



**HAL**  
open science

# A review on some discrete variational techniques for the approximation of essential boundary conditions

Franz Chouly

► **To cite this version:**

Franz Chouly. A review on some discrete variational techniques for the approximation of essential boundary conditions. 2023. ⟨hal-03741809v2⟩

**HAL Id: hal-03741809**

**<https://hal.science/hal-03741809v2>**

Preprint submitted on 11 Sep 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# A review on some discrete variational techniques for the approximation of essential boundary conditions

Franz Chouly<sup>\*1,2,3</sup>

<sup>1</sup>Université Bourgogne Franche-Comté, Institut de Mathématiques de Bourgogne, 21078 Dijon, France

<sup>2</sup>Center for Mathematical Modeling and Department of Mathematical Engineering, University of Chile and IRL 2807 – CNRS, Santiago, Chile

<sup>3</sup>Departamento de Ingeniería Matemática, CI<sup>2</sup>MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile

September 11, 2023

## Abstract

We review different techniques to enforce essential boundary conditions, such as the (nonhomogeneous) Dirichlet boundary condition, within a discrete variational framework, and especially techniques that allow to account for them in a weak sense. Those are of special interest for discretizations such as geometrically unfitted finite elements or high order methods, for instance. Some of them remain primal, and add extra terms in the discrete weak form without adding a new unknown: this is the case of the boundary penalty and Nitsche techniques. Others are mixed, and involve a Lagrange multiplier with or without stabilization terms. For a simple setting, we detail the different associated formulations, and recall what is known about them, with emphasis on their stability and convergence properties.

*Keywords:* essential boundary conditions, Dirichlet boundary condition, finite elements, penalty, Nitsche, mixed finite elements, stabilized finite elements.

*AMS Subject Classification:* 65N12, 65N15, 65N30.

## 1 Introduction

Among the first papers that contributed to the mathematical theory of the finite element method, some already proposed various techniques to handle essential boundary conditions, such as the (nonhomogeneous) Dirichlet boundary condition. Early contributions have been made by I. Babuška in 1972 and 1973 with two seminal papers about Lagrange multipliers [23] and penalty [24] formulations. In the same period, in 1971, J.A. Nitsche published an alternative boundary penalty technique [165], consistent and with no additional unknown.

These pioneering works were followed by studies that allowed to refine the first analysis, to extend these techniques to more complex problems, and to invent alternatives, as well. The first motivation for some improvements in the approximation of

---

\*Corresponding author: [franz.chouly@math.cnrs.fr](mailto:franz.chouly@math.cnrs.fr)

essential boundary conditions concerns discrete variational techniques for which the degrees of freedom on the boundary are not nodal values, and in which it is not direct to approximate a Dirichlet condition. This is notably the case for unfitted finite elements or geometrically nonconforming finite elements, where the mesh boundary and the domain boundary do not match, as it occurs in fictitious domain methods, the eXtended Finite Element Method (XFEM) or the cut finite element method: see, *e.g.*, [59, 92, 99, 122, 134, 161, 169]. It is also the case for IsoGeometric Analysis (IGA) [83] and spline-based finite elements [105] where degrees of freedom are control points of B-spline functions, and are not nodes on the boundary [35]. A similar situation occurs in domain decomposition techniques with non-matching meshes [111, 174], where the mortar method [41, 44, 45, 192] or Nitsche's method [39, 116, 144] have revealed their usefulness. Another motivation has been the design of more and more sophisticated numerical methods to solve efficiently problems involving more complex essential boundary or interface conditions, such as fluid-structure interaction [22, 113, 119], contact and friction [12, 72, 76, 133, 146, 193] or even fluid-structure-contact now [18, 60, 95, 114, 159].

A survey on the topic has been published by R. Stenberg in 1995 [184], that presented some mixed, stabilized and Nitsche's formulations. Notably this survey presented in a simplified form the stabilization technique of H. Barbosa and T.J.R. Hughes [26, 27], and it revealed a fundamental relationship between Barbosa & Hughes stabilization and Nitsche's original symmetric formulation (see also [144] for domain decomposition). The aim of the present survey is to complete and update this previous one with new techniques that emerged since then, as for instance variants of Nitsche's method or new stabilized formulations. Moreover we include a short description of two common techniques: the discrete lifting and boundary penalty formulations.

This paper is outlined as follows. In Section 2 we introduce Poisson's problem with a nonhomogeneous Dirichlet boundary condition as a model problem representing an elliptic partial differential equation with an essential boundary condition. In Section 3 we describe primal methods for which the only unknown is the solution to Poisson's problem, and no extra unknown is introduced. In Section 4 we present methods coming from a duality argument and introducing a Lagrange multiplier that represents the flux on the boundary. Section 5 provides a few details about the practical implementation of the aforementioned methods into modern finite element libraries. We conclude in Section 6.

Let us introduce some useful notations. For  $D$  an open set in  $\mathbb{R}^d$ , we denote by  $H^s(D)$ ,  $s \in \mathbb{R}$ , the Sobolev space of real-valued function on  $D$  (see, *e.g.*, [2, 125, 160, 179]). The standard scalar product (resp. norm) of  $H^s(D)$  is denoted by  $(\cdot, \cdot)_{s,D}$  (resp.  $\|\cdot\|_{s,D}$ ). When  $s = 0$  we drop the index  $s$  and note  $(\cdot, \cdot)_D$  (resp.  $\|\cdot\|_D$ ) the scalar product (resp. norm) in  $L^2(D)$ . The notations  $C > 0$ ,  $c > 0$  and  $\beta > 0$  stand for arbitrary constants independent of the mesh sizes (the domain mesh and the boundary mesh) and of eventual stabilization and regularization parameters, but possibly dependent of  $d$ , of the size of the domain, of the shape-regularity of the finite elements and of the polynomial order of the finite element space(s). The value of these constants may vary from one place to another.

## 2 Setting

Let  $\Omega$  be an open and bounded polytope of  $\mathbb{R}^d$  ( $d \geq 1$ ), of Lipschitz boundary  $\Gamma := \partial\Omega$ . The notation  $\mathbf{n}$  stands for the unit outer normal to  $\Gamma$ . For a smooth enough function  $v : \Omega \rightarrow \mathbb{R}$ ,  $\partial_{\mathbf{n}}v := \nabla v \cdot \mathbf{n}$  stands for its normal derivative on  $\Gamma$ .

## 2.1 Poisson's problem

As a model problem, we focus on Poisson's problem with (nonhomogeneous) Dirichlet boundary condition:

$$\begin{aligned} &\text{Find } u : \Omega \rightarrow \mathbb{R} \text{ solution to} \\ &\begin{cases} -\Delta u = f & \text{in } \Omega, & (i) \\ u = g & \text{on } \Gamma, & (ii) \end{cases} \end{aligned} \quad (1)$$

with given source term  $f \in L^2(\Omega)$  and boundary datum  $g \in H^{\frac{1}{2}}(\Gamma)$ . The equivalent weak form is:

$$\begin{aligned} &\text{Find } u \in H^1(\Omega) \text{ with } u|_{\Gamma} = g \text{ and} \\ &a(u, v) = (f, v)_{\Omega} \quad \text{for all } v \in H_0^1(\Omega), \end{aligned} \quad (2)$$

where  $u|_{\Gamma}$  is the trace of  $u$  on the boundary  $\Gamma$ ,  $H_0^1(\Omega)$  is the subspace of functions in  $H^1(\Omega)$  with vanishing trace on  $\Gamma$ , and with the notation

$$a(v, w) := (\nabla v, \nabla w)_{\Omega}$$

for  $v, w \in H^1(\Omega)$ . Problem (2) admits one unique solution  $u$  in  $H^1(\Omega)$  (see [179, Section 4.4] or [109, Proposition 31.12]) that is also the unique minimizer on  $H^1(\Omega)$  of the quadratic convex functional

$$\mathcal{J} : H^1(\Omega) \ni v \mapsto \frac{1}{2}a(v, v) - (f, v)_{\Omega} \in \mathbb{R}$$

under the equality constraint  $v|_{\Gamma} = g$ . Moreover, with  $\mathcal{L}g \in H^1(\Omega)$  a lifting of  $g$  [160], Problem (2) can be reformulated equivalently:

$$\begin{aligned} &\text{Find } u \in H^1(\Omega) \text{ of the form } u = \mathcal{L}g + u_0, \text{ with } u_0 \in H_0^1(\Omega) \text{ solution to} \\ &a(u_0, v) = (f, v)_{\Omega} - a(\mathcal{L}g, v) \quad \text{for all } v \in H_0^1(\Omega). \end{aligned} \quad (3)$$

## 2.2 Finite element setting

We denote by  $\mathbb{P}_k$  the vector space of  $d$ -variate real polynomials of maximal degree  $k \geq 1$ . Let  $(\mathcal{T}^h)_{h>0}$  be a family of simplicial meshes of the domain  $\Omega$ . For a given  $h$ , each geometrical element  $T \in \mathcal{T}^h$  is a simplex of dimension  $d$ , a triangle for  $d = 2$  and a tetrahedron for  $d = 3$ . It is supposed to be a closed subset of  $\mathbb{R}^d$ :  $\bar{T} = T$  (and the same assumption holds for the edges and faces). We define the mesh size  $h$  as follows:  $h := \max_{T \in \mathcal{T}^h} h_T$ , where  $h_T$  is the diameter of the simplex  $T$ . The mesh is supposed to be regular in Ciarlet's sense: there exists a shape-regularity constant  $\sigma > 0$  such that

$$\frac{h_T}{\rho_T} \leq \sigma, \quad \forall T \in \mathcal{T}^h, \quad (4)$$

where  $\rho_T$  is the radius of the largest ball contained in the simplex  $T$ . We call facets either the edges of the triangles (when  $d = 2$ ) or the faces of the tetrahedra (when  $d = 3$ ). The notation  $E$  is used for a generic facet in  $\mathcal{T}^h$  and the notation  $h_E$  stands for the diameter of the facet  $E$ . Remember that the regularity of the mesh implies there exists  $c, C > 0$  such that

$$ch_T \leq h_E \leq Ch_T$$

for any facet  $E$  that belongs to a simplex  $T \in \mathcal{T}^h$ . The above constants  $c$  and  $C$  are independent of the mesh size, but not of the shape-regularity of the mesh.

We suppose finally that the mesh resolves the boundary  $\Gamma$ : for each simplex, each of its facets that intersects  $\Gamma$  is contained completely in one and only one of the facets of  $\Gamma$ .

The Lagrange finite element space of degree  $k \geq 1$  is (see, *e.g.*, [51, 79, 108, 173]):

$$V^{h,k} := \{v^h \in \mathcal{C}^0(\bar{\Omega}) \mid v^h|_T \in \mathbb{P}_k(T), \forall T \in \mathcal{T}^h\}. \quad (5)$$

We will omit the subscript  $k$  when there is no ambiguity and note simply  $V^h$  instead of  $V^{h,k}$ . We define also

$$V_0^h := V^h \cap H_0^1(\Omega) = \{v^h \in V^h \mid v^h|_\Gamma = 0\}$$

the vector space of functions with vanishing discrete trace.

For the analysis of some methods, it will be convenient to make use of the discrete norms introduced below. For this purpose let us take  $\zeta^\alpha$  ( $\alpha \in \mathbb{R}$ ) as a piecewise constant function on the boundary  $\Gamma$ , that represents locally the power  $\alpha$  of the boundary mesh size, and which restriction to the interior of each boundary facet  $E \subset \Gamma$  is defined as

$$\zeta^\alpha|_E = h_E^\alpha,$$

with  $|_E$  that designs in fact the restriction to the interior of  $E$ . This is a slight abuse of notation and, in fact, the value of  $\zeta$  at the intersection between two facets is of no importance and can be set to 0 for instance. For  $v \in L^2(\Gamma)$  we define

$$\|v\|_{-1/2,h,\Gamma} := \|\zeta^{1/2}v\|_{0,\Gamma_C}, \quad \|v\|_{1/2,h,\Gamma} := \|\zeta^{-1/2}v\|_{0,\Gamma_C}.$$

For  $v \in H^1(\Omega)$  we set

$$\|v\|_h := \left( \|\nabla v\|_{0,\Omega}^2 + \|v\|_{1/2,h,\Gamma}^2 \right)^{1/2}$$

which is an equivalent norm of the  $H^1(\Omega)$ -norm.

For the stability and convergence analysis of some methods, the following discrete trace inequality (often called discrete trace inverse inequality) will be needed.

**Lemma 2.1.** *There exists a constant  $c_I > 0$  independent of the mesh size  $h$  such that, for any  $v^h \in V^h$ :*

$$\|\partial_{\mathbf{n}}v^h\|_{-1/2,h,\Gamma}^2 \leq c_I \|\nabla v^h\|_{0,\Omega}^2. \quad (6)$$

Moreover, for finite elements of degree  $k$  on a simplicial mesh in dimension  $d$  there holds:

$$c_I \leq \frac{c_\rho}{d} (k+1)(k+d) \quad (7)$$

where  $c_\rho$  is a constant that depends only on the shape-regularity constant  $\rho$  and is independent of the polynomial degree  $k$ .

*Proof.* See for instance [76, Lemma 4.1] or [108, Lemma 12.8] for (6) (or also [186, Lemma 2.1] when the mesh is quasi-uniform). The estimate (7) is established in [190] (see also [108, Lemma 12.2]).  $\square$

### 3 Primal methods

This section focuses on primal methods. Primal means that we do not use duality techniques so as to reformulate the problem. Duality techniques introduce some extra unknowns (here associated to the flux on the boundary). So primal methods are

characterized by the fact that there is just one unknown,  $u$ , that is directly approximated. In this framework, there are various possibilities to enforce the nonhomogeneous Dirichlet boundary condition (1)–(ii). We describe first the most standard one, that is presented in various basic textbooks and classnotes, and that is based upon a discrete lifting operator. Then we focus on the penalty method, which is also widespread, and even simpler in some sense than the discrete lifting technique, but with the fundamental issue that consistency is lost. We end this presentation with Nitsche’s technique, in which supplementary terms are added, that allow to recover consistency.

### 3.1 The discrete lifting

We summarize here the standard method for imposing a nonhomogeneous Dirichlet boundary condition, that relies on a discrete lifting or direct nodal imposition [34, 106, 109, 182].

#### 3.1.1 Formulation

To simplify, we suppose that, in Equation (1)–(ii), the Dirichlet data  $g$  is continuous:  $g \in \mathcal{C}^0(\Gamma) \cap H^{\frac{1}{2}}(\Gamma)$ . As a result, we can take its Lagrange interpolant:  $g^h := \mathcal{I}_\Gamma^h(g)$ , where  $\mathcal{I}_\Gamma^h$  denotes the Lagrange interpolant on the trace space of  $V^h$ . A discrete counterpart of Problem (2) is then:

$$\begin{aligned} &\text{Find } u^h \in V^h \text{ that satisfies } u^h|_\Gamma = g^h \text{ and} \\ &a(u^h, v^h) = L(v^h) \quad \text{for all } v^h \in V_0^h. \end{aligned} \quad (8)$$

As for the continuous case, above Problem (8) is equivalent to find the unique minimizer on  $V^h$  of the quadratic convex functional  $\mathcal{J}(\cdot)$  under the equality constraint  $v^h|_\Gamma = g^h$ .

Let  $\mathcal{L}^h g^h \in V^h$  be a discrete lifting of  $g^h$ . It can be obtained for instance by setting  $\mathcal{L}^h g^h(\mathbf{a}_i) = g(\mathbf{a}_i)$  if  $\mathbf{a}_i$  is a boundary node ( $\mathbf{a}_i \in \Gamma$ ), and  $\mathcal{L}^h g^h(\mathbf{a}_i) = 0$  if  $\mathbf{a}_i$  is an interior node ( $\mathbf{a}_i \in \Omega$ ). An equivalent formulation of Problem (8) is:

$$\begin{aligned} &\text{Find } u^h \in V^h \text{ of the form } u^h = \mathcal{L}^h g^h + u_0^h, \text{ with } u_0^h \in V_0^h \text{ solution to} \\ &a(u_0^h, v^h) = (f, v^h)_\Omega - a(\mathcal{L}^h g^h, v^h) \quad \text{for all } v^h \in V_0^h(\Omega). \end{aligned} \quad (9)$$

Problem (9) is a discrete counterpart of Problem (3).

#### 3.1.2 Well-posedness and convergence

Using the same argument [179, Section 4.4] as for the continuous problem, it can be established that the discrete Problem (8) is well-posed and an optimal  $H^1$ -error estimate can be derived [109, Lemma 33.1, Theorem 33.2]. More precisely, let us suppose that the solution  $u$  to Problem (2) belongs to  $H^s(\Omega)$  with a regularity  $s$  that satisfies  $d/2 < s \leq k + 1$ . Let  $u^h$  be the solution to Problem (8), there holds:

$$\|u - u^h\|_{1,\Omega} \leq Ch^{s-1} \|u\|_{s,\Omega},$$

where the constant  $C > 0$  does not depend of the mesh size  $h$ , but depends on the shape-regularity constant  $\sigma$  of the mesh and the polynomial order  $k$ . Moreover, an obvious bound can be obtained for the trace on the boundary. If we suppose that  $g$  is in  $H^\tau(\Gamma)$ , with  $(d-1)/2 < \tau \leq k$  and using the condition  $u^h|_\Gamma = g^h$ , we get

$$\|u - u^h\|_{0,\Gamma} = \|g - g^h\|_{0,\Gamma} = \|g - \mathcal{I}_\Gamma^h(g)\|_{0,\Gamma} \leq Ch^\tau \|g\|_{\tau,\Gamma}$$

from the interpolation error of  $g$ .

Optimal error estimates in the  $L^2$ -norm can be derived using the Aubin-Nitsche duality trick, see for instance [34, 182] for the  $\mathbb{P}_1$  Lagrange finite element, and [109] for  $\mathbb{P}_k$ ,  $k \geq 1$ .

**Remark 3.1.** For the practical solution of the linear system associated to (8), see for instance [109, Section 33.1.3]. As well, for an estimate of the related condition number, followed by numerical experiments, in the case of a quasi-uniform mesh and  $\mathbb{P}_1$  finite elements, see [182] (this issue is critical for cut finite elements [59]).

**Remark 3.2.** The assumption of continuity on the boundary data  $g$  can be alleviated, if one uses for instance a  $L^2$ -projection operator (or local  $L^2$ -projections) instead of Lagrange interpolation. In this case, optimal  $H^1$ - and  $L^2$ -error estimates still hold, see [34] for more details.

This first method, though simple and efficient, is not easy to generalize to more complex boundary conditions. Furthermore, it is closely related to Lagrange finite elements on a conforming mesh, and is not suited to other discretization approaches, in which degrees of freedom are not directly linked to the value of the discrete solution  $u^h$  at some nodes located on the boundary  $\Gamma$ .

### 3.1.3 A posteriori error estimates and adaptive meshing

For the discrete lifting, various studies have been devoted to *a posteriori* error estimates (mostly simple, explicit residual-based estimates), and the interested reader can refer for instance to [4, 34, 176] as basic references. Particularly, the choice of the operator (Lagrange interpolator, Scott-Zhang quasi-interpolator, projection operator) that allows to enforce the essential condition on the boundary has a critical, and counterintuitive influence, on the properties of the estimators and of the adaptive mesh refinement procedure. For references related to the convergence of the adaptive finite element method, see particularly [20, 67, 112, 162]. Let us mention also [54] for least-squares finite element methods of possibly high order, and also [94] for an estimator based on equilibrated fluxes.

## 3.2 Boundary penalty

There are various possibilities to regularize the Dirichlet condition (1)-(ii) and the most simple and widespread one is one of the earliest (see, *e.g.*, [19, 24] for Dirichlet conditions or [146, 147, 166, 167] for Signorini conditions).

### 3.2.1 Continuous formulation

In its most widespread form, the penalty method, or boundary penalty method, for Poisson's problem (2) can be formulated at the continuous level and reads (see, *e.g.*, [19, 24]):

$$\begin{aligned} &\text{Find } u_\varepsilon \in H^1(\Omega) \text{ solution to} \\ &a(u_\varepsilon, v) + \frac{1}{\varepsilon}(u_\varepsilon, v)_\Gamma = (f, v)_\Omega + \frac{1}{\varepsilon}(g, v)_\Gamma \quad \text{for all } v \in H^1(\Omega), \end{aligned} \quad (10)$$

where the penalty parameter is denoted by  $\varepsilon > 0$ . When going back to strong form, we verify that  $u_\varepsilon$  satisfies Poisson's equation (1)-(i) complemented with the following Robin-Fourier boundary condition

$$\partial_{\mathbf{n}} u_\varepsilon = -\frac{1}{\varepsilon}(u_\varepsilon - g) \quad (11)$$

that approximates the Dirichlet boundary condition (1)-(ii) for  $\varepsilon$  small enough. The above weak form (10) is also the first order optimality condition associated to the minimization of the functional

$$\mathcal{J}_\varepsilon : V \ni v \mapsto \mathcal{J}(v) + \frac{1}{2\varepsilon} \|v - g\|_{0,\Gamma}^2 \in \mathbb{R}. \quad (12)$$

So, for a symmetric weak form, we recover the usual technique from convex optimization, which consists in reformulating a constrained minimization problem into an unconstrained problem where the constraint is penalized.

Using a Poincaré-Friedrichs inequality, we check that, for every value of the penalty parameter  $\varepsilon > 0$ , the modified bilinear form in (10) is elliptic on  $H^1(\Omega)$ , so Problem (10) is well-posed by application of the Lax-Milgram lemma (or simply the Riesz representation Theorem here, since the bilinear form is symmetric). Moreover, if we suppose that  $u \in H^2(\Omega)$  we can derive the bound

$$\|u - u_\varepsilon\|_{1,\Omega} \leq C\varepsilon\|u\|_{2,\Omega}$$

from which we conclude that  $u_\varepsilon$  tends to  $u$  when  $\varepsilon$  vanishes [33]. Remark also that Problem (10) remains well-defined and well-posed even if we weaken the regularity assumptions on the boundary datum  $g$ , that needs only to belong to  $L^2(\Gamma)$  instead of  $H^{\frac{1}{2}}(\Gamma)$ .

### 3.2.2 Discrete formulation

The discrete boundary penalty formulation is a direct rewriting of (10) and reads:

$$\begin{cases} \text{Find } u_\varepsilon^h \in V^h \text{ such that:} \\ a(u_\varepsilon^h, v^h) + \frac{1}{\varepsilon}(u_\varepsilon^h, v^h)_\Gamma = (f, v^h)_\Omega + \frac{1}{\varepsilon}(g, v^h)_\Gamma \text{ for all } v^h \in V^h. \end{cases} \quad (13)$$

In such a formulation, it is usual to assume that the mesh is quasi-uniform and that the the penalty parameter is of the form

$$\varepsilon = \varepsilon_0 h^\lambda \quad (14)$$

where the coefficient  $\varepsilon_0 > 0$  and the power  $\lambda \geq 0$  are independent of the mesh and chosen appropriately by the user (of course, for nonuniform meshes,  $\varepsilon$  can be set as a boundary function of the local facet size  $h_E$  for each boundary facet  $E$ , as follows:  $\varepsilon = \varepsilon_0 \zeta^\lambda$ ).

**Remark 3.3.** *Since conditions (11) and (1)–(ii) are not equivalent ( $u_\varepsilon \neq u$  in general for  $\varepsilon > 0$ ), the discrete penalty method (13) is not consistent. Indeed, there holds, for  $u$  solution to (1) and an arbitrary test function  $v^h \in V^h$ :*

$$\begin{aligned} & a(u, v^h) + \frac{1}{\varepsilon}(u, v^h)_\Gamma \\ &= (\partial_{\mathbf{n}} u, v^h)_\Gamma + (f, v^h)_\Omega + \frac{1}{\varepsilon}(g, v^h)_\Gamma \end{aligned}$$

where we applied first integration by parts, and then used that  $u$  solves (1). As a result the extra term  $(\partial_{\mathbf{n}} u, v^h)_\Gamma$  introduces a consistency error.

### 3.2.3 Well-posedness and convergence

The same argument as for the continuous formulation ensures well-posedness of Problem (13) for every value of the penalty parameter  $\varepsilon_0 > 0$  and of the power  $\lambda \geq 0$ . The numerical analysis of the boundary penalty method has been carried out thoroughly in a paper of J.W. Barrett and C.M. Elliott [33] and error estimates for the  $H^1$ - and  $L^2$ -norms on the domain, and for the  $L^2$ -norm on the boundary have been obtained as functions of the power  $\lambda$  in (14), the degree  $k$  of the finite element space and the regularity of the solution  $u$ , in different situations. Notably, when the domain  $\Omega$  is a convex polyhedron, they manage to obtain the following bounds:

- If  $\lambda = k$ , there holds:

$$\|u - u_\varepsilon^h\|_{0,\Omega} + h^{\frac{1}{2}}\|u - u_\varepsilon^h\|_{1,\Omega} + \|u - u_\varepsilon^h\|_{0,\Gamma} \leq C(\varepsilon_0)h^k\|u\|_{1+k,\Omega}, \quad (15)$$

which ensure convergence but is suboptimal.

- If  $\lambda = k + \frac{1}{2}$ , there holds:

$$h^{-\frac{1}{2}}\|u - u_\varepsilon^h\|_{0,\Omega} + \|u - u_\varepsilon^h\|_{1,\Omega} + h^{-\frac{1}{2}}\|u - u_\varepsilon^h\|_{0,\Gamma} \leq C(\varepsilon_0)h^k\|u\|_{1+k,\Omega}, \quad (16)$$

which is optimal for the  $H^1$ -norm but remains suboptimal for the  $L^2$ -norm on the domain.

- If  $\lambda = k + 1$ , there holds:

$$h^{-1}\|u - u_\varepsilon^h\|_{0,\Omega} + \|u - u_\varepsilon^h\|_{1,\Omega} + h^{-1}\|u - u_\varepsilon^h\|_{0,\Gamma} \leq C(\varepsilon_0)h^k\|u\|_{2+k,\Omega}, \quad (17)$$

which are the desired optimal convergence rates.

The constant  $C(\varepsilon_0)$  in the above bounds does not depend on the mesh size  $h$ , but depends on the penalty parameter  $\varepsilon_0$ , and becomes large whenever  $\varepsilon_0$  is large. It may depend also of the polynomial order  $k$  and the shape-regularity constant  $\sigma$ .

At first glance, the bound (17) seems to be satisfying but it requires more regularity on the solution  $u$  than what is usually expected. Moreover, setting the power  $\lambda$  to  $k + 1$  instead of  $k$  has an impact on the conditioning on the linear system and to the convergence of iterative solvers, especially for fine meshes. Nevertheless, it is the best available results in such a configuration to the best of our knowledge, and it improves the first result obtained by I. Babuška in [23], for which no value of  $\lambda$  provided optimal estimates for polynomial degree  $k \geq 2$ . Note however that I. Babuška obtained an almost optimal convergence rate in the  $H^1$  norm for the lowest polynomial degree  $k = 1$  and  $\lambda = 1$ , which is the usual choice in practice. Despite of this, his error estimate in  $L^2$  norm of the domain remained suboptimal. Under some restrictive assumptions on the regularity of the solution, Z.C. Shi [181] provided also an optimal bound for the lower polynomial degree  $k = 1$ .

**Remark 3.4.** *The case of a curved boundary with variational crimes due to the geometric approximation of the boundary and numerical integration is also studied thoroughly in the paper of J.W. Barrett and C.M. Elliott. In such a situation, some improved results have been published recently by I. Dione [88].*

**Remark 3.5.** *In fact, since the boundary penalty method allows to impose weakly the Dirichlet boundary condition, it has been considered in a few works for fictitious domains techniques, see, e.g., [9, 158]. Yet, mixed methods [122, 120] or Nitsche methods [59] are the most widespread within this community.*

In practice, a critical issue about the boundary penalty method is the best choice of the penalty parameter  $\varepsilon_0$ . If its value is too small, the Dirichlet condition is approximated accurately but the conditioning of the global stiffness matrix deteriorates, and if the value is too large, the Dirichlet condition is approximated poorly. As a result, a compromise between these two situations must be adopted.

**Remark 3.6.** *Another issue is an accurate prediction of the boundary flux  $\partial_{\mathbf{n}}u_\varepsilon^h$ , which can be inaccurate and depends on  $\varepsilon_0$ . This is fundamentally due to (11). This point has been object of recent work by V. Garg and S. Prudhomme [117].*

**Remark 3.7.** *Alternatively, a penalty technique can be directly performed at the algebraic level, after assembly of the stiffness matrix corresponding to pure (homogeneous) Neumann boundary conditions. This technique is sometimes called exact penalty: see [137] for more details or [109, Remark 33.5].*

### 3.3 Nitsche

Now we present a consistent boundary penalty technique to incorporate the nonhomogeneous Dirichlet boundary condition (1)–(ii). This technique has been originally proposed by J.A Nitsche in 1971 [165]. It has gained first its popularity in the Discontinuous Galerkin community, see, *e.g.*, [17, 87], and later on in the finite element community as a method to treat various boundary and interface conditions [130]. As we will see, there is in fact a whole family of methods that can be derived systematically and share some common features of well-posedness and optimal accuracy (but with different conditions on the numerical parameter). The most widespread member of this family is the original one of J.A. Nitsche, that derives of a functional and preserves the symmetry of the original weak form (2). Nonsymmetric variants have been derived since then. For the symmetric variant, there are various references that present the method and detail its numerical analysis, particularly the survey of R. Stenberg [184] already mentioned in the introduction, a chapter of P. Hansbo [130] and the book of V. Thomée [186, Chapter 2]. A detailed presentation of two variants, the symmetric and the incomplete ones, can be found in the second volume of A. Ern & J.L. Guermond [109, Chapter 37].

#### 3.3.1 Formulation

Let us first derive the method formally, and for this purpose take  $\gamma > 0$  a positive function on the boundary  $\Gamma$  and  $\theta \in \mathbb{R}$  a fixed parameter.

Let  $u$  be the solution to (1), and  $v$  a test function, and suppose that both functions are regular enough so that the following calculations make sense. From (1)–(i) and using Green formula we get first:

$$a(u, v) - (\partial_{\mathbf{n}}u, v)_{\Gamma} = (f, v)_{\Omega}.$$

Then we use the Dirichlet condition (1)–(ii) to write

$$(u, \gamma v - \theta \partial_{\mathbf{n}}v)_{\Gamma} = (g, \gamma v - \theta \partial_{\mathbf{n}}v)_{\Gamma}.$$

We add the two above equations. We get

$$\begin{aligned} & a(u, v) - (\partial_{\mathbf{n}}u, v)_{\Gamma} - \theta(u, \partial_{\mathbf{n}}v)_{\Gamma} + (u, \gamma v)_{\Gamma} \\ &= (f, v)_{\Omega} + (g, \gamma v - \theta \partial_{\mathbf{n}}v)_{\Gamma}. \end{aligned} \tag{18}$$

Remark first that the boundary term  $-(\partial_{\mathbf{n}}u, v)_{\Gamma}$  above allows to recover consistency (remember Remark 3.3). The above weak form may have no meaning at the continuous level but has a well-defined discrete counterpart. For this purpose let us take  $\gamma$  as a piecewise constant function on the boundary  $\Gamma$ :

$$\gamma = \gamma_0 \zeta^{-1},$$

where  $\gamma_0$  is a positive constant. To lighten the writing of the weak form (18) we introduce the bilinear form

$$A_N(u^h, v^h) := a(u^h, v^h) - (\partial_{\mathbf{n}}u^h, v^h)_{\Gamma} - \theta(u^h, \partial_{\mathbf{n}}v^h)_{\Gamma} + (\gamma u^h, v^h)_{\Gamma}$$

and the linear form

$$l_N(v^h) := (f, v^h)_{\Omega} + (g, \gamma v^h - \theta \partial_{\mathbf{n}}v^h)_{\Gamma}.$$

Nitsche's method for Poisson's problem (1) reads:

$$\begin{cases} \text{Find } u^h \in V^h \text{ solution to} \\ A_N(u^h, v^h) = l_N(v^h), \quad \forall v^h \in V^h. \end{cases} \tag{19}$$

Three notable variants of the method, for different values of the parameter  $\theta$  can be obtained, as for the discontinuous Galerkin Interior Penalty (dGIP) method [87]:

1. For  $\theta = 1$ , the formulation (19) is symmetric and identical to Nitsche's original formulation [165]. It can be obtained as the first order optimality condition associated to the functional

$$\mathcal{J}_N : V^h \ni v^h \mapsto \mathcal{J}(v^h) - (v^h - g, \partial_{\mathbf{n}} v^h)_{\Gamma} + \frac{1}{2} \|\gamma^{\frac{1}{2}}(v^h - g)\|_{\Gamma}^2 \in \mathbb{R}.$$

It can be built alternatively from an augmented Lagrangian formalism, see [64, Section 5.2.2]

Remark that the expression of the Nitsche's functional  $\mathcal{J}_N(\cdot)$  is indeed similar to that of an augmented Lagrangian, and that it includes an extra term in comparison to the functional  $\mathcal{J}_{\varepsilon}(\cdot)$  of the boundary penalty. This term allows to recover consistency and to preserve symmetry.

2. For  $\theta = 0$ , we get the simplest, incomplete, formulation, that has the less terms, and that is presented for instance in [109, Section 37.1].
3. For  $\theta = -1$ , we recover the skew-symmetric formulation of J. Freund and R. Stenberg [115], where discrete ellipticity is ensured whatever the value of  $\gamma_0 > 0$  is.

**Remark 3.8.** *For an abstract viewpoint with a systematic technology to derive Nitsche's method for a large class of, linear and nonlinear, essential boundary or interface conditions and other differential operators, see [139] (and see [76] for contact and friction conditions). For an adaptation to fourth-order differential operator (Kirchhoff plate), see [129]. For an adaptation to nonlinear conditions defined on the domain  $\Omega$ , and not on the boundary, see [65, 127] for the obstacle problem and [73] for the elastoplastic torsion problem.*

For  $u \in H^s(\Omega)$  with  $s > \frac{3}{2}$ , it is direct to check that Nitsche's method (19) is consistent, and, from this point, it differs drastically from the boundary penalty method (13). Fundamentally this difference comes from the fact that it does not regularize the Dirichlet boundary condition.

### 3.3.2 Well-posedness and convergence

Thanks to the discrete trace inequality (6) we deduce that Problem (19) is well-posed provided that

$$\frac{(1 + \theta)^2 c_I}{\gamma_0} \leq 1, \quad (20)$$

where  $c_I$  is the constant of the discrete trace inequality (6). The complete statement for well-posedness and its proof are provided in the Appendix A. See also [70, 109, 130, 165, 184, 186] for specific values of  $\theta$  (and sometimes of  $k$ ), and the generalization to an arbitrary  $\theta$  makes no specific difficulty. For the symmetric variant  $\theta = 1$  this result implies notably the strong convexity of Nitsche's functional  $J_N(\cdot)$  when  $\gamma_0$  is large enough. As a result for the skew-symmetric variant  $\theta = -1$ , well-posedness is ensured irrespectively of the value of  $\gamma_0 > 0$ , and, otherwise,  $\gamma_0$  needs to be large enough.

**Remark 3.9.** *For more complex problems, a similar condition as (20) holds, but with a constant that depends for instance of the physical parameters. It can be interesting to establish precise quantitative bounds for this constant to fix the value of  $\gamma_0$  as low as possible (for  $\theta \neq 1$ ). It can be done through solving eigenvalue problems (see, e.g., [130, 139]) or obtained by other methods (such as dimensional studies). In case of small strain elasticity, see for instance [71, 131], and more recently [16, 36].*

Nitsche's method converges optimally in the  $H^1$ -norm, or equivalently in the discrete norm defined above, provided that  $\gamma_0$  be large enough. So for  $u \in H^s(\Omega)$  with  $\frac{3}{2} < s \leq 1 + k$ , we have

$$\|u - u^h\|_h + \|\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h\|_{-1/2, h, \Gamma} \leq Ch^s \|u\|_{s, \Omega}. \quad (21)$$

Remarkably, and conversely to the boundary penalty method, the constant  $C > 0$  in the above bound (21) does not depend on the Nitsche parameter  $\gamma_0$  provided it be large enough. For the skew-symmetric variant, when  $\theta \neq -1$ , this bounds still hold when  $\gamma_0 > 0$  becomes arbitrarily small but then, the constant  $C > 0$  depends on  $\gamma_0$ . Of course  $C > 0$  does not depend on the mesh size  $h$ , but depends on the shape-regularity constant  $\sigma$  and on the polynomial order  $k$ .

The complete statement of the bound (21) and its proof are given in Appendix B. See also [70, 109, 130, 165, 184, 186] for alternative proofs for some specific values of  $\theta$  (and sometimes of  $k$ ). Remark that the above estimate provides also an optimal convergence rate for the boundary flux  $\partial_{\mathbf{n}}u$ .

**Remark 3.10.** *The regularity assumption  $s > 3/2$  in the above bound (21) is a technical assumption, that makes easier to take into account the consistency into the error analysis (see the Appendix B). It is a delicate issue to weaken this assumption. However, and recently, optimal error estimates in the  $H^1$ -norm have been obtained even for solutions with low (minimal) regularity  $1 \leq s \leq 3/2$ , see, e.g., [107, 128]. They make use of techniques that are nonstandard in the numerical analysis of finite element methods. Additionally, error estimates in the  $L^2$ -norm can be obtained, see, e.g., [109, 186] for statements and proofs. For explicit residual a posteriori error estimates, see for instance [39, 128]: the results are for interface problems but it can be straightforwardly adapted to a Dirichlet boundary condition.*

**Remark 3.11.** *Of course, the value of  $\gamma_0$  influences the condition number of the global stiffness matrix associated to  $A_N(\cdot, \cdot)$ , and for this reason it does not have to be taken too large. Anyway, since the method is consistent, the impact of the numerical parameter  $\gamma_0$  has on the approximation of the Dirichlet condition is not as important as for the boundary penalty method (13). Moreover, since the function  $\gamma$  scales as  $\mathcal{O}(h^{-1})$  whatever the polynomial order  $k$  is, this does not deteriorate too much the conditioning, that remains in  $\mathcal{O}(h^{-2})$  [130]. This is not the case for the boundary penalty (see Section 3.2). This property makes Nitsche’s method particularly relevant for high order discretizations, and for this reason it has been considered for isogeometric analysis (IGA) [13, 69, 139], spline-based finite elements [105], or also for Hybrid High Order (HHO) [68, 71] variational approximations.*

**Remark 3.12.** *As it was already the case for the boundary penalty method, Nitsche’s method enforces weakly the Dirichlet condition and is also a candidate for fictitious domain methods. Various works have been dedicated to this topic: see for instance [59, 61, 155], just to mention a few. In this context of unfitted methods see for instance [85] for the choice of the Nitsche parameter.*

**Remark 3.13.** *Nitsche’s method can be adapted as well for Robin boundary conditions. Of course Robin boundary conditions are not difficult to take into account in a weak formulation, but the Nitsche formulation of [145] is robust irrespectively of the value of the Robin coefficient, that can be arbitrarily large, and this is its main interest. Recently the first analysis of [145] has been improved and extended for solutions with low regularity [157].*

### 3.4 New variants of Nitsche

To end this section, we present here some other Nitsche-type methods that emerged in the last decade. We do not enter as much into details as for the three “canonical” primal methods presented before, but rather try to underline the new possibilities that these methods offer.

### 3.4.1 A penalty free Nitsche method

Note that the skew-symmetric version  $\theta = -1$  still makes sense if we set  $\gamma_0 = 0$  and in this case we get from (19):

$$\begin{cases} \text{Find } u^h \in V^h \text{ solution to} \\ a(u^h, v^h) - (\partial_{\mathbf{n}} u^h, v^h)_{\Gamma} + (u^h, \partial_{\mathbf{n}} v^h)_{\Gamma} = (f, v^h)_{\Omega} + (g, \partial_{\mathbf{n}} v^h)_{\Gamma}, \quad \forall v^h \in V^h. \end{cases}$$

This version has no extra numerical parameter and is called penalty-free Nitsche [57] (see also [115]). In this case the method remains stable, since a discrete uniform inf-sup condition holds, and converges optimally in the  $H^1$  norm, see [57]. In reference [57] an estimate in the  $L^2$  norm is also proven. It is suboptimal of order  $\mathcal{O}(h^{\frac{1}{2}})$  since this nonsymmetric variant is not consistent for the adjoint problem.

**Remark 3.14.** *Various extensions of this variant have been made later on, for fictitious domains with cut elements [49], for elasticity [48], for Signorini [63], for Brinkman [46] and for isogeometric analysis [139, 180].*

### 3.4.2 Other Nitsche-type methods

In [50, 100] a variant of Nitsche has been suggested, that adds an extra term involving the Laplace-Beltrami differential operator on the boundary. In [150] another variant of Nitsche has been suggested, where another variable is added for the approximation of the gradient, and later on eliminated by static condensation. This approach is close in some sense to mixed methods. In [151] modifications of symmetric Nitsche have been made to design parameter-free methods for fitted and unfitted finite element discretizations.

In [37] a variant of Nitsche that does not involve any boundary term has been suggested. The idea, is, roughly speaking, to use a representation of the boundary flux built upon a variational residual. More precisely, the first step is to rewrite  $V^h$  as the following direct sum

$$V^h = V_i^h \oplus V_b^h,$$

where  $V_i^h$  is the subspace of functions with vanishing trace on  $\Gamma$  and  $V_b^h$  is the complementary subspace. Take  $g_b^h \in V_b^h$  an approximation of the boundary data  $g$  (it can be the standard discrete lifting, but alternatively any other kind of reconstruction). We want to find  $u^h = u_i^h + u_b^h$  that minimizes the following modified Nitsche's energy functional

$$\mathcal{J}_B : V^h \ni u^h \mapsto \mathcal{J}(u^h) + (f, u_b^h - g_b^h)_{\Omega} - (\nabla u^h, \nabla(u_b^h - g_b^h))_{\Omega} + \|\nabla(u_b^h - g_b^h)\|_{\Omega}^2 \in \mathbb{R}.$$

Note that it mimics exactly the original functional  $\mathcal{J}_N$  but involves only integral terms on the domain  $\Omega$ . Moreover there is no discrete parameter. The discrete weak problem comes from the first order optimality condition of the above functional and reads:

$$\begin{aligned} & \text{Find } (u_i^h, u_b^h) \in V_i^h \times V_b^h \text{ solution to} \\ & a(u_i^h, v_i^h) = (f, v_i^h)_{\Omega} - a(g_b^h, v_i^h) \quad \text{for all } v_i^h \in V_i^h, \\ & a(u_b^h, v_b^h) = a(g_b^h, v_b^h) \quad \text{for all } v_b^h \in V_b^h. \end{aligned}$$

In complement, some nonsymmetric variants can be obtained (see [37] for the details). The main interest of this method is for singularly perturbed problems such as reaction-diffusion or convection-diffusion, where it becomes nonconforming [37]. Numerical results in [37] illustrate indeed that it performs as Nitsche's method and better than the discrete lifting in the limit cases.

## 4 Methods with Lagrange multipliers

In mixed methods, the boundary condition in (1) is treated weakly thanks to a Lagrange multiplier  $\lambda$  that represents either the normal derivative  $\partial_{\mathbf{n}}u$  or its opposite  $-\partial_{\mathbf{n}}u$  [23, 52, 170, 171] (see also [47, 104, 106, 108, 109, 118] for general references about the discretization of mixed and hybrid methods). The continuous weak problem that serves for further numerical approximations is

$$\begin{aligned} &\text{Find } (u, \lambda) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma) \text{ solution to} \\ &B((u, \lambda), (v, \mu)) = L(v, \mu) \quad \text{for all } (v, \mu) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma), \end{aligned} \quad (22)$$

with the notations

$$B((u, \lambda), (v, \mu)) := a(u, v) + \langle \lambda, v \rangle_{\Gamma} + \langle \mu, u \rangle_{\Gamma}, \quad L(v, \mu) := (f, v)_{\Omega} + \langle \mu, g \rangle_{\Gamma},$$

where  $\langle \cdot, \cdot \rangle_{\Gamma}$  is the duality product  $H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ . Note that we used a sign convention that allows to preserve symmetry, and that corresponds to  $\lambda = -\partial_{\mathbf{n}}u$ . Problem (22) is well-posed. Thanks to the properties of the trace operator on the boundary, a continuous inf-sup condition holds

$$\inf_{\mu \in H^{-\frac{1}{2}}(\Gamma)} \sup_{v \in H^1(\Omega)} \frac{\langle \mu, v \rangle_{\Gamma}}{\|\mu\|_{-\frac{1}{2}, \Gamma} \|v\|_{1, \Omega}} \geq \beta, \quad (23)$$

with  $\beta > 0$  [179, Section 10.3]. Problem (22) is equivalent to the original problem (2). Notably, setting the test function  $v$  to zero in the above weak form 22, we get

$$\langle \mu, u - g \rangle_{\Gamma} = 0$$

for every  $\mu \in H^{-\frac{1}{2}}(\Gamma)$ , which means that the Dirichlet boundary condition (1)–(ii) is imposed weakly. Moreover, the unique solution to Problem (22) is identically the unique saddle-point of the following Lagrangian:

$$\mathcal{L}(v, \mu) := \mathcal{J}(v) + \langle \mu, v - g \rangle_{\Gamma},$$

*i.e.* it verifies

$$\mathcal{L}(u, \mu) \leq \mathcal{L}(u, \lambda) \leq \mathcal{L}(v, \lambda)$$

for all  $(v, \mu) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma)$ . For more detailed explanations about the origin of this setting, particularly how such the above Lagrangian can be obtained using the duality theory, see for instance [47, 76, 146]. Finally, the interested reader can refer for instance to [179, Section 10.3] for a detailed presentation of the above setting.

### 4.1 Compatible pairs of spaces

The most simple technique to approximate Problem (22) consists in choosing a finite element space for the Lagrange multiplier  $\lambda$  on the boundary, that we denote by

$$M^H \subset H^{-\frac{1}{2}}(\Gamma).$$

The notation  $H > 0$  stands for the mesh size for the boundary mesh. The discrete counterpart of (22) reads then:

$$\begin{aligned} &\text{Find } (u^h, \lambda^H) \in V^h \times M^H \text{ solution to} \\ &B((u^h, \lambda^H), (v^h, \mu^H)) = L(v^h, \mu^H) \quad \text{for all } (v^h, \mu^H) \in V^h \times M^H. \end{aligned} \quad (24)$$

From this point, we see the first advantages of this approach: 1) as it was already the case for the boundary penalty and for Nitsche, the weak treatment of the Dirichlet boundary

condition allows extensions for fictitious domains, see for instance [122, 123, 124] or more recently [134] for a framework based on the eXtended Finite Element Method; 2) there is more flexibility to approximate the boundary flux, since it is related to the choice of  $M^H$ . Particularly, the boundary mesh, that we denote by  $\mathcal{E}_H$ , does not need to be the trace mesh of the global mesh  $\mathcal{T}_h$ , and can be chosen independently, as well as the polynomial order for the approximation of  $\lambda^H$ , that can be different from  $k-1$ . Moreover, since  $M^H \subset H^{-\frac{1}{2}}(\Gamma)$ , the dual space for the Lagrange multiplier can be made either of continuous or discontinuous functions while preserving conformity. This flexibility is however limited, since a discrete inf-sup compatibility condition between  $V^h$  and  $M^H$  needs to be satisfied to recover stability and optimal accuracy, and to avoid spurious modes in the solution.

#### 4.1.1 Some simple particular cases

Let us discuss the above point in detail, first in the particular case where the boundary mesh is the trace mesh of  $\mathcal{T}_h$ , *i.e.* when  $h = H$  and

$$\mathcal{E}_H = \mathcal{E}_h = \mathcal{T}_h|_\Gamma.$$

A first natural and simple choice consists in taking  $V^h = V^{h,1}$  (finite elements with polynomial order  $k = 1$ ) and to approximate the dual space with piecewise constant discontinuous functions on the trace mesh:

$$M_0^{h,d} := \{ \mu^h \in L^2(\Gamma) \mid \mu^h|_E \in \mathbb{P}_0(E), \forall E \in \mathcal{E}^h \}. \quad (25)$$

With this choice ( $M^H = M_0^{h,d}$ ) we get a well-posed discrete problem (24). Indeed, since we can easily verify

$$\{ \mu^h \in M_0^{h,d} \mid \langle \mu^h, v^h \rangle_\Gamma = 0, \forall v^h \in V^h \} = \{0\} \quad (26)$$

and since  $a(\cdot, \cdot)$  is elliptic on  $\{v^h \in V^h \mid \langle \mu^h, v^h \rangle_\Gamma = 0, \forall \mu^h \in M_0^{h,d}\} = V_0^h$ , we can apply for instance Proposition 2.42 from [106] to ensure well-posedness. In fact, condition (26) is equivalent to:

$$\inf_{\mu^h \in M_0^h} \sup_{v^h \in V^h} \frac{\langle \mu^h, v^h \rangle_\Gamma}{\|\mu^h\|_{-\frac{1}{2}, \Gamma} \|v^h\|_{1, \Omega}} \geq \beta_h, \quad (27)$$

with  $\beta_h > 0$  a constant that possibly depends of the mesh size  $h$ , and may depends in the wrong way, in the sense  $\beta_h \rightarrow 0$  when  $h \rightarrow 0$ . The above condition is sometimes called *rank condition* [170] or an inf-sup condition nonuniform in  $h$  (or even a weak inf-sup condition). Though it is enough for well-posedness, it leads to suboptimal error estimates unless we can show that  $\beta_h > 0$  can be set independently of  $h$ , *i.e.*, if we can exhibit an inf-sup condition uniform in  $h$  (or strong inf-sup condition) that is a discrete counterpart of (23). Unfortunately, it has been shown that, in two-dimensions and for a uniform boundary mesh,  $\beta_h$  necessarily tends to 0 when  $H$  vanishes [193, Lemma 3.1], so the choice  $M^H = M_0^{h,d}$  does not yields an acceptable solution and particularly one may expect deteriorated convergence rates in this case.

**Remark 4.1.** *Alternatively, other spaces with discontinuous functions can be considered:*

$$M_l^{h,d} := \{ \mu^h \in L^2(\Gamma) \mid \mu^h|_E \in \mathbb{P}_l(E), \forall E \in \mathcal{E}^h \},$$

*with  $l \geq 0$ , but the pair  $V^h \times M_l^{h,d}$  does not satisfy necessarily a discrete inf-sup condition, even weak, especially for the lowest values of  $k$  ( $k = 1, 2$ ).*

Another natural choice is to set the discrete dual space as the trace space of  $V^h$ , and to use continuous piecewise polynomial functions on the trace mesh. So for  $l \geq 1$  the polynomial order of the dual variables, we set

$$M_l^h := \{\mu^h \in \mathcal{C}^0(\bar{\Gamma}) \mid \mu^h|_E \in \mathbb{P}_l(E), \forall E \in \mathcal{E}^h\}. \quad (28)$$

Let us focus first on the simplest situation when  $k = l = 1$ . In this case, it is still straightforward to verify the rank condition

$$\{\mu^h \in M_1^h \mid \langle \mu^h, v^h \rangle_\Gamma = 0, \forall v^h \in V^h\} = \{0\} \quad (29)$$

and thus to verify a non-uniform inf-sup condition and well-posedness of the discrete problem. Moreover, under an extra assumption on the trace mesh, that needs to satisfy the Crouzeix-Thomée criterion [84], it is possible to establish a uniform inf-sup condition (see [43, Lemma 3.1] in two dimensions, and [76, Proposition 7.5] in two and three dimensions, for a quasi-uniform mesh):

$$\inf_{\mu^h \in M_1^h} \sup_{v^h \in V^h} \frac{\langle \mu^h, v^h \rangle_\Gamma}{\|\mu^h\|_{-\frac{1}{2}, \Gamma} \|v^h\|_{1, \Omega}} \geq \beta, \quad (30)$$

where this time,  $\beta > 0$  is independent of  $h$ . Notably, when the trace mesh is quasi-uniform, it satisfies the Crouzeix-Thomée criterion, but this is one of the most restrictive situations, and a large class of non quasi-uniform meshes satisfies this criterion.

#### 4.1.2 The general situation

For the more general case, I. Babuška proved that, under the assumption of quasi-uniform meshes on the boundary and under the geometric condition

$$H \geq Ch,$$

where  $C > 0$  is independent of  $H$  and  $h$ , there holds a strong discrete inf-sup condition

$$\inf_{\mu^H \in M^H} \sup_{v^h \in V^h} \frac{\langle \mu^H, v^h \rangle_\Gamma}{\|\mu^H\|_{-\frac{1}{2}, \Gamma} \|v^h\|_{1, \Omega}} \geq \beta, \quad (31)$$

with  $\beta > 0$  independent of  $h$  and  $H$ , for the pair  $V^h \times M_l^H$ ,  $k, l \geq 1$ , where  $M_l^H$  is the space of continuous piecewise polynomial functions of degree  $l$  on the boundary mesh  $\mathcal{E}_H$  [23].

Since the conditions provided by I. Babuška are only sufficient and may be too restrictive in practice, notably because the constant  $C$  is not easy to determine, much effort has been devoted later on to characterize compatible pairs of spaces, particularly it has been the object of a serie of papers by J. Pitkäranta [170, 171, 172]. Among the contributions of J. Pitkäranta, there is additionally an error analysis using the mesh-dependent norm introduced previously in the finite element setting 2.2, and the following result can be proven [171] (see also [184, Theorem 1B]): assume that the mesh sizes  $h$  and  $H$  are such that  $ch \leq H \leq Ch$ , and assume that

$$\inf_{\mu^H \in M^H} \sup_{v^h \in V^h} \frac{\langle \mu^H, v^h \rangle_\Gamma}{\|\mu^H\|_{-1/2, h, \Gamma} \|v^h\|_h} \geq \beta,$$

with  $\beta > 0$  independent of  $h$ . Then there holds, for  $u \in H^{1+k}(\Omega)$  and  $\lambda \in \Pi_{j=1}^J H^{1+l}(\Gamma_j)$ :

$$\|u - u^h\|_h + \|\lambda - \lambda_h\|_{-1/2, h, \Gamma} \leq C(h^k \|u\|_{1+k, \Omega} + h^{l+\frac{3}{2}} \|\lambda\|_{1+l, \Gamma}).$$

In the above estimate, the constant  $C > 0$  does not depend of the mesh sizes  $h$  and  $H$  but depends of the shape-regularity of the meshes, and possibly of the polynomial orders

$k$  and  $l$ . Above we denoted by  $\Gamma_j$ ,  $j = 1, \dots, J$  the boundary facets of  $\Gamma$ , and  $\|\cdot\|_{1+l,\Gamma}$  should be understood as a broken Sobolev norm (this technical point disappears when the boundary is smooth, but then the estimate does not take into account the approximation error for the curved boundary). A similar result holds when using the natural norms [23]. Note the similarity of the above error estimate with those of Nitsche's method, and that there is again an optimal error bound on the flux  $\partial_{\mathbf{n}}u$  (or equivalently on the Lagrange multiplier  $\lambda$ ).

**Remark 4.2.** *In the case of the fictitious domain formulation [122], and for the pair  $\mathbb{P}_1/\mathbb{P}_0$ , a similar sufficient condition  $H \geq Ch$  (with  $C \geq 3$ ) needs to be satisfied to ensure a strong inf-sup condition, as proven in [120].*

**Remark 4.3.** *There are multiple other possibilities to design compatible pairs of spaces, for instance introducing bubble functions [43, 93, 163] or using biorthogonal basis for the dual space [193]. This last approach can be successfully extended for domain decomposition with nonmatching meshes [192] or contact and friction problems [193]. In the case of contact and friction, an alternative to the mortar method, Local Average Contact (LAC) has been designed recently in [1, 76].*

## 4.2 Barbosa & Hughes stabilization

Stabilized mixed methods start from the mixed formulation (22), and then complement them with extra terms at the discrete level. These terms allow to preserve well-posedness and optimal accuracy, even for pairs of finite element spaces that do not satisfy the discrete inf-sup compatibility condition (31). For the original idea of stabilized mixed methods, see for instance the seminal paper of L. Franca and T.J.R. Hughes [141], where these ideas were applied first to the Stokes equations. In this category, the stabilized method of H. Barbosa and T.J.R. Hughes [26, 27] introduces a residual-least squares stabilization (see also [189] for Navier-Stokes with slip boundary conditions and [25] for domain decomposition). The Barbosa & Hughes method can be formulated as:

$$\begin{aligned} &\text{Find } (u^h, \lambda^H) \in V^h \times M^H \text{ solution to} \\ &B_h((u^h, \lambda^H), (v^h, \mu^H)) = L(v^h, \mu^H) \quad \text{for all } (v^h, \mu^H) \in V^h \times M^H, \end{aligned} \quad (32)$$

where

$$\begin{aligned} B_h((u^h, \lambda^H), (v^h, \mu^H)) &:= B((u^h, \lambda^H), (v^h, \mu^H)) \\ &\quad - (\delta(\lambda^H + \partial_{\mathbf{n}}u^h), \mu^H + \partial_{\mathbf{n}}v^h)_{\Gamma}. \end{aligned}$$

In the above definition,  $\delta$  is a piecewise constant function on the boundary  $\Gamma$ :

$$\delta = \delta_0 \zeta^1,$$

where  $\delta_0 \geq 0$  is the stabilization parameter, that needs to be small enough. The original formulation [26, 27] contains some extra terms, but that are not necessary for stability and optimal convergence. The formulation (32) above comes from R. Stenberg [184]. Problem (32) is the first-order optimality condition of the perturbed Lagrangian:

$$\mathcal{L}_{h,H}(v^h, \mu^H) := \mathcal{L}(v^h, \mu^H) - \frac{1}{2} \|\delta^{\frac{1}{2}}(\mu^H + \partial_{\mathbf{n}}v^h)\|_{0,\Gamma}^2$$

defined for all  $(v^h, \mu^H) \in V^h \times M^H$ .

A stability and convergence analysis of the above formulation is detailed in [184], when the boundary mesh is the trace mesh and when the space of Lagrange multipliers is the space  $M_l^h$  introduced previously. Under the condition

$$0 < \delta_0 < c_I,$$

where  $c_I$  is the constant of the discrete trace inequality (6), already involved in Nitsche's method, well-posedness with a uniform stability bound can be obtained [184, Lemma 6] and the same optimal convergence rates as the mixed method with a compatible pair follow. More precisely, for  $u \in H^{1+k}(\Omega)$  and  $\lambda \in \Pi_{j=1}^J H^{1+l}(\Gamma_j)$ :

$$\|u - u^h\|_h + \|\lambda - \lambda_h\|_{-1/2, h, \Gamma} \leq C(h^k \|u\|_{1+k, \Omega} + h^{l+\frac{3}{2}} \|\lambda\|_{1+l, \Gamma}).$$

The constant  $C > 0$  in the above error bound does not depend on  $h$  but depends in principle of the shape-regularity of the meshes, the polynomial orders  $k$  and  $l$  of the finite element spaces and of the stabilization parameter  $\delta_0$ . When the underlying finite element pair is inf-sup stable, the dependence upon the stabilization parameter  $\delta_0$  can be removed and the error estimate remains robust, even when  $\delta_0$  vanishes, see [36] in the case of frictional contact (this behavior differs somehow from Nitsche's method but is expectable in the context of mixed methods).

Moreover if we eliminate the discrete multiplier through static condensation, we get the symmetric variant of Nitsche's method [184], in which the stabilization parameter  $\delta_0$  plays the same role as the inverse of the Nitsche parameter  $\gamma_0$ . See also Appendix C for a detailed proof in a simple situation.

**Remark 4.4.** *In the case of frictional contact, and for a discontinuous space of multipliers, a similar procedure to eliminate locally the Lagrange multiplier can be applied, and it involves a variant of Nitsche ( $\pi$ -Nitsche) with local  $L^2$ -projections, see [36].*

It is remarkable that the Barbosa & Hughes formulation (32) allows any choice for the pair of discrete spaces, without any modification in the discrete weak formulation: see for instance some numerical examples in [138] in the case of Signorini contact that allow to assess its stability and optimal convergence properties for various pair of spaces.

**Remark 4.5.** *In [32] is suggested a simplification of Barbosa & Hughes stabilization, where some terms have been removed, in the context of a fictitious domain method for a perforated domain. This modification makes the method nonconsistent, but still well-posedness and error estimates can be proven.*

**Remark 4.6.** *The Barbosa & Hughes stabilization technique has been extended for variational inequalities in [28]. This allowed further extensions for Signorini contact [138], friction [153] and contact with extended finite elements for a crack [6] (see also [76]). The relationship between Barbosa and Hughes stabilization and Nitsche's method allowed later on to design Nitsche's methods for contact and friction [72, 74, 75, 76].*

**Remark 4.7.** *Recently augmented Lagrangian formulations have been considered, not as a solution technique, as usual, but as a discretization technique per se [62, 64]. In [64] for instance, the link between augmented Lagrangian formulations and residual-least squares stabilization is studied, and in case of Dirichlet boundary condition, this allows to recover the symmetric variant of Nitsche's formulation (see also Section 3.3.1).*

### 4.3 Minimal stabilization

Many other stabilization techniques have been designed to relax the inf-sup compatibility condition. In the 2000s, minimal stabilization procedures have been proposed and studied, see, *e.g.*, [38, 53, 55]. The adaptation of such techniques to treat essential boundary conditions has been carried out later on, in the 2010s, and has been thoroughly studied by E. Burman [58]. For Problem (22), minimal stabilization methods can be formulated, in a very general form, as:

$$\begin{aligned} &\text{Find } (u^h, \lambda^H) \in V^h \times M^H \text{ solution to} \\ &\tilde{B}_h((u^h, \lambda^H), (v^h, \mu^H)) = L(v^h, \mu^H) \quad \text{for all } (v^h, \mu^H) \in V^h \times M^H, \end{aligned} \quad (33)$$

where

$$\tilde{B}_h((u^h, \lambda^H), (v^h, \mu^H)) := B((u^h, \lambda^H), (v^h, \mu^H)) - \delta_0 s(\lambda_H, \mu_H).$$

Above the numerical parameter  $\delta_0 > 0$  is still the stabilization parameter. Conversely to Barbosa & Hughes stabilization (32) presented previously, the stabilization term  $s(\cdot, \cdot)$  involves solely the Lagrange multiplier  $\lambda^H$ , and does not require the primal variable  $u^h$  or its normal derivative.

Broadly speaking,  $s(\cdot, \cdot)$  is designed to penalize the gap between the Lagrange multiplier and an underlying stable space. More fundamentally this class of methods relies on the following observation [58, Lemma 2.1]:

$$\sup_{v^h \in V^h} \frac{\langle \mu^H, v^h \rangle_\Gamma}{\|v^h\|_{1,\Omega}} \geq \beta \|\mu^H\|_{-\frac{1}{2},\Gamma} - c \|\mu^H - \tilde{\pi} \mu^H\|_{-1/2,h,\Gamma}, \quad (34)$$

where  $\tilde{\pi}$  is the (global)  $L^2$ -projection onto a discrete space,  $\tilde{M}^H$ , coarser than the space  $M^H$  and for which the discrete (strong) inf-sup compatibility condition (31) is verified.

There are various possibilities to write the stabilization term. For instance it can be

$$s(\lambda^H, \mu^H) = (\zeta^1(I - \tilde{\pi}_H)\lambda^H, (I - \tilde{\pi}_H)\mu^H)_\Gamma,$$

where  $\tilde{\pi}_H$  is a local projection operator, easier to compute in practice. When such choice is made, the corresponding method is usually named local projection stabilization (LPS). More conveniently, under appropriate conditions, we can use a term that involves only the jumps of the multiplier and its derivative at the interface between the facets [58, Section III]. Notably, we have seen above that the pair  $V^{h,1} \times M_0^h$  is not uniformly inf-sup stable. In two dimensions, if we denote by  $\mathcal{X}_h$  the set of boundary nodes, at the exception of corner nodes, let us consider the term:

$$s(\lambda^H, \mu^H) = h^2 \sum_{x \in \mathcal{X}_h} \llbracket \lambda^H(x) \rrbracket \llbracket \mu^H(x) \rrbracket,$$

where we used the notation  $\llbracket \cdot(x) \rrbracket$  to denote the jump of a quantity between the two neighbouring edges of each node  $x$ . With such an appropriate stabilization term, still for the pair  $V^{h,1} \times M_0^h$ , formulation (33) is well-posed and the optimal accuracy of the solution can be recovered [58].

**Remark 4.8.** *Some fictitious domains methods have been designed accordingly to this paradigm. From the analysis of V. Girault and R. Glowinski [120] where a geometric condition has been exhibited between the global mesh and the mesh for the Lagrange multiplier (see Remark 4.2 above), a fictitious domain method with the Lagrange multiplier on the trace of the global mesh combined with a local projection stabilization term has been designed in [30] (see also [31] for an extension to time-dependent parabolic problems). A fictitious domain method based on extended finite elements and a minimal stabilization procedure has been made in [7], and later on extended for a crack geometry with Tresca friction condition [8].*

**Remark 4.9.** *In [58, Section C] it is shown that the framework of minimal stabilization allows to interpret the unsymmetric version of Barbosa & Hughes method as a minimal stabilization with respect to the penalty-free Nitsche method, seen as an inf-sup stable mixed method.*

**Remark 4.10.** *In [187] an original stabilized method is proposed, that makes use of a recovery stress field to stabilize the space of Lagrange multipliers on the boundary (in the spirit of the Zienkiewicz-Zhu error estimator [177, 195]).*

## 5 A few remarks about finite element libraries

A long time ago, implementation of finite element methods was a challenging issue in itself (see for instance [168]). Now for a simple use and, at least, for academic problems, there exists a great variety of user-friendly finite element libraries, where finite elements can be implemented directly using their weak form interpreted in a high level symbolic language. To mention just a few ones, are available now FEniCS [154], FreeFEM++ [136], GetFEM [175], MooAFEM [142], or SciKitFEM [126]. As an illustration, we provide in Appendix D a simple example for Nitsche’s method in FEniCS (see [15] for the complete script).

In large scale industrial codes, such as, for instance Code Aster<sup>1</sup> or MEF++<sup>2</sup>, things are not such easy and, because of the code architecture, some of the methods may be simpler to implement than other ones. From the implementation viewpoint, the boundary penalty method may be the simplest one, and is, in fact, still largely used in the industry. Nitsche’s method has gained in popularity within the applied mathematics and computational mechanics communities, but is not as widespread as penalty or nodal methods in industrial codes. However, its implementation requires almost no effort in comparison to boundary penalty. A difficulty may come from the consistency and symmetry boundary terms, that involve the evaluation of the normal flux  $\partial_{\mathbf{n}}u$  on the first layer of simplices near the boundary. This can be an issue in some codes, but can be overcome with appropriate data structures.

## 6 Conclusion

*There are no longer problems that are solved and others that are not, there are only problems that are more or less solved.* Henri Poincaré<sup>3</sup>

### 6.1 Some general remarks

There is now an embarrassingly choice of mathematically sound technologies to incorporate essential boundary conditions within a variational discretization such as the finite element method (or its extensions or its alternatives). An important motivation for the design of most of these methods has been to incorporate essential boundary conditions in a weak sense, in order to mimic in some way what happens with Neumann or Robin boundary conditions. This weak treatment can not be done directly, and this explains why different pathes have been explored to do this. This weak incorporation has however some major advantages in terms of flexibility to account the essential boundary conditions at the discrete level.

At first glance, the boundary penalty method seems to be the most attractive since it may be the easiest to understand and to implement. For this reason, it is still widespread, particularly in some industrial codes. Nevertheless, when applied to some specific problems, its performance in terms of accuracy may not always be satisfactory: see for instance some numerical evidence in [193] for static contact and friction problems or in [77, 96] for contact in elastodynamics. Nitsche method may represent a better alternative, and remains simple to implement. It still involves a numerical parameter, but which does not play the same role as a regularization parameter and is closest to a stabilization parameter. Particularly, it does not need to be very large to ensure the stability and optimal accuracy, but it needs only to be fixed above a threshold. Alternatively, mixed

---

<sup>1</sup>see <https://code-aster.org/V2/spip.php?rubrique2>

<sup>2</sup>see <https://giref.ulaval.ca/>

<sup>3</sup>Il n’y a plus des problèmes résolus et d’autres qui ne le sont pas, il y a seulement des problèmes plus ou moins résolus. (Henri Poincaré, Conférence prononcée au Congrès international des Mathématiciens, Rome, 1908 ; t. I, p. 173 des Actes du Congrès.)

methods with compatible pairs can be relevant to represent accurately the boundary flux, and do not need an extra numerical parameter. However ensuring in practice the strong discrete inf-sup condition needed for stability and accuracy is not always an easy task, particularly for complex geometries and nonlinear problems. Stabilized mixed method allow to circumvent this issue. However, once again an extra numerical parameter appears. Some families of least squares stabilized mixed methods are closely related to Nitsche’s method and perform the same way.

A major field of applications is concerned, broadly speaking, with, on one side, high order discretization techniques such as isogeometric analysis [83] or polytopal methods [40, 80, 86, 81, 152, 156], and, on the other side, geometrically unfitted finite element methods, such as fictitious domains, extended or cut finite element methods. In this last case the interested reader may refer to, *e.g.*, [21, 164] for some surveys with numerical comparisons. In the same fashion, much effort has been dedicated in the computational mechanics community to treat embedded interfaces, see [10, 90, 92, 91, 93, 99, 110, 132, 135, 143, 149, 163, 185], to mention a small sample of references.

In order to make this presentation as simple as possible, we restricted the setting to a mesh conforming to the exact boundary, and did not mentioned the difficulties and specificities of dissociating the physical and the numerical boundaries. These difficulties are mostly related to numerical integration on cut elements, and also to the preservation of a reasonable condition number (see, *e.g.*, [56]). Note that, in the last years, have emerged new methods, called *phi-FEM*, that allow to incorporate very easily essential boundary conditions with help of the level set function that defines the boundary [103]. One major advantage of this strategy is that it requires no numerical integration on cut elements. However it requires special care to take into account Neumann or Robin boundary conditions [102].

Furthermore, various techniques we reviewed have been extended for more challenging problems, particularly problems with nonlinear boundary conditions associated with variational inequalities. For the emblematic Signorini contact problem where boundary conditions are inequations and a complementarity condition, penalized and mixed methods have been suggested very early [133, 146]. In the last two decades, there have been some improvements for mixed techniques [1, 42, 43, 82, 98, 140, 193], but also the first adaptation of stabilized formulations [138] and Nitsche formulations to contact conditions [72, 74, 75]. For variational inequalities, the numerical analysis may be much more challenging, and for instance, optimal error estimates in the  $H^1$ -norm have been obtained only a few years ago for the Signorini problem [74, 76, 97], and the obtention of optimal error estimates in the  $L^2$ -norm is still an open issue [66, 78, 82, 183].

Most of the techniques presented here have in common some kind of genericity and an ability to be extended to many settings. Particularly they do not depend that much of the differential operator under consideration. When the weak form is symmetric and can be obtained as the first order optimality condition associated to the minimization of a functional, there is a systematic way to recover some of the methods thanks to a minimization argument, after having added extra terms to the original functional. Otherwise we shown there is another path for nonsymmetric problems. Most of the discretizations preserve symmetry when needed, but for Nitsche’s method, we presented unsymmetric variants that can be of interest for nonsymmetric problems and allow more freedom to choose the numerical parameter.

## 6.2 A short list of open problems

To end this review, let us just point out a few open problems (the list below may be far from being exhaustive).

- Despite it has a long story now, it seems that the numerical analysis for the boundary penalty technique may still be improved. Nevertheless, it remains very chal-

lenging to do better than [33].

- For the skewsymmetric variant of Nitsche, E. Burman improved the standard analysis, by proving an infsup condition that enables the Nitsche parameter to vanish [57]. It is still unknown if the numerical analysis for the incomplete variant can be improved.
- The constants in the stability and a posteriori error bounds involve in general some physical parameters (this point does not appear in (1)). These constant may degenerate for some limit values of these parameter, as in the emblematic case of locking in small strain elasticity (see the recent work [5] and references therein). For parameter-dependent methods, this point needs to be addressed carefully, see for instance [71] for Nitsche combined with Hybrid High Order methods.
- Very recently mixed methods have been revisited using the concept of T-coercivity [29] and this might help to revisit the analysis in the case of essential boundary conditions.
- Though some papers on a posteriori error analysis cover the discrete lifting technique combined with a residual error estimate [67], it seems that this topic is still largely uncovered for other techniques. This point concerns both the reliability and efficiency of the estimators and the convergence of adaptive finite element methods.
- For numerical methods with an extra parameter, such as the penalty, symmetric Nitsche or Barbosa-Hughes stabilization, still some improvements can be expected to tune finely the parameter. See for instance [3] in the context of stabilized methods combined with a posteriori error estimates.
- Stability estimates and error estimates are numerous for a Dirichlet boundary condition, but for more complex problems, such as complex fluids, fluid-structure problems, contact and friction boundary conditions or fluid-structure-contact problems, the mathematical theory is not that complete. For contact and friction, see for instance [76, 193] and for fluid-structure, see [113]. For models such as second-gradient fluids for instance, some numerical studies exist [148] but no numerical analysis complement them. Also many alternative methods have been proposed in the computational mechanics community (the methods quoted in this review may be only a smallsize sample of what exists), with no counterpart in terms of mathematical analysis.
- For evolution equations, stability of the time-marching schemes can be also an issue and depend on the chosen technique to handle boundary or interface conditions. See again [113] for instationary fluid-structure problems, and [72] and references therein for contact in elastodynamics.
- For generalized Dirichlet boundary conditions, such as slip conditions for Stokes or Navier-Stokes equations [189], methods such as Nitsche, penalty or stabilized mixed methods are attractive from the implementation viewpoint, and is still object of interest. In this context, other open problems appear, due for instance to the infsup condition on the bulk or the approximation of a curved boundary (Babuska-type paradox). See, *e.g.*, [14, 89, 121, 188, 191, 194] for some recent works on this topic.
- Numerical comparisons of the various techniques on relevant model problems, with specific difficulties for the numerical approximation, may be of valuable interest for the practitioners. For embedded and enriched interface, let us mention [11, 178] that compare penalized, Nitsche and mixed techniques. In the same way, for slip conditions, let us mention [188] and for frictional contact in geomechanics, let us quote [36].

## A Well-posedness of Nitsche's method

We provide here the precise statement of well-posedness for Nitsche's method, and a detailed proof.

**Lemma A.1.** *Suppose that*

$$1 \geq \frac{(1 + \theta)^2 c_I}{\gamma_0}, \quad (35)$$

where  $c_I$  is the constant of the discrete trace inequality (6), then the bilinear form  $A_N(\cdot, \cdot)$  is elliptic on  $V^h$ :

$$A_N(v^h, v^h) \geq c \|v^h\|_h^2, \quad (36)$$

for any  $v^h \in V^h$  and where  $c > 0$  does not depend on  $h$ . As a result, Problem (19) is well-posed.

*Proof.* Take  $v^h$  in  $V^h$  and write

$$\begin{aligned} A_N(v^h, v^h) &= a(v^h, v^h) - (1 + \theta)(\partial_{\mathbf{n}} v^h, v^h)_{\Gamma} + (\gamma v^h, v^h)_{\Gamma} \\ &= \|\nabla v^h\|_{0,\Omega}^2 - (1 + \theta)(\gamma^{-\frac{1}{2}} \partial_{\mathbf{n}} v^h, \gamma^{\frac{1}{2}} v^h)_{\Gamma} + \gamma_0 \|v^h\|_{1/2,h,\Gamma}^2 \\ &\geq \|\nabla v^h\|_{0,\Omega}^2 - |1 + \theta| \|\gamma^{-\frac{1}{2}} \partial_{\mathbf{n}} v^h\|_{0,\Gamma} \|\gamma^{\frac{1}{2}} v^h\|_{0,\Gamma} + \gamma_0 \|v^h\|_{1/2,h,\Gamma}^2 \\ &\geq \|\nabla v^h\|_{0,\Omega}^2 - \frac{(1 + \theta)^2}{2\gamma_0} \|\partial_{\mathbf{n}} v^h\|_{-1/2,h,\Gamma}^2 + \frac{1}{2} \gamma_0 \|v^h\|_{1/2,h,\Gamma}^2. \end{aligned}$$

We applied Cauchy-Schwarz inequality on the third line and Young inequality on the fourth. It remains to use the discrete trace inequality (6) to bound the remaining boundary term:

$$A_N(v^h, v^h) \geq \left(1 - \frac{c_I(1 + \theta)^2}{2\gamma_0}\right) \|\nabla v^h\|_{0,\Omega}^2 + \frac{1}{2} \gamma_0 \|v^h\|_{1/2,h,\Gamma}^2.$$

So provided that

$$\left(1 - \frac{c_I(1 + \theta)^2}{2\gamma_0}\right) \geq \frac{1}{2}$$

we get (36). Well-posedness of Problem (19) follows from the Lax-Milgram lemma.  $\square$

## B Error estimate for Nitsche's method

We detail here the convergence result for Nitsche's method.

**Theorem B.1.** *Let  $u \in H^s(\Omega)$  with  $3/2 < s \leq 1 + k$ , be the solution to Problem (2) and let  $u^h \in V^h$  be the solution to Problem (19). Suppose that the Nitsche's parameter  $\gamma_0$  is large, or that  $\gamma_0 > 0$  when  $\theta \neq -1$ . Then there holds:*

$$\|u - u^h\|_h + \|\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h\|_{-1/2,h,\Gamma} \leq Ch^s \|u\|_{s,\Omega}. \quad (37)$$

where the constant  $C > 0$  does not depend on  $h$ .

*Proof.* The proof is in fact an adaptation from [75, Theorem 3.6 and Theorem 3.8] in case of Signorini contact with elasticity. Let  $u \in H^1(\Omega)$  be the solution to Problem (2) and  $u^h \in V^h$  be the solution to Nitsche formulation (19). For any  $v^h \in V^h$  there holds:

$$\begin{aligned} \|\nabla u - \nabla u^h\|_{0,\Omega}^2 &= a(u - u^h, u(-v^h + v^h) - u^h) \\ &= a(u - u^h, u - v^h) + a(u - u^h, v^h - u^h) \\ &\leq \|\nabla u - \nabla u^h\|_{0,\Omega} \|\nabla u - \nabla v^h\|_{0,\Omega} + a(u - u^h, v^h - u^h), \end{aligned}$$

where we used Cauchy-Schwarz inequality at the last line. Then with Young inequality we get

$$\frac{1}{2}\|\nabla u - \nabla u^h\|_{0,\Omega}^2 \leq \frac{1}{2}\|\nabla u - \nabla v^h\|_{0,\Omega}^2 + a(u - u^h, v^h - u^h). \quad (38)$$

We now focus on the discretization error term  $a(u - u^h, v^h - u^h)$  and we use both formulations (1) and (19), the Green formula as well as the properties

$$v^h - u^h \subset V^h \subset H^1(\Omega)$$

to get

$$\begin{aligned} a(u - u^h, v^h - u^h) &= a(u, v^h - u^h) - a(u^h, v^h - u^h) \\ &= (f, v^h - u^h)_\Omega + (\partial_{\mathbf{n}} u, v^h - u^h)_\Gamma \\ &\quad - (f, v^h - u^h)_\Omega - (g, \gamma(v^h - u^h) - \theta(\partial_{\mathbf{n}} v^h - \partial_{\mathbf{n}} u^h))_\Gamma \\ &\quad - (\partial_{\mathbf{n}} u^h, v^h - u^h)_\Gamma - \theta(u^h, \partial_{\mathbf{n}} v^h - \partial_{\mathbf{n}} u^h)_\Gamma + (\gamma u^h, v^h - u^h)_\Gamma. \end{aligned}$$

Note that, because of the regularity assumptions on  $u$ , there holds  $\partial_{\mathbf{n}} u \in L^2(\Gamma)$  and the corresponding boundary term does not need to be a duality product but an integral. We can simplify the above expression as follows

$$\begin{aligned} &a(u - u^h, v^h - u^h) \\ &= \underbrace{(\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h, v^h - u^h)_\Gamma - \theta(u^h - g, \partial_{\mathbf{n}} v^h - \partial_{\mathbf{n}} u^h)_\Gamma}_{\mathcal{T}_1} + \underbrace{(\gamma(u^h - g), v^h - u^h)_\Gamma}_{\mathcal{T}_2}. \end{aligned} \quad (39)$$

The last term is rewritten, using the condition  $u = g$  from (1) and the splitting  $v^h - u^h = v^h - u + u - u^h$ :

$$\mathcal{T}_2 = (\gamma(u^h - u), v^h - u)_\Gamma - (\gamma(u^h - u), u^h - u)_\Gamma.$$

Let  $\omega_1 > 0$  be an arbitrary weight, the value of which will be fixed later on, and let us use Cauchy-Schwarz and Young inequalities:

$$\mathcal{T}_2 \leq \frac{\omega_1}{2} \gamma_0 \|v^h - u\|_{1/2,h,\Gamma}^2 + \gamma_0 \left( \frac{1}{2\omega_1} - 1 \right) \|u^h - u\|_{1/2,h,\Gamma}^2. \quad (40)$$

There remains to treat the other term  $\mathcal{T}_1$  and, using once again the Dirichlet condition  $u = g$  from (1) we transform it as below:

$$\begin{aligned} \mathcal{T}_1 &= (\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h, v^h - u^h)_\Gamma - \theta(u^h - g, \partial_{\mathbf{n}} v^h - \partial_{\mathbf{n}} u^h)_\Gamma \\ &= (\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h, v^h - u)_\Gamma + (\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h, u - u^h)_\Gamma \\ &\quad - \theta(u^h - g, \partial_{\mathbf{n}} v^h - \partial_{\mathbf{n}} u)_\Gamma - \theta(u^h - g, \partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h)_\Gamma \\ &= (\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h, v^h - u)_\Gamma + (1 + \theta)(\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h, u - u^h)_\Gamma \\ &\quad + \theta(u - u^h, \partial_{\mathbf{n}} v^h - \partial_{\mathbf{n}} u)_\Gamma. \end{aligned}$$

With  $\omega_2, \omega_3, \omega_4 > 0$  other weights and still with Cauchy-Schwarz and Young inequalities, we bound:

$$\begin{aligned} \mathcal{T}_1 &\leq \frac{\omega_2}{2} \|v^h - u\|_{1/2,h,\Gamma}^2 + \frac{1}{2\omega_2} \|\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h\|_{-1/2,h,\Gamma}^2 \\ &\quad + \frac{|\theta|\omega_3}{2} \|\partial_{\mathbf{n}} v^h - \partial_{\mathbf{n}} u\|_{-1/2,h,\Gamma}^2 + \frac{|\theta|}{2\omega_3} \|u - u^h\|_{1/2,h,\Gamma}^2 \\ &\quad + \frac{|1 + \theta|\omega_4}{2} \|\partial_{\mathbf{n}} u^h - \partial_{\mathbf{n}} u\|_{-1/2,h,\Gamma}^2 + \frac{|1 + \theta|}{2\omega_4} \|u - u^h\|_{1/2,h,\Gamma}^2. \end{aligned} \quad (41)$$

Now we rewrite

$$\partial_{\mathbf{n}}u - \partial_{\mathbf{n}}u^h = \partial_{\mathbf{n}}u - \partial_{\mathbf{n}}v^h + \partial_{\mathbf{n}}v^h - \partial_{\mathbf{n}}u^h,$$

and then use the discrete trace inequality (6) as well as a triangular inequality as follows

$$\|\partial_{\mathbf{n}}v^h - \partial_{\mathbf{n}}u^h\|_{-1/2,h,\Gamma} \leq c_I^{\frac{1}{2}} \|\nabla v^h - \nabla u^h\|_{0,\Omega} \leq C(\|\nabla v^h - \nabla u\|_{0,\Omega} + \|\nabla u - \nabla u^h\|_{0,\Omega}).$$

We combine the above bounds with (38)–(41) and get:

$$\begin{aligned} & \left( \frac{1}{2} - \frac{C}{2\omega_2} - \frac{C|1+\theta|\omega_4}{2} \right) \|\nabla u - \nabla u^h\|_{0,\Omega}^2 \\ & + \left[ \gamma_0 \left( 1 - \frac{1}{2\omega_1} \right) - \frac{|\theta|}{2\omega_3} - \frac{|1+\theta|}{2\omega_4} \right] \|u^h - u\|_{1/2,h,\Gamma}^2 \\ \leq & \left( \frac{1}{2} + \frac{C}{2\omega_2} + \frac{C|1+\theta|\omega_4}{2} \right) \|\nabla u - \nabla v^h\|_{0,\Omega}^2 \\ & + \left( \frac{\omega_1}{2} \gamma_0 + \frac{\omega_2}{2} \right) \|v^h - u\|_{1/2,h,\Gamma}^2 \\ & + \left( \frac{C}{2\omega_2} + \frac{|\theta|\omega_3}{2} + \frac{C|1+\theta|\omega_4}{2} \right) \|\partial_{\mathbf{n}}u - \partial_{\mathbf{n}}v^h\|_{-1/2,h,\Gamma}^2. \end{aligned}$$

Now take for instance  $\omega_1 = 1$ ,  $\omega_2 = 4C$  and  $\omega_4$  small enough. For  $\theta$  arbitrary, take moreover  $\omega_3 = 1$  for instance and  $\gamma_0$  large enough. For  $\theta = -1$  and any value of  $\gamma_0$ , take  $\omega_3$  large enough. With the above choices, we get the abstract estimate:

$$\begin{aligned} & \|\nabla u - \nabla u^h\|_{0,\Omega}^2 + \|u^h - u\|_{1/2,h,\Gamma}^2 \\ \leq & C \left( \|\nabla u - \nabla v^h\|_{0,\Omega}^2 + \|v^h - u\|_{1/2,h,\Gamma}^2 + \|\partial_{\mathbf{n}}u - \partial_{\mathbf{n}}v^h\|_{-1/2,h,\Gamma}^2 \right). \end{aligned} \quad (42)$$

We then use standard interpolation estimates for the Lagrange operator [101] (for the estimate associated to the interpolation of the normal derivative, see for instance [75, 116]). The estimate on the normal derivative comes from

$$\begin{aligned} & \|\partial_{\mathbf{n}}u - \partial_{\mathbf{n}}u^h\|_{-1/2,h,\Gamma} \\ \leq & \|\partial_{\mathbf{n}}u - \partial_{\mathbf{n}}v^h\|_{-1/2,h,\Gamma} + \|\partial_{\mathbf{n}}v^h - \partial_{\mathbf{n}}u^h\|_{-1/2,h,\Gamma} \\ \leq & \|\partial_{\mathbf{n}}u - \partial_{\mathbf{n}}v^h\|_{-1/2,h,\Gamma} + C(\|\nabla v^h - \nabla u\|_{0,\Omega} + \|\nabla u - \nabla u^h\|_{0,\Omega}). \end{aligned}$$

This ends the proof.  $\square$

## C Barbosa & Hughes and Nitsche

Following [144, 184] we detail the equivalence between the Barbosa & Hughes stabilized method (32) and Nitsche's formulation (19), but in a simple situation where we suppose  $g = 0$  and take the pair  $V^h = V^{h,1}$  ( $k = 1$ ) and  $M^h = M_1^{h,d}$  ( $l = 1$ ) for the solution and the multiplier. In (32) take  $v^h = 0 \in V^h$  and, for an arbitrary boundary facet  $E \in \mathcal{E}_h$ , and an arbitrary polynomial  $\mu_E \in \mathbb{P}_1(E)$ , take  $\mu^h|_E = \mu_E$  and  $\mu^h = 0$  elsewhere. We get then:

$$(\mu_E, u^h)_E - (\delta(\lambda^h + \partial_{\mathbf{n}}u^h), \mu_E)_E = 0.$$

We reformulate

$$(\mu_E, u^h - \delta(\lambda^h + \partial_{\mathbf{n}}u^h))_E = 0.$$

Remark that, since  $u^h - \delta(\lambda^h + \partial_{\mathbf{n}}u^h) \in \mathbb{P}_1(E)$ , this implies the equality

$$u^h - \delta(\lambda^h + \partial_{\mathbf{n}}u^h) = 0$$

or, identically:

$$u^h = \delta(\lambda^h + \partial_{\mathbf{n}}u^h), \quad \lambda^h = \frac{1}{\delta}u^h - \partial_{\mathbf{n}}u^h. \quad (43)$$

Now from (32) which is

$$a(u^h, v^h) + (\lambda^h, v^h)_{\Gamma} + (\mu^h, u^h)_{\Gamma} - (\delta(\lambda^h + \partial_{\mathbf{n}}u^h), \mu^h + \partial_{\mathbf{n}}v^h)_{\Gamma} = (f, v^h)_{\Omega}$$

we use (43) and we recover

$$a(u^h, v^h) + (\lambda^h, v)_{\Gamma} + (\mu^h, u^h)_{\Gamma} - (u^h, \mu^h + \partial_{\mathbf{n}}v^h)_{\Gamma} = (f, v^h)_{\Omega}.$$

We develop and the two terms in  $(\mu^h, u^h)_{\Gamma}$  cancel:

$$a(u^h, v^h) + (\lambda^h, v^h)_{\Gamma} - (u^h, \partial_{\mathbf{n}}v^h)_{\Gamma} = (f, v^h)_{\Omega}.$$

We replace  $\lambda$  with the expression above in (43) and get

$$a(u^h, v^h) - (\partial_{\mathbf{n}}u^h, v^h)_{\Gamma} - (u^h, \partial_{\mathbf{n}}v^h)_{\Gamma} + (\delta^{-1}u^h, v^h)_{\Gamma} = (f, v^h)_{\Omega}.$$

This is exactly the symmetric Nitsche's formulation (19), with  $\theta = 1$  and  $\gamma_0 = \delta_0^{-1}$ . Let us do this the other way round. Let us start from the Nitsche's formulation (19) (with  $\theta = 1$ ):

$$a(u^h, v^h) - (\partial_{\mathbf{n}}u^h, v^h)_{\Gamma} - (u^h, \partial_{\mathbf{n}}v^h)_{\Gamma} + (\gamma u^h, v^h)_{\Gamma} = (f, v^h)_{\Omega}.$$

We define  $\lambda^h$  as in (43) (still with  $\gamma_0 = \delta_0^{-1}$ ), we check that  $\lambda^h \in M^h$  and rewrite (43) as

$$(\delta(\lambda^h + \partial_{\mathbf{n}}u^h), \mu^h + \partial_{\mathbf{n}}v^h)_{\Gamma} = (u^h, \mu^h + \partial_{\mathbf{n}}v^h)_{\Gamma}$$

for all  $\mu^h$  (note that  $\partial_{\mathbf{n}}v^h \in M^h$ ). Nitsche formulation is rewritten

$$a(u^h, v^h) + \underbrace{(\gamma u^h - \partial_{\mathbf{n}}u^h, v^h)_{\Gamma}}_{\lambda^h} - \underbrace{(u^h, \partial_{\mathbf{n}}v^h)_{\Gamma}}_{(\delta(\lambda^h + \partial_{\mathbf{n}}u^h), \mu^h + \partial_{\mathbf{n}}v^h)_{\Gamma} - (u^h, \mu^h)_{\Gamma}} = (f, v^h)_{\Omega}.$$

And we recover

$$a(u^h, v^h) + (\lambda^h, v^h)_{\Gamma} + (u^h, \mu^h)_{\Gamma} - (\delta(\lambda^h + \partial_{\mathbf{n}}u^h), \mu^h + \partial_{\mathbf{n}}v^h)_{\Gamma} = (f, v^h)_{\Omega}.$$

This is exactly Barbosa & Hughes formulation (32). When  $\theta = -1$ , we proceed the same way and we recover another consistent, but skewsymmetric stabilized formulation equivalent to skewsymmetric Nitsche:

$$a(u^h, v^h) + (\lambda^h, v^h)_{\Gamma} - (u^h, \mu^h)_{\Gamma} + (\delta(\lambda^h + \partial_{\mathbf{n}}u^h), \mu^h + \partial_{\mathbf{n}}v^h)_{\Gamma} = (f, v^h)_{\Omega}.$$

For different spaces of functions, or if  $g \neq 0$  the equivalence between the two formulation may not be exact anymore and some projection operators need to be introduced [144, 184].

**Remark C.1.** *Alternatively, for any value of  $\theta$ , we can do as follows: we rewrite (43) as*

$$(\delta(\lambda^h + \partial_{\mathbf{n}}u^h), \mu^h + \theta \partial_{\mathbf{n}}v^h)_{\Gamma} = (u^h, \mu^h + \theta \partial_{\mathbf{n}}v^h)_{\Gamma}$$

for all  $\mu^h$ . Nitsche's formulation is rewritten

$$a(u^h, v^h) + \underbrace{(\gamma u^h - \partial_{\mathbf{n}}u^h, v^h)_{\Gamma}}_{\lambda^h} - \underbrace{(u^h, \theta \partial_{\mathbf{n}}v^h)_{\Gamma}}_{(\delta(\lambda^h + \partial_{\mathbf{n}}u^h), \mu^h + \theta \partial_{\mathbf{n}}v^h)_{\Gamma} - (u^h, \mu^h)_{\Gamma}} = (f, v^h)_{\Omega}.$$

We find

$$a(u^h, v^h) + (\lambda^h, v^h)_{\Gamma} + (u^h, \mu^h)_{\Gamma} - (\delta(\lambda^h + \partial_{\mathbf{n}}u^h), \mu^h + \theta \partial_{\mathbf{n}}v^h)_{\Gamma} = (f, v^h)_{\Omega}.$$

For  $\theta = 1$  it is still the symmetric Barbosa & Hughes formulation, and for other values of  $\theta$ , it is a consistent, but unsymmetric stabilized method.

## D Implementation of Nitsche

Below we display a few lines of Python/FEniCS code that correspond to the Nitsche method 19. The complete script is available online, see [15].

---

```
1 # Bilinear/linear forms:
2 a = inner(grad(uh), grad(vh))*dx - inner(grad(uh),nn)*vh*ds
3 - theta*(inner(grad(vh),nn)*uh*ds) + gamma*uh*vh*ds
4
5 L = f*vh*dx + gamma*u_ex*vh*ds - theta(inner(grad(vh),nn)*u_ex*ds)
6 # Name of the unknown:
7 uh = Function(Hh)
8
9 # Solving the associated linear system:
10 solve(a == L, uh)
```

---

## Acknowledgements

The author would like to thank the CNRS, the CI<sup>2</sup>MA and the CMM for his stay as a visiting researcher, as well as all the collaborators with which he had fruitful discussions on this topic, particularly Gabriel Barrenechea, Roland Becker, Stéphane Bordas, Erik Burman, Alexandre Ern, Miguel Fernández, Patrick Hild, Roland Masson and Yves Renard. The author's work is partially supported by the I-Site BFC project NAANoD and the EIPHI Graduate School (contract ANR-17-EURE-0002). F.C. is grateful of the Center for Mathematical Modeling Grant FB20005.

Finally, the author is grateful to Stéphane Bordas and Carsten Carstensen for their suggestions to improve the manuscript, as well as to Rodolfo Araya, Juan-Pablo Borthagaray, Claude Brezinski, Charbel Farhat and Laura de Lorenzis for their encouragements.

## References

- [1] M. ABBAS, G. DROUET, AND P. HILD, *The local average contact (LAC) method*, Comput. Methods Appl. Mech. Engrg., 339 (2018), pp. 488–513.
- [2] R.-A. ADAMS, *Sobolev spaces*, vol. 65 of Pure and Applied Mathematics, Academic Press, New York-London, 1975.
- [3] M. AINSWORTH, A. ALLENDES, G. R. BARRENECHEA, AND R. RANKIN, *On the adaptive selection of the parameter in stabilized finite element approximations*, SIAM J. Numer. Anal., 51 (2013), pp. 1585–1609.
- [4] M. AINSWORTH AND D. W. KELLY, *A posteriori error estimators and adaptivity for finite element approximation of the non-homogeneous Dirichlet problem*, Adv. Comput. Math., 15 (2001), pp. 3–23 (2002).
- [5] M. AINSWORTH AND C. PARKER, *Unlocking the secrets of locking: finite element analysis in planar linear elasticity*, Comput. Methods Appl. Mech. Eng., 395 (2022), p. 56. Id/No 115034.
- [6] S. AMDOUNI, P. HILD, V. LLERAS, M. MOAKHER, AND Y. RENARD, *A stabilized Lagrange multiplier method for the enriched finite-element approximation of contact problems of cracked elastic bodies*, ESAIM Math. Model. Numer. Anal., 46 (2012), pp. 813–839.

- [7] S. AMDOUNI, M. MOAKHER, AND Y. RENARD, *A local projection stabilization of fictitious domain method for elliptic boundary value problems*, Appl. Numer. Math., 76 (2014), pp. 60–75.
- [8] ———, *A stabilized Lagrange multiplier method for the enriched finite-element approximation of Tresca contact problems of cracked elastic bodies*, Comput. Methods Appl. Mech. Engrg., 270 (2014), pp. 178–200.
- [9] P. ANGOT, *Analysis of singular perturbations on the Brinkman problem for fictitious domain models of viscous flows*, Math. Methods Appl. Sci., 22 (1999), pp. 1395–1412.
- [10] C. ANNAVARAPU, M. HAUTEFEUILLE, AND J. E. DOLBOW, *A robust Nitsche’s formulation for interface problems*, Comput. Methods Appl. Mech. Engrg., 225/228 (2012), pp. 44–54.
- [11] ———, *Stable imposition of stiff constraints in explicit dynamics for embedded finite element methods*, Internat. J. Numer. Methods Engrg., 92 (2012), pp. 206–228.
- [12] ———, *A Nitsche stabilized finite element method for frictional sliding on embedded interfaces. I: Single interface*, Comput. Methods Appl. Mech. Eng., 268 (2014), pp. 417–436.
- [13] A. APOSTOLATOS, R. SCHMIDT, R. WÜCHNER, AND K.-U. BLETZINGER, *A Nitsche-type formulation and comparison of the most common domain decomposition methods in isogeometric analysis*, Internat. J. Numer. Methods Engrg., 97 (2014), pp. 473–504.
- [14] R. ARAYA, A. CAIAZZO, AND F. CHOULY, *Stokes problem with slip boundary conditions using stabilized finite elements combined with Nitsche*. hal-04077986, Apr. 2023.
- [15] R. ARAYA AND F. CHOULY, *Nitsche with a Lagrange Finite Element Method*, (2023). figshare repository. DOI:10.6084/m9.figshare.24082137.v1.
- [16] R. ARAYA AND F. CHOULY, *Residual a posteriori error estimation for frictional contact with Nitsche method*, J. Sci. Comput., 96 (2023). Paper No. 87.
- [17] D. N. ARNOLD, *An interior penalty finite element method with discontinuous elements*, SIAM J. Numer. Anal., 19 (1982), pp. 742–760.
- [18] M. ASTORINO, J.-F. GERBEAU, O. PANTZ, AND K.-F. TRAORÉ, *Fluid-structure interaction and multi-body contact: application to aortic valves*, Comput. Methods Appl. Mech. Engrg., 198 (2009), pp. 3603–3612.
- [19] J.-P. AUBIN, *Approximation of elliptic boundary-value problems*, Pure and Applied Mathematics, Vol. XXVI, Wiley-Interscience, New York-London-Sydney, 1972.
- [20] M. AURADA, M. FEISCHL, J. KEMETMÜLLER, M. PAGE, AND D. PRAETORIUS, *Each  $H^{1/2}$ -stable projection yields convergence and quasi-optimality of adaptive FEM with inhomogeneous Dirichlet data in  $\mathbb{R}^d$* , ESAIM Math. Model. Numer. Anal., 47 (2013), pp. 1207–1235.
- [21] F. AURICCHIO, D. BOFFI, L. GASTALDI, A. LEFIEUX, AND A. REALI, *A study on unfitted 1D finite element methods*, Comput. Math. Appl., 68 (2014), pp. 2080–2102.
- [22] F. P. T. BAAIJENS, *A fictitious domain/mortar element method for fluid-structure interaction*, Internat. J. Numer. Methods Fluids, 35 (2001), pp. 743–761.
- [23] I. BABUŠKA, *The finite element method with Lagrangian multipliers*, Numer. Math., 20 (1972/73), pp. 179–192.
- [24] ———, *The finite element method with penalty*, Math. Comp., 27 (1973), pp. 221–228.

- [25] C. BAIOCCHI, F. BREZZI, AND L. D. MARINI, *Stabilization of Galerkin methods and applications to domain decomposition*, in Future tendencies in computer science, control and applied mathematics (Paris, 1992), vol. 653 of Lecture Notes in Comput. Sci., Springer, Berlin, 1992, pp. 345–355.
- [26] H. J. C. BARBOSA AND T. J. R. HUGHES, *The finite element method with Lagrange multipliers on the boundary: circumventing the Babuška-Brezzi condition*, Comput. Methods Appl. Mech. Engrg., 85 (1991), pp. 109–128.
- [27] ———, *Boundary Lagrange multipliers in finite element methods: error analysis in natural norms*, Numer. Math., 62 (1992), pp. 1–15.
- [28] ———, *Circumventing the Babuška-Brezzi condition in mixed finite element approximations of elliptic variational inequalities*, Comput. Methods Appl. Mech. Engrg., 97 (1992), pp. 193–210.
- [29] M. BARRÉ AND P. CIARLET, *The  $T$ -coercivity approach for mixed problems*. hal-03820910, Oct. 2022.
- [30] G. R. BARRENECHEA AND F. CHOULY, *A local projection stabilized method for fictitious domains*, Appl. Math. Lett., 25 (2012), pp. 2071–2076.
- [31] G. R. BARRENECHEA, F. CHOULY, AND C. GONZALEZ, *A stabilised finite element method for a time-dependent problem solved using a fictitious domain method*. hal-01596106, Sept. 2017.
- [32] G. R. BARRENECHEA AND C. GONZÁLEZ, *A stabilized finite element method for a fictitious domain problem allowing small inclusions*, Numer. Methods Partial Differential Equations, 34 (2018), pp. 167–183.
- [33] J. W. BARRETT AND C. M. ELLIOTT, *Finite element approximation of the Dirichlet problem using the boundary penalty method*, Numer. Math., 49 (1986), pp. 343–366.
- [34] S. BARTELS, C. CARSTENSEN, AND G. DOLZMANN, *Inhomogeneous Dirichlet conditions in a priori and a posteriori finite element error analysis*, Numer. Math., 99 (2004), pp. 1–24.
- [35] Y. BAZILEVS AND T. J. R. HUGHES, *Weak imposition of Dirichlet boundary conditions in fluid mechanics*, Comput. & Fluids, 36 (2007), pp. 12–26.
- [36] L. BEAUDE, F. CHOULY, M. LAAZIRI, AND R. MASSON, *Mixed and Nitsche’s discretizations of Coulomb frictional contact-mechanics for mixed dimensional poromechanical models*, Comput. Methods Appl. Mech. Engrg., 413 (2023), pp. Paper No. 116124, 31.
- [37] R. BECKER, *A variant of Nitsche’s method*, C. R. Math. Acad. Sci. Paris, 356 (2018), pp. 1236–1242.
- [38] R. BECKER AND M. BRAACK, *A finite element pressure gradient stabilization for the Stokes equations based on local projections*, Calcolo, 38 (2001), pp. 173–199.
- [39] R. BECKER, P. HANSBO, AND R. STENBERG, *A finite element method for domain decomposition with non-matching grids*, M2AN Math. Model. Numer. Anal., 37 (2003), pp. 209–225.
- [40] L. BEIRÃO DA VEIGA, F. BREZZI, A. CANGIANI, G. MANZINI, L. D. MARINI, AND A. RUSSO, *Basic principles of virtual element methods*, Math. Models Methods Appl. Sci., 23 (2013), pp. 199–214.
- [41] F. BEN BELGACEM, *The mortar finite element method with Lagrange multipliers*, Numer. Math., 84 (1999), pp. 173–197.
- [42] F. BEN BELGACEM, P. HILD, AND P. LABORDE, *Extension of the mortar finite element method to a variational inequality modeling unilateral contact*, Math. Models Methods Appl. Sci., 9 (1999), pp. 287–303.

- [43] F. BEN BELGACEM AND Y. RENARD, *Hybrid finite element methods for the Signorini problem*, Math. Comp., 72 (2003), pp. 1117–1145.
- [44] C. BERNARDI, Y. MADAY, AND A. T. PATERA, *Domain decomposition by the mortar element method*, in Asymptotic and numerical methods for partial differential equations with critical parameters (Beaune, 1992), vol. 384 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Kluwer Acad. Publ., Dordrecht, 1993, pp. 269–286.
- [45] ———, *A new nonconforming approach to domain decomposition: the mortar element method*, in Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XI (Paris, 1989–1991), vol. 299 of Pitman Res. Notes Math. Ser., Longman Sci. Tech., Harlow, 1994, pp. 13–51.
- [46] L. BLANK, A. CAIAZZO, F. CHOULY, A. LOZINSKI, AND J. MURA, *Analysis of a stabilized penalty-free Nitsche method for the Brinkman, Stokes, and Darcy problems*, ESAIM Math. Model. Numer. Anal., 52 (2018), pp. 2149–2185.
- [47] D. BOFFI, F. BREZZI, AND M. FORTIN, *Mixed finite element methods and applications*, vol. 44 of Springer Series in Computational Mathematics, Springer, Heidelberg, 2013.
- [48] T. BOIVEAU AND E. BURMAN, *A penalty-free Nitsche method for the weak imposition of boundary conditions in compressible and incompressible elasticity*, IMA J. Numer. Anal., 36 (2016), pp. 770–795.
- [49] T. BOIVEAU, E. BURMAN, S. CLAUS, AND M. LARSON, *Fictitious domain method with boundary value correction using penalty-free Nitsche method*, J. Numer. Math., 26 (2018), pp. 77–95.
- [50] J. P. BOUFFLET, M. DAMBRINE, G. DUPIRE, AND P. VILLON, *On the necessity of Nitsche term. Part II: An alternative approach*, Appl. Numer. Math., 62 (2012), pp. 521–535.
- [51] S. C. BRENNER AND L. R. SCOTT, *The mathematical theory of finite element methods*, vol. 15 of Texts in Applied Mathematics, Springer, New York, third ed., 2008.
- [52] F. BREZZI, *On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge, 8 (1974), pp. 129–151.
- [53] F. BREZZI AND M. FORTIN, *A minimal stabilisation procedure for mixed finite element methods*, Numer. Math., 89 (2001), pp. 457–491.
- [54] P. BRINGMANN AND C. CARSTENSEN,  *$h$ -adaptive least-squares finite element methods for the 2D Stokes equations of any order with optimal convergence rates*, Comput. Math. Appl., 74 (2017), pp. 1923–1939.
- [55] E. BURMAN, *Pressure projection stabilizations for Galerkin approximations of Stokes’ and Darcy’s problem*, Numer. Methods Partial Differential Equations, 24 (2008), pp. 127–143.
- [56] ———, *Ghost penalty*, C. R. Math. Acad. Sci. Paris, 348 (2010), pp. 1217–1220.
- [57] ———, *A penalty-free nonsymmetric Nitsche-type method for the weak imposition of boundary conditions*, SIAM J. Numer. Anal., 50 (2012), pp. 1959–1981.
- [58] ———, *Projection stabilization of Lagrange multipliers for the imposition of constraints on interfaces and boundaries*, Numer. Methods Partial Differential Equations, 30 (2014), pp. 567–592.
- [59] E. BURMAN, S. CLAUS, P. HANSBO, M. G. LARSON, AND A. MASSING, *Cut-FEM: discretizing geometry and partial differential equations*, Internat. J. Numer. Methods Engrg., 104 (2015), pp. 472–501.

- [60] E. BURMAN, M. A. FERNÁNDEZ, AND S. FREI, *A Nitsche-based formulation for fluid-structure interactions with contact*, ESAIM Math. Model. Numer. Anal., 54 (2020), pp. 531–564.
- [61] E. BURMAN AND P. HANSBO, *Fictitious domain finite element methods using cut elements: II. A stabilized Nitsche method*, Appl. Numer. Math., 62 (2012), pp. 328–341.
- [62] ———, *Deriving robust unfitted finite element methods from augmented Lagrangian formulations*, in Geometrically unfitted finite element methods and applications, vol. 121 of Lect. Notes Comput. Sci. Eng., Springer, Cham, 2017, pp. 1–24.
- [63] E. BURMAN, P. HANSBO, AND M. G. LARSON, *The penalty-free Nitsche method and nonconforming finite elements for the Signorini problem*, SIAM J. Numer. Anal., 55 (2017), pp. 2523–2539.
- [64] ———, *The augmented Lagrangian method as a framework for stabilised methods in computational mechanics*, Arch. Comput. Methods Eng., 30 (2023), pp. 2579–2604.
- [65] E. BURMAN, P. HANSBO, M. G. LARSON, AND R. STENBERG, *Galerkin least squares finite element method for the obstacle problem*, Comput. Methods Appl. Mech. Eng., 313 (2017), pp. 362–374.
- [66] P. CANTIN AND P. HILD, *Error analysis of the compliance model for the Signorini problem*, Calcolo, 58 (2021), pp. Paper No. 32, 21.
- [67] C. CARSTENSEN, M. FEISCHL, M. PAGE, AND D. PRAETORIUS, *Axioms of adaptivity*, Comput. Math. Appl., 67 (2014), pp. 1195–1253.
- [68] K. L. CASCAVITA, F. CHOULY, AND A. ERN, *Hybrid high-order discretizations combined with Nitsche’s method for Dirichlet and Signorini boundary conditions*, IMA J. Numer. Anal., 40 (2020), pp. 2189–2226.
- [69] T. CHEN, R. MO, N. WAN, AND Z. W. GONG, *Imposing displacement boundary conditions with Nitsche’s method in isogeometric analysis*, Chin. J. Theor. Appl. Mech., 44 (2012), pp. 369–381.
- [70] F. CHOULY, *Sur la prise en compte de quelques conditions aux limites avec la méthode des éléments finis*. Lecture notes. cel-01564693, Jan. 2017.
- [71] F. CHOULY, A. ERN, AND N. PIGNET, *A hybrid high-order discretization combined with Nitsche’s method for contact and Tresca friction in small strain elasticity*, SIAM J. Sci. Comput., 42 (2020), pp. A2300–A2324.
- [72] F. CHOULY, M. FABRE, P. HILD, R. MLIKA, J. POUSIN, AND Y. RENARD, *An overview of recent results on Nitsche’s method for contact problems*, in Geometrically unfitted finite element methods and applications, vol. 121 of Lect. Notes Comput. Sci. Eng., Springer, Cham, 2017, pp. 93–141.
- [73] F. CHOULY, T. GUSTAFSSON, AND P. HILD, *A Nitsche method for the elastoplastic torsion problem*, ESAIM Math. Model. Numer. Anal., 57 (2023), pp. 1731–1746.
- [74] F. CHOULY AND P. HILD, *A Nitsche-based method for unilateral contact problems: numerical analysis*, SIAM J. Numer. Anal., 51 (2013), pp. 1295–1307.
- [75] F. CHOULY, P. HILD, AND Y. RENARD, *Symmetric and non-symmetric variants of Nitsche’s method for contact problems in elasticity: theory and numerical experiments*, Math. Comp., 84 (2015), pp. 1089–1112.
- [76] ———, *Finite element approximation of contact and friction in elasticity*, vol. 48 of Advances in Mechanics and Mathematics / Advances in Continuum Mechanics, Birkhäuser, Springer, 2023. ISBN 978-3-031-31422-3. <https://doi.org/10.1007/978-3-031-31423-0>, xxi+294 pages.

- [77] F. CHOULY AND Y. RENARD, *Explicit Verlet time–integration for a Nitsche–based approximation of elastodynamic contact problems*, Adv. Model. and Simul. in Eng. Sci., 5 (2018), pp. 1–38.
- [78] C. CHRISTOF AND C. HAUBNER, *Finite element error estimates in non-energy norms for the two-dimensional scalar Signorini problem*, Numer. Math., 145 (2020), pp. 513–551.
- [79] P. G. CIARLET, *The finite element method for elliptic problems*, vol. 40 of Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002.
- [80] M. CICUTTIN, A. ERN, AND N. PIGNET, *Hybrid high-order methods—a primer with applications to solid mechanics*, SpringerBriefs in Mathematics, Springer, Cham, 2021.
- [81] B. COCKBURN, D. A. DI PIETRO, AND A. ERN, *Bridging the hybrid high-order and hybridizable discontinuous Galerkin methods*, ESAIM Math. Model. Numer. Anal., 50 (2016), pp. 635–650.
- [82] P. COOREVITS, P. HILD, K. LHALOUANI, AND T. SASSI, *Mixed finite element methods for unilateral problems: convergence analysis and numerical studies*, Math. Comp., 71 (2002), pp. 1–25.
- [83] J. A. COTTRELL, T. J. R. HUGHES, AND Y. BAZILEVS, *Isogeometric analysis*, John Wiley & Sons, Ltd., Chichester, 2009.
- [84] M. CROUZEIX AND V. THOMÉE, *The stability in  $L_p$  and  $W_p^1$  of the  $L_2$ -projection onto finite element function spaces*, Math. Comp., 48 (1987), pp. 521–532.
- [85] F. DE PRENTER, C. LEHRENFELD, AND A. MASSING, *A note on the stability parameter in Nitsche’s method for unfitted boundary value problems*, Comput. Math. Appl., 75 (2018), pp. 4322–4336.
- [86] D. A. DI PIETRO AND J. DRONIOU, *The hybrid high-order method for polytopal meshes*, vol. 19 of MS&A. Modeling, Simulation and Applications, Springer, Cham, 2020.
- [87] D. A. DI PIETRO AND A. ERN, *Mathematical aspects of discontinuous Galerkin methods*, vol. 69 of Mathematics & Applications, Springer, Heidelberg, 2012.
- [88] I. DIONE, *Towards optimal finite element error estimates for the penalized Dirichlet problem in a domain with curved boundary*, Comput. Math. Appl., 71 (2016), pp. 76–84.
- [89] I. DIONE AND J. M. URQUIZA, *Penalty: finite element approximation of Stokes equations with slip boundary conditions*, Numer. Math., 129 (2015), pp. 587–610.
- [90] J. DOLBOW AND T. BELYTSCHKO, *An introduction to programming the meshless Element Free Galerkin method*, Arch. Comput. Methods Eng., 5 (1998), pp. 207–241.
- [91] J. DOLBOW AND I. HARARI, *An efficient finite element method for embedded interface problems*, Internat. J. Numer. Methods Engrg., 78 (2009), pp. 229–252.
- [92] J. DOLBOW, N. MOËS, AND T. BELYTSCHKO, *An extended finite element method for modeling crack growth with frictional contact*, Comput. Methods Appl. Mech. Engrg., 190 (2001), pp. 6825–6846.
- [93] J. E. DOLBOW AND L. P. FRANCA, *Residual-free bubbles for embedded Dirichlet problems*, Comput. Methods Appl. Mech. Engrg., 197 (2008), pp. 3751–3759.
- [94] V. DOLEJŠÍ, A. ERN, AND M. VOHRALÍK, *hp-adaptation driven by polynomial-degree-robust a posteriori error estimates for elliptic problems*, SIAM J. Sci. Comput., 38 (2016), pp. A3220–A3246.

- [95] N. D. DOS SANTOS, J.-F. GERBEAU, AND J.-F. BOURGAT, *A partitioned fluid-structure algorithm for elastic thin valves with contact*, *Comput. Methods Appl. Mech. Engrg.*, 197 (2008), pp. 1750–1761.
- [96] D. DOYEN, A. ERN, AND S. PIPERNO, *Time-integration schemes for the finite element dynamic Signorini problem*, *SIAM J. Sci. Comput.*, 33 (2011), pp. 223–249.
- [97] G. DROUET AND P. HILD, *Optimal convergence for discrete variational inequalities modelling Signorini contact in 2D and 3D without additional assumptions on the unknown contact set*, *SIAM J. Numer. Anal.*, 53 (2015), pp. 1488–1507.
- [98] ———, *An accurate local average contact method for nonmatching meshes*, *Numer. Math.*, 136 (2017), pp. 467–502.
- [99] F. DUBOEUF AND E. BÉCHET, *Embedded solids of any dimension in the X-FEM. Part II – Imposing Dirichlet boundary conditions*, *Finite Elem. Anal. Des.*, 128 (2017), pp. 32–50.
- [100] G. DUPIRE, J. P. BOUFFLET, M. DAMBRINE, AND P. VILLON, *On the necessity of Nitsche term*, *Appl. Numer. Math.*, 60 (2010), pp. 888–902.
- [101] T. DUPONT AND R. SCOTT, *Polynomial approximation of functions in Sobolev spaces*, *Math. Comp.*, 34 (1980), pp. 441–463.
- [102] M. DUPREZ, V. LLERAS, AND A. LOZINSKI, *A new  $\phi$ -fem approach for problems with natural boundary conditions*, *Numer. Methods Partial Differential Equations*, 39 (2023), pp. 281–303.
- [103] M. DUPREZ AND A. LOZINSKI,  *$\phi$ -FEM: a finite element method on domains defined by level-sets*, *SIAM J. Numer. Anal.*, 58 (2020), pp. 1008–1028.
- [104] R. G. DURÁN, *Mixed finite element methods*. Classnotes available online at [http://mate.dm.uba.ar/~rduran/class\\_notes.html](http://mate.dm.uba.ar/~rduran/class_notes.html).
- [105] A. EMBAR, J. DOLBOW, AND I. HARARI, *Imposing Dirichlet boundary conditions with Nitsche’s method and spline-based finite elements*, *Int. J. Numer. Methods Eng.*, 83 (2010), pp. 877–898.
- [106] A. ERN AND J.-L. GUERMOND, *Theory and practice of finite elements*, vol. 159 of *Applied Mathematical Sciences*, Springer-Verlag, New York, 2004.
- [107] ———, *Abstract nonconforming error estimates and application to boundary penalty methods for diffusion equations and time-harmonic Maxwell’s equations*, *Comput. Methods Appl. Math.*, 18 (2018), pp. 451–475.
- [108] ———, *Finite elements. I—Approximation and interpolation*, vol. 72 of *Texts in Applied Mathematics*, Springer, Cham, 2021.
- [109] ———, *Finite elements II—Galerkin approximation, elliptic and mixed PDEs*, vol. 73 of *Texts in Applied Mathematics*, Springer, Cham, 2021.
- [110] C. FARHAT, I. HARARI, AND L. P. FRANCA, *The discontinuous enrichment method*, *Comput. Methods Appl. Mech. Eng.*, 190 (2001), pp. 6455–6479.
- [111] C. FARHAT AND F.-X. ROUX, *A method of finite element tearing and interconnecting and its parallel solution algorithm*, *Internat. J. Numer. Methods Engrg.*, 32 (1991), pp. 1205–1227.
- [112] M. FEISCHL, M. PAGE, AND D. PRAETORIUS, *Convergence and quasi-optimality of adaptive FEM with inhomogeneous Dirichlet data*, *J. Comput. Appl. Math.*, 255 (2014), pp. 481–501.
- [113] M. A. FERNÁNDEZ, *Coupling schemes for incompressible fluid-structure interaction: implicit, semi-implicit and explicit*, *SēMA Journal*, no. 55, (2011), pp. 59–108.
- [114] L. FORMAGGIA, F. GATTI, AND S. ZONCA, *An XFEM/DG approach for fluid-structure interaction problems with contact*, *Appl. Math.*, 66 (2021), pp. 183–211.

- [115] J. FREUND AND R. STENBERG, *On weakly imposed boundary conditions for second order problems*, in Proceedings of the Ninth Int. Conf. Finite Elements in Fluids, Venice, 1995, pp. 327–336.
- [116] A. FRITZ, S. HÜEBER, AND B. I. WOHLMUTH, *A comparison of mortar and Nitsche techniques for linear elasticity*, *Calcolo*, 41 (2004), pp. 115–137.
- [117] V. V. GARG AND S. PRUDHOMME, *Enhanced functional evaluation for the finite element penalty method*, *Comput. Math. Appl.*, 78 (2019), pp. 3821–3840.
- [118] G. N. GATICA, *A simple introduction to the mixed finite element method. Theory and applications*, SpringerBriefs Math., Cham: Springer, 2014.
- [119] A. GERSTENBERGER AND W. A. WALL, *An extended finite element method/Lagrange multiplier based approach for fluid-structure interaction*, *Comput. Methods Appl. Mech. Engrg.*, 197 (2008), pp. 1699–1714.
- [120] V. GIRAULT AND R. GLOWINSKI, *Error analysis of a fictitious domain method applied to a Dirichlet problem*, *Japan J. Indust. Appl. Math.*, 12 (1995), pp. 487–514.
- [121] I. G. GJERDE AND L. R. SCOTT, *Nitsche’s method for Navier-Stokes equations with slip boundary conditions*, *Math. Comp.*, 91 (2022), pp. 597–622.
- [122] R. GLOWINSKI, T. PAN, AND J. PÉRIAUX, *A fictitious domain method for Dirichlet problem and applications*, *Comput. Methods Appl. Mech. Engrg.*, 111 (1994), pp. 283–303.
- [123] R. GLOWINSKI, T.-W. PAN, T. I. HESLA, AND D. D. JOSEPH, *A distributed Lagrange multiplier/fictitious domain method for particulate flows*, *Int. J. Multiph. Flow*, 25 (1999), pp. 755–794.
- [124] R. GLOWINSKI, T. W. PAN, T. I. HESLA, D. D. JOSEPH, AND J. PÉRIAUX, *A fictitious domain approach to the direct numerical simulation of incompressible viscous flow past moving rigid bodies: application to particulate flow*, *J. Comput. Phys.*, 169 (2001), pp. 363–426.
- [125] P. GRISVARD, *Elliptic problems in nonsmooth domains*, vol. 24 of Monographs and Studies in Mathematics, Pitman (Advanced Publishing Program), Boston, MA, 1985.
- [126] T. GUSTAFSSON AND G. D. MCBAIN, *scikit-fem: A Python package for finite element assembly*, *J. Open Source Softw.*, 5 (2020), p. 2369.
- [127] T. GUSTAFSSON, R. STENBERG, AND J. VIDEMAN, *Mixed and stabilized finite element methods for the obstacle problem*, *SIAM J. Numer. Anal.*, 55 (2017), pp. 2718–2744.
- [128] ———, *Error analysis of Nitsche’s mortar method*, *Numer. Math.*, 142 (2019), pp. 973–994.
- [129] ———, *Nitsche’s method for Kirchhoff plates*, *SIAM J. Sci. Comput.*, 43 (2021), pp. a1651–a1670.
- [130] P. HANSBO, *Nitsche’s method for interface problems in computational mechanics*, *GAMM-Mitteilungen*, 28 (2005), pp. 183–206.
- [131] P. HANSBO AND M. G. LARSON, *Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche’s method*, *Comput. Methods Appl. Mech. Engrg.*, 191 (2002), pp. 1895–1908.
- [132] I. HARARI AND J. DOLBOW, *Analysis of an efficient finite element method for embedded interface problems*, *Comput. Mech.*, 46 (2010), pp. 205–211.
- [133] J. HASLINGER, I. HLAVÁČEK, AND J. NEČAS, *Handbook of Numerical Analysis (eds. P.G. Ciarlet and J.L. Lions)*, vol. IV, North Holland, 1996, ch. 2. “Numerical methods for unilateral problems in solid mechanics”, pp. 313–385.

- [134] J. HASLINGER AND Y. RENARD, *A new fictitious domain approach inspired by the extended finite element method*, SIAM J. Numer. Anal., 47 (2009), pp. 1474–1499.
- [135] M. HAUTEFEUILLE, C. ANNAVARAPU, AND J. E. DOLBOW, *Robust imposition of Dirichlet boundary conditions on embedded surfaces*, Internat. J. Numer. Methods Engrg., 90 (2012), pp. 40–64.
- [136] F. HECHT, *New development in freefem++*, J. Numer. Math., 20 (2012), pp. 251–265.
- [137] F. HECHT, *Quelques idées d'utilisation du C++ en calcul scientifique*. 2014.
- [138] P. HILD AND Y. RENARD, *A stabilized Lagrange multiplier method for the finite element approximation of contact problems in elastostatics*, Numer. Math., 115 (2010), pp. 101–129.
- [139] Q. HU, F. CHOULY, P. HU, G. CHENG, AND S. P. A. BORDAS, *Skew-symmetric Nitsche's formulation in isogeometric analysis: Dirichlet and symmetry conditions, patch coupling and frictionless contact*, Comput. Methods Appl. Mech. Engrg., 341 (2018), pp. 188–220.
- [140] S. HÜEBER AND B. I. WOHLMUTH, *A primal-dual active set strategy for non-linear multibody contact problems*, Comput. Methods Appl. Mech. Engrg., 194 (2005), pp. 3147–3166.
- [141] T. J. R. HUGHES AND L. P. FRANCA, *A new finite element formulation for computational fluid dynamics. VII. The Stokes problem with various well-posed boundary conditions: symmetric formulations that converge for all velocity/pressure spaces*, Comput. Methods Appl. Mech. Engrg., 65 (1987), pp. 85–96.
- [142] M. INNERBERGER AND D. PRAETORIUS, *MooAFEM: an object oriented Matlab code for higher-order adaptive FEM for (nonlinear) elliptic PDEs*, Appl. Math. Comput., 442 (2023), pp. Paper No. 127731, 17.
- [143] H. JI AND J. DOLBOW, *On strategies for enforcing interfacial constraints and evaluating jump conditions with the extended finite element method*, Internat. J. Numer. Methods Engrg., 61 (2004), pp. 2508–2535.
- [144] M. JUNTUNEN, *On the connection between the stabilized Lagrange multiplier and Nitsche's methods*, Numer. Math., 131 (2015), pp. 453–471.
- [145] M. JUNTUNEN AND R. STENBERG, *Nitsche's method for general boundary conditions*, Math. Comp., 78 (2009), pp. 1353–1374.
- [146] N. KIKUCHI AND J. T. ODEN, *Contact problems in elasticity: a study of variational inequalities and finite element methods*, vol. 8 of SIAM Studies in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1988.
- [147] N. KIKUCHI AND Y. J. SONG, *Penalty-finite-element approximation of a class of unilateral problems in linear elasticity*, Quart. Appl. Math., 39 (1981), pp. 1–22.
- [148] T.-Y. KIM, J. DOLBOW, AND E. FRIED, *A numerical method for a second-gradient theory of incompressible fluid flow*, J. Comput. Phys., 223 (2007), pp. 551–570.
- [149] T.-Y. KIM, J. DOLBOW, AND T. LAURSEN, *A mortared finite element method for frictional contact on arbitrary interfaces*, Comput. Mech., 39 (2007), pp. 223–235.
- [150] S. KOLLMANNBERGER, A. ÖZCAN, J. BAIGES, M. RUESS, E. RANK, AND A. REALI, *Parameter-free, weak imposition of Dirichlet boundary conditions and coupling of trimmed and non-conforming patches*, Internat. J. Numer. Methods Engrg., 101 (2015), pp. 670–699.
- [151] C. LEHRENFELD, *Removing the stabilization parameter in fitted and unfitted symmetric Nitsche formulations*, arXiv preprint arXiv:1603.00617, (2016).

- [152] S. LEMAIRE, *Bridging the hybrid high-order and virtual element methods*, IMA J. Numer. Anal., 41 (2021), pp. 549–593.
- [153] V. LLERAS, *A stabilized Lagrange multiplier method for the finite element approximation of frictional contact problems in elastostatics*, Math. Model. Nat. Phenom., 4 (2009), pp. 163–182.
- [154] A. LOGG AND G. N. WELLS, *DOLFIN: automated finite element computing*, ACM Trans. Math. Software, 37 (2010), pp. Art. 20, 28.
- [155] A. LOZINSKI, *CutFEM without cutting the mesh cells: a new way to impose Dirichlet and Neumann boundary conditions on unfitted meshes*, Comput. Methods Appl. Mech. Engrg., 356 (2019), pp. 75–100.
- [156] ———, *A primal discontinuous Galerkin method with static condensation on very general meshes*, Numer. Math., 143 (2019), pp. 583–604.
- [157] N. LÜTHEN, M. JUNTUNEN, AND R. STENBERG, *An improved a priori error analysis of Nitsche’s method for Robin boundary conditions*, Numer. Math., 138 (2018), pp. 1011–1026.
- [158] B. MAURY, *Numerical analysis of a finite element/volume penalty method*, SIAM J. Numer. Anal., 47 (2009), pp. 1126–1148.
- [159] U. M. MAYER, A. POPP, A. GERSTENBERGER, AND W. A. WALL, *3D fluid-structure-contact interaction based on a combined XFEM FSI and dual mortar contact approach*, Comput. Mech., 46 (2010), pp. 53–67.
- [160] W. MCLEAN, *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, Cambridge, 2000.
- [161] N. MOËS, E. BÉCHET, AND M. TOURBIER, *Imposing Dirichlet boundary conditions in the extended finite element method*, Internat. J. Numer. Methods Engrg., 67 (2006), pp. 1641–1669.
- [162] P. MORIN, R. H. NOCHETTO, AND K. G. SIEBERT, *Local problems on stars: a posteriori error estimators, convergence, and performance*, Math. Comp., 72 (2003), pp. 1067–1097.
- [163] H. M. MOURAD, J. DOLBOW, AND I. HARARI, *A bubble-stabilized finite element method for Dirichlet constraints on embedded interfaces*, Internat. J. Numer. Methods Engrg., 69 (2007), pp. 772–793.
- [164] V. P. NGUYEN, T. RABCZUK, S. P. A. BORDAS, AND M. DUFLLOT, *Meshless methods: a review and computer implementation aspects*, Math. Comput. Simulation, 79 (2008), pp. 763–813.
- [165] J. NITSCHKE, *Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind*, Abh. Math. Sem. Univ. Hamburg, 36 (1971), pp. 9–15.
- [166] J. ODEN AND N. KIKUCHI, *Finite element methods for constrained problems in elasticity*, Internat. J. Numer. Meth. Engrg., 18 (1982), pp. 701–725.
- [167] J. ODEN AND S. KIM, *Interior penalty methods for finite element approximations of the Signorini problem in elastostatics*, Comp. and Maths. with Appls., 8 (1982), pp. 35–56.
- [168] J. T. ODEN, *Historical comments on finite elements*, in A history of scientific computing, 1990, pp. 152–166.
- [169] C. S. PESKIN, *The immersed boundary method*, Acta Numer., 11 (2002), pp. 479–517.
- [170] J. PITKÄRANTA, *Boundary subspaces for the finite element method with Lagrange multipliers*, Numer. Math., 33 (1979), pp. 273–289.

- [171] ———, *Local stability conditions for the Babuška method of Lagrange multipliers*, Math. Comp., 35 (1980), pp. 1113–1129.
- [172] ———, *The finite element method with Lagrange multipliers for domains with corners*, Math. Comp., 37 (1981), pp. 13–30.
- [173] A. QUARTERONI AND A. VALLI, *Numerical approximation of partial differential equations*, vol. 23 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1994.
- [174] ———, *Domain decomposition methods for partial differential equations*, Numerical Mathematics and Scientific Computation, The Clarendon Press, Oxford University Press, New York, 1999. Oxford Science Publications.
- [175] Y. RENARD AND K. POULIOS, *GetFEM: automated FE modeling of multiphysics problems based on a generic weak form language*, ACM Trans. Math. Software, 47 (2021), pp. Art. 4, 31.
- [176] S. REPIN, S. SAUTER, AND A. SMOLIANSKI, *A posteriori error estimation for the Dirichlet problem with account of the error in the approximation of boundary conditions*, Computing, 70 (2003), pp. 205–233.
- [177] R. RODRÍGUEZ, *Some remarks on Zienkiewicz-Zhu estimator*, Numer. Methods Partial Differ. Equations, 10 (1994), pp. 625–635.
- [178] J. D. SANDERS, J. E. DOLBOW, AND T. A. LAURSEN, *On methods for stabilizing constraints over enriched interfaces in elasticity*, Internat. J. Numer. Methods Engrg., 78 (2009), pp. 1009–1036.
- [179] F.-J. SAYAS, T. S. BROWN, AND M. E. HASSELL, *Variational techniques for elliptic partial differential equations*, CRC Press, Boca Raton, FL, 2019.
- [180] D. SCHILLINGER, I. HARARI, M.-C. HSU, D. KAMENSKY, S. K. F. STOTER, Y. YU, AND Y. ZHAO, *The non-symmetric Nitsche method for the parameter-free imposition of weak boundary and coupling conditions in immersed finite elements*, Comput. Methods Appl. Mech. Engrg., 309 (2016), pp. 625–652.
- [181] Z. C. SHI, *On the convergence rate of the boundary penalty method*, Internat. J. Numer. Methods Engrg., 20 (1984), pp. 2027–2032.
- [182] O. STEINBACH, *Numerical approximation methods for elliptic boundary value problems*, Springer, New York, 2008.
- [183] O. STEINBACH, B. WOHLMUTH, AND L. WUNDERLICH, *Trace and flux a priori error estimates in finite-element approximations of Signorni-type problems*, IMA J. Numer. Anal., 36 (2016), pp. 1072–1095.
- [184] R. STENBERG, *On some techniques for approximating boundary conditions in the finite element method*, J. Comput. Appl. Math., 63 (1995), pp. 139–148.
- [185] N. SUKUMAR, J. DOLBOW, A. DEVAN, J. YVONNET, F. CHINESTA, D. RYCKELYNCK, P. LORONG, I. ALFARO, M.-A. MARTÍNEZ, E. CUETO, AND M. DOBLARÉ, *Meshless methods and partition of unity finite elements*, Int. J. Form. Process., 8 (2005), pp. 409–427.
- [186] V. THOMÉE, *Galerkin finite element methods for parabolic problems*, vol. 25 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1997.
- [187] M. TUR, J. ALBELDA, O. MARCO, AND J. J. RÓDENAS, *Stabilized method of imposing Dirichlet boundary conditions using a recovered stress field*, Comput. Methods Appl. Mech. Engrg., 296 (2015), pp. 352–375.
- [188] J. M. URQUIZA, A. GARON, AND M.-I. FARINAS, *Weak imposition of the slip boundary condition on curved boundaries for Stokes flow*, J. Comput. Phys., 256 (2014), pp. 748–767.

- [189] R. VERFÜRTH, *Finite element approximation of incompressible Navier-Stokes equations with slip boundary condition. II*, Numer. Math., 59 (1991), pp. 615–636.
- [190] T. WARBURTON AND J. S. HESTHAVEN, *On the constants in hp-finite element trace inverse inequalities*, Comput. Methods Appl. Mech. Engrg., 192 (2003), pp. 2765–2773.
- [191] M. WINTER, B. SCHOTT, A. MASSING, AND W. A. WALL, *A Nitsche cut finite element method for the Oseen problem with general Navier boundary conditions*, Comput. Methods Appl. Mech. Engrg., 330 (2018), pp. 220–252.
- [192] B. I. WOHLMUTH, *A mortar finite element method using dual spaces for the Lagrange multiplier*, SIAM J. Numer. Anal., 38 (2000), pp. 989–1012.
- [193] ———, *Variationally consistent discretization schemes and numerical algorithms for contact problems*, Acta Numerica, (2011), pp. 569–734.
- [194] G. ZHOU, I. OIKAWA, AND T. KASHIWABARA, *The Crouzeix-Raviart element for the Stokes equations with the slip boundary condition on a curved boundary*, J. Comput. Appl. Math., 383 (2021), pp. Paper No. 113123, 19.
- [195] O. C. ZIENKIEWICZ AND J. Z. ZHU, *A simple error estimator and adaptive procedure for practical engineering analysis*, Internat. J. Numer. Methods Engrg., 24 (1987), pp. 337–357.