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Students' ideas about variables as generalizers and unknowns: Design Research calling for more explicit comparisons of purposes

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Students' challenges with variables are well documented. Although instructional approaches have been developed to support students' conceptual understanding of variables as generalizers, many students confound variables as generalizers (e.g. in generalization activities) with a second meaning, variables as unknowns (e.g. in equation solving activities). In our design research project, we developed a teaching learning arrangement in which both meanings could be constructed. The design experiments reveal first qualitative evidence that students were able to construct both meanings, but only some students developed explicit awareness of the distinctions of meanings. This calls for more explicit comparison activities with a focus on different algebraic activities and their underlying epistemic purposes, as well as the development of a meaning-related language in which students can express both meanings.

Keywords: Variables as generalizers, variables as unknowns, variation principle, learning from comparison, design research.

Theoretical background

Learning from comparison has proven effective in many contexts for raising awareness of critical differences (Alfieri et al., 2013). In our design research study, we use this design principle to increase students' awareness about critical differences between meanings of the variable (Usiskin, 1988; Malisani & Spagnolo, 2009). The research question for this paper is: *Which ideas do students articulate about variables as generalizers and as unknowns and how do they compare the conceptions?* The qualitative case of a larger design research study reveals that not only local meanings, but also underlying epistemic purposes need to be compared. We outline the theoretical background, the design principle and the methodological framework, before unfolding the case of two students.

Variables as generalizers and unknowns

The concept of variable lies at the heart of most algebraic activities, yet various studies reveal many students' difficulties with understanding variables in all their meanings (e.g., Malisani & Spagnolo, 2009; Küchemann, 1981). Whereas the meaning of *variables as unknowns* (a fixed set of hidden numbers that have to be disclosed, e.g. in equation solving activities) seem to be constructed more easily (Küchemann, 1981), the meaning of the *variable as a generalizer* reveals a critical obstacle in many students' understanding of the variable (Bardini et al., 2005; Malisani & Spagnolo, 2009). The conception of the *variable as generalizer* draws its relevance from generalization activities, in which

the variable is used to express general relationships without any further specification of the numbers for which they apply (Usiskin, 1988).

Although various instructional approaches were established to enhance students' conceptual understanding of the variable as generalizer, students still tend to confound the two conceptions of variable as unknown and as generalizer, even after going through these approaches (Bardini et al., 2005; Prediger & Krägeloh, 2016). At least three reasons can be identified for these confoundations of conceptions: *First*, variables as generalizers and as unknowns share crucial similarities on the surface. Both represent numbers, both are signified by letters, and the same syntactical operations can be performed with them during algebraic transformations. A *second* reason for these confoundations was identified in the unprecise language by which students express their ideas about variables (Prediger & Krägeloh, 2016). Generalization activities require certain phrases to *think* and express generalizations (see Table 1). Hence, to allow students to grasp relevant differences in meanings, contrasting relevant phrases might be useful. *Third*, students do not seem to be aware about the different natures of the algebraic activities in which they are used. They naïvely assume that all variables are used in problem solving activities, in which a variable denotes a hidden number, which has to be disclosed by solving an equation and, thus, assign this meaning also to variables in generalization activities. As Bardini et al. (2005) showed, many of these students impose the meaning of unknowns to the generalization activities by interpreting them as a “*temporally indeterminate number* whose fate is to become determinate at a certain point” (p. 129, emphasis in original). That means, students use variables in activities, which require a generalizer, but act as if they were disclosing unknowns.

Table 1 summarizes similarities and differences of the two conceptions that are potentially overlooked by students. Other authors (Usiskin, 1988; Bardini et al., 2005; Prediger & Krägeloh, 2016) have already identified the surface level, the meanings and the algebraic activities as crucial for students' understanding. Additionally, the underlying epistemic purpose (i.e., what kind of knowledge gain is intended by its use) became apparent during our empirical analysis. Thus, the last row serves as an advance organizer for the later analytical outcomes.

Table 1: Similarities and differences of variables as generalizers and unknowns (of increasing depth)

	Variable as generalizer	Variable as unknown
Similarities on surface level	Both represent numbers Both are signified by letters and are object of equal syntactical operations	
Differences in meanings in examples (Küchemann, 1981)	Variable stands for all possible numbers of a relevant domain (e.g., in equality $3(x+5) = 3x + 15$)	Variable usually stands for a (concrete set of) hidden number of a relevant domain (e.g., in equation $3x+5 = 13$)
Meaning-related phrases for expressing the meaning (Prediger & Krägeloh, 2016)	An arbitrary number, always a different number, the number always changes, ...	The number we are looking for, number is already fixed
Differences in underlying purposes		

Algebraic activity (Usiskin, 1988)	Generalization activities in which variable is used to express general relationships	Problem solving activities in which variable is used as a hidden number that needs to be disclosed
Epistemic purposes (Usiskin, 1988, Bardini et al., 2005)	One wants to express that no further determination of an algebraic statement is needed – there is nothing more to know	One wants to express that something is not yet known, but shall be known in the future

Design Principles: Creating Rich Experiences and Learning from Comparison

The presented research is embedded in a larger design research project (Gravemeijer & Cobb, 2006) that aims at designing a teaching-learning arrangement to enhance students' conceptual understanding of variables as generalizers and as unknowns and at developing an empirically grounded local instruction theory. Here, we focus on sequences in which student explicate the meanings and the purpose of variables, first in rich experiences with generalization and problem solving activities, and later in reflective sequences by the design principle of learning from comparison.

Learning from comparison has proven to be effective in different domains for increasing students' conceptual and procedural knowledge and awareness (cf. meta-analysis by Alfieri et al., 2013). When comparing different uses of a concept or procedure, students have been shown to detect relevant structural features, and to learn to distinguish them from irrelevant surface features (ibid.). In algebra education, learning from comparison proved effective for increasing students' flexibility with different solutions procedures, in particular the Chinese Bianshi tradition also applied it for conceptual learning, e.g. for geometry concepts (Guo & Pang, 2011). Here, we adapt it for our purpose of raising students' awareness for the difference in meanings between generalizer and unknown.

This adaptation is not self-evident as conceptual differences of the generalizer and the unknown seem to lie especially deep, because their meanings ground in different underlying purposes which seem to be confounded by the students (Table 1). Thus, the differences of these two meanings of variables cannot be reduced to pointing out relevant definitional features and contrasting them. In addition, language challenges can be expected in the design experiments (Prediger & Krägeloh, 2016).

Methodological framework of the design research study

Design experiments as method of data gathering

In this paper, we focus on snapshots of students' ideas when *learning from comparison about variables as generalizers and unknowns* in a first design of a teaching-learning arrangement that engages students in generalization activities as well as informal equation solving activities. The data was gathered in design experiments in an online setting, taught by the first author of this paper. For each of the six pairs of students ($n = 12$) with varying mathematical prior knowledge, usually 4-5 sessions were conducted. In total, 33 hours of video were recorded and partly transcribed. This paper focuses on the case of two girls from a German-speaking school for higher tracked students: Anna (Grade 7) was a high achieving student who had not encountered variables signified by letters in her formal education before. Nora (Grade 8) already worked with variables in various algebraic problems

(transforming expressions etc.), but Nora’s teacher considered her in need to refresh and deepen her understanding of variables.

Methods of data analysis

For the qualitative analysis of students’ ideas about the variable as generalizer and unknown, the transcripts were segmented and situations were selected in which students explicitly related the two meanings of variables to another. Then, we applied an open coding procedure to code students’ utterances with respect to their ideas about main differences and similarities, starting from codes in Table 1. The last line of Table 1 emerged during this coding procedure.

Empirical insights into students’ ideas about generalizers and unknowns

Anna and Nora’s understanding of the variable as generalizer

In the first three design experiment sessions (of 90 minutes each), Anna and Nora receive rich possibilities to engage in generalization activities in the context of an e-scooter rental (0.15 € per minute riding time, 1 € to unlock the scooter). The students conduct repeated calculations in the table in Figure 1 and discover that variables can represent all relevant numbers. In Session 1 after filling the table, the teacher asks for an explanation:

Date of e-scooter ride	Driving time (in minutes)	Costs for pure driving time (in Euro)	Expression for complete costs (incl. 1 € unlock)	Total price (in Euro)
May, 16	20	20×0.15	$20 \times 0.15 + 1$	4.00 €
May, 19	12	12×0.15	$12 \times 0.15 + 1$	2.80 €
May, 24	27	27×0.15	$27 \times 0.15 + 1$	5.05 €
May, 25	18	18×0.15	$18 \times 0.15 + 1$	3.70 €
General	x	$x \times 0.15$	$x \times 0.15 + 1$	Not calculatable

Figure 1: Table for re-inventing variables as generalizers in e-scooter context

1.601 Teacher: Um, now we have written x there suddenly. What does the x mean in this calculation? Can you explain that?

1.602 Nora: Um, that is an x -arbitrary number.

1.603 Teacher: Mhm, and what does that mean, “ x -arbitrary number”?

1.604 Nora: Uh, that every number can stand there.

1.605 Teacher: Mhm, very nice. Anna, any extensions?

1.606 Anna: Yes, x -arbitrary just means that only this one number in this calculation, which is represented by this letter x , though, can be every number. This number, 0.15 or 15 Cent, they are not changeable, they stay always. Only x is changeable [...].

Nora describes the meaning of the variable as generalizer by “ x -arbitrary number” and “every number”, which does not necessarily guarantee that she has firmly grasped the idea of a variable as an indeterminate generalizer for generalizing expressions (as the analysis of comparable cases in Prediger & Krägeloh (2016) reveals). In Turn 1.606, Anna compares the changeable nature of x with the unchangeable nature of constants and thereby implies that the variable represents different numbers. Later on, Nora also expresses her understanding by describing that different driving times influence the total cost of the ride:

1.649 Nora: [...] if one drives 10 minutes now, then it is cheaper, as if one drives 20 minutes.

Anna and Nora’s understanding of the epistemic purpose of the generalizer

Even if both girls have built a solid understanding of the meaning of the variable as generalizer, their ideas substantially differ when asked to explain the purpose of describing situations with variables:

- 2.029 Teacher: So, why do we do that, writing down such a general expression [with variables]?
- 2.030 Anna: Yeah, to represent, not so much to know a total price in Euro. Rather, for representation. Okay, um, this information here, the 0.15 and the 1, these are fixed numbers which stay like this in every calculation, but the x , this is the changeable, variable number.
- [...]
- 2.034 Nora: I don't know. Maybe if one calculates this mentally, and then one forgets it, one has to start from the beginning. And here, one has written it down?
- [...]
- 2.039 Teacher: But what is the advantage if I [...] remember this expression [*points at* $33 \times 0.15 + 1$] or this one [*points at* $x \times 0.15 + 1$]? So, what is the difference?
- 2.040 Nora: Uh, that for the one below [*refers to* $x \times 0.15 + 1$] we don't know how long he was driving, and then, we actually don't have to calculate it.
- 2.041 Teacher: [...] Anna, any extensions?
- 2.042 Anna: Yeah, then one just has to look: Okay, I know for sure that the 0.15 and the 1 stay. Thus, I only have to look which information x has been in this calculation. And this would be 19, for example. Then I only have to look: Okay, 19, and then I can calculate again.

In Turn 2.030, Anna describes the purpose of using variables (represented by letters) from a representation perspective: Variables represented by letters allow to differentiate between changeable and constant numbers in a series of similar calculations. Nora, in contrast, points out in Turn 2.034, that algebraic expressions with letters give relief to one's memory capacity. Anna takes up this idea in Turn 2.042 and further develops it into explaining the purpose typical for formulas: Algebraic expressions establish a general sequence of calculations, which guarantees the determination of a number depending on another number. Thus, Anna articulates a purpose that is typical for a "calculator's mind" (Radford, 1996, p. 50): one formula can substitute many similar calculations. In sum, Anna assigns two purposes to variables as generalizers: (a) It signifies the changeable numbers and therefore distinguishes them (visually) from constants, and (b) it allows to establish a formula, which reduces complexity for the evaluation of new numbers.

For Nora, Turn 2.034 does not yet provide evidence that she also thought about Anna's purpose (b) or simply referred to the general idea that written language allows to relief memory capacity. However, in Session 3, after extensive work in a spreadsheet environment, she gives the following answer when asked to describe the use of the variable in algebraic expressions:

- 3.752 Nora: So that we can use the formula for every single number and immediately know how much the solution is.

Thus, at latest the work in the spreadsheet environment allowed Nora to establish the perspective of the calculator's mind and to express Anna's purpose (b). Yet, there might be a crucial difference between Anna and Nora concerning the *epistemic purpose* (i.e. the intended knowledge gain) with which they use the variable. For Nora, the immediate knowledge of a solution seems to be the main intention to establish a formula using variables. Thus, she ascribes an epistemic purpose to generalizers that is more typical for unknowns, namely that the variable needs to be known in the future (Bardini et al., 2005). In contrast, Anna explicitly articulates in Turn 2.022 that the calculation of "a total price in Euro" is not the main goal. Anna's idea (a) allows her to see that the algebraic expression has a function beyond quick calculation and the reduction of complexity, namely to contrast changing quantities from constants. This suggests that Anna is aware of the crucial epistemic purpose of the variable as generalizer, namely to see the variable not just as a temporal misfortune

that needs to be overcome at a certain point (Bardini et al., 2005), but as something that has a value in its own right.

We infer that for these students (and also other students in our design experiments) the understanding of generalizers needs to be expressed also by the underlying epistemic purposes for the use of variables. Therefore, epistemic purposes can reveal subtle but important differences in students' ideas.

Nora's struggle to compare the generalizer and the unknown

In Session 4 of the design experiment series, Nora and Anna are engaged in problem solving activities in which they had to find unknown numbers. For example, they find the unknown price per minute x by informally solving the formal equation $x \times 11 + 1 = 3.20$ that identified two descriptions of the total cost of an e-scooter ride, namely one as a general algebraic expression and the other as a concrete price. Nora manages to interpret both sides of the equation as the total cost of the ride and she also describes the differences of these two descriptions as follows: "one is the total costs in a calculation and the other is the sum of the total costs" (Turn 4.133). Yet, when the teacher points out a difference between x in the general expression $x \times 11 + 1$ and in the equation $x \times 11 + 1 = 3.20$, Nora struggles:

- 4.136 Teacher: [...] If I only have the expression [*writes* $x \times 11 + 1$], Nora, and it stands for the total costs, does it always result in 3.20?
 4.137 Nora: Um, no, because the x , it can be replaced by any arbitrary number.
 4.138 Teacher: Okay, very nice. And here, what is it like [*points at equation* $x \times 11 + 1 = 3.20$]? What does x stand for, here?
 4.139 Nora: Um, what [what do you mean]?
 4.140 Teacher: Okay, does this x still stand for all numbers, somehow arbitrary numbers, like you have said before?
 4.141 Nora: Yes? [*hesitant, asking*].

Afterwards the teacher checks again if Nora really believes that x in the equation $x \times 11 + 1 = 3.20$ still represents any arbitrary number, and Nora again confirms this. It seems that Nora struggles to give the letter x a different meaning than that of an arbitrary number, even though she has profound experiences with tasks on disclosing unknowns from her previous classroom experiences and from the task just solved. For example, she found one concrete number that solves the equation, interpreted this number correctly as the driving price per minute, and, furthermore, in Turn 4.137 explicitly stated that the expression $x \times 11 + 1$ does not always equal 3.20. Hence, for Nora it is not at all obvious that letters need to be reinterpreted in certain algebraic situations.

A possible explanation for Nora's struggle to reinterpret the letter is that she acts like the students observed by Bardini et al. (2005) and understands it as a temporal generalizer that eventually needs to be disclosed (e.g., by transforming the equation to get a solution). Hence, for Nora the determination of an unknown number is part of the *epistemic purpose* that she connects with letters that represent any arbitrary number. If this explanation of Nora's struggle is accurate, it shows that she could profoundly profit from a more explicit comparison of the generalizer

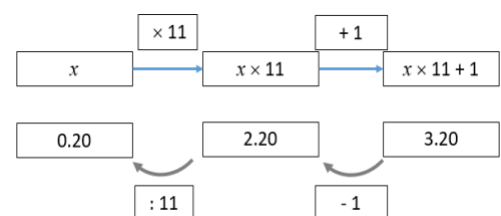


Figure 2: Working backwards for informally solving $x \times 11 + 1 = 3.20$

and the unknown, not only comparing their meanings, but also their purposes, attached to their use in generational and problem solving activities.

Anna's intuitive comparison of the epistemic purposes of generalizer and unknown

In Session 5 of the design experiment series, Nora and Anna are again engaged in informal equation solving activities in which they have to find unknown numbers by undoing, i.e. working backwards in arrow schemes (Figure 2). During a discussion about how the arrow scheme has to be interpreted, Anna provides exactly the comparison of epistemic purposes that Nora needed:

- 5.101 Anna: If I had to describe this by and large, these are two different cases. One is someone who may have to do this calculation several times, because he is somewhat like a salesman or something and has different prices per minute [*x represents the price per minute in the expression $x \times 11+1$*] [...]. Then of course such a calculation [$x \times 11+1$] is a good idea, because one can then simply always insert the *x*, right? So, whether it is 0.1, or 0.2 or 0.3. And the other one is someone who might want to know exactly, uh how - why am I paying 3.20 now? So, he calculates backwards again, looks, uh, what price is coming from where, how is it put together?

In this remarkable statement, Anna describes the upper and the lower part of the arrow scheme as two different cases which represent different meanings of the variable. She explicitly compares the different epistemic purposes of the generalizer and the unknown: For the generalizer, she refers to the calculational ease that comes along with a formula – her purpose (b). Moreover, she enriches it by a meaningful story in which a general expression by means of variable is a “good idea” in itself. Hence, it is quite likely that Anna again uses the variable with the epistemic purpose typical for the generalizer. For the unknown, in contrast, Anna constructs a scenario where someone wants to know a certain unknown quantity. Thus, she explicates the typical epistemic purpose of the unknown.

Discussion and Outlook

In this paper, we investigated which ideas students articulate about variables as generalizers and unknowns before and when being prompted to explicit comparisons. As the presented case study indicates, prompting students to compare the different conceptions of variables seems to be a fruitful approach to explore and develop students' ideas about variables. In the case of Nora, for example, such a comparison-prompt revealed her difficulties in switching from one conception to another. The analysis revealed that Nora's difficulties with confounding meanings might be explained by a misunderstanding located in the purpose that she attributes to the variable as generalizer: Her understanding of the generalizer still used the epistemic purpose of the unknown, namely that variables are only temporally indeterminate and shall be disclosed in the future (Bardini et al., 2005).

From these observations (in the case of Nora and other students in our sample), we infer that the level of comparison should be deepened: *not just the meaning* of generalizer and unknown need to be compared to gain an elaborate and distinctive understanding of these two conceptions, but also the different *underlying epistemic purposes of their use in two different algebraic activities* (what resonates with the epistemological analysis by Usiskin, 1988). The case of Anna showed that some

students intuitively compare the epistemic purposes by themselves. Interestingly, Anna embedded such comparisons in everyday scenarios that made the different algebraic activities and the underlying purposes accessible for her. Since the explication of epistemic purposes proved to enhance the students' deep understanding of the conceptions of variables, we will incorporate this focus of comparison in future design experiment cycles.

In sum, this case study underlines and explicates the diagnosis of Bardini et al. (2005) that students' understanding of the meaning of variable depends on their deep lying ideas about the algebraic activities in which the variable is used, and we add an explicit awareness of the epistemic purpose of this use. Therefore, it seems that a reflection on the purpose of different meanings of variables and, thus, a comparison of different algebraic activities as presented by Usiskin (1988) could also be promising to foster students' understanding of variable.

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