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Is there a number in-between, and if so, how many? Analysis of pre-service primary teachers' knowledge of rational numbers

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Research shows that a considerable number of pre-service primary mathematics teachers enter university with deficient mathematical knowledge, especially on fractions. As part of a design research project that aimed at improving the pre-service education of primary teachers (Grades 1–6), a normative question and a descriptive question arose: What should pre-service mathematics teachers know about rational numbers? And what do they know after completing an arithmetic course at university? The normative question was answered based on mathematical knowledge for teaching models. To answer the descriptive question, a test instrument was designed and administered to 123 pre-service teachers. The results show pre-service teachers' well-developed didactic knowledge, but difficulties in taking a meta-perspective on their knowledge. Some failed to catch up on school knowledge gaps. The results are discussed regarding their theoretical and practical implications.

Keywords: Pre-service teacher education, mathematical knowledge for teaching, school-related content knowledge, teaching fractions, natural number bias.

Motivation: A fractional math problem and pre-service teachers' solutions

Consider the numbers $\frac{1}{9}$ and $\frac{1}{8}$.

- a) Is it possible to find a rational number that is between these two numbers? If so, please provide this number.
- b) Could there be more than one number? If so, include any of the numbers that fall between these two numbers.

This problem might be posed to Grade 6 students. Pre-service teachers, as a matter of course, should be able to solve it correctly. — This is not as self-evident as it first appears. Equivalent tasks were given to all pre-service teachers that were enrolled in the 2020 ($N = 111$) and 2021 ($N = 84$) arithmetic courses in their first year of bachelor degree studies. They had to complete a test on the fractional computing ability (Stampfer et al., 2019) before starting the topic “rational numbers.” The first task was as follows: a) “Enter a number between $\frac{12}{4}$ and $\frac{13}{4}$. If you think there is no such number, click impossible.” In total, 32.4% of the pre-service teachers of the 2021 course stated “impossible,” another 8.1% stated an incorrect intermediate number. The results of a 2020 survey were similar: 33.3% stated “impossible,” and even 28.6% an incorrect intermediate. The latter result can be explained by the fact that this time there were fractions given with the same numerator and different denominators ($\frac{6}{18}$ and $\frac{6}{19}$), whereby it was apparently more difficult for the pre-service teachers to calculate an intermediate number. The second task was as follows: b) “How many numbers are there between $\frac{10}{4}$ and $\frac{10}{8}$?” It is distressing that 69.4% of the pre-service teachers of the 2021 course gave a wrong solution. Particularly often (21.6%) the solutions 4, 3 or 2 were provided. This

can be interpreted as an indicator of a natural number bias (Stampfer et al., 2019; van Hoof et al., 2015). Presumably, the participants simply counted how many fractions there were with the same numerator 10, and various denominators between 4 and 8. Again, the 2020 survey results were similar with 58.3% of wrong answers. These incorrect answers, and the underlying misconceptions, remained persistent even after six weeks of intensive study of rational numbers and the didactics of fractions. Concretely, in the 2021 post-test with similar tasks, still 21.6% of the pre-service teachers (compared to 32.4% in the pre-test) answered “impossible” for task a), and 24.3% for task b) (compared to 69.4% in the pre-test) gave incorrect answers. These results were used as a challenging starting point for a design research project which aimed to improve, among others, the arithmetic course for pre-service teachers (Grades 1–6), who will be teaching fractions in their future classes. Many of them, as our data showed, enter university with rather poor mathematical school knowledge, and more than a fifth of the total number of pre-service teachers do not manage to catch up on their knowledge gaps concerning fractions during their studies, as we found out with several further tests.

In this paper, we will first investigate what knowledge pre-service teachers should acquire, especially concerning fractions. This analysis formed the basis for the design of new university courses, in particular the arithmetic course mentioned above. The second part of the paper deals with the effectiveness of the newly designed university courses with respect to the first goal.

Theoretical background: What should teachers know?

Broadly speaking, mathematics teachers should know the subject they teach and they should know how to help learners to acquire that same knowledge in the setting of school lessons. When describing the content knowledge teachers need to carry out their work, Shulman (1986) discerned between *subject matter knowledge* (SMK) and *pedagogical content knowledge* (PCK). In the course of several decades of research on teachers’ knowledge, Shulman’s foundational work was differentiated in various directions. Ball et al. (2008) developed a framework of *mathematical knowledge for teaching* (MKT), distinguishing, inter alia, two subdomains within PCK, *knowledge of content and students* and *knowledge of content and teaching*. Furthermore, they recognized a type of SMK which was exclusive to teachers, namely *specialized content knowledge* (SCK). Another approach to define the content knowledge specific for the mathematics teacher profession is Dreher’s et al. (2018) model of *school-related content knowledge* (SRCK). This model is based on the fundamental distinction between academic and school mathematics¹, and describes the knowledge teachers need about the interrelations between the two in top-down and in bottom-up directions, as well as knowledge of the curricular structure and its legitimation. The latter includes knowledge about fundamental ideas of mathematics, and the significance of certain topics regarding these ideas (e.g., Why are fractions taught at school?) (Dreher et al., 2018). When we deal with content knowledge in this paper, then with the focus on SRCK. In the area of PCK or, to use the term common in most parts of Europe, *subject didactic knowledge*, we are particularly interested in pedagogical content knowledge that derives from mathematics itself (see Carrillo-Yañez et al., 2018). Thereby, we will follow the distinction of a learning and a teaching perspective (Ball et al., 2008; Carrillo-Yañez et al., 2018).

¹ ... rather than that between common and specialized content knowledge as defined by Ball et al. (2008).

We exemplify the different relevant types of knowledge using the initial example. Starting with the *teaching* perspective of PCK, pre-service teachers need to acquire the knowledge of how to explain mathematical content to students, for instance a strategy to find or calculate an intermediate number. And not only one strategy, but several strategies, as their students might suggest different approaches, and they should be able to support each of them. Concerning the *learning* perspective of PCK, they need knowledge about students' typical errors, misconceptions, and difficulties in understanding in order to assist them in their learning. As to our example, a typical misconception might be that there is no (natural) number between 8 and 9, and subsequently, there is also no number between $\frac{1}{9}$ and $\frac{1}{8}$. An algorithmic hurdle could be that students come up with the strategy of expanding to the main denominator, but still cannot find another fraction between $\frac{8}{72}$ and $\frac{9}{72}$. Or students might find a few fractions between $\frac{1}{8}$ and $\frac{1}{9}$, but have no idea of how to provide all the numbers.

In order to explain the task correctly, when considering different levels of students' knowledge, pre-service teachers need to understand the mathematics involved deeper than their students. What mathematical content knowledge exactly, above mere school knowledge, should they possess? With respect to our example, we would argue that they need to know specific properties of rational numbers as distinct from those of integers, in particular the properties of order and density (Freudenthal, 2002; Padberg et al., 1995). For instance, a rational number, unlike an integer, does not have a clearly defined successor and one can always specify an intermediate between any two rational numbers (i.e., arithmetic mean). Pre-service teachers also need to know how rational numbers are mathematically constructed and defined as equivalence classes of pairs of integers (Padberg et al., 1995). This knowledge would allow them to flexibly shift between representatives within equivalence classes. This is academic mathematical knowledge, and pre-service teachers will not teach fractions in this way to their students. Sometimes, as we experience, pre-service teachers use this as an argument why they do not need university mathematics. We, however, consider important for them to possess this knowledge and we regard it to be close to or even coincide with what was described as SRCK above.

The idea of using equivalence relations to construct new number systems can be regarded as a fundamental idea or "big idea" (Kuntze et al., 2011) of mathematics, and the knowledge about fundamental ideas is part of SRCK (Dreher et al., 2018). Pre-service teachers should have answers to the above mentioned question "Why are fractions taught at school?" (Dreher et al., 2018) or more specifically, "Why is it interesting to know whether there is a number between two fractions and, if so, how many?", as the reasoning on the legitimation of curricular structure and content belongs to the SRCK.

Constructing new number systems as extensions of natural numbers by preserving some number properties and altering others is a typical mathematical procedure. We consider the knowledge about typical procedures, strategies for solving problems and paths of generating mathematical knowledge as an important (meta-)knowledge for teachers and would also assign it to SRCK. In a similar way, Carrillo-Yañez et al. (2018) distinguished the *knowledge of practices in mathematics* as an essential subdomain of teachers' mathematical knowledge in their *mathematics teacher's specialized knowledge* (MTSK) model. In order to convey an adequate image of mathematics, pre-service teachers should *do* mathematics themselves, and experience mathematics as a science. Such expectations are aligned with the demands paced on modern school mathematics: In the German

educational standards for mathematics education (Ständige Konferenz der Kultusminister der Länder in der Bundesrepublik Deutschland, 2005), great importance is given to process-related competencies, such as communicating, arguing, problem-solving and modeling. In this sense, not only mathematics lessons, but also mathematics teacher education needs to meet these general educational requirements. Doing mathematics is an integral part of human culture, and aspiring mathematics teachers should embody that culture.

Different conceptualizations of MKT (Ball et al., 2008; Carrillo-Yañez et al., 2018; Dreher et al., 2018) together with the above-mentioned considerations, led to a completely new design of courses for pre-service teacher education at the primary level which connect subject content and subject didactics at the curricular level (Mayer et al., 2018). The intended facets of MKT were mapped into specific design principles for these courses (Reitz-Koncebovski et al., 2020), namely *follow fundamental ideas of mathematics*, *connect fundamental ideas to main principles of mathematics didactics*, and *experience the process nature of mathematics*. Moreover, the principles of mathematics didactics are not only being taught, but are also implemented while teaching following the idea of a “pedagogical biplane” (Wahl, 2013), and by *making learning processes of students experienceable*. Lastly, a *cross-sectional principle* demands the design principles as well as the processes of teaching be made explicit and reflected with pre-service teachers *on a meta-level* (Reitz-Koncebovski et al., 2020). At the end of each course, we research to what extent this has been achieved. Especially for the arithmetic course, we were interested in the following research questions: To what extent have the pre-service teachers acquired the desired knowledge of rational numbers? What understanding (and what misconceptions) can we identify at the end of the two-semester arithmetic course?

Test instrument and methods of analysis: What do pre-service teachers know?

To evaluate the effectiveness of the arithmetic course and to answer the descriptive research questions just mentioned, we developed a knowledge test. This included, among others, an item on pre-service teachers’ knowledge of the density of rational numbers in the real numbers which was an attempt to operationalize the aspired content knowledge of fractions with its different facets as was described earlier in the paper. The item comprised four subitems with respect to our introductory math problem (see above):

- (1) Solve the problem parts a) and b) in the way you expect students in Grade 6 might solve it.
- (2) Now give an explanation for a student how the problem can be solved correctly.
- (3) What difficulties may arise when a student is working on the problem?
- (4) Name two essential mathematical ideas that you as a teacher need to understand in order to use the problem in class.

Subitem (1) concerns mathematical school knowledge, the other subitems relate to different facets of MKT: (2) PCK in the teaching perspective, (3) PCK in the learning perspective, and (4) SRCK in the form of knowledge about the interrelations between school mathematics and academic mathematics in bottom-up direction: Starting point is a school mathematics problem (1), and the last subitem focuses on essential or “fundamental” mathematical ideas, in particular on the academic mathematics

behind it. Both mathematical content and didactic knowledge are necessary for answering the test item. These were discussed in detail in the course, so the task appeared to be reasonable.

The knowledge test was carried out at the end of the two-semester arithmetic course as part of the final examination. The written answers to the exemplary item of all 123 pre-service primary teachers enrolled in the course were analyzed using the *qualitative content analysis* method (Kuckartz, 2019; Mayring, 2014). For it, a deductive-inductive category system was employed. Main categories were developed deductively (i.e., concept-driven) from a sample solution to the task which was derived from the desired teacher knowledge presented in the theoretical part. The inductive categories emerged from the data (i.e., pre-service teachers' answers) building a basis for subcategories. For an easier understanding, more details about the categories are reported below, together with the results. We created a coding guide which was supplemented by definitions and anchor examples, and used it for the coding of the entire written material. This approach made it possible to analyze the pre-service teachers' answers in great detail and to make visible different, often unpredictable, expressions of knowledge and understanding. In a subsequent step, a quantitative analysis was carried out using descriptive statistics to determine the frequency of occurrence of certain categories or subcategories (Kuckartz, 2019) as well as of correct and incorrect, complete or missing answers per participant.

Results

Table 1 shows that between 15% and 27% of the pre-service teachers did not give an answer to one or more subitems. Only very few (4.7%) answered all subitems correctly. The first subitem (1a), which requested school knowledge alone, was answered incorrectly, inadequately, or not at all by nearly 40% of the participants. The results to subitem (2) relating to teaching as a facet of PCK as well as subitem (4) relating to SRCK were particularly weak: nearly half of the explanations for Grade 6 students (2) were incorrect or missing, and less than 20% of the test participants were able to name adequate essential mathematical ideas.

Table 1: Rough quantitative evaluation of the test item on rational numbers (total N = 123)

Item with subitems	Completely correct answer	Incomplete or partially correct answer	Incorrect answer	No answer
(1a) „Is there a number between ...“	61.8%	1.6%	21.1%	15.5%
(1b) „are there more ... how many“	43.9%	26.0%	6.5%	23.6%
(2) Explain to a student	15.4%	36.6%	21.1%	26.9%
(3) Difficulties for a student	47.2%	27.6%	6.5%	18.7%
(4) Essential mathematical ideas	18.7%	56.9%	4.9%	19.5%

Qualitative content analysis allowed a more in-depth investigation of pre-service teachers' answers. In subitem (1a), the pre-service teachers chose different strategies to specify a number between $\frac{1}{9}$ and $\frac{1}{8}$, namely determining the arithmetic mean, expanding the fractions to the same denominator or the same numerator (possibly several times), the “wrong addition” (divide the sum of the numerators by the sum of the denominators) or converting into decimal numbers. Almost a quarter of the test participants chose the latter way. This preference of converting to decimal numbers was

also seen in pre-service teachers' explanations for subitem (2), whereby most of them did not explain the procedure in detail. In general, a lack of accuracy and concreteness was the greatest shortcoming of many of the explanations for subitem (2). In addition to referring to decimal numbers, there was often a reference to a visual model (i.e., circle, rectangle, number line), but without an explanation of how exactly this was supposed to help. The didactically impeccable explanations (15.4%) used the strategies of arithmetic mean or expansion, and described in detail how this can be done several times in order to find more and more numbers in-between.

Subitem (3) that focused on possible students' difficulties tended to be answered much better than subitem (2). Many test participants considered problem part a) and mentioned difficulties at the procedural level (e.g., expanding and shortening, calculating the arithmetic mean, converting to a decimal number), but also typical misconceptions about the order of fractions (e.g., no number between $\frac{1}{9}$ and $\frac{1}{8}$, as there is no natural number between 8 and 9, or $\frac{1}{8}$ is smaller than $\frac{1}{9}$) were mentioned by 51.2% of the participants. Only 7.3% of them mentioned misconceptions concerning the density of fractions or lack of understanding of the concept of infinity required in part b).

Less than 20% of the test participants succeeded in naming essential mathematical ideas in subitem (4). Test participants could get partial credits for this subitem by naming suitable keywords. Most of those keywords were quite unspecific like "knowledge on basic ideas about fractions" (66.7%), without specifying which basic ideas, "general understanding of fractions" (37.4%) or "knowledge of number systems" (13.8%). Here again many test participants referred to procedural knowledge (60.2%). The intended answers regarding knowledge on the equivalence of fractions (7.3%), understanding of the order (10.6%), density (4.0%) or infinity of rational numbers (5.7%) were mentioned to a limited extent. Moreover, some test participants (4.9%) described knowledge requirements of the students, and not those of the teachers.

Discussion and Conclusion

In our study, we investigated what kind of knowledge of rational numbers the pre-service teachers had acquired at the end of the two-semester arithmetic course. Thereby, we focused on mathematical content knowledge at school and at university level, in particular SRCK, as well as subject didactic knowledge (i.e., PCK). Two major difficulties faced by the pre-service teachers stood out:

The first major difficulty concerns mathematical school knowledge. Obviously, a sizable group of pre-service teachers have the same or similar problems with the task as school students and thus lack basic subject matter knowledge (Dreher et al., 2018; Shulman, 1986). This can be derived from the incorrect and missing answers to subitem (1), incorrect explanations for students in subitem (2), and also from the results of the arithmetic test reported in the beginning. Hence, the pre-service teachers obviously need to catch up on school knowledge and skills in fractions. The arithmetic course alone seems not to suffice to compensate for their deficits. We might also ask: What are the causes of these deficits? Is it pre-service teachers' lack of motivation for their mathematics studies in general? Have there been obstacles as a result of mathematics anxiety since school? For pre-service teachers in the Inclusive Pedagogy Teacher Education Program, mathematics is a compulsory subject, and some of them have reported about own mathematics anxiety. Or do they perceive the mathematics university course having low relevance for their future professional work, which resulted in poor performance?

The second major difficulty concerns perspective adoption. As to different perspectives of the PCK considered in theory (Ball et al., 2008; Carrillo-Yañez et al., 2018), it seems to be easier for pre-service teachers to adopt the learning perspective addressed in subitem (3) than the teaching perspective addressed in subitem (2). This may be due to the fact that they are only in the second semester of their bachelor's degree. The results showed that it was particularly difficult for the pre-service teachers to adopt a meta-perspective, as was required in subitem (4). This finding is consistent with those by Kuntze et al. (2011) who reported that the pre-service teachers in their study often were unable to link mathematical content to the big ideas of mathematics. In view of this difficulty, the design principles for our course (Reitz-Koncebovski et al., 2020) require the lecturers to occasionally adopt the meta-level, and to make explicit what “school-related content knowledge” means in certain contexts, as outlined in the theory chapter above. Consequently, the lecturers explicitly motivated their audience to take a meta-perspective themselves. The participants' poor results regarding the latter aspect raise a number of questions: Why was subitem (4) so difficult? Is it generally difficult to adopt a meta-perspective? Do pre-service teachers lack a deeper knowledge about rational numbers (Freudenthal, 2002; Padberg et al., 1995) which we consider an essential part of SRCK? Or do they actually possess this knowledge, but rather have difficulties in articulating it in a written form?

A critical view of the test results implies further development of the course, in particular regarding the implementation of the *cross-sectional design principle* (Reitz-Koncebovski et al., 2020), namely adopting the meta-level and making explicit what SRCK (Dreher et al., 2018) means in certain contexts, and why this knowledge is important and useful for pre-service teachers. Our assertion is that it is not only necessary to allow pre-service teachers to question the usefulness of the content of the mathematics courses at university, but also to explicitly discuss their usefulness in seminars, and to a greater extent than has been done in the past. Some arguments for this kind of discourse with pre-service teachers can be drawn from the lines of argument in the theoretical considerations above. Another, perhaps even more difficult endeavor, is to address the level of attitudes and emotions. We assume that some pre-service teachers' poor performance on fraction tasks was related to specific aspects of the affective domain since the MTSK model of mathematics teacher knowledge (Carrillo-Yañez et al., 2018) places beliefs at the center of the various facets of knowledge.

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References

- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389–407. <https://doi.org/10.1177/0022487108324554>
- Carrillo-Yañez, J., Climent, N., Montes, M., Contreras, L. C., Flores-Medrano, E., Escudero-Ávila, D., Vasco, D., Rojas, N., Flores, P., Aguilar-González, Á., Ribeiro, M., & Muñoz-Catalán, M. C. (2018). The mathematics teacher's specialised knowledge (MTSK) model*. *Research in Mathematics Education*, 20(3), 236–253. <https://doi.org/10.1080/14794802.2018.1479981>

- Dreher, A., Lindmeier, A., Heinze, A., & Niemand, C. (2018). What kind of content knowledge do secondary mathematics teachers need? *Journal für Mathematik-Didaktik*, 39(2), 319–341. <https://doi.org/10.1007/s13138-018-0127-2>
- Freudenthal, H. (1986). *Didactical phenomenology of mathematical structures*. Springer.
- Kuckartz, U. (2019). Qualitative text analysis: A systematic approach. In G. Kaiser & N. Presmeg (Eds.), *Compendium for early career researchers in mathematics education* (pp. 181–197). Springer International Publishing. https://doi.org/10.1007/978-3-030-15636-7_8
- Kuntze, S., Lerman, S., Murphy, B., Kurz-Milcke, E., Siller, H.-S., & Winbourne, P. (2011). Professional knowledge related to big ideas in mathematics. An empirical study with pre-service teachers. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp. 2717–2726). University of Rzeszów.
- Mayer, J., Ziepprecht, K., & Meier, M. (2018). Vernetzung fachlicher, fachdidaktischer und bildungswissenschaftlicher Studienelemente in der Lehrerbildung. In M. Meier, K. Ziepprecht, & J. Mayer (Eds.), *Lehrerausbildung in vernetzten Lernumgebungen* (pp. 9–20). Waxmann.
- Mayring, P. (2014). *Qualitative content analysis: Theoretical foundation, basic procedures and software solution*. Social Science Open Access Repository. <https://nbn-resolving.org/urn:nbn:de:0168-ssoar-395173>
- Padberg, F., Danckwerts, R., & Stein, M. (1995). *Zahlbereiche*. Springer Spektrum.
- Reitz-Koncebovski, K., Hermanns, J., Kortenkamp, U., & Kuzle, A. (2020). Qualitätsoffensive Lehrerbildung an der Universität Potsdam. Projekt SPIES zur Professionalisierung der Lehrerbildung Mathematik. *Mitteilungen der Gesellschaft für Didaktik der Mathematik*, 46(109), 26–30.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4–14. <https://doi.org/10.3102/0013189X015002004>
- Stampfer, F., Reitz-Koncebovski, K., & Hell, T. (2019). Feststellung und Entwicklung des Natural Number Bias bei Lehramtsstudierenden in der fachdidaktischen Ausbildung. In A. Frank, S. Krauss, & K. Binder (Eds.), *Beiträge zum Mathematikunterricht 2019* (Vol. 2, pp. 781–784). WTM-Verlag.
- Ständige Konferenz der Kultusminister der Länder in der Bundesrepublik Deutschland. (2005). *Bildungsstandards im Fach Mathematik für den Primarbereich*. Luchterhand.
- van Hoof, J., Verschaffel, L., & van Dooren, W. (2015). Inappropriately applying natural number properties in rational number tasks: characterizing the development of the natural number bias through primary and secondary education. *Educational Studies in Mathematics*, 90(1), 39–56. <https://doi.org/10.1007/s10649-015-9613-3>
- Wahl, D. (2013). *Lernumgebungen erfolgreich gestalten* (3rd ed.). Klinkhardt.