



**HAL**  
open science

## Regular directed path and Moore flow

Philippe Gaucher

► **To cite this version:**

| Philippe Gaucher. Regular directed path and Moore flow. 2022. hal-03741876v2

**HAL Id: hal-03741876**

**<https://hal.science/hal-03741876v2>**

Preprint submitted on 31 Oct 2022 (v2), last revised 3 Jan 2024 (v5)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# REGULAR DIRECTED PATH AND MOORE FLOW

PHILIPPE GAUCHER

ABSTRACT. Using the notion of regular  $d$ -path of the topological  $n$ -cube, we construct the regular realization of a precubical set as a multipointed  $d$ -space. Its execution paths correspond to the tame regular  $d$ -paths of the precubical set between two vertices. By considering the associated Moore flow, we obtain a colimit-preserving functor from precubical sets to Moore flows. As a consequence, we obtain that the associated Moore flow is always  $m$ -cofibrant. This provides a model category interpretation of the known fact that the space of tame regular  $d$ -paths of a precubical set between two vertices is homotopy equivalent to a CW-complex. Finally, we prove that the flow associated with the regular realization of a precubical set coincides with the natural realization of a precubical set as a flow.

## CONTENTS

|   |    |
|---|----|
| 1. Introduction   | 1  |
| 2. The regular realization of a precubical set            | 4  |
| 3. $L_1$ -arc length of $d$ -paths of precubical sets     | 7  |
| 4. The $\{q,h,m\}$ -model structures of Moore flows       | 10 |
| 5. Moore flow of a regular realization                    | 13 |
| 6. Relation with the natural realization functor          | 18 |
| Appendix A. Is the regular realization $m$ -cofibrant ?   | 20 |
| Appendix B. The category $\text{Ch}_{\alpha,\beta}(K, n)$ | 21 |
| References  | 22 |

## 1. INTRODUCTION

**Presentation.** It is described in [16] a way of realizing a precubical set as a flow without any non-canonical choice in the construction, by introducing a natural realization functor. The latter terminology comes from the fact that it uses the notion of natural  $d$ -path of the topological  $n$ -cube. This work is devoted to the study of a  $d$ -space version of the notion of natural realization of a precubical set as a flow. It is therefore a sequel of [16]. However, the reading of the latter paper is not required to read this one. Precubical sets are a model for concurrency theory [6]. The  $n$ -cube represents the concurrent execution of  $n$  actions. This idea is even further developed in [17] in which it is established that the non-positively curved precubical sets contain most of the useful examples. However, we do not need this concept and we keep working with general precubical sets like in [16]. The  $d$ -paths of a precubical set, i.e. of the Grandis  $d$ -space in the sense of [18] associated

---

2020 *Mathematics Subject Classification.* 55U35,68Q85.

*Key words and phrases.* directed path, precubical set, directed homotopy, combinatorial model category, accessible category, mixed model structure.

with the precubical set, are the continuous paths which are concatenations of continuous paths included in cubes of the geometric realization which are locally nondecreasing with respect to each axis of coordinates. These particular continuous paths represent the possible execution paths of the concurrent process modelled by the precubical set. The local nondecreasingness represents time irreversibility. The notion of regular  $d$ -path of a precubical set is introduced in [5, Definition 1.1]. The terminology comes from elementary differential geometry where a differentiable maps  $\gamma : [0, 1] \rightarrow \mathbb{R}^n$  is called regular if  $p'(t) \neq 0$  for all  $t \in ]0, 1[$ . Intuitively, it is a  $d$ -path without zero speed point.

By starting from the topological  $n$ -cube equipped with the set of regular  $d$ -paths between two vertices of the  $n$ -cube, we introduce the regular realization  $|K|_{reg}$  of a precubical set  $K$  as a multipointed  $d$ -space. Since the concatenation of two regular  $d$ -paths is still regular, all execution paths of this multipointed  $d$ -space are regular  $d$ -paths. However, the regular realization of a precubical set does not contain as execution paths all regular  $d$ -paths between two vertices of the precubical set, but only the tame ones, i.e. the regular  $d$ -paths which are concatenations of  $d$ -paths going from the initial state to the final state of a cube of the precubical set. Thanks to the tamification theorem [32, Theorem 6.1], this point of view is not restrictive.

It is not clear how the multipointed  $d$ -space  $|K|_{reg}$  interacts with the three known model category structures (q,h,m) of the category of multipointed  $d$ -spaces constructed in [15, Theorem 6.14]: see Appendix A for a discussion about this problem. On the other hand, the Moore flow  $\mathbb{M}^{\mathcal{G}}|K|_{reg}$  which is obtained by forgetting the underlying topological space of the multipointed  $d$ -space  $|K|_{reg}$  and by keeping only the execution paths and the information about their reparametrization, is always m-cofibrant. From this information only, and by fiddling a little bit with homotopy theory, we deduce immediately that the space of tame regular  $d$ -paths between two vertices of a precubical set is m-cofibrant, and therefore that it is homotopy equivalent to a CW-complex: see Corollary 5.12. We obtain therefore a purely model category interpretation of this known fact. The technical results of the paper are now listed.

**Theorem.** (Corollary 5.9) *The composite functor*

$$\mathbb{M}^{\mathcal{G}}| - |_{reg} : \square^{op}\mathbf{Set} \rightarrow \mathcal{G}\mathbf{dTop} \rightarrow \mathcal{G}\mathbf{Flow}$$

*which takes a precubical set to the Moore flow associated with its regular realization is colimit-preserving.*

The right adjoint  $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \rightarrow \mathcal{G}\mathbf{Flow}$  from multipointed  $d$ -spaces to Moore flows is not colimit-preserving in general. As any right adjoint between locally presentable category, it is accessible. However it is not finitely accessible. This is due to the fact that the convenient category of topological spaces we are working with (see the beginning of Section 2) is locally  $(2^{\aleph_0})^+$ -presentable and not locally finitely presentable. However, it does commute with some particular transfinite compositions [12, Theorem 6.14] [13, Theorem 5.7]. The right adjoint  $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \rightarrow \mathcal{G}\mathbf{Flow}$  is also part of a Quillen equivalence  $\mathbb{M}_!^{\mathcal{G}} \dashv \mathbb{M}^{\mathcal{G}}$  between the q-model structures of multipointed  $d$ -spaces and Moore flows by [13, Theorem 8.1] which satisfies a very specific property: the unit and the counit of this Quillen equivalence induce isomorphisms between the q-cofibrant objects [13, Theorem 7.6 and Corollary 7.9]. It means that the right adjoint  $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \rightarrow \mathcal{G}\mathbf{Flow}$  preserves the

colimits used to build the cellular multipointed  $d$ -spaces. These facts and Corollary 5.9 are an illustration of the following informal conjecture: the functor  $\mathbb{M}^{\mathcal{G}} : \mathcal{GdTop} \rightarrow \mathcal{GFlow}$  preserves good enough colimits.

The proof of Corollary 5.9 has some similarities with the one of [13, Theorem 7.3]. In both cases, it must be proved that some continuous bijection is a homeomorphism because it is a  $\Delta$ -inclusion. The proof is simpler in the cubical setting of this paper than in the globular setting of [13] thanks to the key geometric property of the precubical sets: the fact that each  $d$ -path has a  $L_1$ -arc length which is the same on a path-connected component of the space of  $d$ -paths. It is not true in the globular setting. It is the reason why an important technical part of the proof of [13, Theorem 7.3] consists of verifying that the analogue in the globular setting of the  $L_1$ -arc length is bounded on a path-connected component of the space of execution paths of a cellular multipointed  $d$ -space: it is the role of [13, Theorem 5.19].

We do not know any proof of [13, Theorem 7.3] and most of its consequences, of Theorem 5.8 and of Corollary 5.9 outside the category of  $\Delta$ -generated spaces. The key topological facts in all these proofs are that  $\Delta$ -generated spaces are generated by the segment  $[0, 1]$  which is path-connected and that all  $\Delta$ -generated spaces are sequential. It is not true for  $k$ -spaces: the Cantor space is not path-connected and  $[0, 1]^{[0, 1]}$  equipped with the product topology is not sequential [31, Example 29 and Example 105].

We then deduce, as an application, the two following theorems.

**Theorem.** (Theorem 5.11) *For every precubical set  $K$ , the Moore flow  $\mathbb{M}^{\mathcal{G}}|K|_{reg}$  is m-cofibrant.*

**Theorem.** (Theorem 6.5) *Let  $cat : \mathcal{GdTop} \rightarrow \mathbf{Flow}$  be the categorization functor taking a multipointed  $d$ -space to the associated flow (obtained by quotienting by the reparametrization). Then, for all precubical sets  $K$ , there is the natural isomorphism of flows*

$$cat(|K|_{reg}) \cong |K|_{nat}$$

where  $|K|_{nat}$  is the natural realization of  $K$  as a flow introduced in [16].

The notion of natural  $d$ -path of a precubical set enables us in [16] to find a way to realize any precubical set as a m-cofibrant flow without using any non-canonical choice of any cofibrant replacement functor on the category of flows. In this paper, the notion of regular  $d$ -path of a precubical set enables us to obtain a realization of any precubical set as a multipointed  $d$ -space without using any non-canonical choice of any cofibrant replacement functor on the category of multipointed  $d$ -spaces either. In both cases, we obtain much better results than in [9] because all constructions become canonical thanks to the geometric properties of the cubes. However, the associated Moore flow of the regular realization is m-cofibrant, and not q-cofibrant. It is the counterpart of these improvements: unlike in [9], we have to deal with accessible model categories which are unlikely to be combinatorial.

**Outline of the paper.** Section 2 introduces the regular realization of a precubical set as a multipointed  $d$ -space. It is built using the notion of regular  $d$ -path of the topological  $n$ -cube. The execution paths of the regular realization are exactly the tame regular  $d$ -paths of the precubical set between two vertices. Section 3 is devoted to the important

notion of  $L_1$ -arc length of a  $d$ -path. We adapt some results coming from [27] to our setting of  $\Delta$ -generated spaces. Theorem 3.12 is even an improved statement of a similar statement in [27]: a homotopy equivalence is replaced by a homeomorphism. In Section 4, we construct the  $\{q, m, h\}$ -model structures of Moore flows. It is an easy adaptation of the case of flows treated in [15, Theorem 7.4]. Section 5 is the core of the paper. It proves that the Moore flow associated with the regular realization of a precubical set yields a colimit-preserving functor, despite the fact that the functor from multipointed  $d$ -spaces to Moore flows is a right adjoint, and not a left adjoint. In the same section, it is proved as an application that the Moore flow of the regular realization of any precubical set is m-cofibrant. Finally Section 6 proves that the flow associated with the regular realization of a precubical set coincides with its natural realization constructed in [16].

**Prerequisites and notations.** We refer to [1] for locally presentable categories, to [29] for combinatorial model categories. We refer to [22] and to [21] for more general model categories, and to [7, 20, 30] for accessible model categories. We work with the category **Top** of  $\Delta$ -generated spaces or of  $\Delta$ -Hausdorff  $\Delta$ -generated spaces (cf. [14, Section 2 and Appendix B]). As explained in the introduction, we do not know how to prove Theorem 5.8 without using  $\Delta$ -generated spaces. It is not even clear that it still holds e.g. for  $k$ -spaces. The category **Top** is equipped with its  $q$ -model structure (we use the terminology of [25]). The  $m$ -model structure [4] and the  $h$ -model structure [2] of **Top** are also used in various places of the paper. Compact means quasicompact Hausdorff (French convention). The terminal object of a category is denoted by **1**. We summarize some basic properties of **Top** for the convenience of the reader:

- **Top** is locally presentable.
- All objects of **Top** are sequential topological spaces.
- A closed subset of a  $\Delta$ -generated space equipped with the relative topology is not necessarily  $\Delta$ -generated (e.g. the Cantor set), but it is always sequential.
- The inclusion functor from the full subcategory of  $\Delta$ -generated spaces to the category of general topological spaces together with the continuous maps has a right adjoint called the  $\Delta$ -kelleyfication functor. The latter functor does not change the underlying set.
- The colimit in **Top** is given by the final topology in the following situations:
  - A transfinite compositions of one-to-one maps.
  - A pushout along a closed inclusion.
  - A quotient by a closed subset or by an equivalence relation having a closed graph.
In these cases, the underlying set of the colimit is therefore the colimit of the underlying sets. In particular, the CW-complexes are equipped with the final topology, and therefore are Hausdorff.
- **Top** is cartesian closed. The internal hom **TOP**( $X, Y$ ) is given by taking the  $\Delta$ -kelleyfication of the compact-open topology on the set  $\mathcal{TOP}(X, Y)$  of all continuous maps from  $X$  to  $Y$ .

## 2. THE REGULAR REALIZATION OF A PRECUBICAL SET

**2.1. Notation.** *The notations  $\ell, \ell', \ell_i, L, \dots$  mean a strictly positive real number unless specified something else.*

**2.2. Notation.** Let  $\ell > 0$ . Let  $\mu_\ell : [0, \ell] \rightarrow [0, 1]$  be the homeomorphism defined by  $\mu_\ell(t) = t/\ell$ .

**2.3. Notation.** The notation  $[0, \ell_1] \cong^+ [0, \ell_2]$  means a nondecreasing homeomorphism from  $[0, \ell_1]$  to  $[0, \ell_2]$  with  $\ell_1, \ell_2 > 0$ . Let

$$\mathcal{G}(\ell_1, \ell_2) = \{[0, \ell_1] \cong^+ [0, \ell_2]\}$$

for  $\ell_1, \ell_2 > 0$ .

The sets  $\mathcal{G}(\ell_1, \ell_2)$  are equipped with the  $\Delta$ -kelleyfication of the compact-open topology, which coincides with the compact-open topology, and with the pointwise convergence topology by [13, Proposition 2.5].

**2.4. Definition.** Let  $\gamma_1$  and  $\gamma_2$  be two continuous maps from  $[0, 1]$  to some topological space such that  $\gamma_1(1) = \gamma_2(0)$ . The composite defined by

$$(\gamma_1 *_N \gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \gamma_2(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is called the *normalized composition*.

A *multipointed  $d$ -space*  $X$  [10] is a triple  $(|X|, X^0, \mathbb{P}^{\mathcal{G}}X)$  where

- The pair  $(|X|, X^0)$  is a multipointed space. The space  $|X|$  is called the *underlying space* of  $X$  and the set  $X^0$  the *set of states* of  $X$ .
- The set  $\mathbb{P}^{\mathcal{G}}X$  is a set of continuous maps from  $[0, 1]$  to  $|X|$  called the *execution paths*, satisfying the following axioms:
  - For any execution path  $\gamma$ , one has  $\gamma(0), \gamma(1) \in X^0$ .
  - Let  $\gamma$  be an execution path of  $X$ . Then any composite  $\gamma\phi$  with  $\phi : [0, 1] \cong^+ [0, 1]$  is an execution path of  $X$ .
  - Let  $\gamma_1$  and  $\gamma_2$  be two composable execution paths of  $X$ ; then the normalized composition  $\gamma_1 *_N \gamma_2$  is an execution path of  $X$ .

A map  $f : X \rightarrow Y$  of multipointed  $d$ -spaces is a map of multipointed spaces from  $(|X|, X^0)$  to  $(|Y|, Y^0)$  such that for any execution path  $\gamma$  of  $X$ , the map  $\mathbb{P}^{\mathcal{G}}f : \gamma \mapsto f.\gamma$  is an execution path of  $Y$ . The category of multipointed  $d$ -spaces is denoted by  $\mathcal{GdTop}$ . It is locally presentable by [10, Theorem 3.5]. The subset of execution paths from  $\alpha$  to  $\beta$  is the set of  $\gamma \in \mathbb{P}^{\mathcal{G}}X$  such that  $\gamma(0) = \alpha$  and  $\gamma(1) = \beta$ ; it is denoted by  $\mathbb{P}_{\alpha, \beta}^{\mathcal{G}}X$ :  $\alpha$  is called the *initial state* and  $\beta$  the *final state* of such a  $\gamma$ . The set  $\mathbb{P}_{\alpha, \beta}^{\mathcal{G}}X$  is equipped with the  $\Delta$ -kelleyfication of the relative topology induced by the inclusion  $\mathbb{P}_{\alpha, \beta}^{\mathcal{G}}X \subset \mathbf{TOP}([0, 1], |X|)$ .

**2.5. Notation.** Let  $[0] = \{()\}$  and  $[n] = \{0, 1\}^n$  for  $n \geq 1$ . By convention, one has  $\{0, 1\}^0 = [0] = \{()\}$ . The set  $[n]$  is equipped with the product ordering  $\{0 < 1\}^n$ . Let  $0_n = (0, \dots, 0) \in \{0, 1\}^n$  and  $1_n = (1, \dots, 1) \in \{0, 1\}^n$

Let  $\delta_i^\alpha : [n-1] \rightarrow [n]$  be the coface map defined for  $1 \leq i \leq n$  and  $\alpha \in \{0, 1\}$  by  $\delta_i^\alpha(\epsilon_1, \dots, \epsilon_{n-1}) = (\epsilon_1, \dots, \epsilon_{i-1}, \alpha, \epsilon_i, \dots, \epsilon_{n-1})$ . The small category  $\square$  is by definition the subcategory of the category of sets with the set of objects  $\{[n], n \geq 0\}$  and generated by the morphisms  $\delta_i^\alpha$ . They satisfy the cocubical relations  $\delta_j^\beta \delta_i^\alpha = \delta_i^\alpha \delta_{j-1}^\beta$  for  $i < j$  and for all  $(\alpha, \beta) \in \{0, 1\}^2$ . If  $p > q \geq 0$ , then the set of morphisms  $\square([p], [q])$  is empty. If  $p = q$ ,

then the set  $\square([p], [p])$  is the singleton  $\{\text{Id}_{[p]}\}$ . For  $0 \leq p \leq q$ , all maps of  $\square$  from  $[p]$  to  $[q]$  are one-to-one. A good reference for presheaves is [24].

**2.6. Definition.** [3] *The category of presheaves over  $\square$ , denoted by  $\square^{op}\mathbf{Set}$ , is called the category of precubical sets. A precubical set  $K$  consists of a family of sets  $(K_n)_{n \geq 0}$  and of set maps  $\partial_i^\alpha : K_n \rightarrow K_{n-1}$  with  $1 \leq i \leq n$  and  $\alpha \in \{0, 1\}$  satisfying the cubical relations  $\partial_i^\alpha \partial_j^\beta = \partial_{j-1}^\beta \partial_i^\alpha$  for any  $\alpha, \beta \in \{0, 1\}$  and for  $i < j$ . An element of  $K_n$  is called a  $n$ -cube. Let  $\dim(x) = n$  if  $x \in K_n$ . Let*

$$K_{\leq n} = \varinjlim_{\substack{\square[p] \rightarrow K \\ p \leq n}} \square[p].$$

Let  $\square[n] = \square(-, [n])$ . The boundary of  $\square[n]$  is the precubical set denoted by  $\partial\square[n]$  defined by removing the interior of  $\square[n]$ :  $(\partial\square[n])_k = (\square[n])_k$  for  $k < n$  and  $(\partial\square[n])_k = \emptyset$  for  $k \geq n$ . In particular, one has  $\partial\square[0] = \emptyset$ .

**2.7. Definition.** *A cocubical object of a category  $\mathcal{C}$  is a functor  $\square \rightarrow \mathcal{C}$ .*

**2.8. Notation.** *Let  $\mathcal{C}$  be a cocomplete category. Let  $X : \square \rightarrow \mathcal{C}$  be a functor. Let*

$$\widehat{X}(K) = \varinjlim_{\square[n] \rightarrow K} X(\square[n]).$$

**2.9. Proposition.** [9, Proposition 2.3.2] *Let  $\mathcal{C}$  be a cocomplete category. The mapping  $X \mapsto \widehat{X}$  induces an equivalence of categories between the category of cocubical objects of  $\mathcal{C}$  and the colimit-preserving preserving functors from  $\square^{op}\mathbf{Set}$  to  $\mathcal{C}$ .*

**2.10. Definition.** *Let  $U$  be a topological space. A (Moore) path of  $U$  consists of a continuous map  $\gamma : [0, \ell] \rightarrow U$  with  $\ell > 0$ . The real number  $\ell > 0$  is called the length of the path. It can be extended to a continuous map  $\gamma : [0, +\infty[ \rightarrow U$  such that  $\gamma(t) = \gamma(\ell)$  for all  $t \geq \ell$ .*

The notions of stop interval and regular path appear in [5, Definition 1.1].

**2.11. Definition.** *Let  $U$  be a Hausdorff topological space<sup>1</sup>. Let  $\gamma : [0, \ell] \rightarrow U$  be a Moore path of  $U$  with  $\ell > 0$ . A  $\gamma$ -stop interval is an interval  $[a, b] \subset [0, \ell]$  with  $a < b$  such that the restriction  $\gamma \upharpoonright_{[a, b]}$  is constant and such that  $[a, b]$  is maximal for this property. The set of  $\gamma$ -stop intervals is denoted by  $\Delta_\gamma$ . The path  $\gamma$  is regular if  $\Delta_\gamma = \emptyset$ .*

**2.12. Definition.** *Let  $\gamma_1 : [0, \ell_1] \rightarrow U$  and  $\gamma_2 : [0, \ell_2] \rightarrow U$  be two paths of a topological space  $U$  such that  $\gamma_1(\ell_1) = \gamma_2(0)$ . The Moore composition  $\gamma_1 * \gamma_2 : [0, \ell_1 + \ell_2] \rightarrow U$  is the Moore path defined by*

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{for } t \in [0, \ell_1] \\ \gamma_2(t - \ell_1) & \text{for } t \in [\ell_1, \ell_1 + \ell_2]. \end{cases}$$

*The Moore composition of Moore paths is strictly associative. The Moore composition of two regular paths of a Hausdorff space is regular.*

There is the obvious proposition:

<sup>1</sup>This definition will be used only in the case of  $U$  being the geometric realization of a precubical set which is Hausdorff: see Notation 3.1; the Hausdorff condition is used here only to ensure that the point of  $U$  are closed, which implies that the  $\gamma$ -stop intervals are closed segments.

**2.13. Proposition.** *Let  $n \geq 1$ . The following data assemble into a multipointed  $d$ -space  $|\square[n]|_{reg}$  called the regular  $n$ -cube:*

- *The underlying space is the topological  $n$ -cube  $[0, 1]^n$ .*
- *The set of states is  $\{0, 1\}^n \subset [0, 1]^n$ .*
- *The set of execution paths from  $\underline{a}$  to  $\underline{b}$  with  $\underline{a} < \underline{b} \in \{0 < 1\}^n$  is the set of regular paths  $[0, 1] \rightarrow [0, 1]^n$  from  $\underline{a}$  to  $\underline{b}$  which are nondecreasing with respect to each axis of coordinates; in the other cases  $\underline{a} \geq \underline{b}$ , it is empty.*

*Let  $|\square[0]|_{reg} = \{()\}$ . Let  $\delta_i^\alpha : [0, 1]^{n-1} \rightarrow [0, 1]^n$  be the continuous map defined for  $1 \leq i \leq n$  and  $\alpha \in \{0, 1\}$  by  $\delta_i^\alpha(\epsilon_1, \dots, \epsilon_{n-1}) = (\epsilon_1, \dots, \epsilon_{i-1}, \alpha, \epsilon_i, \dots, \epsilon_{n-1})$ . By convention, let  $[0, 1]^0 = \{()\}$ . The mapping  $[n] \mapsto |\square[n]|_{reg}$  yields a well-defined cocubical multipointed  $d$ -space.*

Using Proposition 2.9, this leads to the definition:

**2.14. Definition.** *Let  $K$  be a precubical set. The regular realization of  $K$  is the multipointed  $d$ -space*

$$|K|_{reg} = \varinjlim_{\square[n] \rightarrow K} |\square[n]|_{reg}.$$

*It yields a colimit-preserving functor from precubical sets to multipointed  $d$ -spaces.*

### 3. $L_1$ -ARC LENGTH OF $d$ -PATHS OF PRECUBICAL SETS

**3.1. Notation.** *The cocubical topological space  $[n] \mapsto [0, 1]^n$  gives rise to a colimit-preserving functor from precubical sets to topological spaces denoted by*

$$|K|_{geom} = \varinjlim_{\square[n] \rightarrow K} [0, 1]^n.$$

Since the underlying space functor from multipointed  $d$ -spaces to spaces is colimit-preserving, the topological space  $|K|_{geom}$  is the underlying space of the multipointed  $d$ -space  $|K|_{reg}$ . The topological space  $|K|_{geom}$  is a CW-complex. It is equipped with the final topology which is  $\Delta$ -generated and Hausdorff. We introduce the following definition:

**3.2. Definition.** *Let  $K$  be a precubical set. Let  $(\alpha, \beta) \in K_0 \times K_0$ . Let  $\vec{R}_{\alpha, \beta}^\ell(K)$  be the subspace of continuous maps from  $[0, \ell]$  to  $|K|_{geom}$  defined by*

$$\vec{R}_{\alpha, \beta}^\ell(K) = \{t \mapsto \gamma \mu_\ell \mid \gamma \in \mathbb{P}_{\alpha, \beta}^\mathcal{G} |K|_{reg}\}.$$

*Its elements are called in this paper the (tame regular)  $d$ -paths of  $K$  (of length  $\ell$ ) from  $\alpha$  to  $\beta$ . Let*

$$\vec{R}^\ell(K) = \coprod_{(\alpha, \beta) \in K_0 \times K_0} \vec{R}_{\alpha, \beta}^\ell(K).$$

The definition of  $\vec{R}_{\alpha, \beta}^\ell(K)$  is not restrictive. Indeed, we have:

**3.3. Proposition.** *Let  $K$  be a precubical set. Let  $\phi : [0, \ell] \cong^+ [0, \ell]$ . Let  $\gamma \in \vec{R}_{\alpha, \beta}^\ell(K)$ . Then  $\gamma \phi \in \vec{R}_{\alpha, \beta}^\ell(K)$ .*

*Proof.* By definition of  $\vec{R}_{\alpha, \beta}^\ell(K)$ , there exists  $\bar{\gamma} \in \mathbb{P}_{\alpha, \beta}^\mathcal{G} |K|_{reg}$  such that  $\gamma = \bar{\gamma} \mu_\ell$ . We obtain  $\gamma \phi = \bar{\gamma} \mu_\ell \phi \mu_\ell^{-1} \mu_\ell$ . Since  $\mu_\ell \phi \mu_\ell^{-1} \in \mathcal{G}(1, 1)$ , we deduce that  $\bar{\gamma} \mu_\ell \phi \mu_\ell^{-1} \in \mathbb{P}_{\alpha, \beta}^\mathcal{G} |K|_{reg}$  and that  $\gamma \phi \in \vec{R}_{\alpha, \beta}^\ell(K)$ .  $\square$



**3.4. Notation.** Let  $\underline{x} = (x_1, \dots, x_n)$  and  $\underline{x}' = (x'_1, \dots, x'_n)$  be two elements of  $[0, 1]^n$  with  $n \geq 1$ . Let

$$d_1(\underline{x}, \underline{x}') = \sum_{i=1}^n |x_i - x'_i|.$$

An important feature shared by all  $d$ -paths (regular or not, tame or not) of a precubical set  $K$  is that they have a well-defined  $L_1$ -arc length [27, Section 2.2.1] [28, Section 2.2]. It is defined as follows. Consider a  $d$ -path  $\gamma : [0, \ell] \rightarrow [0, 1]^n$ . Define the  $L_1$ -arc length between  $\gamma(t)$  and  $\gamma(t')$  by  $d_1(\gamma(t), \gamma(t'))$ . The  $L_1$ -arc length of a  $d$ -path of  $[0, 1]^n$  between  $0_n$  and  $1_n$  is therefore  $n$ . The  $L_1$ -arc length of a Moore composition of  $d$ -paths is defined by adding the  $L_1$ -arc length of each  $d$ -path.

**3.5. Remark.** The length of a  $d$ -path  $\gamma : [0, \ell] \rightarrow [0, 1]^n$  between two vertices of  $\{0, 1\}^n$  is  $\ell$ , whereas its  $L_1$ -arc length is an integer belonging to  $\{1, \dots, n\}$ .

**3.6. Proposition.** Let  $K$  be a precubical set. Let  $(\alpha, \beta) \in K_0 \times K_0$ . Two execution paths of  $\mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|K|_{reg}$  which are in the same path-connected component have the same  $L_1$ -arc length.

*Proof.* Write  $(\mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|K|_{reg})_{co}$  for the underlying set of  $\mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|K|_{reg}$  endowed with the compact-open topology. Let  $\vec{P}_{\alpha, \beta}(K)_{co}$  be the set of all  $d$ -paths of  $K$  from  $\alpha$  to  $\beta$  equipped with the compact-open topology. By [28, Proposition 2.2], a composite map of the form  $[0, 1] \rightarrow (\mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|K|_{reg})_{co} \subset \vec{P}_{\alpha, \beta}(K)_{co} \rightarrow \mathbb{R}$  is constant, where the right-hand map is the  $L_1$ -arc length. The  $\Delta$ -kelleyfication functor does not change the path-connected components. Hence the proof is complete.  $\square$

Due to the variety of the terminologies used in the mathematical literature, we recall the following definition for the convenience of the reader.

**3.7. Definition.** A pseudometric space  $(X, d)$  is a set  $X$  equipped with a map  $d : X \times X \rightarrow [0, +\infty]$  called a pseudometric such that:

- $\forall x \in X, d(x, x) = 0$
- $\forall (x, y) \in X \times X, d(x, y) = d(y, x)$
- $\forall (x, y, z) \in X \times X \times X, d(x, y) \leq d(x, z) + d(z, y)$ .

A map  $f : (X, d) \rightarrow (Y, d)$  of pseudometric spaces is a set map  $f : X \rightarrow Y$  which does not increase distance, i.e.  $\forall (x, y) \in X \times X, d(f(x), f(y)) \leq d(x, y)$ .

Every metric space is a pseudometric space. The interest of this category for Proposition 3.8 lies in the following two facts. At first, it is bicomplete, being a coreflective subcategory of the bicomplete category of Lawvere metric spaces<sup>2</sup> [23]. Secondly, the family of balls  $B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$  with  $x \in X$  and  $\epsilon > 0$  generates a topology called the underlying topology of  $(X, d)$ . This gives rise to a functor from pseudometric spaces to general topological spaces.

**3.8. Proposition.** Let  $K$  be a precubical set. Let  $\ell > 0$ . The set map

$$L : \vec{R}^{\ell}(K) \times [0, \ell] \longrightarrow [0, +\infty[$$

which takes  $(\gamma, t)$  to the  $L_1$ -arc length between  $\gamma(0)$  and  $\gamma(t)$  is continuous.

<sup>2</sup>which are pseudometric spaces with the symmetry condition dropped

*Proof.* By equipping each topological  $n$ -cube  $[0, 1]^n$  for  $n \geq 0$  with the metric  $d_1$ , we obtain a cocubical pseudometric space, and by Proposition 2.9, a colimit-preserving functor from precubical sets to pseudometric spaces denoted by  $K \mapsto |K|_{d_1}$ . Since there is a homeomorphism  $|\square[n]|_{geom} \cong |\square[n]|_{d_1}$  for all  $n \geq 0$ , the set of tame regular  $d$ -paths in  $|K|_{geom}$  coincides with the set of tame regular  $d$ -paths in  $|K|_{d_1}$  and the universal property of the colimit proves that the identity induces a continuous map  $|K|_{geom} \rightarrow |K|_{d_1}$ . Let  $\vec{P}(K)_{d_1}$  be the set of all  $d$ -paths of  $K$  equipped with the compact-open topology associated with the underlying topological space of  $|K|_{d_1}$ . Using [27, Lemma 2.13], the set map  $L : \vec{P}(K)_{d_1} \times [0, +\infty[ \rightarrow [0, +\infty[$  taking a pair  $(\gamma, t)$  to the  $L_1$ -arc length between  $\gamma(0)$  and  $\gamma(t)$  is continuous. Let  $(\vec{R}^\ell(K))_{d_1}$  ( $(\vec{R}^\ell(K))_{co}$  resp.) be the underlying set of the space  $\vec{R}^\ell(K)$  equipped with the compact-open topology associated with the underlying topological space of  $|K|_{d_1}$  (associated with the topological space  $|K|_{geom}$  resp.). Since the identity induces a continuous map  $|K|_{geom} \rightarrow |K|_{d_1}$ , we obtain a continuous map

$$L : (\vec{R}^\ell(K))_{co} \times [0, \ell] \rightarrow (\vec{R}^\ell(K))_{d_1} \times [0, \ell] \subset \vec{P}(K)_{d_1} \times [0, +\infty[ \rightarrow [0, +\infty[.$$

Finally, take the image by the  $\Delta$ -kelleyfication functor. The latter is a right adjoint therefore it preserves binary products. Besides,  $[0, \ell]$  and  $[0, +\infty[$  are already  $\Delta$ -generated. Hence the proof is complete.  $\square$

**3.9. Notation.** *By adjunction, we obtain a continuous map*

$$\hat{L} : \vec{R}^\ell(K) \longrightarrow \mathbf{TOP}([0, \ell], [0, +\infty[).$$

**3.10. Lemma.** *For all  $r \in \vec{R}^\ell(K)$ , for all  $\phi \in \mathcal{G}(\ell', \ell)$ , and for all  $t \in [0, \ell']$ , one has*

$$\hat{L}(r\phi)(t) = \hat{L}(r)(\phi(t)).$$

*Proof.* The  $L_1$ -arc length between  $r(0) = r(\phi(0))$  and  $r(\phi(t))$  for the  $d$ -path  $r$  is equal to the  $L_1$ -arc length between  $(r\phi)(0)$  and  $(r\phi)(t)$  for the  $d$ -path  $r\phi$ .  $\square$

Intuitively, the natural  $d$ -paths are the  $d$ -paths whose speed corresponds to the  $L_1$ -arc length.

**3.11. Definition.** [27, Definition 2.14] *Let  $K$  be a precubical set. Let  $(\alpha, \beta) \in K_0 \times K_0$ . A  $d$ -path  $\gamma$  of  $\vec{R}_{\alpha, \beta}^\ell(K)$  is natural if  $\hat{L}(\gamma)(t) = t$  for all  $t \in [0, \ell]$ . This implies that  $\ell$  is an integer. The subset of natural  $d$ -paths of length  $n \geq 1$  from  $\alpha$  to  $\beta$  equipped with the  $\Delta$ -kelleyfication of the relative topology is denoted by  $\vec{N}_{\alpha, \beta}^n(K)$ . Let*

$$\vec{N}^\ell(K) = \coprod_{(\alpha, \beta) \in K_0 \times K_0} \vec{N}_{\alpha, \beta}^\ell(K).$$

The following theorem is an improvement and an adaptation in our topological setting of [27, Proposition 2.16]: the homotopy equivalence is replaced by a homeomorphism thanks to the spaces  $\mathcal{G}(\ell, n)$ .

**3.12. Theorem.** *Let  $K$  be a precubical set. Let  $(\alpha, \beta) \in K_0 \times K_0$ . There is a homeomorphism*

$$\Psi^\ell : \vec{R}_{\alpha, \beta}^\ell(K) \xrightarrow{\cong} \prod_{n \geq 1} \mathcal{G}(\ell, n) \times \vec{N}_{\alpha, \beta}^n(K).$$

*Proof.* By Proposition 3.6, the space  $\vec{R}_{\alpha,\beta}^\ell(K)$  is the direct sum of the subspaces  $\vec{R}_{\alpha,\beta}^{\ell,n}(K)$  of regular  $d$ -paths of  $L_1$ -arc length  $n$  for  $n \geq 1$ . It then suffices to prove the homeomorphism  $\vec{R}_{\alpha,\beta}^{\ell,n}(K) \cong \mathcal{G}(\ell, n) \times \vec{N}_{\alpha,\beta}^n(K)$  for all  $n \geq 1$ . Consider the continuous map  $\Phi^\ell : \mathcal{G}(\ell, n) \times \vec{N}_{\alpha,\beta}^n(K) \rightarrow \vec{R}_{\alpha,\beta}^{\ell,n}(K)$  defined by

$$\Phi^\ell(\phi, \gamma) = \gamma\phi.$$

Since  $\widehat{L}(\gamma\phi) = \widehat{L}(\gamma)\phi = \phi$ , the first equality by Lemma 3.10 and the second equality since  $\gamma$  is natural, the  $d$ -path  $\gamma\phi$  has no stop-interval and it is therefore regular. We want to define a continuous map  $\Psi^\ell : \vec{R}_{\alpha,\beta}^{\ell,n}(K) \rightarrow \mathcal{G}(\ell, n) \times \vec{N}_{\alpha,\beta}^n(K)$ . Let  $r \in \vec{R}_{\alpha,\beta}^{\ell,n}(K)$ . Then  $\widehat{L}(r) : [0, \ell] \rightarrow [0, n]$  is surjective and nondecreasing. Let  $t, t' \in [0, 1]$  such that  $\widehat{L}(r)(t) = \widehat{L}(r)(t')$ . Since  $r$  is regular, it has no stop interval. It implies that  $t = t'$ . Thus,  $\widehat{L}(r) \in \mathcal{G}(\ell, n)$ . Let

$$\Psi^\ell(r) = (\widehat{L}(r), r\widehat{L}(r)^{-1}).$$

The map  $\Psi^\ell$  is continuous by Proposition 3.8 and by [13, Lemma 6.2]. Then one has

$$\widehat{L}(r\widehat{L}(r)^{-1})(t) = \widehat{L}(r)(\widehat{L}(r)^{-1}(t)) = t$$

for all  $0 \leq t \leq n$ , the first equality by Lemma 3.10 and the second equality by algebraic simplification. It means that  $r\widehat{L}(r)^{-1} \in \vec{N}_{\alpha,\beta}^n(K)$ : the path  $r\widehat{L}(r)^{-1}$  is called the naturalization of  $r$  in [27, Definition 2.14]. One has

$$\Phi^\ell \Psi^\ell(r) = \Phi^\ell(\widehat{L}(r), r\widehat{L}(r)^{-1}) = r\widehat{L}(r)^{-1}\widehat{L}(r) = r,$$

the first equality by definition of  $\Psi^\ell$ , the second equality by definition of  $\Phi^\ell$  and the last equality by algebraic simplification. Let  $(\phi, \gamma) \in \mathcal{G}(\ell, n) \times \vec{N}_{\alpha,\beta}^n(K)$ . Since  $\widehat{L}(\gamma\phi) = \phi$ , we obtain

$$\Psi^\ell \Phi^\ell(\phi, \gamma) = \Psi^\ell(\gamma\phi) = (\widehat{L}(\gamma\phi), \gamma\phi\widehat{L}(\gamma\phi)^{-1}) = (\phi, \gamma),$$

the first equality by definition of  $\Phi^\ell$ , the second equality by definition of  $\Psi^\ell$ , and the last equality by algebraic simplification.  $\square$

#### 4. THE $\{\mathbf{Q}, \mathbf{H}, \mathbf{M}\}$ -MODEL STRUCTURES OF MOORE FLOWS

The enriched small category  $\mathcal{G}$  is defined as follows:

- The set of objects is the open interval  $]0, \infty[$ .
- The space  $\mathcal{G}(\ell_1, \ell_2)$  is the set  $\{[0, \ell_1] \cong^+ [0, \ell_2]\}$  for all  $\ell_1, \ell_2 > 0$  equipped with the  $\Delta$ -kelleyfication of the relative topology induced by the set inclusion  $\mathcal{G}(\ell_1, \ell_2) \subset \mathbf{TOP}([0, \ell_1], [0, \ell_2])$ . By [13, Proposition 2.5], this topology coincides with the compact-open topology and the pointwise topology.
- For every  $\ell_1, \ell_2, \ell_3 > 0$ , the composition map

$$\mathcal{G}(\ell_1, \ell_2) \times \mathcal{G}(\ell_2, \ell_3) \rightarrow \mathcal{G}(\ell_1, \ell_3)$$

is induced by the composition of continuous maps.

The enriched category  $\mathcal{G}$  is an example of a reparametrization category in the sense of [12, Definition 4.3] which is different from the terminal category. It is introduced in [12, Proposition 4.9].

**4.1. Definition.** *The enriched category of enriched presheaves from  $\mathcal{G}$  to  $\mathbf{Top}$  is denoted by  $[\mathcal{G}^{op}, \mathbf{Top}]$ . The underlying set-enriched category of enriched maps of enriched*

presheaves is denoted by  $[\mathcal{G}^{op}, \mathbf{Top}]_0$ . The objects of  $[\mathcal{G}^{op}, \mathbf{Top}]_0$  are called the  $\mathcal{G}$ -spaces. Let

$$\mathbb{F}_\ell^{\mathcal{G}^{op}} U = \mathcal{G}(-, \ell) \times U \in [\mathcal{G}^{op}, \mathbf{Top}]_0$$

where  $U$  is a topological space and where  $\ell > 0$ .

The category  $[\mathcal{G}^{op}, \mathbf{Top}]_0$  is locally presentable by [11, Proposition 5.1].

**4.2. Proposition.** [11, Proposition 5.3 and Proposition 5.5] *The category  $[\mathcal{G}^{op}, \mathbf{Top}]_0$  is a full reflective and coreflective subcategory of  $\mathbf{Top}^{\mathcal{G}^{op}}$ . For every  $\mathcal{G}$ -space  $F : \mathcal{G}^{op} \rightarrow \mathbf{Top}$ , every  $\ell > 0$  and every topological space  $X$ , we have the natural bijection of sets*

$$[\mathcal{G}^{op}, \mathbf{Top}]_0(\mathbb{F}_\ell^{\mathcal{G}^{op}} X, F) \cong \mathbf{Top}(X, F(\ell)).$$

**4.3. Theorem.** ([12, Theorem 5.14]) *Let  $D$  and  $E$  be two  $\mathcal{G}$ -spaces. Let*

$$D \otimes E = \int^{(\ell_1, \ell_2)} \mathcal{G}(-, \ell_1 + \ell_2) \times D(\ell_1) \times E(\ell_2).$$

*The pair  $([\mathcal{G}^{op}, \mathbf{Top}]_0, \otimes)$  has the structure of a biclosed semimonoidal category.*

**4.4. Notation.** *Let  $D$  be a  $\mathcal{G}$ -space. Let  $\phi : \ell \rightarrow \ell'$  be a map of  $\mathcal{G}$ . Let  $x \in D(\ell')$ . We will use the notation*

$$x.\phi = D(\phi)(x).$$

*Intuitively,  $x$  is a path of length  $\ell'$  and  $x.\phi$  is a path of length  $\ell$  which is the reparametrization by  $\phi$  of  $x$ .*

By [26, Theorem 6.5(ii)], since all topological spaces are fibrant and since  $(\mathbf{Top}, \times, \{0\})$  is a locally presentable base by [11, Corollary 3.3], the category of  $\mathcal{G}$ -spaces  $[\mathcal{G}^{op}, \mathbf{Top}]_0$  can be endowed with the projective model structure associated with one of the three model structures  $\mathbf{Top}_q, \mathbf{Top}_m, \mathbf{Top}_h$ . They are called the projective q-model structure (m-model structure, h-model structure resp.) and denoted by  $[\mathcal{G}^{op}, \mathbf{Top}_q]_0^{proj}$  ( $[\mathcal{G}^{op}, \mathbf{Top}_m]_0^{proj}$ ,  $[\mathcal{G}^{op}, \mathbf{Top}_h]_0^{proj}$  resp.). The three model structures are accessible. The fibrations are the objectwise fibrations of the corresponding model structure of  $\mathbf{Top}$ . They are called the projective q-fibrations (projective m-fibrations, projective h-fibrations resp.). The weak equivalences are the objectwise weak equivalence of the corresponding model structure of  $\mathbf{Top}$ . Since the projective m-fibrations of spaces are the projective h-fibrations of spaces, and since the weak equivalences of the projective m-model structure of  $\mathcal{G}$ -spaces are the weak equivalences of the projective q-model structure of  $\mathcal{G}$ -spaces, it implies that the projective m-model structure of  $\mathcal{G}$ -spaces is the mixing in the sense of [4, Theorem 2.1] of the projective q-model structure of  $\mathcal{G}$ -spaces and the projective h-model structure of  $\mathcal{G}$ -spaces. All  $\mathcal{G}$ -spaces are fibrant for these three model structures.

When the reparametrization category  $\mathcal{G}$  is replaced by the terminal category, these model structures coincide with the q-model structure, the m-model structure and the h-model structure of  $\mathbf{Top}$ .

**4.5. Definition.** [12, Definition 6.2] *A Moore flow is a small semicategory enriched over the biclosed semimonoidal category  $([\mathcal{G}^{op}, \mathbf{Top}]_0, \otimes)$  of Theorem 4.3. The corresponding category is denoted by  $\mathcal{G}\mathbf{Flow}$ .*

The category  $\mathcal{G}\mathbf{Flow}$  is locally presentable by [12, Theorem 6.11].

**4.6. Definition.** A  $[\mathcal{G}^{op}, \mathbf{Top}]_0$ -graph  $X$  consists of a pair  $(X^0, (\mathbb{P}_{\alpha,\beta}X)_{(\alpha,\beta) \in X^0 \times X^0})$  such that  $X^0$  is a set and such that each  $\mathbb{P}_{\alpha,\beta}X$  is a  $\mathcal{G}$ -space. A map of  $[\mathcal{G}^{op}, \mathbf{Top}]_0$ -graphs  $f : X \rightarrow Y$  consists of a set map  $f^0 : X^0 \rightarrow Y^0$  (called the underlying set map) together with a map  $\mathbb{P}_{\alpha,\beta}X \rightarrow \mathbb{P}_{f^0(\alpha),f^0(\beta)}Y$  of  $\mathcal{G}$ -spaces for all  $(\alpha, \beta) \in X^0 \times X^0$ . The composition is defined in an obvious way. The corresponding category is denoted by  $\mathbf{Gph}([\mathcal{G}^{op}, \mathbf{Top}]_0)$ .

**4.7. Theorem.** Let  $(\mathcal{C}, \mathcal{F}, \mathcal{W})$  be either the projective  $q$ -model structure, or the projective  $m$ -model structure, or the projective  $h$ -model structure of  $\mathcal{G}$ -spaces. Then the category Moore flows can be endowed with an accessible model structure characterized as follows:

- A map of Moore flows  $f : X \rightarrow Y$  is a weak equivalence if and only if  $f^0 : X^0 \rightarrow Y^0$  is a bijection and  $\mathbb{P}f : \mathbb{P}_{\alpha,\beta}X \rightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$  belongs to  $\mathcal{W}$ .
- A map of Moore flows  $f : X \rightarrow Y$  is a fibration if and only if  $f^0 : X^0 \rightarrow Y^0$  is a bijection and  $\mathbb{P}f : \mathbb{P}_{\alpha,\beta}X \rightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$  belongs to  $\mathcal{F}$ .

All objects are fibrant. These three model structures are called the  $q$ -model structure, the  $m$ -model structure and the  $h$ -model structure of  $\mathcal{G}\mathbf{Flow}$ . The  $m$ -model structure of Moore flows is the mixing in the sense of [4, Theorem 2.1] of the  $q$ -model structure and the  $h$ -model structure of Moore flows. The  $q$ -model structure coincides with the one of [12, Theorem 8.8]. Every  $q$ -cofibration of Moore flows is a  $m$ -cofibration of Moore flows and every  $m$ -cofibration of Moore flows is a  $h$ -cofibration of Moore flows.

*Sketch of proof.* This theorem is already proved in [15, Theorem 7.4] for the particular case of the reparametrization category  $\mathcal{G}$  being the terminal category. It is the reason why the proof is only sketched. Choose  $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ . By [15, Corollary 5.6], there exists a unique model structure on the category  $\mathbf{Gph}([\mathcal{G}^{op}, \mathbf{Top}]_0)$  such that the weak equivalences are the maps of enriched graphs which induce a bijection between the sets of vertices and which are objectwise weak equivalences and such that the fibrations are the objectwise fibrations. This model structure is accessible and all enriched graphs are fibrant. There is a forgetful functor  $\mathbf{Gph} : \mathcal{G}\mathbf{Flow} \rightarrow \mathbf{Gph}([\mathcal{G}^{op}, \mathbf{Top}]_0)$  which forgets the composition law. This functor is a right-adjoint. The left adjoint is the free Moore flow generated by the enriched graph. The semicategorical description of Moore flows is crucial: see [15, Proposition 7.3] for a description of the left adjoint. Like in [15, Theorem 7.4], the model structure on enriched graphs of  $\mathcal{G}$ -spaces can be right-lifted along this right adjoint by using [15, Theorem 2.1], namely the Quillen path object argument in model categories where all objects are fibrant, with the path functor  $\mathbf{Path} : \mathcal{G}\mathbf{Flow} \rightarrow \mathcal{G}\mathbf{Flow}$  defined on objects by  $\mathbf{Path}(X)^0 = X^0$ , and for all  $(\alpha, \beta) \in X^0 \times X^0$  and for all  $\ell > 0$  by  $\mathbb{P}_{\alpha,\beta}^\ell \mathbf{Path}(X) = \mathbf{TOP}([0, 1], \mathbb{P}_{\alpha,\beta}^\ell X)$  with an obvious definition of the composition law. The hypotheses of [15, Theorem 2.1] are satisfied because they are satisfied objectwise. Indeed, for all  $(\alpha, \beta) \in X^0 \times X^0$  and for all  $\ell > 0$ , the diagonal of the topological space  $\mathbb{P}_{\alpha,\beta}^\ell X$  factors as a composite  $\mathbb{P}_{\alpha,\beta}^\ell X \rightarrow \mathbf{TOP}([0, 1], \mathbb{P}_{\alpha,\beta}^\ell X) \rightarrow \mathbb{P}_{\alpha,\beta}^\ell X \times \mathbb{P}_{\alpha,\beta}^\ell X$  where the left-hand map is a homotopy equivalence and the right-hand map is a  $r$ -fibration for  $r \in \{q, m, h\}$ . The  $q$ -model structure of this theorem coincides with the one of [12, Theorem 8.8] because the two model structures have the same class of fibrations and weak equivalences. The last sentence is a general fact about mixed model structures (see [4, Theorem 2.1 and Proposition 3.6]).  $\square$

**4.8. Notation.** These three model categories are denoted by  $\mathcal{G}\mathbf{Flow}_q, \mathcal{G}\mathbf{Flow}_m, \mathcal{G}\mathbf{Flow}_h$ .

## 5. MOORE FLOW OF A REGULAR REALIZATION

Let  $X$  be a multipointed  $d$ -space. Let  $\mathbb{P}_{\alpha,\beta}^\ell X$  be the subspace of continuous maps from  $[0, \ell]$  to  $|X|$  defined by  $\mathbb{P}_{\alpha,\beta}^\ell X = \{t \mapsto \gamma\mu_\ell \mid \gamma \in \mathbb{P}_{\alpha,\beta}^\mathcal{G} X\}$ . By [13, Theorem 4.12], there exists a Moore flow  $\mathbb{M}^\mathcal{G}(X)$  such that:

- The set of states  $X^0$  of  $X$
- For all  $\alpha, \beta \in X^0$  and all real numbers  $\ell > 0$ ,  $\mathbb{P}_{\alpha,\beta}^\ell \mathbb{M}^\mathcal{G}(X) = \mathbb{P}_{\alpha,\beta}^\ell X$ .
- For all maps  $[0, \ell] \cong^+ [0, \ell']$ , a map  $f : [0, \ell'] \rightarrow |X|$  of  $\mathbb{P}_{\alpha,\beta}^{\ell'} \mathbb{M}^\mathcal{G}(X)$  is mapped to the map  $[0, \ell] \cong^+ [0, \ell'] \xrightarrow{f} |X|$  of  $\mathbb{P}_{\alpha,\beta}^\ell \mathbb{M}^\mathcal{G}(X)$
- For all  $\alpha, \beta, \gamma \in X^0$  and all real numbers  $\ell, \ell' > 0$ , the composition maps

$$* : \mathbb{P}_{\alpha,\beta}^\ell \mathbb{M}^\mathcal{G}(X) \times \mathbb{P}_{\beta,\gamma}^{\ell'} \mathbb{M}^\mathcal{G}(X) \rightarrow \mathbb{P}_{\alpha,\gamma}^{\ell+\ell'} \mathbb{M}^\mathcal{G}(X)$$

is the Moore composition.

This mapping induces a functor  $\mathbb{M}^\mathcal{G} : \mathcal{G}\mathbf{dTop} \rightarrow \mathcal{G}\mathbf{Flow}$  which is a right adjoint. The following notations coincide with [16, Definition 4.8 and Definition 4.12] by [16, Proposition 4.10 and Proposition 4.14].

**5.1. Notation.** Let  $N_n = \overrightarrow{N}_{0_n, 1_n}^n(\square[n])$  and  $\partial N_n = \overrightarrow{N}_{0_n, 1_n}^n(\partial\square[n])$  for  $n \geq 0$ . These spaces are first countable,  $\Delta$ -generated and  $\Delta$ -Hausdorff.

The map  $[0, 1]^{m_1} \sqcup [0, 1]^{m_2} \rightarrow [0, 1]^{m_1+m_2}$  defined by taking  $(t_1, \dots, t_{m_1})$  to  $(t_1, \dots, t_{m_1}, 0_{m_2})$  and  $(t'_1, \dots, t'_{m_2})$  to  $(1_{m_1}, t'_1, \dots, t'_{m_2})$  induces a continuous map  $N_{m_1} \times N_{m_2} \rightarrow N_{m_1+m_2}$  by using the fact that the Moore composition of two natural  $d$ -paths is still a natural  $d$ -path.

**5.2. Proposition.** The mapping  $(\phi\gamma) \mapsto \gamma\phi$  yields the homomorphisms

$$\begin{aligned} \mathcal{G}(\ell, n) \times N_n &\cong \mathbb{P}_{0_n, 1_n}^\ell \mathbb{M}^\mathcal{G}|\square[n]|_{reg} \\ \mathcal{G}(\ell, n) \times \partial N_n &\cong \mathbb{P}_{0_n, 1_n}^\ell \mathbb{M}^\mathcal{G}|\partial\square[n]|_{reg} \end{aligned}$$

for all  $n \geq 1$  and all  $\ell > 0$ . In particular, for  $\ell = 1$ , we obtain the homomorphisms

$$\begin{aligned} \mathcal{G}(1, n) \times N_n &\cong \mathbb{P}_{0_n, 1_n}^\mathcal{G}|\square[n]|_{reg} \\ \mathcal{G}(1, n) \times \partial N_n &\cong \mathbb{P}_{0_n, 1_n}^\mathcal{G}|\partial\square[n]|_{reg} \end{aligned}$$

for all  $n \geq 1$ . Note that for  $n = 1$ ,  $\partial N_1 = \emptyset = \mathbb{P}_{0_n, 1_n}^\mathcal{G}|\partial\square[n]|_{reg}$ .

*Proof.* The first part is a consequence of Theorem 3.12 applied to the regular  $d$ -paths of length  $\ell$  and of  $L_1$ -arc length  $n$  of the precubical sets  $\square[n]$  and  $\partial\square[n]$ . The particular case  $\ell = 1$  is a consequence of the definition of the functor  $\mathbb{M}^\mathcal{G} : \mathcal{G}\mathbf{dTop} \rightarrow \mathcal{G}\mathbf{Flow}$   $\square$

**5.3. Corollary.** For all  $n \geq 1$ , there are the isomorphisms of  $\mathcal{G}$ -spaces

$$\begin{aligned} \mathbb{F}_n^{\mathcal{G}op} N_n &\cong \mathbb{P}_{0_n, 1_n} \mathbb{M}^\mathcal{G}|\square[n]|_{reg}, \\ \mathbb{F}_n^{\mathcal{G}op} \partial N_n &\cong \mathbb{P}_{0_n, 1_n} \mathbb{M}^\mathcal{G}|\partial\square[n]|_{reg}. \end{aligned}$$

*Proof.* It is a consequence of Proposition 5.2, of the fact that all regular  $d$ -paths from  $0_n$  to  $1_n$  of  $|\square[n]|$  are of  $L_1$ -arc length  $n$ , and of the definition of the functor  $\mathbb{M}^\mathcal{G} : \mathcal{G}\mathbf{dTop} \rightarrow \mathcal{G}\mathbf{Flow}$ .  $\square$

**5.4. Definition.** The cocubical Moore flow  $\mathbb{M}^{\mathcal{G}}|\square[*]|_{reg}$  gives rise by Proposition 2.9 to a colimit-preserving functor  $[-]_{reg} : \square^{op}\mathbf{Set} \rightarrow \mathcal{G}\mathbf{Flow}$  defined by

$$[K]_{reg} = \varinjlim_{\square[n] \rightarrow K} \mathbb{M}^{\mathcal{G}}|\square[n]|_{reg}$$

Before expounding the main result of the paper, we need to recall a basic fact about  $\Delta$ -inclusions.

**5.5. Proposition.** [13, Proposition 2.2 and Corollary 2.3] A continuous bijection  $f : U \rightarrow V$  of  $\mathbf{Top}$  is a homeomorphism if and only if it is a  $\Delta$ -inclusion, i.e. a set map  $[0, 1] \rightarrow A$  is continuous if and only if the composite set map  $[0, 1] \rightarrow A \rightarrow B$  is continuous.

We give a useful example of  $\Delta$ -inclusion in what follows.

**5.6. Proposition.** Let  $K \subset L$  be two precubical sets. Let  $(\alpha, \beta) \in K_0 \times K_0$ . Then there is the equality  $\mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|L|_{reg} \cap \mathbf{TOP}([0, 1], |K|_{geom}) = \mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|K|_{reg}$  and the continuous map  $\mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|K|_{reg} \rightarrow \mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|L|_{reg}$  is a  $\Delta$ -inclusion.

*Proof.* Every execution path of  $\mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|L|_{reg} \cap \mathbf{TOP}([0, 1], |K|_{geom})$  is a Moore composition of regular  $d$ -paths of cubes of  $K$ . It means that

$$\mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|L|_{reg} \cap \mathbf{TOP}([0, 1], |K|_{geom}) \subset \mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|K|_{reg},$$

hence the equality. The underlying topological space of the multipointed  $d$ -space  $|K|_{reg}$  ( $|L|_{reg}$  resp.) is the CW-complex  $|K|_{geom}$  ( $|L|_{geom}$  resp.). Let  $s : [0, 1] \rightarrow \mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|K|_{reg}$  be a set map such that the composite set map  $[0, 1] \rightarrow \mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|K|_{reg} \rightarrow \mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|L|_{reg}$  is continuous. By adjunction, one obtain a continuous map  $[0, 1] \times [0, 1] \rightarrow |L|_{geom}$  such that the image is included in  $|K|_{geom}$ . The point is that the map  $|K|_{geom} \rightarrow |L|_{geom}$  is a q-cofibration of topological spaces, and therefore a closed inclusion. Thus the continuous map  $[0, 1] \times [0, 1] \rightarrow |L|_{geom}$  factors uniquely as a composite of continuous maps  $[0, 1] \times [0, 1] \rightarrow |K|_{geom} \rightarrow |L|_{geom}$ . By adjunction, we obtains a continuous map

$$[0, 1] \rightarrow \mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|L|_{reg} \cap \mathbf{TOP}([0, 1], |K|_{geom}) = \mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|K|_{reg}.$$

Thus, the set map  $[0, 1] \rightarrow \mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|K|_{reg}$  is continuous.  $\square$

There is a technical issue due to the fact that the one-point compactification  $\mathbb{N} \cup \{\infty\}$  of the discrete space  $\mathbb{N}$  which encodes the notion of convergent sequence is not  $\Delta$ -generated<sup>3</sup>. Therefore, we need to recall the following lemma.

**5.7. Lemma.** [13, Lemma 6.10] Let  $U_1, \dots, U_p$  be  $p$  first-countable  $\Delta$ -Hausdorff  $\Delta$ -generated spaces with  $p \geq 1$ . Let  $(u_n^i)_{n \geq 0}$  be a sequence of  $U_i$  for  $1 \leq i \leq p$  which converges to  $u_\infty^i \in U_i$ . Then the sequence  $((u_n^1, \dots, u_n^p))_{n \geq 0}$  converges to  $(u_\infty^1, \dots, u_\infty^p) \in U_1 \times \dots \times U_p$  for the product calculated in  $\mathbf{Top}$ .

We can now prove the main theorem of this paper.

**5.8. Theorem.** For all precubical sets  $K$ , there is a natural isomorphism of Moore flows

$$[K]_{reg} \cong \mathbb{M}^{\mathcal{G}}(|K|_{reg}).$$

<sup>3</sup>But it is a  $k$ -space; thus this technical issue does not exist in the setting of  $k$ -spaces.

*Proof.* From the cocone

$$\left( \mathbb{M}^{\mathcal{G}}|\square[n]|_{reg} \right)_{\square[n] \rightarrow K} \xrightarrow{\bullet} \mathbb{M}^{\mathcal{G}}|K|_{reg}$$

we deduce the natural map of Moore flows  $[K]_{reg} \longrightarrow \mathbb{M}^{\mathcal{G}}|K|_{reg}$ . It is bijective on states. The point is to prove that for all  $(\alpha, \beta) \in K_0 \times K_0$ , it induces a homeomorphism  $\mathbb{P}^{\ell}[K]_{reg} \cong \mathbb{P}^{\ell}\mathbb{M}^{\mathcal{G}}|K|_{reg}$  for all  $\ell > 0$ . In fact, it suffices to prove the homeomorphism for  $\ell = 1$  since all objects of the reparametrization category  $\mathcal{G}$  are isomorphic. It means that it suffices to prove that the map  $g : \mathbb{P}^1[K]_{reg} \rightarrow \mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|K|_{reg}$  is a homeomorphism.

Let  $(\alpha, \beta) \in K_0 \times K_0$ . Consider the small diagram of spaces (the definition of the small category  $\text{Ch}_{\alpha, \beta}(K, n)$  is recalled in Appendix B)

$$\mathcal{D}_{\alpha, \beta}(K, n) : \text{Ch}_{\alpha, \beta}(K, n) \longrightarrow \mathbf{Top}$$

defined on objects by

$$\mathcal{D}_{\alpha, \beta}(K, n)(\square[\underline{n}] \rightarrow K) = N_{n_1} \times \dots \times N_{n_p}$$

and on morphisms by taking the map  $\phi_{A, B} : \square[m_1] * \square[m_2] \rightarrow \square[m_1 + m_2]$  to the map  $N_{m_1} \times N_{m_2} \rightarrow N_{m_1 + m_2}$  given by the Moore composition of natural  $d$ -paths. By [16, Theorem 6.3], the Moore composition of natural  $d$ -paths yields a continuous bijection  $\varinjlim \mathcal{D}_{\alpha, \beta}(K, n) \rightarrow \vec{N}_{\alpha, \beta}^n(K)$ . The latter map is a homeomorphism by [16, Theorem 7.5], the latter theorem proving that  $\varinjlim \mathcal{D}_{\alpha, \beta}(K, n)$  has the ‘‘correct’’ topology, namely the  $\Delta$ -kelleyfication of the compact-open topology, which is precisely the topology equipping  $\vec{N}_{\alpha, \beta}^n(K)$ . Consider the following diagram of topological spaces

$$\begin{array}{ccc} \prod_{n \geq 1} \mathcal{G}(1, n) \times \varinjlim \mathcal{D}_{\alpha, \beta}(K, n) & \xrightarrow{\cong} & \prod_{n \geq 1} \mathcal{G}(1, n) \times \vec{N}_{\alpha, \beta}^n(K) \\ \downarrow h & & \downarrow \Phi^1 \\ \mathbb{P}_{\alpha, \beta}^1[K]_{reg} & \xrightarrow{g} & \mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|K|_{reg} = \vec{R}_{\alpha, \beta}^1(K) \end{array}$$

where the map  $\Phi^1$  is the homeomorphism of Theorem 3.12. The map  $h : \mathcal{G}(1, n) \times \varinjlim \mathcal{D}_{\alpha, \beta}(K, n) \rightarrow \mathbb{P}_{\alpha, \beta}^1[K]_{reg}$  is defined as follows. Consider at first the map

$$\mathcal{G}(1, n) \times \mathcal{D}_{\alpha, \beta}(K, n)(\square[\underline{n}] \rightarrow K) \rightarrow \mathcal{G}(1, n) \times N_n \rightarrow \mathbb{P}_{0_n, 1_n}^1 \mathbb{M}^{\mathcal{G}}|\square[n]|_{reg} \rightarrow \mathbb{P}_{\alpha, \beta}^1[K]_{reg}$$

the left-hand map being induced by the Moore composition of natural paths  $N_{n_1} \times \dots \times N_{n_p} \rightarrow N_n$ , the middle map by taking  $(\phi, \gamma) \in \mathcal{G}(1, n) \times N_n$  to  $\gamma\phi \in \mathbb{P}_{0_n, 1_n}^{\mathcal{G}}|\square[n]|_{reg} = \mathbb{P}_{0_n, 1_n}^1 \mathbb{M}^{\mathcal{G}}|\square[n]|_{reg}$  by Proposition 5.2, and the right-hand map being induced by the map  $\square[\underline{n}] \rightarrow K$ . It is routine to check that the diagram of spaces above is commutative. We deduce that the composite

$$gh : \prod_{n \geq 1} \mathcal{G}(1, n) \times \varinjlim \mathcal{D}_{\alpha, \beta}(K, n) \rightarrow \mathbb{P}_{\alpha, \beta}^1[K]_{reg} \rightarrow \mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|K|_{reg}$$



is a homeomorphism. It implies that  $h$  is one-to-one and that  $g$  is onto. For any map of precubical sets  $\square[n] \rightarrow K$  taking  $0_n$  to  $\alpha$  and  $1_n$  to  $\beta$  in the following diagram:

$$\begin{array}{ccc} & & \coprod_{n \geq 1} \mathcal{G}(1, n) \times \varinjlim \mathcal{D}_{\alpha, \beta}(K, n) \\ & \nearrow j & \downarrow h \\ \mathbb{P}_{0_n, 1_n}^{\mathcal{G}} |\square[n]|_{reg} = \mathbb{P}_{0_n, 1_n}^1 \mathbb{M}^{\mathcal{G}} |\square[n]|_{reg} & \longrightarrow & \mathbb{P}_{\alpha, \beta}^1 [K]_{reg} \end{array}$$

the lift  $j$  exists by Proposition 5.2 again. The Moore flow  $[K]_{reg}$  is a colimit of Moore flows of the form  $\mathbb{M}^{\mathcal{G}} |\square[n]|_{reg}$ . Recall that a Moore flow is a semicategory enriched over the semimonoidal category of  $\mathcal{G}$ -spaces  $([\mathcal{G}^{op}, \mathbf{Top}], \otimes)$ . It implies that  $h$  is onto and, consequently, that  $g$  is one-to-one.

It remains to prove that the continuous bijection  $g : \mathbb{P}_{\alpha, \beta}^1 [K]_{reg} \rightarrow \mathbb{P}_{\alpha, \beta}^{\mathcal{G}} |K|_{reg}$  is a homeomorphism. Let  $s : [0, 1] \rightarrow \mathbb{P}_{\alpha, \beta}^1 [K]_{reg}$  be a set map such that the composite map  $gs : [0, 1] \rightarrow \mathbb{P}_{\alpha, \beta}^{\mathcal{G}} |K|_{reg}$  is continuous. By adjunction, the latter map induces a continuous map  $\hat{g} : [0, 1] \times [0, 1] \rightarrow |K|_{geom}$  since  $|K|_{geom}$  is the underlying space of the multipointed  $d$ -space  $|K|_{reg}$ . Since  $|K|_{geom}$  is a CW-complex, the compact  $\hat{g}([0, 1] \times [0, 1])$  intersects finitely many interiors of cubes and vertices by [19, Proposition A.1] which defines, by adding all their faces, a finite precubical set  $K_f \subset K$ . There is therefore the commutative diagram of set maps

$$\begin{array}{ccccc} [0, 1] & \xrightarrow{s_f} & \mathbb{P}_{\alpha, \beta}^1 [K_f]_{reg} & \xrightarrow{g_f} & \mathbb{P}_{\alpha, \beta}^{\mathcal{G}} |K_f|_{reg} \\ \parallel & & \downarrow & & \downarrow \\ [0, 1] & \xrightarrow{s} & \mathbb{P}_{\alpha, \beta}^1 [K]_{reg} & \xrightarrow{g} & \mathbb{P}_{\alpha, \beta}^{\mathcal{G}} |K|_{reg} \end{array}$$

Since the set map  $\mathbb{P}_{\alpha, \beta}^{\mathcal{G}} |K_f|_{reg} \rightarrow \mathbb{P}_{\alpha, \beta}^{\mathcal{G}} |K|_{reg}$  is a  $\Delta$ -inclusion by Proposition 5.6, we deduce that the composite set map  $g_f s_f$  is continuous. It then suffices to prove that the set map  $s_f : [0, 1] \rightarrow \mathbb{P}_{\alpha, \beta}^1 [K_f]_{reg}$  is continuous to complete the proof because the vertical map  $\mathbb{P}_{\alpha, \beta}^1 [K_f]_{reg} \rightarrow \mathbb{P}_{\alpha, \beta}^1 [K]_{reg}$  is continuous by functoriality. We can therefore suppose that  $K_f = K$  is a finite precubical set. The  $L_1$ -arc length of an execution path of  $\mathbb{P}_{\alpha, \beta}^{\mathcal{G}} |K|_{reg}$ , i.e. of a regular tame  $d$ -path of  $K$  from  $\alpha$  to  $\beta$ , mapped to an element of  $\mathcal{G}(1, n) \times \varinjlim \mathcal{D}_{\alpha, \beta}(K, n)$  is  $n$ . Consequently, since  $[0, 1]$  is path-connected and using Proposition 3.6, there exists  $n_0 \geq 1$  and a commutative diagram of spaces

$$\begin{array}{ccccccc} [0, 1] & \xrightarrow{s} & \mathbb{P}_{\alpha, \beta}^1 [K]_{reg} & \xrightarrow{g} & \mathbb{P}_{\alpha, \beta}^{\mathcal{G}} |K|_{reg} & \xrightarrow{\cong} & \coprod_{n \geq 1} \mathcal{G}(1, n) \times \varinjlim \mathcal{D}_{\alpha, \beta}(K, n) \\ \parallel & & & & & & \uparrow \subset \\ [0, 1] & \xrightarrow{\quad} & & & & & \mathcal{G}(1, n_0) \times \varinjlim \mathcal{D}_{\alpha, \beta}(K, n_0) \end{array}$$

Let  $(t_k)_{k \geq 0}$  be a sequence of  $[0, 1]$  which converges to  $t_\infty \in [0, 1]$ . Since  $K$  is finite, the diagram  $\mathcal{D}_{\alpha, \beta}(K, n_0)$  is finite as well. By extracting a subsequence, one can suppose that there exists  $(n_1, \dots, n_p) \in \text{Seq}(n_0)$  such that  $gs(t_k) = (\phi_k, \gamma_k^1, \dots, \gamma_k^p) \in \mathcal{G}(1, n_0) \times N_{n_1} \times \dots \times N_{n_p}$  for all  $k \geq 0$ . By continuity of  $gs$ , the sequence  $(\phi_k, \gamma_k^1, \dots, \gamma_k^p) \in$

$\mathcal{G}(1, n_0) \times \varinjlim \mathcal{D}_{\alpha, \beta}(K, n_0)$  is convergent. By using the projection map on the first factor, we deduce that the sequence  $(\phi_k)_{k \geq 0}$  of  $\mathcal{G}(1, n_0)$  converges to some  $\phi_\infty \in \mathcal{G}(1, n_0)$ <sup>4</sup>. By [16, Proposition 4.10], the space  $N_{n_j}$  is sequentially compact for  $1 \leq j \leq p$ . Thus by extracting a subsequence, one can suppose that each sequence  $(\gamma_k^j)_{k \geq 0}$  of  $N_{n_j}$  converges to some  $\gamma_\infty^j$  for all  $1 \leq j \leq p$ . By Lemma 5.7, it implies that the sequence  $(\phi_k, \gamma_k^1, \dots, \gamma_k^p) \in \mathcal{G}(1, n_0) \times N_{n_1} \times \dots \times N_{n_p}$  converges to  $(\phi_\infty, \gamma_\infty^1, \dots, \gamma_\infty^p)$ . We deduce that the sequence  $(s(t_k))_{k \geq 0}$  converges to  $h(\phi_\infty, \gamma_\infty^1, \dots, \gamma_\infty^p) \in \mathbb{P}_{\alpha, \beta}^1[K]_{reg}$ . In fact, we have proved that every subsequence of  $(s(t_k))_{k \geq 0}$  has a subsequence which has a limit point which is necessarily  $g^{-1}(gs(t_\infty)) = s(t_\infty)$ . Suppose that the sequence  $(s(t_k))_{k \geq 0}$  does not converge to  $s(t_\infty)$ . Then there exists an open neighborhood  $V$  of  $s(t_\infty)$  in  $\mathbb{P}_{\alpha, \beta}^1[K]_{reg}$  such that for some  $M \geq 0$ , and for all  $k \geq M$ ,  $s(t_k) \in V^c$ , the complement of  $V$ , the latter being closed in  $\mathbb{P}_{\alpha, \beta}^1[K]_{reg}$ . Thus,  $(s(t_k))_{k \geq M}$  cannot have a limit point: contradiction. We deduce that  $s : [0, 1] \rightarrow \mathbb{P}_{\alpha, \beta}^1[K]_{reg}$  is sequentially continuous, hence continuous. We deduce that the continuous bijection  $g : \mathbb{P}_{\alpha, \beta}^1[K]_{reg} \rightarrow \mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|K|_{reg}$  is a  $\Delta$ -inclusion. Hence,  $g$  and  $h$  are homeomorphisms by Proposition 5.5.  $\square$

**5.9. Corollary.** *The functor  $K \mapsto \mathbb{M}^{\mathcal{G}}(|K|_{reg})$  from precubical sets to Moore flows is colimit-preserving.*

**5.10. Proposition.** *For all  $n \geq 1$ , there is the pushout diagram of Moore flows*

$$\begin{array}{ccc} \text{Glob}(\mathbb{F}_n^{\mathcal{G}^{op}} \partial N_n) & \longrightarrow & \mathbb{M}^{\mathcal{G}}|\partial \square[n]|_{reg} \\ \downarrow & & \downarrow \\ \text{Glob}(\mathbb{F}_n^{\mathcal{G}^{op}} N_n) & \longrightarrow & \mathbb{M}^{\mathcal{G}}|\square[n]|_{reg} \end{array}$$

*Proof.* By Corollary 5.3, there is the commutative diagram of  $\mathcal{G}$ -spaces

$$\begin{array}{ccc} \mathbb{F}_n^{\mathcal{G}^{op}} \partial N_n & \xrightarrow{\cong} & \mathbb{P}_{0_n, 1_n} \mathbb{M}^{\mathcal{G}}|\partial \square[n]|_{reg} \\ \downarrow & & \downarrow \\ \mathbb{F}_n^{\mathcal{G}^{op}} N_n & \xrightarrow{\cong} & \mathbb{P}_{0_n, 1_n} \mathbb{M}^{\mathcal{G}}|\square[n]|_{reg} \end{array}$$

where the two horizontal maps are isomorphisms of  $\mathcal{G}$ -spaces. This commutative diagram of  $\mathcal{G}$ -spaces is therefore a pushout diagram.  $\square$

**5.11. Theorem.** *For all precubical sets  $K$ , the Moore flow  $\mathbb{M}^{\mathcal{G}}|K|_{reg}$  is  $m$ -cofibrant.*

*Proof.* A map of Moore flows  $f : X \rightarrow Y$  satisfies the RLP with respect to the map of Moore flows  $\text{Glob}(\mathbb{F}_n^{\mathcal{G}^{op}} \partial N_n) \rightarrow \text{Glob}(\mathbb{F}_n^{\mathcal{G}^{op}} N_n)$  if and only if for each  $(\alpha, \beta) \in K_0 \times K_0$ , the map of  $\mathcal{G}$ -spaces  $f : \mathbb{P}_{\alpha, \beta} X \rightarrow \mathbb{P}_{f(\alpha), f(\beta)} Y$  satisfies the RLP with respect to the map of  $\mathcal{G}$ -spaces  $\mathbb{F}_n^{\mathcal{G}^{op}} \partial N_n \rightarrow \mathbb{F}_n^{\mathcal{G}^{op}} N_n$ . Therefore, by Proposition 4.2, a map of Moore flows  $f : X \rightarrow Y$  satisfies the RLP with respect to the map of Moore flows  $\text{Glob}(\mathbb{F}_n^{\mathcal{G}^{op}} \partial N_n) \rightarrow \text{Glob}(\mathbb{F}_n^{\mathcal{G}^{op}} N_n)$  if and only if the map of topological spaces  $\mathbb{P}_{\alpha, \beta}^n X \rightarrow \mathbb{P}_{f(\alpha), f(\beta)}^n Y$  satisfies the RLP with respect to the map of topological spaces  $\partial N_n \rightarrow N_n$ . By [16, Theorem 5.9], the

<sup>4</sup>This point is not trivial. Indeed,  $\mathcal{G}(1, n_0)$  is not sequentially compact, and not even relatively sequentially compact as explained in [13, Figure 2].

latter map is a m-cofibration of spaces. By definition of the m-model structures of Moore flows, we then deduce that the map of Moore flows  $\text{Glob}(\mathbb{F}_n^{\mathcal{G}^{op}} \partial N_n) \rightarrow \text{Glob}(\mathbb{F}_n^{\mathcal{G}^{op}} N_n)$  is a m-cofibration of Moore flows for  $n \geq 1$ . Using Proposition 5.10, and since  $[\square[0]]_{reg} = \{0\}$  (the Moore flow without execution paths and one state 0) is m-cofibrant, we deduce that for all precubical sets  $K$ , the Moore flow  $[K]_{reg}$  is m-cofibrant. The proof is complete thanks to Theorem 5.8.  $\square$

**5.12. Corollary.** *Let  $K$  be a precubical set. Let  $(\alpha, \beta) \in K_0 \times K_0$ . The topological space  $\mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|K|_{reg}$  is homotopy equivalent to a CW-complex.*

*Proof.* The m-cofibrant Moore flow  $\mathbb{M}^{\mathcal{G}}|K|_{reg}$  is weakly equivalent in the h-model structure of Moore flows to a q-cofibrant Moore flow  $X$  by [4, Corollary 3.7]. The latter has projective q-cofibrant  $\mathcal{G}$ -spaces of execution paths by [12, Theorem 9.11]. It implies that  $\mathbb{P}_{\alpha, \beta} X$  is injective m-cofibrant by [11, Corollary 7.2]. Thus  $\mathbb{P}_{\alpha, \beta}^1 X$  is a m-cofibrant topological space. Since weak equivalences of the projective h-model structure of  $\mathcal{G}$ -spaces are objectwise homotopy equivalences, it implies that  $\mathbb{P}_{\alpha, \beta}^1 \mathbb{M}^{\mathcal{G}}|K|_{reg}$  is homotopy equivalent to  $\mathbb{P}_{\alpha, \beta}^1 X$ . By definition of the functor  $\mathbb{M}^{\mathcal{G}}$ , there is the equality  $\mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|K|_{reg} = \mathbb{P}_{\alpha, \beta}^1 \mathbb{M}^{\mathcal{G}}|K|_{reg}$ . It implies that  $\mathbb{P}_{\alpha, \beta}^{\mathcal{G}}|K|_{reg}$  is homotopy equivalent to  $\mathbb{P}_{\alpha, \beta}^1 X$  which is homotopy equivalent to a CW-complex.  $\square$

## 6. RELATION WITH THE NATURAL REALIZATION FUNCTOR

**6.1. Definition.** [8, Definition 4.11] *A flow is a small semicategory enriched over the closed monoidal category  $(\mathbf{Top}, \times)$ . The corresponding category is denoted by  $\mathbf{Flow}$ .*

A flow  $X$  consists of a topological space  $\mathbb{P}X$  of *execution paths*, a discrete space  $X^0$  of *states*, two continuous maps  $s$  and  $t$  from  $\mathbb{P}X$  to  $X^0$  called the source and target map respectively, and a continuous and associative map  $*$  :  $\{(x, y) \in \mathbb{P}X \times \mathbb{P}X; t(x) = s(y)\} \rightarrow \mathbb{P}X$  such that  $s(x * y) = s(x)$  and  $t(x * y) = t(y)$ . Let  $\mathbb{P}_{\alpha, \beta} X = \{x \in \mathbb{P}X \mid s(x) = \alpha \text{ and } t(x) = \beta\}$ : it is the space of execution paths from  $\alpha$  to  $\beta$ ,  $\alpha$  is called the initial state and  $\beta$  is called the final state. Note that the composition is denoted by  $x * y$ , not by  $y \circ x$ . The category  $\mathbf{Flow}$  is locally presentable by [12, Theorem 6.11].

Let  $X$  be a flow. The Moore flow  $\mathbb{M}(X)$  is the enriched semicategory over the semi-monoidal category of  $\mathcal{G}$ -spaces defined as follows:

- The set of states is  $X^0$ .
- The  $\mathcal{G}$ -space  $\mathbb{P}_{\alpha, \beta} \mathbb{M}(X)$  is the  $\mathcal{G}$ -space  $\Delta_{\mathcal{G}^{op}}(\mathbb{P}_{\alpha, \beta} X)$  (the constant diagram  $\mathbb{P}_{\alpha, \beta} X$ ).
- The composition law is defined, using [12, Proposition 5.17], as the composite map

$$\Delta_{\mathcal{G}^{op}}(\mathbb{P}_{\alpha, \beta} X) \otimes \Delta_{\mathcal{G}^{op}}(\mathbb{P}_{\beta, \gamma} X) \cong \Delta_{\mathcal{G}^{op}}(\mathbb{P}_{\alpha, \beta} X \times \mathbb{P}_{\beta, \gamma} X) \xrightarrow{\Delta_{\mathcal{G}^{op}}(*)} \Delta_{\mathcal{G}^{op}}(\mathbb{P}_{\alpha, \gamma} X).$$

The construction above yields a well-defined functor  $\mathbb{M} : \mathbf{Flow} \rightarrow \mathcal{G}\mathbf{Flow}$ . Consider a Moore flow  $Y$ . For all  $\alpha, \beta \in Y^0$ , let  $Y_{\alpha, \beta} = \varinjlim \mathbb{P}_{\alpha, \beta} Y$ . Let  $(\alpha, \beta, \gamma)$  be a triple of states of  $Y$ . The composition law of the Moore flow  $Y$  induces, using [12, Proposition 5.18], a continuous map

$$Y_{\alpha, \beta} \times Y_{\beta, \gamma} \cong \varinjlim (\mathbb{P}_{\alpha, \beta} Y \otimes \mathbb{P}_{\beta, \gamma} Y) \longrightarrow \varinjlim \mathbb{P}_{\alpha, \gamma} Y \cong Y_{\alpha, \gamma}$$

which is associative. We obtain a flow  $\mathbb{M}_!(Y)$  such that

- The set of states is  $Y^0$

- For all  $\alpha, \beta \in Y^0$ , let  $\mathbb{P}_{\alpha, \beta} \mathbb{M}_! Y = Y_{\alpha, \beta}$
- For all  $\alpha, \beta, \gamma \in Y^0$ , the composition law  $Y_{\alpha, \beta} \times Y_{\beta, \gamma} \rightarrow Y_{\alpha, \gamma}$ .

We obtain a well-defined functor  $\mathbb{M}_! : \mathcal{G}\mathbf{Flow} \rightarrow \mathbf{Flow}$  and there is an adjunction  $\mathbb{M}_! \dashv \mathbb{M}$  by [12, Proposition 10.7].

We define a flow  $|\square[n]|_{nat}$  for  $n \geq 0$  called *the natural  $n$ -cube* as follows (see [16, Section 5] for more details). The set of states is  $\{0, 1\}^n$ . Let  $n \geq 1$  and  $\alpha, \beta \in \{0, 1\}^n$ . Let  $\alpha = (\alpha_1, \dots, \alpha_n)$  and  $\beta = (\beta_1, \dots, \beta_n)$ . Let

$$\mathbb{P}_{\alpha, \beta} |\square[n]|_{nat} = \begin{cases} N_m & \text{if } d_1(\alpha, \beta) = m \geq 1 \text{ and } \alpha < \beta \\ \emptyset & \text{if } \alpha \geq \beta. \end{cases}$$

The continuous map  $N_{m_1} \times N_{m_2} \rightarrow N_{m_1+m_2}$  induced by the Moore composition of two natural  $d$ -paths yields the associative composition law of the flow  $|\square[n]|_{nat}$ . We obtain a well-defined cocubical flow  $|\square[*]|_{nat}$  by [16, Corollary 5.2]. Using Proposition 2.9, we obtain:

**6.2. Definition.** ([16, Definition 5.3]) *Let  $K$  be a precubical set. Consider the colimit-preserving functor*

$$|K|_{nat} = \varinjlim_{\square[n] \rightarrow K} |\square[n]|_{nat}.$$

*It is called the natural realization of  $K$  as a flow.*

Let  $X$  be a multipointed  $d$ -space. Consider for every  $(\alpha, \beta) \in X^0 \times X^0$  the coequalizer of spaces

$$\mathbb{P}_{\alpha, \beta} X = \varinjlim \left( \mathbb{P}_{\alpha, \beta}^{\mathcal{G}} X \times \mathcal{G}(1, 1) \rightrightarrows \mathbb{P}_{\alpha, \beta}^{\mathcal{G}} X \right)$$

where the two maps are  $(c, \phi) \mapsto c$  and  $(c, \phi) \mapsto c \cdot \phi$ . Let  $[-]_{\alpha, \beta} : \mathbb{P}_{\alpha, \beta}^{\mathcal{G}} X \rightarrow \mathbb{P}_{\alpha, \beta} X$  be the canonical map.

**6.3. Theorem.** [10, Theorem 7.2] [13, Theorem 8.11] *Let  $X$  be a multipointed  $d$ -space. Then there exists a flow  $\text{cat}(X)$  with  $\text{cat}(X)^0 = X^0$ ,  $\mathbb{P}_{\alpha, \beta} \text{cat}(X) = \mathbb{P}_{\alpha, \beta} X$  and the composition law  $*$  :  $\mathbb{P}_{\alpha, \beta} X \times \mathbb{P}_{\beta, \gamma} X \rightarrow \mathbb{P}_{\alpha, \gamma} X$  is for every triple  $(\alpha, \beta, \gamma) \in X^0 \times X^0 \times X^0$  the unique map making the following diagram commutative:*

$$\begin{array}{ccc} \mathbb{P}_{\alpha, \beta}^{\mathcal{G}} X \times \mathbb{P}_{\beta, \gamma}^{\mathcal{G}} X & \xrightarrow{*_N} & \mathbb{P}_{\alpha, \gamma}^{\mathcal{G}} X \\ \downarrow [-]_{\alpha, \beta} \times [-]_{\beta, \gamma} & & \downarrow [-]_{\alpha, \gamma} \\ \mathbb{P}_{\alpha, \beta} X \times \mathbb{P}_{\beta, \gamma} X & \longrightarrow & \mathbb{P}_{\alpha, \gamma} X \end{array}$$

where  $*_N$  is the normalized composition (cf. Definition 2.4). The mapping  $X \mapsto \text{cat}(X)$  induces a functor from  $\mathcal{G}\mathbf{dTop}$  to  $\mathbf{Flow}$ . There is the isomorphism of functors

$$\text{cat} \cong \mathbb{M}_! \mathbb{M}^{\mathcal{G}}.$$

**6.4. Definition.** *The functor  $\text{cat} : \mathcal{G}\mathbf{dTop} \rightarrow \mathbf{Flow}$  is called the categorization functor.*

**6.5. Theorem.** *For every precubical set  $K$ , there is the natural isomorphism of flows*

$$\text{cat}(|K|_{reg}) \cong |K|_{nat}.$$

*Proof.* Let  $n \geq 0$ . By Proposition 5.2, there is the isomorphism of flows

$$\text{cat}(|\square[n]|_{reg}) \cong |\square[n]|_{nat}.$$

The definition of the cocubical multipointed  $d$ -space  $|\square[*]|_{reg}$  expounded in Proposition 2.13 shows that a regular  $d$ -path  $\gamma$  between two vertices of  $\square[n]$  is taken by a coface map  $\delta_i^\alpha : [n] \rightarrow [n+1]$  to the regular  $d$ -path  $\delta_i^\alpha \gamma$  of  $\square[n+1]$ . Thus, the repametrization  $\gamma\phi$  with  $\phi \in \mathcal{G}(1,1)$  is taken to the regular  $d$ -path  $\delta_i^\alpha \gamma\phi$  of  $\square[n+1]$ . It means that the above isomorphism of flows is natural with respect to the maps of the small category  $\square$ . We deduce the sequence of isomorphisms of flows

$$\begin{aligned} \text{cat}(|K|_{reg}) &\cong \mathbb{M}_! \mathbb{M}^{\mathcal{G}}(|K|_{reg}) \\ &\cong \mathbb{M}_! \left( \varinjlim_{\square[n] \rightarrow K} \mathbb{M}^{\mathcal{G}}|\square[n]|_{reg} \right) \\ &\cong \varinjlim_{\square[n] \rightarrow K} \mathbb{M}_! \mathbb{M}^{\mathcal{G}}|\square[n]|_{reg} \\ &\cong \varinjlim_{\square[n] \rightarrow K} \text{cat}(|\square[n]|_{reg}) \\ &\cong \varinjlim_{\square[n] \rightarrow K} |\square[n]|_{nat} \\ &\cong |K|_{nat}, \end{aligned}$$

the first and fourth isomorphisms since  $\text{cat} \cong \mathbb{M}_! \mathbb{M}^{\mathcal{G}}$  by Theorem 6.3, the second isomorphism by Corollary 5.9, the third isomorphism since  $\mathbb{M}_!$  is a left adjoint by [12, Proposition 10.7], the fifth isomorphism by the above natural isomorphism of flows, and the last isomorphism by definition of the natural realization.  $\square$

## APPENDIX A. IS THE REGULAR REALIZATION M-COFIBRANT ?

Despite all our efforts, we are unable to prove or to disprove the fact that the multipointed  $d$ -space  $|K|_{reg}$  is h-cofibrant. The example of non h-cofibrant multipointed  $d$ -space provided in [15, Proposition 6.19] suggests that the h-cofibrant objects are the objects without algebraic relations. According to this intuition, it means that  $|K|_{reg}$  is likely to be h-cofibrant. If it is the case, this section then shows that  $|K|_{reg}$  is a m-cofibrant multipointed  $d$ -space. First of all, let us recall the three model structures of multipointed  $d$ -spaces.

**A.1. Theorem.** [15, Theorem 6.14] *Let  $r \in \{q, m, h\}$ . There exists a unique model structure on  $\mathcal{GdTop}$  such that:*

- *A map of multipointed  $d$ -spaces  $f : X \rightarrow Y$  is a weak equivalence if and only if  $f^0 : X^0 \rightarrow Y^0$  is a bijection and for all  $(\alpha, \beta) \in X^0 \times X^0$ , the continuous map  $\mathbb{P}_{\alpha, \beta}^{\mathcal{G}} X \rightarrow \mathbb{P}_{f(\alpha), f(\beta)}^{\mathcal{G}} Y$  is a weak equivalence of the  $r$ -model structure of  $\mathbf{Top}$ .*
- *A map of flows  $f : X \rightarrow Y$  is a fibration if and only if for all  $(\alpha, \beta) \in X^0 \times X^0$ , the continuous map  $\mathbb{P}_{\alpha, \beta}^{\mathcal{G}} X \rightarrow \mathbb{P}_{f(\alpha), f(\beta)}^{\mathcal{G}} Y$  is a fibration of the  $r$ -model structure of  $\mathbf{Top}$ .*

*This model structure is accessible and all objects are fibrant.*

**A.2. Notation.** *These three model categories are denoted by  $\mathcal{GdTop}_q, \mathcal{GdTop}_m, \mathcal{GdTop}_h$ .*

**A.3. Theorem.** *The right adjoint  $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \rightarrow \mathcal{G}\mathbf{Flow}$  yields a right Quillen equivalence between the  $m$ -model structures  $\mathcal{G}\mathbf{dTop}_m$  and  $\mathcal{G}\mathbf{Flow}_m$ .*

*Proof.* The functor  $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \rightarrow \mathcal{G}\mathbf{Flow}$  takes (trivial resp.)  $m$ -fibrations of multipointed  $d$ -spaces to (trivial resp.)  $m$ -fibrations of Moore flows by definition of the  $m$ -model structures. Thus, the functor  $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \rightarrow \mathcal{G}\mathbf{Flow}$  is a right Quillen adjoint between the  $m$ -model structures. We have the commutative diagram of right Quillen adjoints

$$\begin{array}{ccc} \mathcal{G}\mathbf{dTop}_m & \xrightarrow{\mathbb{M}^{\mathcal{G}}} & \mathcal{G}\mathbf{Flow}_m \\ \text{Id}_{\mathcal{G}\mathbf{dTop}} \downarrow & & \downarrow \text{Id}_{\mathcal{G}\mathbf{Flow}} \\ \mathcal{G}\mathbf{dTop}_q & \xrightarrow{\mathbb{M}^{\mathcal{G}}} & \mathcal{G}\mathbf{Flow}_q \end{array}$$

The bottom horizontal arrow is a right Quillen equivalence by [13, Theorem 8.1]. It is a general fact about mixed model structures that the two vertical arrows are right Quillen equivalences. Thus, the top right Quillen adjoint is a right Quillen equivalence.  $\square$

**A.4. Notation.** *Denote by  $\mathbb{M}_!^{\mathcal{G}} : \mathcal{G}\mathbf{Flow} \rightarrow \mathcal{G}\mathbf{dTop}$  the left adjoint of  $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \rightarrow \mathcal{G}\mathbf{Flow}$ . An explicit description is expounded in [13, Appendix B].*

**A.5. Theorem.** *Let  $K$  be a precubical set. Then the multipointed  $d$ -space  $|K|_{reg}$  is  $m$ -cofibrant if it is  $h$ -cofibrant.*

*Proof.* Using Theorem 5.11 and Theorem A.3, we deduce that the counit map

$$\mathbb{M}_!^{\mathcal{G}} \mathbb{M}^{\mathcal{G}} |K|_{reg} \longrightarrow |K|_{reg}$$

is a weak equivalence of the  $m$ -model structure of multipointed  $d$ -spaces and,  $\mathbb{M}_!^{\mathcal{G}}$  being a left Quillen adjoint, the multipointed  $d$ -space  $\mathbb{M}_!^{\mathcal{G}} \mathbb{M}^{\mathcal{G}} |K|_{reg}$  is  $m$ -cofibrant. For all  $(\alpha, \beta) \in K_0 \times K_0$ , we deduce the weak homotopy equivalence

$$\mathbb{P}_{\alpha, \beta}^{\mathcal{G}} \mathbb{M}_!^{\mathcal{G}} \mathbb{M}^{\mathcal{G}} |K|_{reg} \longrightarrow \mathbb{P}_{\alpha, \beta}^{\mathcal{G}} |K|_{reg}.$$

By [15, Theorem 8.6], the topological space  $\mathbb{P}_{\alpha, \beta}^{\mathcal{G}} \mathbb{M}_!^{\mathcal{G}} \mathbb{M}^{\mathcal{G}} |K|_{reg}$  is  $m$ -cofibrant. By Corollary 5.12, the space  $\mathbb{P}_{\alpha, \beta}^{\mathcal{G}} |K|_{reg}$  is a  $m$ -cofibrant space. Thus the weak homotopy equivalence

$$\mathbb{P}_{\alpha, \beta}^{\mathcal{G}} \mathbb{M}_!^{\mathcal{G}} \mathbb{M}^{\mathcal{G}} |K|_{reg} \longrightarrow \mathbb{P}_{\alpha, \beta}^{\mathcal{G}} |K|_{reg}$$

is a weak homotopy equivalence between  $m$ -cofibrant spaces. By [4, Corollary 3.4], the latter map is therefore a homotopy equivalence. It means that the counit map  $\mathbb{M}_!^{\mathcal{G}} \mathbb{M}^{\mathcal{G}} |K|_{reg} \rightarrow |K|_{reg}$  is a weak equivalence of the  $h$ -model structure of  $\mathcal{G}\mathbf{dTop}$ . It implies by [4, Corollary 3.7] that the multipointed  $d$ -space  $|K|_{reg}$  is  $m$ -cofibrant if it is  $h$ -cofibrant.  $\square$

## APPENDIX B. THE CATEGORY $\text{Ch}_{\alpha, \beta}(K, n)$

Cube chains are a powerful notion introduced by Ziemiański. We use the presentation given in [32, Section 7]. Let  $\text{Seq}(n)$  be the set of sequences of positive integers  $\underline{n} = (n_1, \dots, n_p)$  with  $n_1 + \dots + n_p = n$ . Let  $\underline{n} = (n_1, \dots, n_p) \in \text{Seq}(n)$ . Then  $|\underline{n}| = n$  is the length of  $\underline{n}$  and  $\ell(\underline{n}) = p$  is the number of elements of  $\underline{n}$ . Let  $K$  be a precubical set and

$A = a_1 < \dots < a_k \subset \{1, \dots, n\}$  and  $\epsilon \in \{0, 1\}$ . The *iterated face map* is defined by

$$\partial_A^\epsilon = \partial_{a_1}^\epsilon \partial_{a_2}^\epsilon \dots \partial_{a_k}^\epsilon.$$

**B.1. Definition.** Let  $\underline{n} \in \text{Seq}(n)$ . The  $\underline{n}$ -cube is the precubical set

$$\square[\underline{n}] = \square[n_1] * \dots * \square[n_p]$$

where the notation  $*$  means that the final state  $1_{n_i}$  of the precubical set  $\square[n_i]$  is identified with the initial state  $0_{n_{i+1}}$  of the precubical set  $\square[n_{i+1}]$  for  $1 \leq i \leq p - 1$ .

Let  $K$  be a precubical set. Let  $\alpha, \beta \in K_0$ . Let  $n \geq 1$ . The category  $\text{Ch}_{\alpha, \beta}(K, n)$  is defined as follows. The objects are the maps of precubical sets  $\square[\underline{n}] \rightarrow K$  with  $|\underline{n}| = n$  where the initial state of  $\square[n_1]$  is mapped to  $\alpha$  and the final state of  $\square[n_p]$  is mapped to  $\beta$ . Let  $A \sqcup B = \{1, \dots, m_1 + m_2\}$  be a partition with the cardinal of  $A$  equal to  $m_1 > 0$  and the cardinal of  $B$  equal to  $m_2 > 0$ . Let

$$\phi_{A, B} : \square[m_1] * \square[m_2] \longrightarrow \square[m_1 + m_2]$$

be the unique map of precubical sets such that

$$\begin{aligned} \phi_{A, B}(\text{Id}_{[m_1]}) &= \partial_B^0(\text{Id}_{[m_1 + m_2]}), \\ \phi_{A, B}(\text{Id}_{[m_2]}) &= \partial_A^1(\text{Id}_{[m_1 + m_2]}). \end{aligned}$$

For  $i \in \{1, \dots, \ell(\underline{n})\}$  and a partition  $A \sqcup B = \{1, \dots, n_i\}$ , let

$$\delta_{i, A, B} = \text{Id}_{\square[n_1]} * \dots * \text{Id}_{\square[n_{i-1}]} * \phi_{A, B} * \text{Id}_{\square[n_{i+1}]} * \dots * \text{Id}_{\square[n_{\ell(\underline{n})}]}.$$

The morphisms are the commutative diagrams

$$\begin{array}{ccc} \square[\underline{n}_a] & \xrightarrow{a} & K \\ \downarrow & & \parallel \\ \square[\underline{n}_b] & \xrightarrow{b} & K \end{array}$$

where the left vertical map is a composite of maps of precubical sets of the form  $\delta_{i, A, B}$ .

## REFERENCES

- [1] J. Adámek and J. Rosický. *Locally presentable and accessible categories*. Cambridge University Press, Cambridge, 1994. <https://doi.org/10.1017/cbo9780511600579.004>.
- [2] T. Barthel and E. Riehl. On the construction of functorial factorizations for model categories. *Algebraic & Geometric Topology*, 13(2):1089–1124, 2013. <https://doi.org/10.2140/agt.2013.13.1089>.
- [3] R. Brown and P. J. Higgins. On the algebra of cubes. *Journal of Pure and Applied Algebra*, 21(3):233–260, 1981. [https://doi.org/10.1016/0022-4049\(81\)90018-9](https://doi.org/10.1016/0022-4049(81)90018-9).
- [4] M. Cole. Mixing model structures. *Topology and its Applications*, 153(7):1016–1032, 2006. <https://doi.org/10.1016/j.topol.2005.02.004>.
- [5] U. Fahrenberg and M. Raussen. Reparametrizations of continuous paths. *Journal of Homotopy and Related Structures*, 2(2):93–117, 2007.
- [6] L. Fajstrup, E. Goubault, E. Haucourt, S. Mimram, and M. Raussen. *Directed algebraic topology and concurrency*. With a foreword by Maurice Herlihy

- and a preface by Samuel Mimram. Springer International Publishing, 2016. <https://doi.org/10.1007/978-3-319-15398-8>.
- [7] R. Garner, M. Kędziołek, and E. Riehl. Lifting accessible model structures. *Journal of Topology*, 13(1):59–76, 2020. <https://doi.org/10.1112/topo.12123>.
- [8] P. Gaucher. A model category for the homotopy theory of concurrency. *Homology, Homotopy and Applications*, 5(1):p.549–599, 2003. <https://doi.org/10.4310/hha.2003.v5.n1.a20>.
- [9] P. Gaucher. Globular realization and cubical underlying homotopy type of time flow of process algebra. *The New York Journal of Mathematics*, 14:101–137, 2008.
- [10] P. Gaucher. Homotopical interpretation of globular complex by multipointed d-space. *Theory and Applications of Categories*, 22(22):588–621, 2009.
- [11] P. Gaucher. Enriched diagrams of topological spaces over locally contractible enriched categories. *New York Journal of Mathematics*, 25:1485–1510, 2019.
- [12] P. Gaucher. Homotopy theory of Moore flows (I). *Compositionality*, 3(3), 2021. <https://doi.org/10.32408/compositionality-3-3>.
- [13] P. Gaucher. Homotopy theory of Moore flows (II). *Extracta Mathematicae*, 36(2):157–239, 2021. <https://doi.org/10.17398/2605-5686.36.2.157>.
- [14] P. Gaucher. Left properness of flows. *Theory and Applications of Categories*, 37(19):562–612, 2021.
- [15] P. Gaucher. Six model categories for directed homotopy. *Categories and General Algebraic Structures with Applications*, 15(1):145–181, 2021. <https://doi.org/10.52547/cgasa.15.1.145>.
- [16] P. Gaucher. Comparing cubical and globular directed paths, 2022. <https://doi.org/10.48550/arxiv.2207.01378>.
- [17] E. Goubault and S. Mimram. Directed homotopy in non-positively curved spaces. *Log. Methods Comput. Sci.*, 16(3):55, 2020. Id/No 4. [https://doi.org/10.23638/LMCS-16\(3:4\)2020](https://doi.org/10.23638/LMCS-16(3:4)2020).
- [18] M. Grandis. Directed homotopy theory. I. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 44(4):281–316, 2003.
- [19] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
- [20] K. Hess, M. Kędziołek, E. Riehl, and B. Shipley. A necessary and sufficient condition for induced model structures. *Journal of Topology*, 10(2):324–369, 2017. <https://doi.org/10.1112/topo.12011>.
- [21] P. S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003. <https://doi.org/10.1090/surv/099>.
- [22] M. Hovey. *Model categories*. American Mathematical Society, Providence, RI, 1999. <https://doi.org/10.1090/surv/063>.
- [23] F. W. Lawvere. Metric spaces, generalized logic, and closed categories. *Reprints in Theory and Applications of Categories*, 2002(1):1–37, 2002.
- [24] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic*. Universitext. Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition. <https://doi.org/10.1007/978-1-4612-0927-0>.
- [25] J. P. May and J. Sigurdsson. *Parametrized homotopy theory*, volume 132 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI,



2006. <https://doi.org/10.1090/surv/132>.
- [26] L. Moser. Injective and projective model structures on enriched diagram categories. *Homology, Homotopy and Applications*, 21(2):279–300, 2019. <https://doi.org/10.4310/hha.2019.v21.n2.a15>.
- [27] M. Raussen. Trace spaces in a pre-cubical complex. *Topology and its Applications*, 156(9):1718–1728, 2009. <https://doi.org/10.1016/j.topol.2009.02.003>.
- [28] M. Raussen. Simplicial models for trace spaces II: General higher dimensional automata. *Algebraic and Geometric Topology*, 12(3):1745–1765, August 2012. <https://doi.org/10.2140/agt.2012.12.1745>.
- [29] J. Rosický. On combinatorial model categories. *Applied Categorical Structures*, 17(3):303–316, 2009. <https://doi.org/10.1007/s10485-008-9171-2>.
- [30] J. Rosický. Accessible model categories. *Appl. Categ. Struct.*, 25(2):187–196, 2017. <https://doi.org/10.1007/s10485-015-9419-6>.
- [31] L. A. Steen and J. A. Seebach. *Counterexamples in topology*. Mineola, NY: Dover Publications, reprint of the 2nd edition published 1978 by Springer edition, 1995. <https://doi.org/10.1007/978-1-4612-6290-9>.
- [32] K. Ziemiański. Spaces of directed paths on pre-cubical sets II. *Journal of Applied and Computational Topology*, 4(1):45–78, 2020. <https://doi.org/10.1007/s41468-019-00040-z>.

UNIVERSITÉ PARIS CITÉ, CNRS, IRIF, F-75013, PARIS, FRANCE  
 URL: <http://www.irif.fr/~gaucher>