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# Analysis of children's generalisations with a focus on patterns and with a focus on structures

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*Generalisation processes belong to the core of algebraic thinking. The development of these processes begins as early as the primary school classroom, when children describe and reason about mathematical patterns. Our theoretical framework includes a distinction between patterns and structures. The focus on structures is important for children to engage in reasoning about patterns rather than just describing them on the surface. In an exploratory study with 45 third and fourth graders, we use an epistemological perspective to examine how this distinction is reflected in children's generalisations. The results create awareness of the algebraic quality of justifications.*

*Keywords: Early algebra, generalisation, description, explanation.*

## Introduction

Algebraic thinking can evolve around a wide range of mathematical topics. Blanton's research group addresses equations, generalised arithmetic, and function (Blanton et al., 2019). Twohill (2020) collapses the first two into one branch. In line with various researchers, we regard arithmetic as a starting point and interpret arithmetic tasks as potentially algebraic (Malara & Navarra, 2003; Britt & Irwin, 2008; Steinweg, 2017). This potential can emerge as early as arithmetic lessons in primary school if the underlying mathematical structure is focused on. Kieran et al. (2017) indicate structure as "one of the central pillars in the development of early algebraic thinking" (p. 423). The relation between structure and pattern, therefore, is essential and focussed on in the first part of the theory.

Structural thinking in the outlined sense expects generalisations which can be characterised as "seeing the general in the particular" (Mason & Pimm, 1984). The process of generalisation includes a certain shift of attention (Mason, 1989) from calculating given examples to thinking about the objects and their properties and relations (Mason et al, 2009). This paper provides a suggestion for a more in-depth analysis of generalisation processes and gives insight into a pilot study.

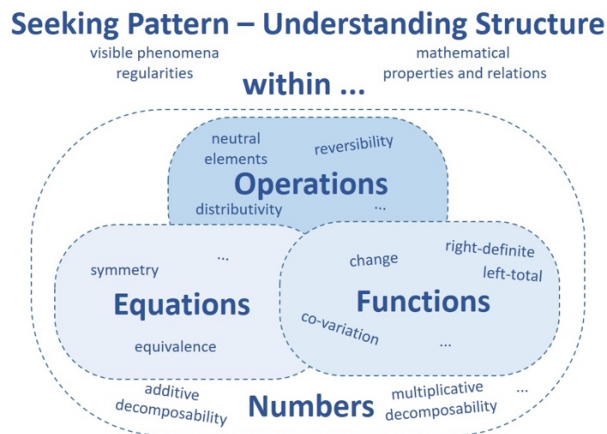
## Theoretical framework

First, we provide an overview of key ideas of algebra arising from arithmetic and thereby clarify the difference we assume between mathematical structures and patterns. Taking a closer look at generalisations, this distinction leads to two categories of generalisations. Finally, the research questions of the study presented here are outlined.

Our research is embedded in a specific perspective on the relationship between *patterns* and *structures* (Steinweg et al., 2018). In line with Schifter (2018) and Mason et al. (2009) *structure* is defined by the mathematical properties and relations. In primary school arithmetic four realms of objects with specific properties can be identified and ordered in key ideas. The earlier used key ideas of algebraic thinking (Steinweg, 2017; Steinweg et al., 2018) have been fundamentally revised and

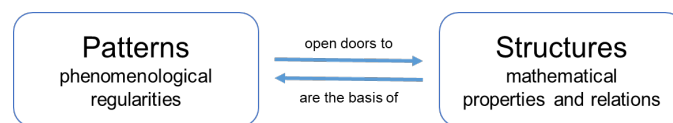
enhanced based on theoretical and empirical findings. We consider properties and relations of numbers, operations, equations, and functions as key ideas (Figure 1).

Each key idea holds very specific mathematical structure to be explored, even though the realms are interwoven, e.g., equations require operations and numbers. The properties that are explored in one key idea differ from the ones that are focused on and explored in another. The focus of attention enables different generalisations of specific structural aspects. For example, working on the key idea of operations requires specially tailored tasks and questions that allow to recognise commutativity, distributivity, et cetera.



**Figure 1: Key ideas of algebraic thinking**

The mathematical cores of the objects associated with the key ideas are the abstract *structures*. Understanding structures and reasoning with these properties requires advanced relational or structural knowledge (Schwarzkopf et al., 2018) in the sense of profound background information. Access to mathematical structures is not easily gained and should be mediated. *Patterns* can literally function as mediators and thus as door openers to the respective mathematical background. The abstract mathematical structure can be made accessible and particularly visible in patterns and, vice versa, the mathematical structure reveals itself in patterns (Figure 2).



**Figure 2: The distinction and relation of patterns and structures**

*Patterns* offer perceptible regularities in visible phenomena (Schwarzkopf et al., 2018; Mulligan & Mitchelmore, 2009). Seeking for pattern is the first step to find the door behind the scenes. Pattern as representations hold information about the underlying structure (Küchemann, 2020). The visibility of pattern can be realised in various representations, i.e., tables, graphs, number sentences, and so forth.

Pattern tasks that aim for understanding structure allow for four different activities summarized as ReCoDE (Recognise-Continue-Describe-Explain) (Steinweg, 2019). Descriptions and explanations of a certain pattern require expressions, which can be verbal or symbolic. Even when the use of variables is not available or not yet developed, research documents various effective linguistic means such as referring to examples, phrasing conditional sentences, quasi-variables (Akinwunmi, 2012; Bastable & Schifter, 2008; Cooper & Warren, 2011; Schliemann et al., 2007).

Children's reactions to given patterns may include both descriptive and explanatory parts. More importantly, *generalisations* may occur in both parts. Strømskag (2011) provides theoretical foundation for regarding algebra as generalisation of patterns. Blanton et al. (2019) differentiate between four types of thinking: generalise, represent generalisations, justify generalisations, reason with generalisations. Although this distinction enables teachers to focus on different aspects of generalisations activities, these four aspects are not disjunctive. For example, representations may support justification, or justification may include reasoning with the newly gained objects (properties, relations). Mason et al. (2009) identifies a disposition to think structural and to integrate the structural ideas into one's mathematical thinking. Therefore, there is an opportunity to open the doors to the abstract mathematics behind a pattern, to realise structures by exploring patterns. "Generalization of a phenomenon involves the analysis of visible instances of the phenomenon, and the application of conclusions to cases that are not observable" (Twohill, 2018, p. 216). The step through the pattern door into the mathematical realm –towards mathematical thinking and away from given examples– does not emerge automatically.

In our view, *generalisations* in *descriptions* generally focus on the visible regularities, i.e., the patterns. *Explanations* provide the possibility to focus on the structure behind a pattern. However, this possibility is not automatically perceived by every child. Explanations justify certain regularities and may refer either to patterns or to structures.

Our research project addresses qualitative analysis of children's responses, considering the outlined distinction between patterns and structures. The analysis categorises:

- *generalisations* which focus on the *pattern*, i.e., on the visible regularities, and
- *generalisations* which focus on the *structure*, i.e., on the mathematical properties or relations.

Furthermore, the *means of generalisations* in both categories are studied on. The processes of generalisation –starting from local findings and combining them into more general arguments– appear to be linguistically similar at first. Some national studies document that children –without further impulses of the teacher– typically recognize regularities only on the surface of the given patterns and justify their findings with empirical knowledge without looking beyond the pattern (Häsel-Weide, 2016; Link, 2012). However, a closer look at the two categories reveals that the objects of generalisation, and thus the awareness, differs substantially. Accordingly, the distinction between pattern and structure as two kinds of context of reference (Steinbring, 2005) may shed light on different –if any– forms of generalisations.

Finally, the in-depth analysis can be beneficial to the classroom interaction and provide adequate and helpful ideas to facilitate approaches to mathematical structures. Supporting children's development of algebraic thinking depends on deliberate attention of the teachers. The teachers need awareness of the given references in explanations and should invoke appropriate impulses and further questions.

According to this research desideratum described above, we addressed the following research questions:

- Which focus of generalisation can be identified in children's explanations in pattern tasks?
- Which impulses do support the development of generalisations that focus on structure?

## Methodology and Design

In this paper we present results from an interview study with 45 participating 3<sup>rd</sup> to 4<sup>th</sup> grade primary school children. The one-on-one interviews are conducted in separated rooms at the school, video-taped and transcribed. The content focuses on patterns based on the properties of operations.

In the first part of the interviews the participants are confronted with patterns in symbolic representation (e.g., left side of Figure 3; task type A). The arithmetic context is deliberately no challenge to calculate for 3<sup>rd</sup> and 4<sup>th</sup> graders, as the study focuses on generalisations. These kind of pattern tasks are common in German textbooks. Children are, therefore, used to the typical classroom questions to continue and describe the pattern.

In the second part, a special impulse is designed: A picture showing a child using counters in a common material (field of 20) to solve the second task of the given pattern (the right side of Figure 3 shows a simplified version of the picture which is presented to the pupils; task type B). Counters are on the table throughout the interview. The aim is that the change of representation from symbolic to iconic stimulates the children to change the focus of generalisations.

Task type A	Task type B	
18 - 5 = ___	9 - 4 =	
17 - 4 = ___	10 - 5 =	
16 - 3 = ___	11 - 6 =	
15 - 2 = ___	12 - 7 =	
_____		

**Figure 3: Task sheets on exploring the constant difference**

The qualitative analysis of the data follows the interaction analysis according to Steinbring (2005). Following this method in children's interaction and verbal responses mediation processes between the given *signs/symbols* to be interpreted and the *reference contexts* that the children use for interpretation are identified. The revealed mediation in turn provides information about the children's so called epistemological basis of mathematical knowledge, i.e., the *concepts*. The triadic relationship, usually represented in a triangle (cf. Fig. 4–6), thus allows the re-construction of the child's focus in the generalisation process in order to answer the raised research question.

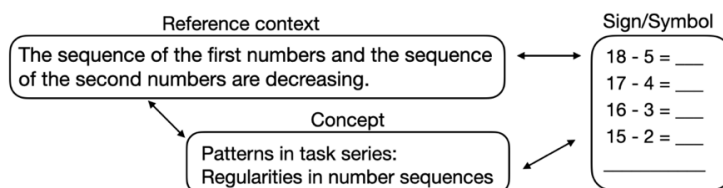
## Results and Discussion

Due to place restrictions, only a summary of the overall findings of the qualitative analysis of all 45 cases can be given at the end of this section. The in-depth analysis is exemplarily set out in two cases, Katie and Nicole. These cases are chosen because they exemplify two substantially different arguments used in the generalisation processes.

Katie is asked to explain the pattern (task type A) and then to justify the constant result. Like all the participating children in our study, Katie generalises the symbolic patterns in the first part of the interview by describing the variation of the numbers separately.

Katie: Here, the numbers are getting smaller and here they are also getting smaller. [...] For plus [addition tasks] these would have to get bigger. For minus [subtraction tasks] it is also getting smaller, because otherwise it wouldn't work.

Katie refers to the given numbers as if minuends and subtrahends were separate sequences [‘here’ and ‘here’]. In the further comment, she describes the increase and decrease for all addition or subtraction tasks in a more general term. The use of the subjunctive indicates the process of abstraction away from given (numerical) examples towards general statements. Katie gives evidence to her generalisation referring to (classroom) pattern experiences and, therefore, empirical knowledge (Schwarzkopf et al., 2018). Hence, her justification does not yet include a structural argument. Using the epistemological triangle (Steinbring, 2005), Katie’s generalisation process, her interpretation of the signs mediated through a reference context, can be expressed as follows (Figure 4):

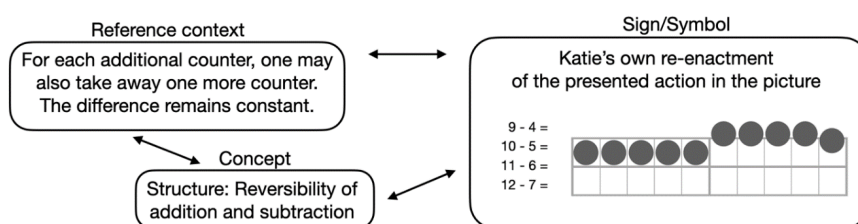


**Figure 4: Katie’s interpretation of the task series**

Given the iconic impulse in the second part (task type B), Katie immediately decides to reconstruct the tasks using counters and lines up 9 counters in front of her.

Katie: So, 9 minus 4. 1, 2, 3, 4 (counts aloud while removing 4 counters successively) is 5. Now we have one more here (takes another counter from the material box and fills the line up to 10 counters), because now it’s 10 (points to the task sheet), but in return we are allowed to take away one more (removes 5 counters). Again it’s 5.  
 (Fills up the line to 10 counters.) Now again we take one more (adds one counter from the material box to the line) and we are allowed to take away one more. 2, 3, 4, 5, 6 (counts aloud while removing 6 counters then regarding the remaining counters). And therefore, the result is always the same.

Working with the counters, Katie’s arguments change fundamentally. It is the change of representation that creates a new interpretive challenge that leads Katie to consider and generalise the effects of her actions. She now focuses on the effect of changing both numbers simultaneously and she reasons, that ‘in return’ to add one more (increasing the minuend), one is allowed to also take away one more (increasing the subtrahend). This expression focuses on the reversibility of addition and subtraction and the context of reference in her generalisation is, therefore, now one of the important properties of those operations. Her reasoning with structural knowledge exceeds the numerical examples and includes a valid argument for an indeterminate ‘we’. She intends not only to convince the interview partner, but points to a general validity. Her conclusion [‘therefore’] holds universally [‘always’]. The difference in arguments in her generalisation process is also visible in the epistemological triangle (Figure 5).

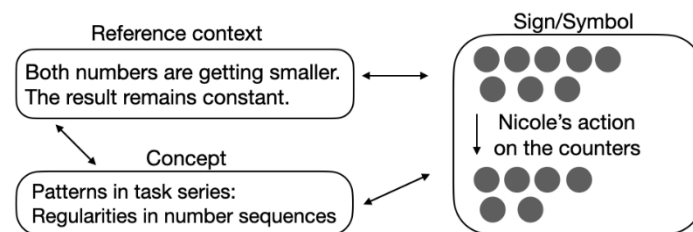


**Figure 5: Katie’s interpretation of her own re-enactment of the action in the picture**

In the second case, Nicole's reaction in the first part of the interview is very similar to Katie's reaction and is therefore not presented here. In the second part (task type B) her reasoning is quite different from Katie's, although she also decides to use counters to explain why the result remains constant.

Nicole: Now if you had five minus three (puts 5 blue counters in a first row and 3 red counters in a second row without one-to-one correspondence), that would be 2 and that is the starting task. And if you then remove this one here (takes away one from the blue counters) and this (takes away one from the red counters), then immediately you see, that both are getting smaller. And for minus it has to be smaller on each side (looks up thoughtfully as she speaks), so that the result stays the same.

The analysis of Nicole's argument shows some generalisations with the focus on patterns. First, she presents a self-selected task  $5 - 3$  that is not part of the task sheet. She describes the regularities of the pattern in a general way using some general expressions ["both" and "each side"]. Nicole is convinced that the simultaneous decrease of the two numbers will always lead to the same results. Her generalisation refers to known empirical facts ['it has to be'] without giving evidence, why this rule holds. In her justification, she does not focus on the properties of the operations. It can neither clearly be said, whether she interprets the space of the missing counters in the second row as the difference of the two amounts, nor does she express the effect (the shift) on the space by decreasing minuend and subtrahend simultaneously. Moreover, she does not mention the reversibility of the two operations. Instead, she refers to her existing empirical knowledge to justify the pattern, as illustrated in Figure 6. The lack of reference to the properties of the operations in the given explanation in this second case gives no discernible indication that Nicole understands the underlying structure that generates the pattern.



**Figure 6: Nicole's interpretation of her own action on the counters**

Through such analyses as exemplified above, insights could be gained into the generalisation processes of the 45 participating children, some of which we will summarize here:

- Both foci of generalisation processes with the focus on patterns and with the focus on structure were identified in the interviews but not in equal proportions during both interview sections (with task type A and B).
- In the first encounter (task type A) none of the children focused on the properties (of the operations) that are the structural basis for the emergence of the pattern. Instead, they used generalisations of patterns by referring to empirical knowledge (classroom experiences or known facts) or they used one discovered regularity in the pattern to justify another (see Nicole's case).

- In the second part of the interview (task type B), the iconic impulse and the counters initiated reasoning processes in which almost all children started to focus on structure, i.e., the properties of operations (see Katie's case as an example).

## Closing Remarks

Working on pattern tasks is an established approach to algebraic thinking in primary school. The distinction between *patterns* as visible regularities and *structures* as abstract properties and relations that generate the patterns provides opportunities for deeper analysis of children's reactions to pattern tasks.

In our view, it is useful to distinguish between these generalisations theoretically as well as empirically. Researchers and even more teachers need to be aware of both kinds of arguments in order to provide appropriate follow-up-questions or stimuli –such as changing representations– to help children access the mathematical background behind patterns.

Teachers who are themselves explicitly aware of structural relationships, who are aware of perceiving situations as instances of properties (rather than as surprising and unique events), are in a position to promote similar awareness in their learners (Mason et al., 2009, p. 29).

To identify children's understanding of structure is a complex endeavour. With the theoretical framework and the methodology presented in this paper we contribute a possible approach to deeper analysis of generalisations in algebraic thinking processes and provide some exemplary insights. In particular, generalisation processes with the focus on structure can be identified as references to properties of mathematical objects.

Generalisations do not necessarily imply or include structural justifications. The context of reference provides information about the concept focused on and the depth of a generalisation. The focus on patterns can pave the way to the underlying structures, i.e., the mathematical background. Sticking to the surface of patterns only and teacher's acceptance (or expectation) of solely generalisations of patterns hinders deeper understanding and ultimately algebraic argumentations.

This study also shed light on the impulses which support the development of generalisations that focus on structure. It became apparent that, paradoxically, symbolic representations of tasks entice children to remain on the surface of the pattern and stick to concrete (numerical) examples, meanwhile iconic or material-based representations provide the opportunity to find more structural arguments. Therefore, representations and processes of representing play a crucial role for generalisations with the focus on structure. However, they are not a sufficient criterion for structural arguments as can be seen in Nicole's case.

The qualitative results of our study are of course limited to first indications but also pave way for further studies. They are only a first step, and further studies on the other key aspects of algebraic thinking (numbers, equalities, and functions) or different age groups could be conducted.



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