared using the example of the processing of tasks B4 and C1 (see Table 1) by three children. But they also show the influence of adding another, indefinite variable to exercise C1. It should be noted that the change in task types was not communicated to the children. In task type B, it was possible to answer with specific numerical values. Immediately afterwards, task C1 was set, whereby no specific numerical values could be given. Thus, some problems for the children are to be expected. But the answers to task C1 also reflect the spontaneous approaches of the children without being influenced by learning effects.

### **Leonie, 4<sup>th</sup>- grader**

In task B4, Leonie states that there must be a marble in each of the red and green boxes. She justifies this with the fact that both children then each have four marbles. When asked whether she needs to know the contents of the red boxes, she seemed unsettled. This shows that in addition to the number-oriented approach described, no structure-oriented approach is conceivable for her, in which the content of the red boxes is not determined.

In task C1, Leonie spontaneously gives numerical values for the two boxes given. The interviewer then names various other numerical values as the contents of the girl's box. Leonie can give the correct amount of marbles for the boy's box. This shows that she does not accept the aforementioned numerical values as fixed actual content, but can imagine different values. She is able to deal with changing numerical values and recognizes the dependency. However, she does not explicitly succeed in generalizing the relationship between the indefinite quantities.

### **Matteo, 4<sup>th</sup>-grader**

In task B4, Matteo structures subsets as equivalent to one another without having to determine their specific content. Based on the individually lying marbles, he deduces the contents of the green boxes. This approach can be characterized as structural:

Matteo: There must be one marble in a green box, because here (*points to the red boxes*) there are the same numbers and here (*points to a green box on the boy's side and on the girl's side*) ... In order to make this one marble difference (*points to the girl's marble*), one must also be in here (*points to the girl's front green box*).

In task C1 Matteo states that there is a dependency between the indefinite quantities of marbles in the two differently colored boxes. He says correctly that the contents of the boy's box cannot be determined because the content of the girl's box is not known. When asked by the interviewer, Matteo can then state the general relationship between the quantities of marbles in both boxes and refers to the marble lying individually in his argumentation. In addition, a structuring of the given quantities becomes clear:

Matteo: Here (*points to the boy's box*) there is one more marble than here (*points to the girl's box*), because here (*points to the girl's marble*) there is a single marble.

### **Julius, 2<sup>nd</sup> grader**

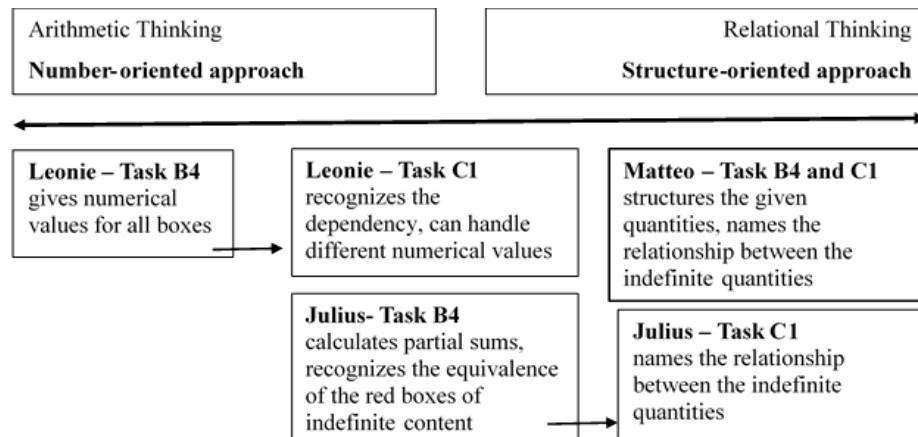
In task B4, the second grader Julius names the value of one marble as the content of the green box. He claims to have checked this by calculating subtotals: the contents of the green boxes and the individually lying marbles. He doesn't say anything about the red boxes. He may have excluded these because they contain the same amount of marbles. Although his approach by calculating partial sums is to be regarded as number-oriented, this already represents a transition to the structure-oriented approach. He has recognized partial amounts of equal amount (the red boxes) that will be disregarded for further consideration.

In task C1 Julius succeeds in specifying the relationship between the indefinite quantities:

Interviewer: How many marbles must there be in a green box so that both children have the same amount of marbles? [...]

Julius: Because one thing (*taps the girl's marble*) and then you still have to calculate that here (*taps the girl's box*), there can be one, two or three (*waving his hand rhythmically in the air*) and there would always have to be one more (*taps on the boy's box*) to be in it than in the box (*points to the girl's box*) then it would be right.

Julius example shows impressively that he is able to establish relationships and to support this with gestures. At this point he shows a structure-oriented view of the task.



**Figure 1: Approaches of the three children in comparison of tasks B4 and C1**

Figure 1 shows how Leoneie and Julius move from a number-oriented approach to a structure-oriented approach in the transition from task types B to C. This suggests that the task design, and especially task type C, can encourage children to think relationally. Since not all values are given, the children have no way of calculating specific numbers. They are encouraged to take a look at the whole and to establish relationships between the indefinite quantities. Thus, the results of the study by Stephens and Wang (2008) can also be confirmed with regard to the handling of real material and with primary school children.

## Conclusion

The analysis of the entire interviews showed that there are children who mainly proceed in a number-oriented approach. To do this, they name numbers for additional boxes in task type B (like the red boxes in task B4) and give arithmetic reasoning. In task type C, these children are primarily tied to numerical values in order to find access to the task. This is shown in the case study of Leoneie. However, she already manages to deal with various numerical examples in task C1 but she cannot indicate a static relationship between the indefinite quantities.

In contrast, there are children who show mainly structure-oriented approaches across all of the 12 tasks. Matteo's explanations of tasks B4 and C1 serve as an example. He is able to take a structure-oriented view of the whole from above across all tasks. This can be characterized as relational thinking (e.g. Molina & Ambrose, 2008).

In addition to these cases, however, there are also children who switch approaches. Second grader Julius described in task type B mainly number-oriented procedures. This may be due to the fact that it is possible to operate with specific numerical values. But also in his processing of task B4 it should be noted that he is already on the way to a structure-oriented approach in that he only calculates partial sums instead of the whole sum of marbles, as Leoneie did. In task C1, Julius shows that he is able to

describe relationships between the sets. The tasks of type C thus stimulated him to move from a number-oriented approach to structure-oriented approach and thus relational thinking.

## Discussion

The empirical study shows that relational thinking as a sub-area of algebraic thinking can be stimulated in elementary school children. The study also showed that some kindergarten children are also capable of relational thinking, which, due to the length of this article, could not be taken up. The skills of the kindergarten children are to be regarded as prior knowledge and at the same time the starting point for further mathematical learning with regard to early algebra. With the help of the task design in its non-formal representation, instead of starting from procedures and calculating result values, many children succeed in recognizing relationships between the elements with the help of a look at the whole and using them to find solutions. It is precisely this “view from above” on the elements of a mathematical situation and their relationship to one another that constitutes relational thinking and thus also an essential aspect of algebraic thinking.

In particular, the type C tasks, which contain indefinite quantities, encourage children of primary school age to use relational thinking. This is already indicated by the small insight into the processing by the second grader Julius within the case studies and also corresponds with the explanations of Stephens and Wang (2008). As a result, the inclusion of unknown and indefinite quantities in the sense of a spiral curriculum also appears profitable in elementary school lessons.

The analyses give a little insight into the differences in the approaches of the children. With regard to the use of the task design in school, very different approaches by the children are to be expected. A classroom discussion about children’s ways of thinking used can lead the children to focus on both number-oriented approaches and structure-oriented approaches. The real material can be used to clarify your own ways of thinking to others - in particular the established relationships between the quantities.

Nevertheless, it is important to break away from a real representation and later perceive the variable as a thought object instead of a real object in the form of the box.

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