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A UNIFYING APPROACH TO RELATIVISTIC DIFFUSIONS AND H -THEOREMS

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A new, wide class of relativistic stochastic processes is introduced. All relativistic processes considered so far in the literature (the Relativistic Ornstein–Uhlenbeck Process as well as the Franchi–Le Jan and the Dunkel–Hänggi processes) are members of this class. The stochastic equations of motion and the associated forward Kolmogorov equations are obtained for each process in the class. The corresponding manifestly covariant transport equation is also obtained. In particular, the manifestly covariant equations for the Franchi–Le Jan and the Dunkel–Hänggi processes are derived here for the first time. Finally, the manifestly covariant approach is used to prove a new H -theorem for all processes in the class.

Keywords: Relativistic stochastic processes; H -theorem; transport equation.

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1. Introduction

It is probably fair to say that Stochastic Process Theory originated with Einstein’s 1905 study on Brownian motion.¹² The theory has since developed into a full-grown branch of Mathematics^{20,21} and its current applications include Physics and Chemistry,^{25,15} Biology^{1,17} and Economics.^{23,24} As far as Physics is concerned, one had to wait until the 70’s to see what started as an attempt to describe non-quantum Galilean diffusions being extended to include Galilean quantum processes.^{25,16} But the wait for a relativistic extension was even longer, since the first paper dealing with a relativistic stochastic process of clear physical interpretation was only published in 1997.⁵ Other relativistic stochastic processes have recently been considered by various authors,^{13,10,11,14,7,2} sometimes with extremely different and seemingly irreconcilable points of view. The aim of the present letter is to propose a unified approach towards all processes already considered in the literature. It turns out that these processes are particular members of a wide class of generalized relativistic Ornstein–Uhlenbeck processes. This class is characterized by a certain simple property obeyed by the stochastic force entering the definition of the processes.

More precisely, for each process in the class, and for each point on the phase-space trajectory of the diffusing particle, there exists a Lorentz frame in which the stochastic force acting on the particle is a Gaussian white noise.

The gaps in the present literature are filled by providing, for all processes in the class, the stochastic equations of motion in an arbitrary Lorentz frame and the associated transport equations, in both standard $(3 + 1)$ and manifestly covariant form. In particular, a manifestly covariant treatment of both the Franchi–Le Jan and the Dunkel–Hänggi processes is proposed here for the first time. Finally, the general manifestly covariant approach is used to prove a new H -theorem, valid for all processes in the class.

2. Definition of the Class

We work in the special relativistic framework. The space–time is thus flat, equipped with the Minkowski metric η . We only consider Lorentz frames, where the components of η read $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. There is no simple, direct special relativistic analog of the usual Galilean Brownian motion. The simplest relativistic diffusion processes are analogs of the Ornstein–Uhlenbeck process.⁵ In these processes, the force acting on the particle is made up of two distinct contributions. The first one is deterministic and represents the mean force acting on the diffusing particle; the second contribution is stochastic and is the source of noise for the motion. This letter deals with a new class C of relativistic processes. This class is characterized by the fact that, for each process P in C , there exists at any point Z of the phase-space trajectory of the diffusing particle, a P - and possibly Z -dependent Lorentz frame $\mathcal{R}^*(Z)$ in which the noise force acting on the particle is a usual Gaussian white noise. Each process P in the class C is thus fully determined by a choice of deterministic force and a choice of Lorentz frame $\mathcal{R}^*(Z)$ for each Z .

Let \mathcal{R} be an arbitrary, fixed Lorentz frame, with coordinates (t, \mathbf{x}) and let \mathbf{p} be the three-momentum of a point particle of unit mass in \mathcal{R} . A point on the phase-space trajectory of the diffusing particle can be labeled by $Z = (t, \mathbf{x}, \mathbf{p})$. Let us fix a process P in the class C and a point Z_0 in phase-space. The Lorentz frame $\mathcal{R}^*(Z_0)$ can be represented by its three-velocity field $\mathbf{u}(Z_0)$ with respect to \mathcal{R} . Let $Z^* = (t^*, \mathbf{x}^*, \mathbf{p}^*)$ be coordinates attached to $\mathcal{R}^*(Z_0)$. These coordinates naturally depend on Z_0 but are defined over all Minkowski space–time. Their values at point Z_0 will be denoted by $Z_0^* = (t_0^*, \mathbf{x}_0^*, \mathbf{p}_0^*)$. The process P is represented, in $\mathcal{R}^*(Z_0)$, by stochastic equations of the form:

$$\begin{aligned} dx_{t^*}^{*i} &= \eta^{ij} \frac{p_j, t^*}{\gamma(\mathbf{p}_{t^*}^*)} dt^* \\ dp_{i, t^*}^* &= {}^{(S)}\phi^*(Z_{t^*}^*) dt^* + \sigma_{ij}^*(Z_{t^*}^*) dB_{t^*}^j, \end{aligned} \quad (1)$$

where $\gamma(\mathbf{p}) = \sqrt{1 + \mathbf{p}^2}$ is the Lorentz factor of the particle, ${}^{(S)}\phi^*$ is the deterministic three-force acting on the diffusing particle in $\mathcal{R}^*(Z_0)$ and the superscript (S) indicates that these equations are to be understood in the Stratonovich sense.²¹

The fundamental property tracing the fact that P belongs to the class C is that the noise entering (1) is, at point Z_0^* , a Gaussian white noise:

$$\sigma_{ij}^*(Z_0^*) = \sqrt{2D} \eta_{ij}. \quad (2)$$

3. Equations in an Arbitrary Reference Frame

The stochastic equations of motions describing the process P in the original, fixed Lorentz frame \mathcal{R} can be deduced from Eqs. (1) and (2) by carrying out, for all Z_0 , a Lorentz boost from $\mathcal{R}^*(Z_0)$ to \mathcal{R} , followed by a random time change from t^* to t . One obtains equations of the form:

$$\begin{aligned} dx_t^i &= v^i(Z_t) dt, \\ dp_{i,t} &= {}^{(I)}\phi_i(Z_t) dt + \sigma_{ij}(Z_t) dB_t^j, \end{aligned} \quad (3)$$

where

$$v^i(Z) = \eta^{ij} \frac{p_j}{\gamma(\mathbf{p})}, \quad (4)$$

$${}^{(I)}\phi_i(Z) = H_i(Z) + I_i(Z) + J_i(Z), \quad (5)$$

$$H_i(Z) = \frac{p \cdot U}{\gamma(\mathbf{p})} {}^{(S)}\phi_i^* - \frac{u_i}{\gamma(\mathbf{p})} \left[\frac{\gamma(\mathbf{p})(1-\Gamma)}{\mathbf{u}^2} {}^{(S)}\phi_j^* u^j + \Gamma {}^{(S)}\phi_j^* p^j \right], \quad (6)$$

$$I_i(Z) = D \frac{\Gamma}{\gamma(\mathbf{p})(p \cdot U)^2} \left[u_i(1+2(p \cdot U)^2) - \Gamma^2 \gamma(\mathbf{p}) u_j \frac{\partial u_i}{\partial p_j} + \Gamma \gamma(\mathbf{p})(p \cdot U) p_j \frac{\partial u_i}{\partial p_j} \right], \quad (7)$$

$$J_i(Z) = -\frac{D}{2} \frac{\Gamma^2}{\gamma(\mathbf{p})(p \cdot U)^2} p \cdot \frac{\partial U}{\partial p_j} \left[\left(\frac{p \cdot U}{\Gamma} \right)^2 \eta_{ij} - \left(\frac{p \cdot U}{\Gamma} \right) (u_i p_j + u_j p_i) + u_i u_j \right], \quad (8)$$

and, finally,

$$\sigma_{ij}(Z) = -\sqrt{2D} \sqrt{\frac{1}{\gamma(\mathbf{p})(p \cdot U)}} \left[\frac{\gamma(\mathbf{p})(1-\Gamma)}{\mathbf{u}^2} u_i u_j + \Gamma u_i p_j - (p \cdot U) \eta_{ij} \right]. \quad (9)$$

In Eqs. (6) to (9), we have introduced the four-velocity U , along with the Lorentz factor $\Gamma = (1 - \mathbf{u}^2)^{-1/2}$, associated to the three-velocity \mathbf{u} . The associated forward Kolmogorov equation reads²¹:

$$\frac{\partial \Pi}{\partial t} + \frac{\partial}{\partial x^i} \left(\frac{p^i}{\gamma(\mathbf{p})} \Pi \right) = \frac{\partial}{\partial p_i} (\psi_i \Pi) + \frac{\partial}{\partial p_i} \left(A_{ij} \frac{\partial \Pi}{\partial p_j} \right), \quad (10)$$

with

$$A_{ij}(t, \mathbf{x}, \mathbf{p}) = -\frac{D\Gamma^2}{\gamma(\mathbf{p})} \frac{1}{p \cdot U} \left[\left(\frac{p \cdot U}{\Gamma} \right)^2 \eta_{ij} - \left(\frac{p \cdot U}{\Gamma} \right) (u_i p_j + u_j p_i) + u_i u_j \right], \quad (11)$$

$$\psi_i(t, \mathbf{x}, \mathbf{p}) = \frac{\partial A_{ij}}{\partial p_j} - {}^{(I)}\phi_i(t, \mathbf{x}, \mathbf{p}), \quad (12)$$

and where Π is the t -dependent distribution function in (\mathbf{x}, \mathbf{p}) -space, equipped with the Lebesgue measure $d^3x d^3p$.

In \mathcal{R} , each process of the class C is fully determined by fixing the deterministic three-force $\psi(t, \mathbf{x}, \mathbf{p})$ and the velocity field $\mathbf{u}(t, \mathbf{x}, \mathbf{p})$.

4. Manifestly Covariant Treatment

Equation (3) or, alternately, Eq. (10), fully characterize the class C . They are however rather cumbersome and, therefore, difficult to use in practice. This problem can be remedied by lifting the Kolmogorov equation off the mass-shell and treating p_0 as independent of \mathbf{p} .¹⁹ One thus introduces a new, unphysical distribution f , which depends on $(t, \mathbf{x}, p_0, \mathbf{p})$ and whose restriction to the mass-shell coincides with Π . The Kolmogorov equation is itself replaced by a so-called manifestly covariant transport equation i.e. by a manifestly covariant equation obeyed by the distribution f and whose validity is a sufficient condition for the validity of the Kolmogorov equation. Making use of the standard techniques of relativistic statistical physics, one finds that the simplest manifestly covariant transport equation associated to (10) reads:

$$\mathcal{L}(f) = 0, \quad (13)$$

with

$$\mathcal{L}(f) = \frac{\partial}{\partial x^\mu} (\eta^{\mu\nu} p_\nu f) + \frac{\partial}{\partial p_\mu} (\Xi_\mu f) + \frac{\partial}{\partial p_\rho} \left(DK^\mu{}_\rho{}^\beta{}_\nu \frac{p_\mu p_\beta}{p \cdot V} \frac{\partial f}{\partial p_\nu} \right), \quad (14)$$

where V is an off-shell four-velocity whose restriction on the mass-shell is U ,

$$K^{\mu\rho\beta\nu} = V^\mu V^\beta \Delta^{\rho\nu} - V^\mu V^\nu \Delta^{\rho\beta} + V^\rho V^\nu \Delta^{\mu\beta} - V^\rho V^\beta \Delta^{\mu\nu}, \quad (15)$$

and Δ is the projector on the orthogonal to V :

$$\Delta_{\mu\nu} = \eta_{\mu\nu} - V_\mu V_\nu. \quad (16)$$

Let ξ be any off-shell three-force whose restriction on the mass-shell is the three-force ψ . In Eq. (14), Ξ is an off-shell four-force acting on the particle, related to ξ by:

$$\begin{aligned} \Xi_0 &= p^i \xi_i, \\ \Xi_i &= -p_0 \xi_i. \end{aligned} \quad (17)$$

Note that $\Xi \cdot p = 0$, even for off-shell momenta. The operator \mathcal{L} is clearly Lorentz invariant. It can be proven by a very general argumentation that both distributions f and Π also are scalars.^{18,19,6}

5. Particular Processes

5.1. The ROUP

The ROUP³⁻⁵ is the first relativistic process to have been introduced in the physical literature. It models the stochastic motion of a point particle diffusing through its interaction with an isotropic fluid. The fluid is taken to be in a state of global thermodynamical equilibrium, with rest-frame \mathcal{R}_W and four-velocity W . The coordinates associated to \mathcal{R}_W are $(T, \mathbf{X}, \mathbf{P})$. The isotropy of the fluid leads to the choice $\mathcal{R}^*(Z) = \mathcal{R}_W$ for all Z . The deterministic force acting on the particle is a frictional force. The expression of the corresponding three-force in \mathcal{R}_W is⁵:

$$({}^{(S)}\phi_W)_i(T, \mathbf{X}, \mathbf{P}) = -\alpha \frac{P_i}{\gamma(\mathbf{P})}, \quad (18)$$

where α is a friction coefficient. The Kolmogorov equation describing the transport in \mathcal{R}_W reads:

$$\frac{\partial \Pi}{\partial T} + \frac{\partial}{\partial \mathbf{X}} \cdot \left(\frac{\mathbf{X}}{\gamma(\mathbf{P})} \Pi \right) + \frac{\partial}{\partial \mathbf{P}} \cdot \left(-\alpha \frac{\mathbf{P}}{\gamma(\mathbf{P})} \Pi \right) = D \Delta_{\mathbf{P}} \Pi. \quad (19)$$

The Kolmogorov equation in an arbitrary Lorentz frame $\mathcal{R}^{3,4}$ is given by Eq. (10) with

$$A_{ij}(t, \mathbf{x}, \mathbf{p}) = -\frac{D \Gamma^2}{\gamma(\mathbf{p})} \frac{1}{p \cdot W} \left[\left(\frac{p \cdot W}{\Gamma} \right)^2 \eta_{ij} - \left(\frac{p \cdot W}{\Gamma} \right) (w_i p_j + w_j p_i) + w_i w_j \right], \quad (20)$$

and

$$\psi_i(t, \mathbf{x}, \mathbf{p}) = -\frac{p \cdot W}{\gamma(\mathbf{p})} ({}^{(S)}\phi_W)_i - \frac{w_i}{\gamma(\mathbf{p})} \left[\frac{\gamma(\mathbf{p})(1-\Gamma)}{\mathbf{w}^2} ({}^{(S)}\phi_W)_j w^j + \Gamma ({}^{(S)}\phi_W)_j p^j \right]. \quad (21)$$

The manifestly covariant transport equation is (13), along with (14), where $V = W$, and⁴

$$\Xi_\mu = -\lambda_\mu^\nu p_\nu p \cdot p + \lambda^{\alpha\beta} p_\alpha p_\beta p_\mu, \quad (22)$$

with

$$\lambda_\nu^\mu = \frac{\alpha}{(p \cdot W)^2} \Delta_\nu^\mu, \quad (23)$$

where Δ denotes the projector on the orthogonal to V (see Eq. (16)). Equation (13) admits the off-shell Jüttner distribution¹⁹

$$f_J(p) = \frac{1}{4\pi} \frac{\beta}{K_2(\beta)} \exp(-\beta W^\mu p_\mu), \quad (24)$$

as invariant measure in p -space, provided the inverse temperature β , the friction coefficient α and the noise coefficient D are linked by the relation $\alpha = \beta D$. This constitutes a special relativistic fluctuation theorem.

5.2. The Franchi–Le Jan process

This process^{13,14,2} is obtained by choosing, for all Z , the instantaneous proper frame of the diffusing particle as the Lorentz frame $\mathcal{R}^*(Z)$ and by setting to zero the force acting on the particle. This process does not describe the diffusion of a particle interacting with a usual surrounding fluid. It has been suggested that the Franchi–Le Jan process is a model for diffusions induced by interaction with quantum gravitational degrees of freedom.^{8,7} The Kolmogorov equation describing the Franchi–Le Jan process in an arbitrary Lorentz frame \mathcal{R} is Eq. (10) with

$$A_{ij}(t, \mathbf{x}, \mathbf{p}) = \frac{D}{\gamma(\mathbf{p})} [p_i p_j - \eta_{ij}], \quad (25)$$

and

$$\psi_i(t, \mathbf{x}, \mathbf{p}) = 0. \quad (26)$$

It can be rewritten as

$$\frac{\partial \Pi}{\partial t} + \frac{\partial}{\partial x^i} \left(\frac{p^i}{\gamma(\mathbf{p})} \Pi \right) = \frac{D}{\gamma(\mathbf{p})} \Delta_m \Pi, \quad (27)$$

where m is the metric induced by the Minkowski metric η on the mass-shell and Δ_m is the Laplace–Beltrami operator⁹ associated to m . The components of the inverse metric m^{-1} are given by $m^{ij} = p^i p^j - \eta^{ij}$ and the Laplace–Beltrami operator is defined by

$$\Delta_m = \frac{1}{\sqrt{\det m_{ij}}} \frac{\partial}{\partial p^i} \left(\sqrt{\det m_{ij}} m^{ij} \frac{\partial}{\partial p^j} \right), \quad (28)$$

where $\det m_{ij} = \gamma^{-2}(\mathbf{p})$. The off-shell transport equation is given by Eq. (13) along with Eq. (14), where $\Xi_\mu = 0$. The noise term simplifies greatly because one can choose $V = p$ for this process, and the corresponding manifestly covariant transport equation reads:

$$\frac{\partial}{\partial x^\mu} (\eta^{\mu\nu} p_\nu f) + \frac{\partial}{\partial p_\rho} \left(D(\eta_{\rho\nu} - p_\rho p_\nu) \frac{\partial f}{\partial p_\nu} \right) = 0. \quad (29)$$

5.3. The Dunkel–Hänggi process

This process^{10,11} mixes characteristics of both the ROUP and the Franchi–Le Jan process. The Dunkel–Hänggi process is best presented as describing the interaction of a point particle with a fluid. A proper frame \mathcal{R}_W and a four-velocity W for the fluid are introduced and the deterministic force is a frictional force. The expression of the associated three-force in \mathcal{R}_W reads:

$$({}^{(I)}\phi_W)_i(T, \mathbf{X}, \mathbf{P}) = -\nu P_i + \frac{3DP_i}{\gamma(\mathbf{P})}, \quad (30)$$

where ν is a positive friction coefficient. The noise term, on the other hand, is identical to the Franchi–Le Jan process. It is thus isotropic in the proper frame of

the diffusing particle, and not in the proper frame \mathcal{R}_W of the fluid. It therefore seems that the Dunkel–Hänggi process does not describe diffusions in standard isotropic fluids.

The Kolmogorov equation in \mathcal{R}_W is given by

$$\frac{\partial \Pi}{\partial T} + \frac{\partial}{\partial X^i} \left(\frac{P^i}{\gamma(\mathbf{P})} \Pi \right) = \frac{\partial}{\partial P_i} (\nu P_i \Pi) + \frac{\partial}{\partial P_i} \left[\left(\frac{D}{\gamma(\mathbf{P})} (P_i P_j - \eta_{ij}) \right) \frac{\partial \Pi}{\partial P_j} \right]. \quad (31)$$

The Kolmogorov equation in an arbitrary Lorentz frame \mathcal{R} is Eq. (10), with

$$A_{ij} = -\frac{1}{2} \sigma_{il} \sigma_{kj}^T \eta^{kl}, \quad (32)$$

and

$$\psi_i = \frac{\partial A_{ij}}{\partial p_j} - {}^{(I)}\phi_i, \quad (33)$$

where

$$\sigma_{ij}(t, \mathbf{x}, \mathbf{p}) = \sqrt{\frac{p \cdot W}{\gamma(\mathbf{p})}} (\sigma_W)_{kj} \left[\delta_i^k - \frac{w_i}{p \cdot W} \left(\frac{(1 - \Gamma(\mathbf{w})) \gamma(\mathbf{p})}{\mathbf{w}^2} w^k + \Gamma(\mathbf{w}) p^k \right) \right], \quad (34)$$

$${}^{(I)}\phi_i(t, \mathbf{x}, \mathbf{p}) = \frac{p \cdot W}{\gamma(\mathbf{p})} \left({}^{(S)}\phi_i - \frac{1}{2} \eta^{jl} \sigma_{kl} \frac{\partial \sigma_{ij}}{\partial p_k} \right), \quad (35)$$

$${}^{(S)}\phi_i(t, \mathbf{x}, \mathbf{p}) = {}^{(S)}\phi_W)_i - \frac{w_i}{p \cdot W} \left(\frac{(1 - \Gamma(\mathbf{w})) \gamma(\mathbf{p})}{\mathbf{w}^2} w^k {}^{(S)}\phi_W)_k + \Gamma(\mathbf{w}) p^k {}^{(S)}\phi_W)_k \right), \quad (36)$$

$${}^{(S)}\phi_W)_i = {}^{(I)}\phi_W)_i + \frac{1}{2} \eta^{jl} (\sigma_W)_{kl} \frac{\partial (\sigma_W)_{ij}}{\partial p_k} \quad (37)$$

and

$$(\sigma_W)_{ij}(T, \mathbf{X}, \mathbf{P}) = \sqrt{\frac{2D}{\gamma(\mathbf{P})}} \left[\frac{1 - \gamma(\mathbf{P})}{\mathbf{P}^2} P_i P_j + \eta_{ij} \right]. \quad (38)$$

The manifestly covariant transport equation is (13), with (14), where

$$\Xi_\mu = -\tilde{\lambda}_\mu^\nu p_\nu p \cdot p + \tilde{\lambda}^{\alpha\beta} p_\alpha p_\beta p_\mu, \quad (39)$$

$$\tilde{\lambda}_\nu^\mu = \frac{\nu}{(p \cdot W)} \tilde{\Delta}_\nu^\mu, \quad (40)$$

and $\tilde{\Delta}$ is the projector on the orthogonal to W :

$$\tilde{\Delta}_{\mu\nu} = \eta_{\mu\nu} - W_\mu W_\nu. \quad (41)$$

The noise term can be simplified into the noise term of Eq. (29).

6. *H*-Theorem

The structure of the manifestly covariant equation makes it possible to prove an *H*-theorem common to all processes in the class *C*. Let *f* and *g* be two solutions of Eq. (13). The four-current $S_{[f|g]}$ of the conditional entropy of *f* with respect to *g* is defined by:

$$S_{[f|g]}^\mu(x) = - \int p^\mu f(x, p) \ln \left(\frac{f(x, p)}{g(x, p)} \right) D^4p, \quad (42)$$

with $D^4p = \theta(p_0)\delta(p^2 - 1)d^4p$, where θ is the Heaviside function and δ is the Dirac distribution. The first step in the proof of the *H*-theorem is to evaluate the four-divergence of this current and to use the manifestly covariant transport equation to convert all space–time derivatives into momentum derivatives. An integration by parts then leads to:

$$\begin{aligned} \partial_\mu S_{[f|g]}^\mu &= \int \left\{ \partial_{p_\nu} (J_{\mu\nu} f) - \partial_{p_\nu} (J_{\mu\nu} g) \frac{f}{g} \right\} \partial_{p_\mu} \ln \left(\frac{f}{g} \right) D^4p \\ &+ \int \left\{ \mathcal{K}_\mu(g) \frac{f}{g} - \mathcal{K}_\mu(f) \left[1 + \ln \left(\frac{f}{g} \right) \right] \right\} \partial_{p_\mu} (D^4p), \end{aligned} \quad (43)$$

with

$$J_{\mu\nu} = -DK^\alpha{}_\mu{}^\beta{}_\nu \frac{p_\alpha p_\beta}{p \cdot V}, \quad (44)$$

$$\mathcal{K}_\mu(f) = I_\mu f - \frac{\partial}{\partial p_\nu} (J_{\mu\nu} f), \quad (45)$$

where

$$I_\mu = -DK^\alpha{}_\mu{}^\beta{}_\nu \frac{\partial}{\partial p_\nu} \left(\frac{p_\alpha p_\beta}{p \cdot V} \right) + \Xi_\mu. \quad (46)$$

A direct calculation shows that $\partial_{p_\mu} D^4p = 2p^\mu \theta(p_0) \delta'(p^2 - m^2 c^2) d^4p$. The second integral in Eq. (43) thus involves contractions of the form $p^\mu \mathcal{K}_\mu(u)$, where the function *u* is either *f* or *g*. Replacing \mathcal{K} by its definition (45), one obtains:

$$\begin{aligned} p^\mu \mathcal{K}_\mu(u) &= p^\mu \left\{ I_\mu u - \frac{\partial}{\partial p_\nu} (J_{\mu\nu} u) \right\} \\ &= DK^\alpha{}_\mu{}^\beta{}_\nu p^\mu \frac{p_\alpha p_\beta}{p \cdot V} \frac{\partial u}{\partial p_\nu} + p^\mu \Xi_\mu u. \end{aligned} \quad (47)$$

The tensor $K^{\alpha\mu\beta\nu}$ is antisymmetric upon exchange of the indices μ and α , entailing that $K^{\alpha\mu\beta\nu} p_\alpha p_\mu p_\beta = 0$. Moreover, the deterministic four-force *F* is orthogonal to the momentum *p*, i.e. $p^\mu \Xi_\mu = 0$ (see Eq. (17)). Equation (47) therefore simply reduces to

$$p^\mu \mathcal{K}_\mu(u) = 0. \quad (48)$$

Equation (43) thus simplifies into

$$\partial_\mu S_{[f|g]}^\mu(x) = \int J_{\mu\nu} \mathcal{D}^\mu[f/g] \mathcal{D}^\nu[f/g] D^4 p, \quad (49)$$

where J is defined by Eq. (44) and the functional \mathcal{D} is given by

$$\mathcal{D}^\mu[f/g] = \frac{\partial}{\partial p_\mu} (\ln(f/g)). \quad (50)$$

The value of the scalar $J_{\mu\nu} \mathcal{D}^\mu[f/g] \mathcal{D}^\nu[f/g]$ at point $z = (x, p)$ is best computed in the Lorentz frame of four-velocity $V(z)$. By definition, the components of $V(z)$ in this frame are $(1, 0, 0, 0)$ and one finds by a direct calculation that:

$$J_{\mu\nu} \mathcal{D}^\mu[f/g] \mathcal{D}^\nu[f/g] = -\frac{D}{p_0} \eta^{ij} q_i q_j, \quad (51)$$

with $q_i = p_i \mathcal{D}_0[f/g] - p_0^2 \mathcal{D}_i[f/g]$. This proves that $J_{\mu\nu} \mathcal{D}^\mu[f/g] \mathcal{D}^\nu[f/g]$ is non-negative at all points z of the extended phase-space. The integral (49) is thus non-negative, which proves the H -theorem.

7. Conclusion

We have introduced a new, wide class of relativistic stochastic processes. Processes in this class are generalizations of the standard Ornstein–Uhlenbeck process and are characterized by a certain simple property of the stochastic force acting on the diffusing particle. All relativistic stochastic processes considered so far in the literature^{13,10,11,14,7,2} belong to this class. We have obtained, for each process in the class, the stochastic equations describing the diffusion in an arbitrary Lorentz frame and the associated forward Kolmogorov equation. The corresponding manifestly covariant transport equation has been obtained as well. In particular, a manifestly covariant treatment of both the Franchi–Le Jan and the Dunkel–Hänggi processes is given here for the first time. We have used the manifestly covariant formulation to prove a new H -theorem, valid for all the processes in the class. Let us conclude by mentioning some directions in which the work presented in this letter should be extended. The calculations presented here have been performed in flat space–time only. A necessary extension is therefore to perform all calculations in curved space–time as well. The construction of relativistic stochastic processes proposed in this letter also makes it apparent that the class of processes considered here can be extended into even more general ones to describe, for example, diffusions in non-isotropic media. Such extensions will be addressed in forthcoming publications.

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