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# An optimization process for extension spring design

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## Abstract

This paper presents a calculation process to optimize the design of helical extension springs whatever the nature of the specifications. This process can be used even in the early design stages as data are set with interval values. It includes a large set of constraints, which can express the designer's requirements, but also the standards and the technical capability limits of spring manufacturers. The problem is resolved using the Excel solver. The required starting point is automatically calculated by a dedicated algorithm, which uses interval arithmetic. Two examples are presented.

## Key-words

Extension springs, Optimization, Interval arithmetic, Industrial application.

## Nomenclature

Do : outside diameter in mm

D : mean diameter in mm

Di : inside diameter in mm

d : wire diameter in mm

R : spring rate in N/mm

L0 : free length in mm

n : number of active coils

z : helix angle in degrees

P1, P2 : spring load in N

L1, L2 : spring length in mm

sh : spring travel in mm,  $sh = L1 - L2$

Nc : number of cycles

Rm : Ultimate Tensile Strength of the material in  $N/mm^2$

## Subscripts

S : from the specification sheet

M : from the manufacturer constraints

IST : from the Institute of Spring Technology requirements

## Superscripts

U : Upper limit

L : Lower limit

## Introduction

The design of mechanical devices imposes the dimensioning of numbers of common components (gears, cams, shafts). Helical extension springs are among these components in use today. Literature concerning numerical methods provides several examples of spring optimization processes for custom spring design. Deb and Goyal [1], Kannan and Kramer [2], Sandgren [3] use compression springs to illustrate their optimization methods in mixed variables. These problems are used as mathematical illustrations and have thus been oversimplified. Qian [4] also envisaged an optimization problem where all the operating points are given. The obligation to define completely the operating points limits its practical use. None of these methods can thus be applied directly. The main industrial software available to a designer during the extension spring definition work use exhaustive calculations from standards to make a full check of the proposed spring's compatibility with the given specifications : "FED2" from Hexagon® [5], "Spring CAD Software Packages" from IST® [6], "Spring Design Software" from SMI [7]. "Spring Design Software" can carry out minimum mass optimization to adjust one design parameter but three other design parameters have to be known beforehand. As far as we are aware, the following capability is not commonly presented i.e. to carry out exhaustive calculations from standards and to provide the most suitable spring design from global specifications. This paper proposes an approach to present this capability by linking both industrial and mathematical knowledge. A method is given to determine the optimal extension spring design from a specification sheet where data is set with interval values. Setting data in this way has been tested for stock compression springs by Paredes [8] where it provides a powerful and efficient means of expression. The associated mixed variables optimization problem has been implemented on Excel for easy use in industry as proposed by Anselmetti [9]. The resolution method suggested has the advantage of providing in all circumstances a point pertaining to the solution area but requires a starting point close to the solution area [9,10]. So, an algorithm that automatically determines the calculation starting point has been developed. It uses interval arithmetic [11] to analyse the parameters of the specification sheet. It has been coded in Visual Basic and directly linked to the solver. Finally, two examples are presented. In this study, only stainless steel helical extension springs with machine loops are considered.

# 1 Definitions

## 1.1 Spring design parameters

The parameters which define the extension spring geometry are :  $D_o$ ,  $d$ ,  $D_i$ ,  $R$ ,  $L_0$ ,  $d$ ,  $n$ ,  $P_0$ . Figure 1 illustrates these parameters which characterize the intrinsic properties of the spring. Four independent design parameters have to be known to calculate the others.

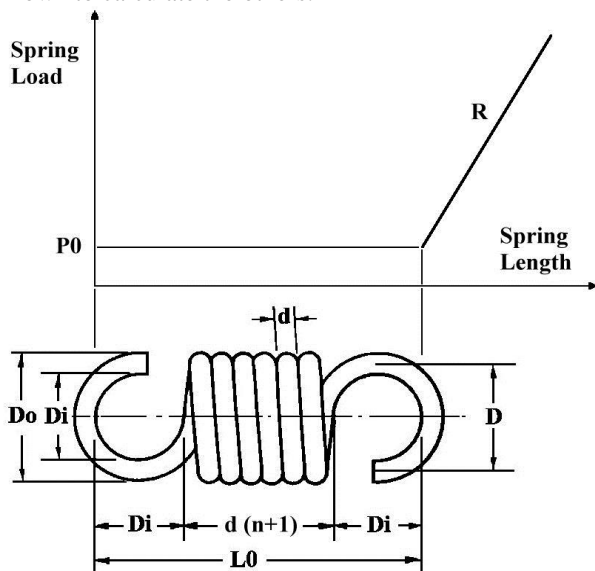


Figure 1 Design parameters

The initial tension ( $P_0$ ) is often defined as a percentage of  $R_m$  ( $T_0$ ), see Appendix 1.

The number of active coils ( $n$ ) can take discrete values. It is defined from  $N$  (integer variable) depending on the angle between end loops as described in Figure 2.

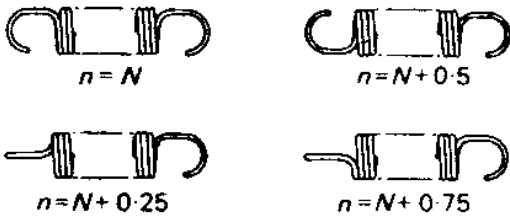


Figure 2 Angles between end loops

## 1.2 Spring operating parameters

A spring works traditionally between two configurations, one corresponding to the least extended state W1, the other corresponding to the most extended state W2. The parameters which define the use of a spring are : P1, P2, L1, L2 and sh (see Figure 3).

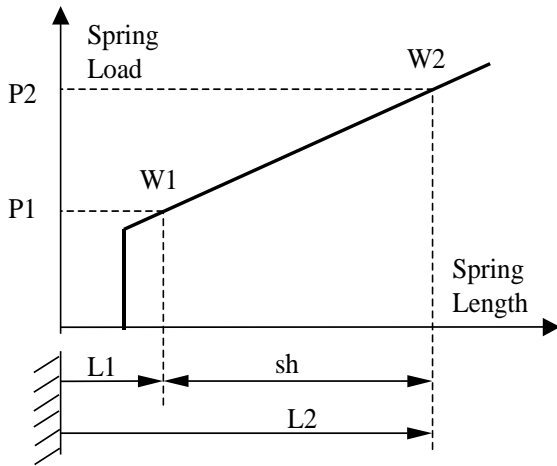


Figure 3 Operating parameters

When the design parameters are known, two independent operating parameters (to be taken among P1, P2, L1, L2 and sh) are necessary to determine the two points W1 and W2.

## 1.3 Specification sheet

The goal here is to find an easy way to define global specifications. For this reason, data is defined with interval values.

Design and operating parameters can be set by giving their bounds (lower and/or upper limits :  $L0_S^L$ ,  $L0_S^U$ , ...  $P1_S^L$ ,  $P1_S^U$  ...).

Moreover, to be as global as possible, many other characteristics to be defined (with interval values) are proposed :

- Natural Frequency of surge waves
- Internal energy during operating travel
- Spring mass
- Overall space taken up when  $L=L0$  (VolL0)
- Overall space taken up when  $L=L2$  (VolL2)

The number of cycles  $N_c$  can also be provided to calculate the fatigue life factor (and to check that it is higher than unity).

The designer can also specify the required angle between end loops. Finally, to be able to calculate the best spring, the objective function F (maximum fatigue life, minimum mass...) has to be given.

To help the designer with his specifications, a sheet from Excel has been used, as shown in Figure 4.

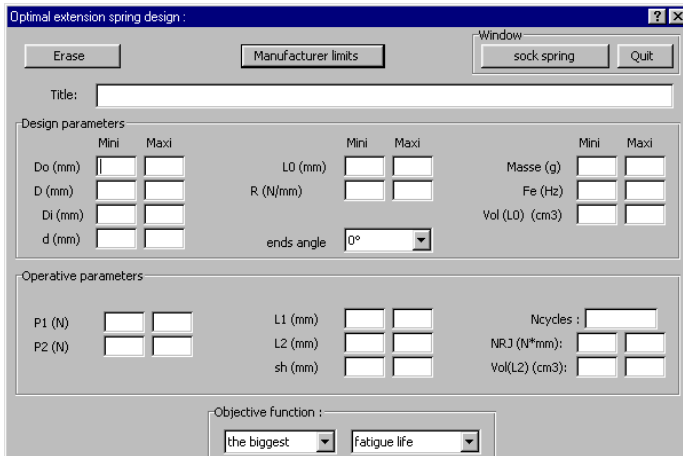


Figure 4 Specification sheet

Of course, real-life industrial problems often imply the provision of only some of the proposed limits.

For example, in the early design stages, there are always many parameters, which have not been already fixed. Given the difficulty of giving fixed values for a spring, it is more convenient to define parameters through their possible lower and upper limits.

On the other hand, to fix the value of a parameter, it must be placed in both the lower and upper limits defining that parameter in the specification sheet (see Figure 4). For example, although a standardized wire range exists, any wire diameter can be manufactured. When using standardized wire, each wire diameter ( $d$ ) considered implies placing  $d_s^L = d_s^U = d$  in the specification sheet.

Finally, each value not defined in the user specification sheet is automatically set to a default value : 0 for a lower limit and  $10^7$  for an upper limit.

Then the optimization problem is constructed automatically.

## 2 The optimization problem

### 2.1 Defining the problem

The discovery of the extension spring design offering the best possible value of the objective function and satisfying the required angle between end loops is a mixed variables optimization problem.

Four design parameters ( $D_o$ ,  $d$ ,  $T_0$ ,  $N$ ) and two operating parameters ( $L_1$  and  $L_2$ ) are stored in the « vector of design variables »  $X$ , so  $X = [D_o, d, T_0, N, L_1, L_2]^T$ . There are 5 continuous variables ( $D_o, d, T_0, L_1, L_2$ ) and one integer variable ( $N$ ).

The objective function is expressed using the conventional form :

Minimize or Maximize  $F(X)$

where  $F(X)$  is either the mass, or the fatigue life, or the maximum operating load  $P_2$ , or the minimum operating length  $L_2$ ...

Table 1 enumerates the large set of constraints that have been considered in addition to this objective. They can express not only the designer's specifications but also the standards and the technical capability limits of the spring manufacturer. All the necessary formulae for coding these equations are given in Appendix 1.

An optimization problem with 6 variables (5 continuous and one integer variable) and a maximum of 44 constraints is thus obtained. It is solved using a Branch and bound process (to find the integer variable  $N$ ) added to a Generalised Reduced Gradient solution process (to find continuous optima), as proposed within the Excel solver.

Two methods are offered : the conjugate gradient method or a quasi-Newton approach (BFGS). Taking into account the low number of variables, the BFGS method has been used [10].

Constraint about :	Upper limit	Lower limit
Do	$g1(X): Do - \text{Min} (Do_s^U, Do_M^U) \leq 0$	$g2(X): \text{Max} (Do_s^L, Do_M^L) - Do \leq 0$
d	$g3(X): d - \text{Min} (d_s^U, d_M^U) \leq 0$	$g4(X): \text{Max} (d_s^L, d_M^L) - d \leq 0$
T0	$g5(X): T0 - T0_{IST}^U \leq 0$	$g6(X): T0_{IST}^L - T0 \leq 0$
L1	$g7(X): L1 - L1_s^U \leq 0$	$g8(X): L1_s^L - L1 \leq 0$
L2	$g9(X): L2 - L2_s^U \leq 0$	$g10(X): L2_s^L - L2 \leq 0$
n	$g11(X): n - n_{IST}^U \leq 0$	$g12(X): n_{IST}^L - n \leq 0$
D	$g13(X): D - D_s^U \leq 0$	$g14(X): D_s^L - D \leq 0$
Di	$g15(X): Di - Di_s^U \leq 0$	$g16(X): Di_s^L - Di \leq 0$
L0	$g17(X): L0 - L0_s^U \leq 0$	$g18(X): L0_s^L - L0 \leq 0$
R	$g19(X): R - R_s^U \leq 0$	$g20(X): R_s^L - R \leq 0$
Helix angle z at L2	$g21(X): z - z^U \leq 0$	-----
C	$g22(X): C - C_{IST}^U \leq 0$	$g23(X): C_{IST}^L - C \leq 0$
P1	$g24(X): P1 - P1_s^U \leq 0$	$g25(X): P1_s^L - P1 \leq 0$
P2	$g26(X): P2 - P2_s^U \leq 0$	$g27(X): P2_s^L - P2 \leq 0$
sh	$g28(X): sh - sh_s^U \leq 0$	$g29(X): sh_s^L - sh \leq 0$
Minimum operating length	-----	$g30(X): L1_{IST}^L - L1 \leq 0$
Maximum operating length	$g31(X): L2_{IST}^U - L2 \leq 0$	-----
Fe	$g32(X): Fe - Fe_s^U \leq 0$	$g33(X): Fe_s^L - Fe \leq 0$
Energy	$g34(X): \text{Energy} - \text{Energy}_s^U \leq 0$	$g35(X): \text{Energy}_s^L - \text{Energy} \leq 0$
Spring mass	$g36(X): M - M_s^U \leq 0$	$g37(X): M_s^L - M \leq 0$
VolL0	$g38(X): \text{VolL0} - \text{VolL0}_s^U \leq 0$	$g39(X): \text{VolL0}_s^L - \text{VolL0} \leq 0$
VolL2	$g40(X): \text{VolL2} - \text{VolL2}_s^U \leq 0$	$g41(X): \text{VolL2}_s^L - \text{VolL2} \leq 0$
End loop constraint	-----	$g42(X): 1 - \alpha E \leq 0$
Fatigue life	-----	$g43(X): 1 - \alpha F \leq 0$
N	$g44(X): N \text{ is an integer}$	

Table 1 Constraints

Before running, this kind of process needs the variables to be initialized, defining the calculation starting point.

### 2.3 Automatic determination of the calculation starting point

The choice of the starting point is very important. Indeed, the closer this is to the final solution, the more likely is the algorithm to converge towards the optimal solution [9,10]. This is particularly true here in viewing the high number of constraints. To determine the starting point automatically, data is analyzed in order to define an initial extension spring design which respects the majority of the problem constraints.

Figure 5 illustrates the different steps of the proposed method. The precise calculations are detailed in appendix 2. Each step includes a calculation of parameter limits based on interval arithmetic [11].

In step 1, L1, L2, sh, R, L0 limits are refined considering L1, L2, sh, R, L0, P1 and P2 limits. Then d limits are calculated.

In step 2, d has been chosen and Do limits are refined.

In step 3, d and Do being fixed, P0 limits are calculated from the IST requirements. Then N limits are calculated considering the limits of R, L0, mass, Fe, vol0 and the specifications upon the operating points W1 and W2.

In step 4, the method proposed by Paredes [8] to find optimal values of L1 and L2 for compression springs, has been extended to find P0, L1 and L2 satisfying the specifications and maximizing the fatigue life of the spring.

Then a comparison is made to see if the tested configuration is better than the one previously retained. A configuration is considered better if it respects the specifications more closely. If both respect the specifications, the configuration with the best-associated objective function value is chosen.

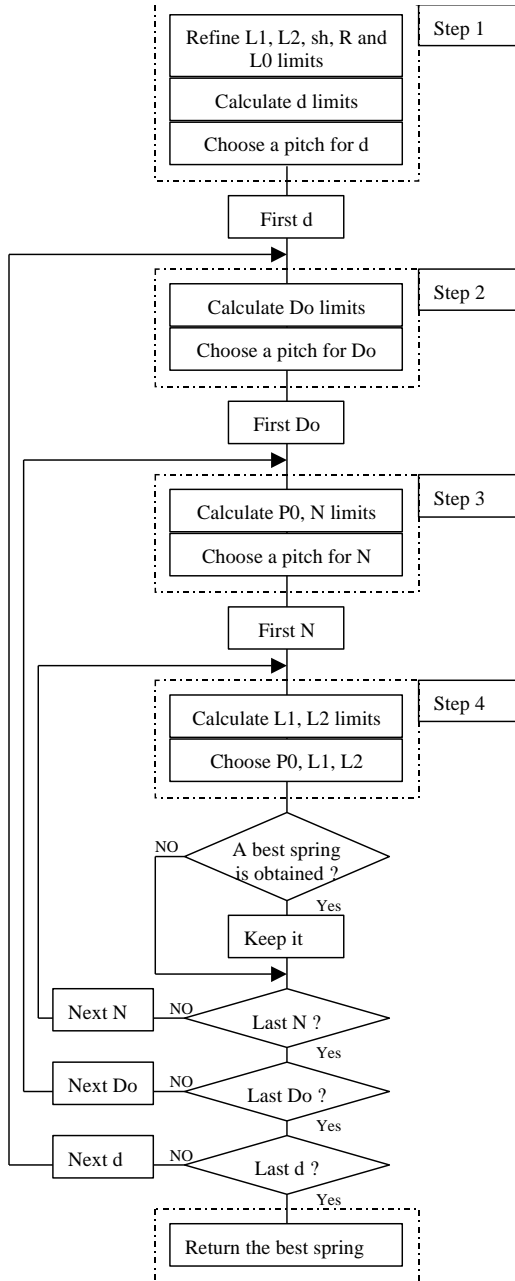


Figure 5 : Calculation starting point algorithm

### 3 Examples

A spring maker has successfully tested this method. All the calculations below have been obtained from the automatic determination of the starting point presented in section 2.3.

Maximum calculation time on a personal computer (200Mhz) is about 10 seconds. All results have been confirmed by using several starting points and other optimization algorithms.

### 3.1 Fatigue life application

In this example, based on a real industrial problem, the spring must have a minimum life of  $10^7$  cycles and the goal is to obtain the greatest fatigue life factor.

The required angle between end loops implies  $n = N$  (see Figure 2). The study of the system defines the spring load limits ( $40 < P1 < 50$  N and  $80 < P2 < 90$  N). The maximum outside diameter ( $D_o$ ) is 30 mm. Spring travel  $sh$  is constant (40 mm) and the maximum operating length ( $L2$ ) has to be less than 200 mm.

In this design stage, there is no other geometrical constraint and the final design of the contiguous parts will be adjusted to fit with the optimal spring design.

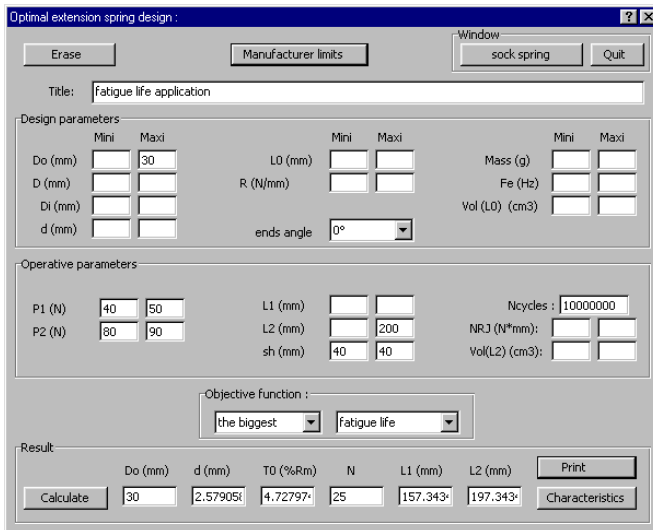


Figure 6 Greatest fatigue life factor

Problem resolution in mixed variables provides the following result (Figure 6):

$D_o = 30.00$  mm,  $d = 2.58$  mm,  $T_0 = 4.73$ ,  $N = 25$ ,

$L_1 = 157.34$  mm,  $L_2 = 197.34$  mm.

The associated fatigue life factor is 1.29.

The algorithm can also search for optimal springs made from standardized wires. It is sufficient to specify  $d^L = d^U = 2.5$  mm to get the best design related to this wire diameter. Tests with the closest standardized wire diameters from AMIC [12] standards (2.2 / 2.5 / 2.8 / 3.0 mm) provide the following results (Table 2) :

Ø wire (mm)	Fatigue life factor
2.2	1.17
<b>2.5</b>	<b>1.28</b>
2.8	no solution
3.0	no solution

Table 2 Maximum fatigue life with standardized wires

The solution adopted with a standardized wire is therefore the one with a wire diameter of 2.5 mm :

$D_o = 27.84$  mm,  $d = 2.5$  mm,  $T_0 = 4.95$ ,  $N = 28$ ,

$L_1 = 157.93$  mm,  $L_2 = 197.93$  mm.

The optimal spring design with a continuous wire diameter offers only 1% more fatigue life factor than the best spring design with a standard wire diameter, so the last solution is retained.

### 3.2 Mass production application



In this example, the spring is intended for mass production. In order to reduce the production costs, the goal is to find the spring with the smallest mass that respects the following specifications.

Free length L0 must be less than 200 mm. The minimum operating load P1 is to be 200 N. The operating lengths are defined : L1 = 245 mm and L2 = 345 mm. The required angle between end loops is 0°.

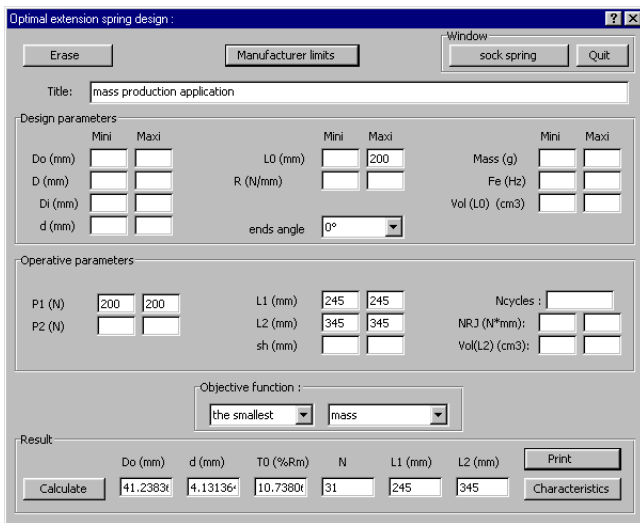


Figure 7 Smallest mass

Problem resolution in mixed variables provides the following result (Figure 7):

Do = 41.24 mm, d = 4.13 mm, T0 = 10.74, N = 31,

L1 = 245.00 mm, L2 = 345.00 mm.

The associated mass is 407g.

Tests with the closest standardized wire diameters from AMIC [12] standards (3.6 / 4.0 / 4.5 / 5 mm) provide the following results (Table 3) :

∅ wire (mm)	Mass (g)
3.6	no solution
4	no solution
<b>4.5</b>	<b>415</b>
5	437

Table 3 Smallest mass with standardized wires

The solution adopted with a standardized wire is therefore the one with a wire diameter of 4.5 mm :

Do = 50.25 mm, d = 4.5 mm, T0 = 7.22, N = 21,

L1 = 245.00 mm, L2 = 345.00 mm.

These two solutions have close mass values inducing similar production costs. But the optimal spring design with a continuous wire diameter offers an overall space of 276 cm<sup>3</sup> (volL0) whereas the other is 382 cm<sup>3</sup>. This would lead to significant difference in packaging costs, so the first solution (d = 4.13 mm) is retained.

## Conclusion

A comprehensive process for extension spring optimization is presented. Its main advantage is to link both industrial and mathematical knowledge in order to propose a solution starting directly from global specifications.

Furthermore, the specification sheet can accept data tolerances for use in the early design stages. The short calculation time allows the designer to test several configurations in order to determine which is the best (optimal spring design or optimal spring design with a standardized wire diameter). This study shows that extension springs can now be designed to solve most real-life industrial problems.

Spring manufacturers or designers can easily implement this process using any standard optimization code, or a spreadsheet endowed with an optimization solver, to find the optimal solution of extension spring design problems.

## Acknowledgements

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## Appendix 1

Details of all the formulae used to define the problem constraints.

### Constraints related to variables

◇ Specifications of the spring manufacturer upon  $D_0$  and  $d$  ( $D_{0M}^U, D_{0M}^L, d_M^U, d_M^L$ ) have to be taken into account. Here are the values used in our study for stainless steel :  $D_{0M}^U=315\text{mm}$ ,  $D_{0M}^L=1.6\text{mm}$ ,  $d_M^U=15\text{mm}$ ,  $d_M^L=0.15\text{mm}$

◇ Initial tension  $P_0$  is often defined as a percentage of  $R_m$ . The equations of the limit curves for the preferred range proposed by IST are (see Figure 8) :

$$T0_{IST}^U = 21.361 e^{\frac{[0.5329 - \ln(D/d)]^2}{4.82}}$$

$$T0_{IST}^L = 6.5 \cdot D/d/6$$

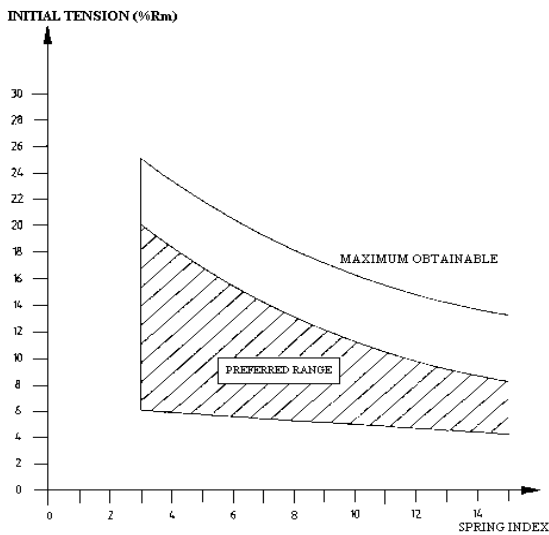


Figure 8 T0 limits from IST

### Constraints related to design parameters

◇ Number of active coils :  $n = N + n_e$

where  $n_e = 0$  or  $0.25$  or  $0.5$  or  $0.75$  depending of the angle required between end loops (see Figure 2).

Standards explain that all the formulae used are available if the number of coils lies between  $n_{IST}^L = 3$  and  $n_{IST}^U = 200$ .

◇ Mean diameter :  $D = D_o - d$

◇ Inside diameter :  $D_i = D_o - 2d$

◇ Free length :  $L_0 = 2D_i + d(n + 1)$

◇ Spring rate :  $R = Gd^4 / (8D^3n)$

$G$  : torsion modulus of the material in  $N/mm^2$

$G = 70000 N/mm^2$  for stainless steel.

◇ Helix angle at L2 :  $z = 180 \arctan(p / \pi / D) / \pi$  where  $p = (L_2 - 2D_i - d) / n$

The helix angle has to be lower than  $z^U = 7,5^\circ$  so that the calculations remain valid.

◇ Spring index :  $C = D / d$ . The standard imposes the spring index lies between  $C_{IST}^L = 4$  and  $C_{IST}^U = 20$ .

### Constraints related to operating parameters

◇ Minimum spring load :  $P_1 = R(L_1 - L_0) + P_0$

where  $P_0 = \frac{T_0 R m \pi d^3}{800 D}$

and  $R_m = 1918 - 255.86 Ln(d)$  : AISI 302 from AMIC (see Figure 9).

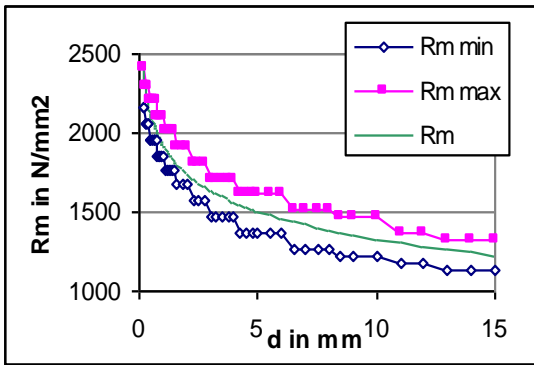


Figure 9 AMIC AISI 302

◇ Maximum spring load :  $P2 = R (L2 - L0) + P0$

◇ Spring travel :  $sh = L2 - L1$

◇ The minimum and maximum operating lengths are defined by IST using the maximum allowable travel

$$\text{Travel} = \frac{Re \pi d^3}{8 R D Kts} - \frac{P0}{R}$$

Where  $Re = 0.48 Rm$  for stainless steel

and  $Kts = 1.6 / (D / d)^{0.14}$ : stress correction factor (other formulations of this factor exist, all with equivalent results [13,14]).

$$L2_{IST}^U = L0 + 0.8 \text{ Travel}$$

$$L1_{IST}^L = L0 + 0.2 \text{ Travel}$$

◇ Natural Frequency of surge waves

$$Fe = 0.5 \sqrt{\frac{R}{\pi^2 \rho n D d^2}} \text{ with } \rho = 7900 \text{ kg/m}^3$$

◇ Internal energy :  $\text{Energy} = 0.5 R sh^2$

◇ Total mass :  $M = \pi^2 \rho (n + 2) D d^2 / 4$

◇ Free overall space taken up

$$\text{VolL0} = (2d + L0) \pi D d^2 / 4$$

◇ Operating overall space taken up

$$\text{VolL2} = (2d + L2) \pi D d^2 / 4$$

◇ Constraint in machine loops. The spring is automatically designed to ensure operating length L2 achieves a maximum 70 % of  $Rm$  in the machine loops.

$$\alpha E = 0.7 Rm / \sigma$$

$$\text{with } \sigma = \frac{16 D P2}{\pi d^3} \left( \frac{C}{C-1} \right): \text{ stress for } L = L2 .$$

◇ Fatigue life factor calculated from the Haigh diagram [14] (see Figure 10).

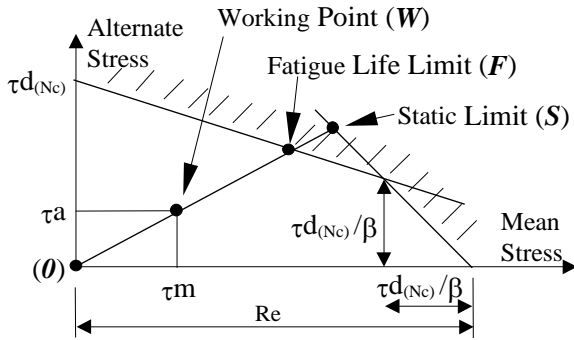


Figure 10 Haigh diagram

$$\alpha F = 0.75 * \text{Min} (T1, T2)$$

with :

The 0.75 factor is required by IST for fatigue life applications of extension springs.

$$T1 = \frac{OF}{OW} = \frac{\tau d_{(Nc)} (\beta Re - \tau d_{(Nc)})}{\tau a (\beta Re - \tau d_{(Nc)}) + (\beta - 1) \tau m \tau d_{(Nc)}}$$

$$T2 = \frac{OS}{OW} = \frac{Re}{\tau a + \tau m}$$

Where :

$\tau m$  : mean shear stress in  $N/mm^2$

$$\tau m = 4D [2 P0 + R (L1 + L2 - 2 L0)] Kts / (\pi d^3)$$

$\tau a$  : alternate shear stress in  $N/mm^2$

$$\tau a = 4DR(L2 - L1) Kts / (\pi d^3)$$

$\beta = 3$  (from DIN standards).

$\tau d_{(Nc)}$  is calculated from the Wohler curve :

$$\tau d_{(Nc)} = \begin{cases} \bullet \tau d & \text{if } Nc \geq 10^7 \text{ (fatigue limit)} \\ \bullet \left[ \frac{(\tau d - Re) \ln Nc}{\ln 10} + 7 Re - 4 \tau d \right] / 3 & \text{if } 10^4 < Nc < 10^7 \\ \bullet Re & \text{if } Nc \leq 10^4 \text{ (static limit)} \end{cases}$$

$$\tau d = \frac{303}{d^{0.268}} N / mm^2 \text{ (from DIN standards).}$$

## Appendix 2

### Step 1 : calculate d limits

The upper and lower limits of L1, L2, sh, R, L0 and d will be calculated using interval arithmetic [11]. Let us calculate the limits of L1, L2 and sh.

$$L1^U = \text{Min} (L1_S^U ; L2_S^U - sh_S^L)$$

$$L1^L = \text{Max} (L1_S^L ; L2_S^L - sh_S^U)$$

$$L2^U = \text{Min} (L2_s^U ; L1_s^U + sh_s^U)$$

$$L2^L = \text{Max} (L2_s^L ; L1_s^L + sh_s^L)$$

$$sh^U = \text{Min} (sh_s^U ; L2_s^U - L1_s^L)$$

$$sh^L = \text{Max} (sh_s^L ; L2_s^L - L1_s^U)$$

Then the limits of R and L0 due to the operating parameters can be calculated.

$$R^U = \text{Min} (R_s^U ; \frac{P2_s^U - P1_s^L}{sh^L} ; \frac{P1_s^U}{L1^L - L0_s^U} \text{ when}$$

$$L1^L > L0_s^U ; \frac{P2_s^U}{L1^L - L0_s^U} \text{ when } L2^L > L0_s^U)$$

$$R^L = \text{Max} (R_s^L ; \frac{P2_s^L - P1_s^U}{sh^U} ; \frac{P2_s^L - P1_s^U}{L2^U - L0_s^L} \text{ when } L2^U > L0_s^L$$

$$\text{and } P2_s^L > P1_s^U)$$

$$L0^U = \text{Min} (L0_s^U ; L1^U ; L2^U - (P2_s^L - P1_s^U) / R^U$$

$$\text{when } P2_s^L > P1_s^U)$$

$$L0^L = \text{Max} (L0_s^L ; L1^L - P1_s^U / R^L ; L2^L - P2_s^U / R^L)$$

From the following different relations available to calculate d :

$$d = Do - D = D - Di = (Do - Di) / 2 = Do / (C + 1)$$

$$= D / C = Di / (C-1).$$

d limits can be deduced :

$$d^U = \text{Min} (Do_s^U - D_s^L ; D_s^U - Di_s^L ; (Do_s^U - Di_s^L) / 2 ;$$

$$Do_s^U / (C^L + 1) ; D_s^U / C^L ; Di_s^U / (C^L - 1) ; d_s^U)$$

$$d^L = \text{Max} (Do_s^L - D_s^U ; D_s^L - Di_s^U ; (Do_s^L - Di_s^U) / 2 ;$$

$$Do_s^L / (C^U + 1) ; D_s^L / C^U ; Di_s^L / (C^U - 1) ; d_s^L)$$

### Step 2 : Calculate Do limits

In the same way :

$$Do^U = \text{Min} (D_s^U + d_s^U ; Di_s^U + 2 d_s^U ; 2 D_s^U - Di_s^L ;$$

$$D_s^U (1+1/C^L) ; Di_s^U (1+2/(C^L - 1)) ; d_s^U (C^U+1) ;$$

$$Do_s^U)$$

$$Do^L = \text{Max} (D_s^L + d_s^L ; Di_s^L + 2 d_s^L ; 2 D_s^L - Di_s^U ;$$

$$D_s^L (1+1/C^U) ; Di_s^L (1 + 2 / (C^U - 1)) ; d_s^L (C^L+1) ;$$

$$Do_s^L)$$

### Step 3 : Calculate P0 limits and N limits

At this step d and Do are known. Thus Di, D are also known.

Step 3-1 Calculation of P0 limits according to IST

The equations of the limit curves of Figure 8 are :

$$T0_{IST}^U = 21.361 e^{-\frac{[0.5329 - \ln(D/d)]^2}{4.82}}$$

$$T0_{IST}^L = 6.5 - D/d/6$$

Thus P0 limits can be calculated.

$$P0^U = \pi d^3 Rm T0_{IST}^U / (800 D)$$

$$P0^L = \pi d^3 Rm T0_{IST}^L / (800 D)$$

Step 3-2 Calculation of N limits

The limits values from the specification sheet are calculated as follow :

$$N^U = \text{Min} (n_{IST}^U ; n1^U ; n2^U ; n3^U ; n4^U ; n5^U ; n6^U ; n7^U) - ne$$

$$N^L = \text{Min} (n_{IST}^L ; n1^L ; n2^L ; n3^L ; n4^L ; n5^L ; n6^L ; n7^L) - ne$$

where :

$n1^U$  and  $n1^L$  are calculated to satisfy R limits :

$$n1^U = \frac{G d^4}{8 R^L D^3}$$

$$n1^L = \frac{G d^4}{8 R^U D^3}$$

$n2^U$  and  $n2^L$  are calculated to satisfy L0 limits :

$$n2^U = \frac{L0^U - 2 Di}{d} - 1$$

$$n2^L = \frac{L0^L - 2 Di}{d} - 1$$

$n3^U$  and  $n3^L$  are calculated to satisfy the requirements upon the operating point W1 :

$$n3^U = \frac{G d^4 (L1^U - 2 Di - d)}{8 D^3 (P1_s^L - P0^U) + G d^5} \text{ when } P1_s^L > P0^U$$

$$n3^L = \frac{G d^4 (L1^L - 2 Di - d)}{8 D^3 (P1_s^U - P0^L) + G d^5} \text{ when } P1_s^U > P0^L$$

$n4^U$  and  $n4^L$  are calculated to satisfy the requirements upon the operating point W2 :

$$n4^U = \frac{G d^4 (L2^U - 2 Di - d)}{8 D^3 (P2_s^L - P0^U) + G d^5} \text{ when } P2_s^L > P0^U$$

$$n4^L = \frac{G d^4 (L2^L - 2 Di - d)}{8 D^3 (P2_s^U - P0^L) + G d^5} \text{ when } P2_s^U > P0^L$$

$n5^U$  and  $n5^L$  are calculated to satisfy the mass limits :

$$n5^U = \frac{4 mass_s^U}{\rho \pi^2 D d^2} - 2$$

$$n5^L = \frac{4 mass_s^L}{\rho \pi^2 D d^2} - 2$$

$n6^U$  and  $n6^L$  are calculated to satisfy vol0 limits :

$$n6^U = \frac{4 vol0_s^U}{\pi d D o^2} - \frac{2Di + 3d}{d}$$

$$n6^L = \frac{4 vol0_s^L}{\pi d D o^2} - \frac{2Di + 3d}{d}$$

$n7^U$  and  $n7^L$  are calculated to satisfy Fe limits :

$$n7^U = \frac{0.5 d}{\pi Fe_s^L D^2} \sqrt{\frac{G}{2 \rho}}$$

$$n7^L = \frac{0.5 d}{\pi Fe_s^U D^2} \sqrt{\frac{G}{2 \rho}}$$

#### Step 4 : Calculation of P0, L1 and L2

P0 is initially set to its medium possible value :

$$P0 = (P0^U + P0^L) / 2$$

Then P0 is adjusted to fit with the limits according to the operating parameter limits :

$$P0_{op}^U = \text{Min}(P1_s^U - R (L1^L - L0); P2_s^U - R (L2^L - L0))$$

$$P0_{op}^L = \text{Min}(P1_s^L - R (L1_s^U - L0); P2_s^L - R (L2_s^U - L0))$$

$$\text{IF } P0 > P0_{op}^U \text{ THEN } P0 = P0_{op}^U$$

$$\text{IF } P0 < P0_{op}^L \text{ THEN } P0 = P0_{op}^L$$

Design parameters being known, the calculation of the operating points is obtained to find the maximum fatigue life, which implies finding the maximum value of L2 and then the minimum value of L1 in order to reduce maximum shear stress and alternate stress [8].

Here are the new sh, L1 and L2 limits:

$$sh^U = \text{Min} ( sh_s^U ; \sqrt{\frac{\text{energy}_s^U}{0.5 R}} )$$

$$sh^L = \text{Max} ( sh_s^L ; \sqrt{\frac{\text{energy}_s^L}{0.5 R}} )$$

$$L1^U = \text{Min} ( L1_s^U ; L0 + \frac{P1_s^U - P0}{R} )$$

$$L1^L = \text{Max} ( L1_s^L ; L1_{IST}^L ; L0 + \frac{P1_s^L - P0}{R} )$$

$$L2^U = \text{Min} ( L2_s^U ; L2_{IST}^U ; LP2 ; Lvol2 ; Lz ; L2ends )$$

Where :

$$LP2 = L0 + \frac{P2_s^U - P0}{R}$$

$$Lvol2 = \frac{4 vol2_s^U}{\pi Do^2} - 2d$$

$$Lz = 2Di + d + \pi n D \tan(z^U)$$

$$L2ends = L0 + \left[ \frac{0.7 Rm \pi d^3}{16D} \left( \frac{C-1}{C} \right) - P0 \right] / R$$

$$L2^L = \text{Max} ( L2_s^L ; L0 + \frac{P2_s^L - P0}{R} ; \frac{4 vol2_s^L}{\pi Do^2} - 2d )$$

Finally L1 and L2 values can be calculated :



$$L2 = \text{Max} (L2^L; L1^L + sh^L)$$

$$\text{IF } L2 = L1^L + sh^L \text{ THEN } L1 = L1^L$$

$$\text{ELSE } L1 = \text{Min} (L1^U; L2^L - sh^L)$$