A FOURIER ANALYSIS OF QUADRATIC RIEMANN SUMS AND LOCAL INTEGRALS OF $\zeta(s)$

Michel Weber

A FOURIER ANALYSIS OF QUADRATIC RIEMANN SUMS AND LOCAL INTEGRALS OF $\zeta(s)$

MICHEL J. G. WEBER

Abstract. Let $\zeta(s)$, $s=\sigma+it$, be the Riemann zeta function. We use Fourier analysis to obtain, after a preliminary study of quadratic Riemann sums, a precise formula of the local integrals $\int_n^{n+1}|\zeta(\sigma+it)|^2\mathrm{d}t$, for $\frac{1}{2}<\sigma<1$. We also study related $\mathcal{S}^2$-Stepanov norms of $\zeta(s)$ in connection with the strong Voronin Universality Theorem.

1. Introduction-Results

In this work we study local integrals on *bounded* intervals of the Riemann zeta function $\zeta(s)$, $s=\sigma+it$, namely their asymptotic behavior near infinity. For results concerning local integrals of $\zeta(s)$ on short intervals (but of growing size near infinity) we can cite [2], [12] where the Riemann Hypothesis (RH) is assumed, and refer to the remarkable survey [11], (see notably (5.15), (7.7) and estimates p. 14), which is sufficient for our purpose. The methods used in mean square theory are quite elaborated and are mostly ultimate refinements of previous papers, also very old ones as the theory is. Although the works made are really impressive, they unfortunately don't give the impression, in view of the conjectures of the theory notably, that more can be expected from new refinements of these ones. Moreover from the considerable amount of works and results concerning the meanvalue of $\zeta(s)$, for instance

$$\int_0^T|\zeta(\sigma+it)|^2\mathrm{d}t,$$

T large, we clarify that nothing can be drawn to measure the size's order of the local integrals we consider, because of the size of the error term. Estimates of the local integrals

$$\int_a^{a+H}|\zeta(\sigma+it)|^2\mathrm{d}t$$

in general require H to be large with a. We couldn't find in the related literature estimates corresponding to the case a large and H bounded.

The approach we purpose in this paper is new and can be developed. It is also completing the recent work [14]. We use the natural link existing between the zeta local integrals

$$\int_n^{n+1}|\zeta(\sigma+it)|^2\mathrm{d}t, \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary 42A75, 11M06 ; Secondary 42A24, 42B25.

Key words and phrases: Riemann sums, Quadratic Riemann sums, Riemann zeta function, Stepanov norm, weighted amalgams $\ell^q(L^2,w)$.

August 1, 2022

and quadratic Riemann sums. Recall that the Riemann sum of order ℓ of a function f defined at rational points of $[0,1]$ is defined by

$$R_f(\ell) = \frac{1}{\ell}\sum_{1\le k\le \ell} f\Big(\frac{k}{\ell}\Big), \qquad \ell\ge 1. \tag{1.2}$$

These sums converge to $\int_0^1 f(t)\mathrm{d}t$ if for instance f has bounded derivative. We refer to [16], Chap. XI, which is devoted to Riemann sums and their link to Number Theory, as well as for convergence properties of (Wintner's equidistant) Riemann sums.

The quadratic Riemann sums are defined as follows,

$$S_{n,\sigma}(f) \;=\; \sum_{1\le k\le \ell\le n} \frac{1}{(k\ell)^\sigma} f\Big(\frac{k}{\ell}\Big), \qquad 0\le\sigma<1. \tag{1.3}$$

These sums are easily seen as weightings of Riemann sums of associated with $f_\sigma(x)=f(x)/x^\sigma$,

$$S_{n,\sigma}(f) \;=\; \sum_{1\le k\le \ell\le n} \frac{1}{\ell^{2\sigma}} f_\sigma\Big(\frac{k}{\ell}\Big) \;=\; \sum_{1\le \ell\le n} \ell^{1-2\sigma} R_{f_\sigma}(\ell). \tag{1.4}$$

When the Riemann sums $R_f(\ell)$ converge, these weightings converge to the same limit and a speed of convergence can be exhibited ([14], Proposition 2.1).

Their link with ζ-local integrals expresses as follows (see (3.4)):

$$\int_n^{n+1} |\zeta(\sigma+it)|^2\mathrm{d}t \;=\; \textit{Quadratic Riemann sum} + \textit{Error term}. \tag{1.5}$$

The main result of this paper is the Theorem below which provides an explicit unconditional formula for the local integrals (1.1).

Theorem 1.1. (i) *For* $\frac12<\sigma<1$,

$$\int_n^{n+1} |\zeta(\sigma+it)|^2\mathrm{d}t = \zeta(2\sigma) + 4\sum_{k\in\mathbb{Z}_*} \Re\big(c_{g_{n,\sigma}}(k)\big)\Big(\sum_{1\le\ell\le n} \ell^{-2\sigma}\varepsilon_k(\ell)\Big) + \mathcal{O}_\sigma\big(n^{1-2\sigma}\big),$$

as $n\to\infty$, *where* $g_n(x)$ *is defined by*

$$g_n(x) = \frac{\cos((n+\frac12)\log x)\sin\frac12\log x}{\log x} \quad 0<x<1,\ g_n(1)=\tfrac12,\ g_n(0)=0, \tag{1.6}$$

and $(c_{g_{n,\sigma}}(k))_{k\in\mathbb{Z}}$ *are the Fourier coefficients of* $g_{n,\sigma}(x)=g_n(x)/x^\sigma$, $n\ge1$, *and*

$$\varepsilon_k(\ell) = \begin{cases} \ell-1 & \text{if } \ell|k,\\ -1 & \text{if } \ell\not| k.\end{cases} \tag{1.7}$$

Further for any $k\in\mathbb{Z}*$,

$$\Re(c_{g_{n,\sigma}}(k)) \;=\; \sum_{m=0}^\infty \frac{(-1)^m(2\pi k)^{2m}}{(2m)!}\Big(\frac{\arctan\frac{n+1}{2m+1-\sigma}-\arctan\frac{n}{2m+1-\sigma}}{2}\Big). \tag{1.8}$$

(ii) *Furthermore,*

$$\Big|\Re(c_{g_{n,\sigma}}(k)) - \sum_{m<m(n,\sigma)\vee\lceil\pi ek\rceil} (-1)^m\frac{(2\pi k)^{2m}}{(2m)!}\Big(\frac{\mathrm{Arctan}\,\frac{n+1}{2m+1-\sigma}-\mathrm{Arctan}\,\frac{n}{2m+1-\sigma}}{2}\Big)\Big|$$

$$\leq \frac{C_\sigma}{k^{3/2}}\,.$$

where C_σ depends on σ only and $m(n,\sigma) = (\sigma + \sqrt{3n(n+1)/2+2}\,)/2$.

Remarks 1.2. (i) Note that by Riesz's criterion, [3, Vol. II], p. 184, and Young's inequality,

$$\sum_{\nu\in\mathbb{Z}_*} |c_{g_{n,\sigma}}(\nu)| = \infty \quad \text{and} \quad \sum_{\nu\in\mathbb{Z}_*} |c_{g_{n,\sigma}}(\nu)|^q < \infty, \quad (q > \frac{1-\sigma}{\sigma},\ n \geq 1). \tag{1.9}$$

(ii) The arithmetical factor of $\Re(c_{g_{n,\sigma}}(k))$ is easily estimated. We have

$$\sum_{1\leq\ell\leq n} \ell^{-2\sigma}\varepsilon_k(\ell) = \sum_{\substack{1\leq\ell\leq n\\ \ell|k}} \ell^{1-2\sigma} - \sum_{1\leq\ell\leq n} \ell^{-2\sigma} = \sum_{\substack{1\leq\ell\leq n\\ \ell|k}} \ell^{1-2\sigma} - C_\sigma + \mathcal{O}_\sigma\big(n^{1-2\sigma}\big).$$

and if $n \geq k$, it reduces to $\tau_{1-2\sigma}(k) - C_\sigma + \mathcal{O}_\sigma(n^{1-2\sigma})$ where $\tau_h(n) = \sum_{d|n} d^h$ is the usual sum-divisor function. By Gronwall's estimate, ([7] p. 119–122),

$$\limsup_{k\to\infty} \frac{\log \tau_{1-2\sigma}(k)}{\frac{(\log k)^{2(1-\sigma)}}{\log\log k}} = \frac{1}{2(1-\sigma)}. \tag{1.10}$$

Theorem 1.1 is deduced from a preliminary Fourier analysis of the quadratic Riemann sums (1.3).

The control of Fourier coefficients $(c_{g_{n,\sigma}}(k))_k$ is a key question. Section 3 contains a thorough study of their properties.

Introduce the class $\mathcal{C}$ of functions $f : [0,1] \to \mathbb{R}$ such that for all $x \in \mathbb{Q}\cap]0,1[$ and some $\eta > 0$,

$$f(x-0) = f(x+0) \ \textit{and} \ \int_0^\eta \tfrac{|f(x+u)+f(x-u)-2f(x)|}{u}\,\mathrm{d}u < \infty. \tag{1.11}$$

This defines a fairly wide class and one has the following obvious inclusions: $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}$, where

$$\begin{aligned} \mathcal{C}_1 &= \big\{f : [0,1] \to \mathbb{R} : f(x) \text{ is derivable for each } x \in \mathbb{Q}\cap]0,1[\big\}\\ \mathcal{C}_2 &= \big\{f : [0,1] \to \mathbb{R} : f(x) \text{ satisfies a Lipschitz condition of order } \alpha > 0,\\ &\qquad \text{in the neighbourhood of each } x \in \mathbb{Q}\cap]0,1[\big\}. \end{aligned} \tag{1.12}$$

Functions from the class $\mathcal{C}$ are not necessarily absolutely continuous.

We prove the following unconditional results. We identify the limiting term of the quadratic Riemann sums and provide a precise control of the error term.

Theorem 1.3. *Let $f : [0,1] \to \mathbb{R}$. Let $\frac{1}{2} < \sigma < 1$ and assume that $f_\sigma \in \mathcal{C}$. Further assume that the series*

$$\sum_{\nu\in\mathbb{Z}} \Re\big(c_{f_\sigma}(\nu)\big), \qquad \sum_{\nu\in\mathbb{Z}_*} \Re\big(c_{f_\sigma}(\nu)\big)\,\sigma_{1-2\sigma}(\nu)$$

are convergent, where $c_{f_\sigma}(\ell)$, $\ell \in \mathbb{Z}$ are the Fourier coefficients of f_σ, and $\sigma_{1-2\sigma}(\nu)$ denotes the sum of the $(1-2\sigma)$-th powers of the divisors of ν. Then,

$$S_{n,\sigma}(f) \ = \ \Big(\int_0^1 f_\sigma(x)\mathrm{d}x\Big)\Big(\frac{n^{2(1-\sigma)}}{2(1-\sigma)} + \zeta(2\sigma-1) - 1\Big)$$

$$+\sum_{\nu\in\mathbb{Z}_*}\Re\big(c_{f_\sigma}(\nu)\big)\big(\sigma_{1-2\sigma}(\nu)-1\big)+o(1),$$

as $n\to\infty$, *recalling that* $\zeta(s)=\lim_{x\to\infty}\big(\sum_{n\le x}\frac{1}{n^s}-\frac{x^{1-s}}{1-s}\big)$, $0<s<1$.

Theorem 1.3 is obtained as a consequence of the following result.

Theorem 1.4. *Let* σ *be a real number. Let* $f:[0,1]\to\mathbb{R}$ *and assume that* $f_\sigma\in\mathcal{C}$.

(1) *We have*

$$S_{n,\sigma}(f) \;=\; \Big(\int_0^1 f_\sigma(x)\mathrm{d}x\Big)\Big(\sum_{1\le\ell\le n}(\ell-1)\ell^{-2\sigma}\Big)+\sum_{\nu\in\mathbb{Z}_*}c_{f_\sigma}(\ell)\Big(\sum_{1\le\ell\le n}\ell^{-2\sigma}\varepsilon_\nu(\ell)\Big),$$

for each positive n, *where* $\varepsilon_\ell(d)$ *is defined in Theorem 1.1.*

(2) *Assume that the series* $\sum_{\nu\in\mathbb{Z}}c_{f_\sigma}(\nu)$ *is convergent. Then,*

$$\begin{aligned}S_{n,\sigma}(f) \;=\; &\Big(\int_0^1 f_\sigma(x)\mathrm{d}x\Big)\Big(\sum_{1\le\ell\le n}\ell^{1-2\sigma}\Big)+\sum_{\nu\in\mathbb{Z}_*}c_{f_\sigma}(\nu)\Big(\sum_{\substack{1\le\ell\le n\\ \ell|\nu}}\ell^{1-2\sigma}\Big)\\ &-\sum_{\nu\in\mathbb{Z}}c_{f_\sigma}(\nu)\Big(\sum_{1\le\ell\le n}\ell^{-2\sigma}\Big),\end{aligned}$$

for each positive n.

Remark 1.5. A study of the corresponding quadratic Farey sums – however with no connection with $\zeta(s)$ as in (1.5) – is made in [15] where unconditional results for Farey sums are also proved.

The paper is organized as follows. In the next section, we give the proofs of Theorems 1.3 and 1.4. In Section 3 we prove Theorem 1.1. Finally we discuss in Section 4 the finiteness of the related $\mathcal{S}^2$-Stepanov norm of $\zeta(s)$.

2. Proofs of Theorems 1.3 and 1.4.

2.1. Proof of Theorem 1.4. Recall that

$$S_{n,\sigma}(f) \;=\; \sum_{1\le k\le\ell\le n}\frac{1}{\ell^{2\sigma}}\,f_\sigma\Big(\frac{k}{\ell}\Big) \;=\; \sum_{1\le\ell\le n}\ell^{1-2\sigma}\,R_{f_\sigma}(\ell).$$

Dini's test [3, Vol. I], implies in view of the assumption made, that the Fourier series of f_σ,

$$\sigma_{f_\sigma}(x)=\sum_{\nu\in\mathbb{Z}}c_{f_\sigma}(\nu)e^{2\mathrm{i}\pi\nu x}\quad\text{converges to } f_\sigma(x), \tag{2.1}$$

that is, the partials sums of $\sigma_{f_\sigma}(x)$ are converging to $f_\sigma(x)$, and this for any $x\in\mathbb{Q}\cap]0,1[$.

Taking successively $x=\frac{k}{\ell}$, $k=1,2,\dots,\ell-1$ in (2.1), gives

$$f_\sigma\Big(\frac{k}{\ell}\Big) \;=\; c_{f_\sigma}(0)+\sum_{\nu\in\mathbb{Z}_*}c_{f_\sigma}(\nu)e^{2\mathrm{i}\pi\nu\frac{k}{\ell}},\qquad k=1,2,\dots,\ell-1.$$

Let $D_{f_\sigma}(\ell)=\ell R_{f_\sigma}(\ell)$. Then

$$D_{f_\sigma}(\ell) \;:=\; (\ell-1)c_{f_\sigma}(0)+\sum_{\nu\in\mathbb{Z}_*}c_{f_\sigma}(\nu)\Big(\sum_{k=1}^{\ell-1}e^{2\mathrm{i}\pi\nu\frac{k}{\ell}}\Big)$$

$$= \quad (\ell-1)c_{f_\sigma}(0) + \sum_{\nu\in\mathbb{Z}_*} c_{f_\sigma}(\nu)\varepsilon_\ell(\nu). \tag{2.2}$$

Therefore

$$\begin{aligned} S_{n,\sigma}(f) &= \sum_{1\le\ell\le n} \ell^{-2\sigma}\, D_{f_\sigma}(\ell) \\ &= c_{f_\sigma}(0)\Big(\sum_{1\le\ell\le n}(\ell-1)\ell^{-2\sigma}\Big) + \sum_{\nu\in\mathbb{Z}_*} c_{f_\sigma}(\ell)\Big(\sum_{1\le\ell\le n}\ell^{-2\sigma}\varepsilon_\nu(\ell)\Big), \end{aligned}$$

whence the first assertion.

Remark 2.1. Dini's test does not imply that the series $\sum_{\ell\in\mathbb{Z}} c_{f_\sigma}(\ell)$ is convergent. However from (2.2) follows that for any $\ell\ge 2$, the difference of series

$$(\ell-1)\sum_{\substack{\nu\in\mathbb{Z}_*\\ \ell|\nu}} c_{f_\sigma}(\nu) - \sum_{\substack{\nu\in\mathbb{Z}_*\\ \ell\nmid\nu}} c_{f_\sigma}(\nu) \tag{2.3}$$

converges in the sense that the limit below

$$\lim_{N\to\infty}\Big\{(\ell-1)\sum_{\substack{\nu\in\mathbb{Z}_*\,,\,|\nu|\le N\\ \ell|\nu}} c_{f_\sigma}(\nu) - \sum_{\substack{\nu\in\mathbb{Z}_*\,,\,|\nu|\le N\\ \ell\nmid\nu}} c_{f_\sigma}(\nu)\Big\}, \tag{2.4}$$

exists.

Now if the series $\sum_{\ell\in\mathbb{Z}} c_{f_\sigma}(\ell)$ is convergent, we can write

$$\begin{aligned} D_{f_\sigma}(\ell) &= (\ell-1)\,c_{f_\sigma}(0) + \sum_{\nu\in\mathbb{Z}_*} c_{f_\sigma}(\nu)\Big(\sum_{k=1}^{\ell} e^{2\mathrm{i}\pi\nu\frac{k}{\ell}} - 1\Big) \\ &= \ell\, c_{f_\sigma}(0) + \ell\sum_{\substack{\nu\in\mathbb{Z}_*\\ \ell|\nu}} c_{f_\sigma}(\nu) - \sum_{\nu\in\mathbb{Z}} c_{f_\sigma}(\nu) \end{aligned}$$

since $\sum_{k=1}^{\ell} e^{2\mathrm{i}\pi\nu\frac{k}{\ell}}$ equals to ℓ or 0 according to $\ell|\nu$ or not. Thus

$$\begin{aligned} S_{n,\sigma}(f) &= \sum_{1\le\ell\le n} \ell^{-2\sigma}\, D_{f_\sigma}(\ell) \\ &= c_{f_\sigma}(0)\Big(\sum_{1\le\ell\le n}\ell^{1-2\sigma}\Big) + \sum_{\nu\in\mathbb{Z}_*} c_{f_\sigma}(\nu)\Big(\sum_{\substack{1\le\ell\le n\\ \ell|\nu}}\ell^{1-2\sigma}\Big) - \sum_{\nu\in\mathbb{Z}} c_{f_\sigma}(\nu)\Big(\sum_{1\le\ell\le n}\ell^{-2\sigma}\Big). \end{aligned}$$

2.2. Proof of Theorem 1.3. As f is real-valued

$$\begin{aligned} S_{n,\sigma}(f) &= c_{f_\sigma}(0)\Big(\sum_{1\le\ell\le n}\ell^{1-2\sigma}\Big) + \sum_{\nu\in\mathbb{Z}_*}\Re\big(c_{f_\sigma}(\nu)\big)\Big(\sum_{\substack{1\le\ell\le n\\ \ell|\nu}}\ell^{1-2\sigma}\Big) \\ &\quad - \sum_{\nu\in\mathbb{Z}}\Re\big(c_{f_\sigma}(\nu)\big)\Big(\sum_{1\le\ell\le n}\ell^{-2\sigma}\Big). \end{aligned}$$

For each $N\ge 1$,

$$\lim_{n\to\infty}\sum_{\substack{-N\le\nu\le N\\ \nu\ne 0}}\Re\big(c_{f_\sigma}(\nu)\big)\Big(\sum_{\substack{1\le\ell\le n\\ \ell|\nu}}\ell^{1-2\sigma}\Big) = \sum_{\substack{-N\le\nu\le N\\ \nu\ne 0}}\Re\big(c_{f_\sigma}(\nu)\big)\sigma_{1-2\sigma}(\nu).$$

From the assumptions made, by using a standard approximation argument, we deduce

$$\lim_{n\to\infty}\sum_{\nu\in\mathbb{Z}_*} c_{f_\sigma}(\nu)\Big(\sum_{\substack{1\le\ell\le n\\ \ell|\nu}}\ell^{1-2\sigma}\Big) \;=\; \sum_{\nu\in\mathbb{Z}_*} c_{f_\sigma}(\nu)\sigma_{1-2\sigma}(\nu).$$

Using Theorem 1.4 and elementary estimate

$$\sum_{1\le\ell\le n}\ell^{1-2\sigma} = \frac{n^{2(1-\sigma)}}{2(1-\sigma)} + \zeta(2\sigma-1) + \mathcal{O}\left(n^{1-2\sigma}\right) \qquad (1/2<\sigma<1), \tag{2.5}$$

we get,

$$S_{n,\sigma}(f) \;=\; c_{f_\sigma}(0)\Big(\frac{n^{2(1-\sigma)}}{2(1-\sigma)} + C_\sigma\Big) + \sum_{\nu\in\mathbb{Z}_*}\Re\big(c_{f_\sigma}(\nu)\big)\sigma_{1-2\sigma}(\nu) - \sum_{\nu\in\mathbb{Z}}\Re\big(c_{f_\sigma}(\nu)\big) + o(1),$$

as $n\to\infty$.

3. Proof of Theorem 1.1.

Let $1\le a<b<\infty$, $a\ge cb$ for some positive $c<1$. We first note that for $\sigma>\frac12$,

$$\begin{aligned}\int_a^b\Big|\sum_{k=1}^b\frac{1}{k^{\sigma+it}}\Big|^2\mathrm{d}t \;&=\; (b-a)\sum_{k=1}^b\frac{1}{k^{2\sigma}}\\ &\qquad + 2\sum_{1\le k<\ell\le b}\frac{1}{(k\ell)^\sigma}\;\frac{\sin b\log\frac{k}{\ell}-\sin a\log\frac{k}{\ell}}{\log\frac{k}{\ell}}\\ &= (b-a)\sum_{k=1}^b\frac{1}{k^{2\sigma}} + 4\sum_{1\le k<\ell\le b}\frac{1}{(k\ell)^\sigma}\;\frac{\sin(\frac12(b-a)\log\frac{k}{\ell})\cos(\frac12(a+b)\log\frac{k}{\ell})}{\log\frac{k}{\ell}}\\ &= -(b-a)\sum_{k=1}^b\frac{1}{k^{2\sigma}} + 4S_{b,\sigma}(g(a,b)),\end{aligned} \tag{3.1}$$

where

$$g(a,b)(x) = \frac{\sin(\frac12(b-a)\log x)\cos(\frac12(a+b)\log x)}{\log x} \qquad 0<x<1,$$

and $g(a,b)(1)=\frac12(b-a)$, $g(a,b)(0)=0$.

By the classical approximation formula ([13, p. 77]), given $\sigma_0>0$, we have uniformly for $\sigma\ge\sigma_0$, $0<|t|\le\pi n$

$$\zeta(\sigma+it) = \sum_{k\le n}\frac{1}{k^{\sigma+it}} - \frac{n^{1-\sigma-it}}{1-\sigma-it} + \mathcal{O}_{\sigma_0}(x^{-\sigma}). \tag{3.2}$$

We deduce from (3.1) that

$$\int_a^b|\zeta(\sigma+it)|^2\mathrm{d}t = \zeta(2\sigma)(b-a) + 4S_{b,\sigma}(g(a,b)) + \mathcal{O}_{\sigma,c}\left(b^{1-2\sigma}\right). \tag{3.3}$$

Choose $\sigma_0=1/2$. As for $a\le t\le b$,

$$\Big|\frac{b^{1-\sigma-it}}{1-\sigma-it}\Big| = \frac{b^{1-\sigma}}{\sqrt{(1-\sigma)^2+t^2}} \le \frac{b^{-\sigma}}{c},$$

it follows that uniformly for $\sigma \geq 1/2$, $a \leq t \leq b$,

$$\zeta(\sigma+it) - \sum_{k=1}^{b} \frac{1}{k^{\sigma+it}} \;=\; \mathcal{O}_{1/2,c}\,(b^{-\sigma}).$$

Thus

$$\int_a^b |\zeta(\sigma+it)|^2 \mathrm{d}t = -(b-a)\zeta(2\sigma) + 4S_{b,\sigma}(g(a,b)) + \mathcal{O}_{\sigma,c}\,(b^{1-2\sigma}).$$

Letting $a = n$ and $b = n+1$ gives

$$\int_n^{n+1} |\zeta(\sigma+it)|^2 \mathrm{d}t \;=\; -\zeta(2\sigma) + 4S_{n,\sigma}(g_n) + \mathcal{O}_\sigma(n^{1-2\sigma}), \tag{3.4}$$

where g_n, $n \geq 1$ is defined in (1.6). Recall that $g_{n,\sigma}(x) = g_n(x)/x^\sigma$. Since for $0 < x < 1$,

$$\begin{aligned} g'_{n,\sigma}(x) \;=\; \tfrac{1}{x^{1+\sigma}(\log x)}\Big\{ & -n\sin(n+\tfrac{1}{2})\log x \cdot \sin\tfrac{1}{2}\log x + \tfrac{1}{2}\cos(n+1)\log x \\ & -\cos(n+\tfrac{1}{2})\log x \cdot \tfrac{\sin\frac{1}{2}\log x}{\log x} - \sigma\cos(n+\tfrac{1}{2})\log x \cdot \sin\tfrac{1}{2}\log x\Big\}, \end{aligned} \tag{3.5}$$

it follows that $g_{n,\sigma} \in \mathcal{C}_1 \subset \mathcal{C}$.

The Riemann sums $R_{g_{n,\sigma}}(d)$ cannot be directly estimated since $g_{n,\sigma}(x)$ oscillates wildly near 0_+, with peaks increasing to infinity with n. The Fourier coefficients can however be computed.

By (3.4),

$$\int_n^{n+1} |\zeta(\sigma+it)|^2 \mathrm{d}t \;=\; \zeta(2\sigma) + 4S_{n,\sigma}(g_n) + \mathcal{O}_\sigma(n^{1-2\sigma}).$$

Thus by Theorem 1.4,

$$S_{n,\sigma}(g_n) \;=\; c_{g_{n,\sigma}}(0)\Big(\sum_{1\leq\ell\leq n}(\ell-1)\ell^{-2\sigma}\Big) + \sum_{k\in\mathbb{Z}_*}\Re\big(c_{g_{n,\sigma}}(k)\big)\Big(\sum_{1\leq\ell\leq n}\ell^{-2\sigma}\varepsilon_k(\ell)\Big).$$

Whence

$$\begin{aligned} \int_n^{n+1} |\zeta(\sigma+it)|^2 \mathrm{d}t = \zeta(2\sigma) + 4c_{g_{n,\sigma}}(0)\Big(\sum_{1\leq\ell\leq n}(\ell-1)\ell^{-2\sigma}\Big) + \\ 4\sum_{k\in\mathbb{Z}_*}\Re\big(c_{g_{n,\sigma}}(k)\big)\Big(\sum_{1\leq\ell\leq n}\ell^{-2\sigma}\varepsilon_k(\ell)\Big) + \mathcal{O}_\sigma(n^{1-2\sigma}). \end{aligned}$$

As ([6, 863.4])

$$\int_0^\infty \frac{\sin t}{t} e^{-xt}\mathrm{d}t \;=\; \frac{\pi}{2} - \operatorname{Arctan} x \;=\; \operatorname{Arctan}\frac{1}{x} \qquad\qquad x > 0, \tag{3.6}$$

we have

$$c_{g_{n,\sigma}}(0) \;=\; \int_0^1 g_{n,\sigma}(t)\mathrm{d}t \;=\; \frac{\operatorname{Arctan}\frac{n+1}{1-\sigma} - \operatorname{Arctan}\frac{n}{1-\sigma}}{2} \sim \frac{C_\sigma}{n^2}.$$

Thus

$$c_{g_{n,\sigma}}(0)\Big(\sum_{1\leq\ell\leq n}(\ell-1)\ell^{-2\sigma}\Big) = \mathcal{O}_\sigma\big(n^{-2\sigma}\big).$$

Therefore

$$\int_n^{n+1} |\zeta(\sigma+it)|^2 \mathrm{d}t = \zeta(2\sigma) + 4\sum_{k\in\mathbb{Z}_*} \Re\big(c_{g_{n,\sigma}}(k)\big)\Big(\sum_{1\le\ell\le n} \ell^{-2\sigma}\varepsilon_k(\ell)\Big) + \mathcal{O}_\sigma\big(n^{1-2\sigma}\big),$$

as claimed.

Now we compute the Fourier coefficients of $g_{n,\sigma}$. We have

$$\begin{aligned} c_{g_{n,\sigma}}(k) &= \int_0^1 g_{n,\sigma}(x)e^{-2\mathrm{i}\pi kx}\mathrm{d}x \\ &= \sum_{j=0}^\infty \frac{(-2\mathrm{i}\pi k)^j}{j!}\int_0^1 \frac{\sin(n+1)\log x - \sin n\log x}{2x^\sigma \log x}\, x^j\,\mathrm{d}x \\ &= \sum_{j=0}^\infty \frac{(-2\mathrm{i}\pi k)^j}{j!}\int_0^\infty \Big(e^{-(\frac{j+1-\sigma}{n+1})v} - e^{-(\frac{j+1-\sigma}{n})v}\Big)\frac{\sin v}{2v}\mathrm{d}v \\ &= \sum_{j=0}^\infty \frac{(-2\mathrm{i}\pi k)^j}{j!}\Big(\frac{\mathrm{Arctan}\,\frac{n+1}{j+1-\sigma} - \mathrm{Arctan}\,\frac{n}{j+1-\sigma}}{2}\Big). \end{aligned}$$

Thus

$$\Re(c_{g_{n,\sigma}}(k)) = \sum_{m=0}^\infty \frac{(-1)^m(2\pi k)^{2m}}{(2m)!}\Big(\frac{\mathrm{Arctan}\,\frac{n+1}{2m+1-\sigma} - \mathrm{Arctan}\,\frac{n}{2m+1-\sigma}}{2}\Big). \tag{3.7}$$

This proves the first part of Theorem 1.1. The second part relies upon a study of Fourier coefficients $c_{g_{n,\sigma}}(k)$, and is a direct consequence of Corollary 3.3.

For each k, two sequences appear, both are in turn unimodal. Let for $m \ge 0$,

$$\begin{cases} \beta_m = \beta_m(k) = \frac{(2\pi k)^{2m}}{(2m)!}, \\ b_m = b_m(n) = \frac{\mathrm{Arctan}\frac{n+1}{2m+1-\sigma} - \mathrm{Arctan}\frac{n}{2m+1-\sigma}}{2}, \end{cases} \tag{3.8}$$

so that

$$\Re(c_{g_{n,\sigma}}(k)) = \sum_{m=0}^\infty (-1)^m \beta_m(k)\, b_m(n). \tag{3.9}$$

Theorem 3.1. *The following properties are satisfied.*

(i) $\lim_{M\to\infty}\sum_{j=1}^M (-1)^j\beta_j(k) = 0$.

(ii) *Let* $m(n,\sigma) = (\sigma + \sqrt{3n(n+1)/2+2}\,)/2$. *The sequence* $\{b_m(n), m \ge 0\}$ *is non-decreasing on* $[0, m(n,\sigma)[$ *and non-increasing on* $[m(n,\sigma),\infty)$.

(iii) *Let* $m_k = -3/4 + \sqrt{(\pi k)^2 - 1/2}$. *The sequence* $\{\beta_m(k), m \ge 1\}$ *is non-decreasing on* $[1, m_k]$ *and non-increasing on* $[m_k,\infty)$.

(iv) *Let* $\overline{m}(k,n,\sigma) = \max\big(m_k, m(n,\sigma)\big)$. *For* $\mu \ge \overline{m}(k,n,\sigma)$,

$$\Big|\sum_{m\ge\mu}(-1)^m\beta_m(k)b_m\Big| \le \beta_\mu b_\mu(n).$$

Remark 3.2. We have

$$\beta_{\lfloor m_k\rfloor}(k) = \frac{(2\pi k)^{2\lfloor m_k\rfloor}}{(2\lfloor m_k\rfloor)!} = \Big(\frac{\pi e k}{\lfloor m_k\rfloor}\Big)^{2\lfloor m_k\rfloor}\frac{e^{-\varepsilon_{2\lfloor m_k\rfloor}}}{2\sqrt{\lfloor m_k\rfloor\pi}} \sim\ C_0\frac{e^{2\lfloor m_k\rfloor}}{\sqrt{\lfloor m_k\rfloor}}.$$

Thus $\beta_{\lfloor m_k\rfloor}(k)$ is large, more precisely $\beta_{\lfloor m_k\rfloor}(k) \sim e^{C_1 k}$ where C_1 is numerical.

Proof. (i) is immediate since $\cos 2\pi k = \sum_{j=0}^{\infty}(-1)^j\beta_j(k)$.

(ii) Recall that [1, Formula 4.4.34],

$$\operatorname{Arctan} x \pm \operatorname{Arctan} y \ = \ \operatorname{Arctan}\frac{x\pm y}{1\mp xy}. \tag{3.10}$$

Thus if $n(n+1) < (2m-1-\sigma)^2$, namely

$$3n(n+1)+2 \le 8m^2 - 8\sigma m + 2\sigma^2,$$

we have

$$\begin{aligned}
\operatorname{Arctan}\frac{n+1}{2m+1-\sigma} - \operatorname{Arctan}\frac{n}{2m+1-\sigma} &= \operatorname{Arctan}\frac{2m+1-\sigma}{(2m+1-\sigma)^2+n(n+1)},\\
\operatorname{Arctan}\frac{n+1}{2m-1-\sigma} - \operatorname{Arctan}\frac{n}{2m-1-\sigma} &= \operatorname{Arctan}\frac{2m-1-\sigma}{(2m-1-\sigma)^2+n(n+1)}.
\end{aligned}$$

Next

$$\begin{aligned}
&\operatorname{Arctan}\frac{2m-1-\sigma}{(2m-1-\sigma)^2+n(n+1)} - \operatorname{Arctan}\frac{2m+1-\sigma}{(2m+1-\sigma)^2+n(n+1)}\\
=\ &\operatorname{Arctan}\Bigg[\Big(\frac{2m-1-\sigma}{(2m-1-\sigma)^2+n(n+1)} - \frac{2m+1-\sigma}{(2m+1-\sigma)^2+n(n+1)}\Big)\\
&\qquad\times\frac{1}{1+\frac{(2m-1-\sigma)(2m+1-\sigma)}{[(2m-1-\sigma)^2+n(n+1)]\,[(2m+1-\sigma)^2+n(n+1)]}}\Bigg].
\end{aligned}$$

We have

$$\begin{aligned}
&(2m-1-\sigma)\big[(2m+1-\sigma)^2+n(n+1)\big] - (2m+1-\sigma)\big[(2m-1-\sigma)^2+n(n+1)\big]\\
&= -3n(n+1)+8m^2-2(1+\sigma^2)-2\sigma(4m-2\sigma).
\end{aligned}$$

We get

$$\begin{aligned}
&\operatorname{Arctan}\frac{2m-1-\sigma}{(2m-1-\sigma)^2+n(n+1)} - \operatorname{Arctan}\frac{2m+1-\sigma}{(2m+1-\sigma)^2+n(n+1)}\\
=\ &\operatorname{Arctan}\Bigg[\Big(\frac{-3n(n+1)+8m^2-2(1+\sigma^2)-2\sigma(4m-2\sigma)}{\big[(2m-1-\sigma)^2+n(n+1)\big]\big[(2m+1-\sigma)^2+n(n+1)\big]}\Big)\\
&\qquad\times\frac{1}{1+\frac{(2m-1-\sigma)(2m+1-\sigma)}{[(2m-1-\sigma)^2+n(n+1)]\,[(2m+1-\sigma)^2+n(n+1)]}}\Bigg]\\
=\ &\operatorname{Arctan}\Big(\frac{A}{B}\Big),
\end{aligned}$$

where

$$\begin{aligned}
A &= 8m^2-8\sigma m+2\sigma^2-3n(n+1)-2\\
B &= \big[(2m-1-\sigma)^2+n(n+1)\big]\big[(2m+1-\sigma)^2+n(n+1)\big]\\
&\qquad + (2m-1-\sigma)(2m+1-\sigma).
\end{aligned}$$

Therefore

$$\mathrm{Arctan}\,\frac{2m-1-\sigma}{(2m-1-\sigma)^2+n(n+1)} \geq \mathrm{Arctan}\,\frac{2m+1-\sigma}{(2m+1-\sigma)^2+n(n+1)},$$

if and only if $A \geq 0$, and thus if and only if

$$m \geq \frac{\sigma+\sqrt{3n(n+1)/2+2}}{2} = m(n,\sigma).$$

Consequently the sequence $\{b_m(n), m \geq 0\}$ is non-decreasing on $[0, m(n,\sigma))$ and non-increasing on $[m(n,\sigma),\infty)$. (iii) We have,

$$\beta_m(k) \geq \beta_{m+1}(k) \Leftrightarrow \frac{(2\pi k)^{2m}}{(2m)!} \geq \frac{(2\pi k)^{2m+2}}{(2m+2)!} \Leftrightarrow (m+\frac{1}{2})(m+1) \geq (\pi k)^2,$$

or $m^2+\frac{3}{2}m-((\pi k)^2-\frac{1}{2}) \geq 0$, namely $m \geq m_k$.

(iv) Let $\overline{m}(k,n,\sigma) = \max\big(m_k, m(n,\sigma)\big)$. It follows from (ii) and (iii) that

$$\beta_m(k)\, b_m(n) \;\geq\; \beta_{m+1}(k)\, b_{m+1}(n), \tag{3.11}$$

for every $m \geq \overline{m}(k,n,\sigma)$. By using a classical bound for alternate series with monotone terms derived from Abel's summation, we get for all $\mu \geq \overline{m}(k,n,\sigma)$

$$\Big|\sum_{m\geq\mu}(-1)^m\beta_m(k)b_m(n)\Big| \leq \beta_\mu(k)b_\mu(n). \tag{3.12}$$

Whence also (iv). □

Corollary 3.3. *We have the following estimate*

$$\Big|\Re(c_{g_{n,\sigma}}(k)) - \sum_{m<m(n,\sigma)\vee\lceil\pi e k\rceil}(-1)^m\frac{(2\pi k)^{2m}}{(2m)!}\Big(\frac{\mathrm{Arctan}\,\frac{n+1}{2m+1-\sigma}-\mathrm{Arctan}\,\frac{n}{2m+1-\sigma}}{2}\Big)\Big| \leq \frac{C_\sigma}{k^{3/2}}.$$

Proof. Note the obvious bound

$$|b_m(n)| \leq \frac{1}{(m+(1-\sigma)/2)(1+(\frac{n}{2m+(1-\sigma)})^2)} \leq \frac{C_\sigma\, m}{m^2+n^2}. \tag{3.13}$$

It follows from (iv) that for $\mu \geq \overline{m}(k,n,\sigma)$,

$$\begin{aligned}
&\Big|\sum_{m\geq\mu}(-1)^m\frac{(2\pi k)^{2m}}{(2m)!}\,(\frac{\mathrm{Arctan}\,\frac{n+1}{2m+1-\sigma}-\mathrm{Arctan}\,\frac{n}{2m+1-\sigma}}{2})\Big| \\
\leq\;& \frac{(2\pi k)^{2\mu}}{(2\mu)!}\Big(\frac{\mathrm{Arctan}\,\frac{n+1}{2\mu+1-\sigma}-\mathrm{Arctan}\,\frac{n}{2\mu+1-\sigma}}{2}\Big) \\
\leq\;& \frac{(2\pi k)^{2\mu}}{(2\mu)!}\;\frac{1}{\mu+(1-\sigma)/2}\Big(\frac{1}{1+(\frac{n}{2\mu+(1-\sigma)})^2}\Big) \;\leq\; C_\sigma\;\frac{(2\pi k)^{2\mu}}{\mu(2\mu)!}.
\end{aligned}$$

By Stirling's formula,

$$n! = n^{n+\frac{1}{2}}e^{-n+\varepsilon_n}\sqrt{2\pi} \qquad \text{with} \qquad \frac{1}{12n+1} < \varepsilon_n < \frac{1}{12n}, \tag{3.14}$$

we have

$$\frac{(2\pi k)^{2\mu}}{(2\mu)!} = \Big(\frac{\pi e k}{\mu}\Big)^{2\mu}\,\frac{e^{-\varepsilon_{2\mu}}}{2\sqrt{\mu\pi}}. \tag{3.15}$$

Thus for every $\mu \geq \pi e k$, $(\pi e \sim 8,53973)$

$$\beta_\mu(k) = \frac{(2\pi k)^{2\mu}}{(2\mu)!} \;\leq\; \frac{C}{\sqrt{\mu}}. \tag{3.16}$$

Note that $\lceil \pi e k \rceil \geq m_k$. Taking $\mu = m(n,\sigma) \vee \lceil \pi e k \rceil$ we finally get

$$\Big| \sum_{m \geq m(n,\sigma)\vee\lceil \pi e k\rceil} (-1)^m \frac{(2\pi k)^{2m}}{(2m)!} \Big(\frac{\mathrm{Arctan}\,\frac{n+1}{2m+1-\sigma} - \mathrm{Arctan}\,\frac{n}{2m+1-\sigma}}{2} \Big) \Big| \;\leq\; \frac{C_\sigma}{k^{3/2}}\,. \tag{3.17}$$

□

Lemma 3.4. *Let M_0 be any integer such that $M_0 \geq m_n(\sigma)$, and let*

$$\delta(M_0) = \sup_{M \geq M_0} \Big| \sum_{j=1}^{M} (-1)^j \beta_j(k) \Big|. \tag{3.18}$$

Then we have,

$$\Big| \Re(c_{g_{n,\sigma}}(k)) - \sum_{m=1}^{M_0} \Big(\sum_{j=1}^{m} (-1)^j \beta_j(k) \Big) (b_m(n) - b_{m+1}(n)) \Big| \;\leq\; \delta(M_0)\, b_{M_0}(n).$$

Proof. Let $M \geq 1$ and let $A_0 = 0$, $A_m = \sum_{j=1}^{m} (-1)^j \beta_j(k)$, $m \geq 1$. By Abel summation,

$$\begin{aligned}
\sum_{m=1}^{M} (-1)^m \beta_m(k) b_m(n) &= \sum_{m=1}^{M} (A_m - A_{m-1}) b_m(n) \\
&= \sum_{m=1}^{M-1} A_m (b_m(n) - b_{m+1}(n)) + A_M b_M(n) \\
&= \sum_{m=1}^{M-1} \Big(\sum_{j=1}^{m} (-1)^j \beta_j(k) \Big) (b_m(n) - b_{m+1}(n)) \\
&\qquad + \Big(\sum_{j=1}^{M} (-1)^j \beta_j(k) \Big) b_M(n).
\end{aligned} \tag{3.19}$$

Now as $M_0 \geq m_n(\sigma)$, the sequence $\{b_m(n), m \geq M_0\}$ is by Theorem 3.1-(ii) non-increasing, so that for $M > M_0$

$$\begin{aligned}
&\Big|\sum_{m=M_0}^{M-1}\Big(\sum_{j=1}^{m}(-1)^j\beta_j(k)\Big)(b_m(n)-b_{m+1}(n))\Big| \\
&\quad\leq \sum_{m=M_0}^{M-1}\Big|\sum_{j=1}^{m}(-1)^j\beta_j(k)\Big|(b_m(n)-b_{m+1}(n)) \\
&\quad\leq \delta(M_0)\sum_{m=M_0}^{M-1}\big(b_m(n)-b_{m+1}(n)\big) \;=\; \delta(M_0)\big(b_{M_0}(n)-b_M(n)\big).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\Big|\sum_{m=1}^{M}(-1)^m\beta_m(k)b_m(n)-\sum_{m=1}^{M_0}\Big(\sum_{j=1}^{m}(-1)^j\beta_j(k)\Big)(b_m(n)-b_{m+1}(n))\Big| \\
&\qquad\leq\; \delta(M_0)\big(b_{M_0}(n)-b_M(n)\big)+\Big|\sum_{j=1}^{M}(-1)^j\beta_j(k)\Big|\,b_M(n). \qquad (3.20)
\end{aligned}$$

Letting M tend to infinity, gives

$$\Big|\Re(c_{g_{n,\sigma}}(k))-\sum_{m=1}^{M_0}\Big(\sum_{j=1}^{m}(-1)^j\beta_j(k)\Big)(b_m(n)-b_{m+1}(n))\Big| \;\leq\; \delta(M_0)\,b_{M_0}(n).$$

□

Remark 3.5. Theorem 3.1, Corollary 3.3 and Lemma 3.4 lead to study the sums

$$\sum_{k\in K_n}\Re\big(c_{g_{n,\sigma}}(k)\big)\Big(\sum_{1\leq\ell\leq n}\ell^{-2\sigma}\varepsilon_k(\ell)\Big),\qquad \sum_{k\in K^n}\Re\big(c_{g_{n,\sigma}}(k)\big)\Big(\sum_{1\leq\ell\leq n}\ell^{-2\sigma}\varepsilon_k(\ell)\Big),$$

where

$$K_n=\{k\,{:}\,m(n,\sigma)\geq\lceil\pi ek\rceil\},\qquad K^n=\{k\,{:}\,m(n,\sigma)<\lceil\pi ek\rceil\}.$$

This will be made in a subsequent work.

4. The $\mathcal{S}^2$-Stepanov norm of $\zeta(s)$.

We conclude this paper with relating the zeta local integrals previously investigated with the $\mathcal{S}^2$-Stepanov norm of ζ. The Stepanov space $\mathcal{S}^2$ is defined as the sub-space of functions f of $L^2_{\rm loc}(\mathbb{R})$ verifying the following analogue of Bohr almost periodicity property: *For all $\varepsilon>0$, there exists $K_\varepsilon>0$ such that for any $x_0\in\mathbb{R}$, there exists $\tau\in[x_0,x_0+K_\varepsilon]$ such that* $\|f(.+\tau)-f(.)\|_{\mathcal{S}^2}\leq\varepsilon$. The Stepanov norm in $\mathcal{S}^2$ is equivalent to the "amalgam" norm

$$\|f\|_{\mathcal{S}^2}=\sup_{x\in\mathbb{R}}\Big(\int_x^{x+1}|f(t)|^2{\rm d}t\Big)^{1/2}.$$

The almost everywhere convergence properties of almost periodic Fourier series in $\mathcal{S}^2$ and of corresponding series of dilates were recently studied in [5], where a new form of Carleson's theorem for almost periodic Fourier series was proved.

A natural question arising from this study concerns the Riemann zeta function $\zeta(s)$, $s = \sigma + it$, and more precisely the evaluation of its Stepanov's norm, namely the supremum over all n of the local integrals $\int_n^{n+1} |\zeta(\sigma+it)|^2 \mathrm{d}t$. It is clear from the classical mean value estimate $\int_0^T |\zeta(\frac{1}{2}+it)|^2 \mathrm{d}t \sim T \log T$, ([13] p. 176), that

Lemma 4.1.

$$\big\|\zeta\big(\frac{1}{2} + i\,\cdot\,\big)\big\|_{\mathcal{S}^2} \;=\; \infty.$$

Note that

$$\int_n^{n+1} \big|\zeta\big(\frac{1}{2}+it\big)\big|^2 \mathrm{d}t = \Omega\big(\log n\big). \tag{4.1}$$

It is interesting to observe that the above Ω-result is in a sense optimal when t is modelled by a Cauchy random walk. The behavior of the Riemann zeta-function on the critical line, along the Cauchy random walk, was studied in [9]. The approach used is based on the well-known, simplest and old approximation formula (3.2) due to Hardy and Littlewood which suffices to prove the fine estimates (4.3) below. Let $X_1, X_2, \ldots$ denote an infinite sequence of independent Cauchy distributed random variables (with characteristic function $\varphi(t) = e^{-|t|}$), and consider the partial sums $S_n = X_1 + \ldots + X_n$. We shall prove the following precise result.

Proposition 4.2. *We have*

$$\mathbb{E}\,\Big|\zeta\big(\frac{1}{2} + iS_n\big)\Big|^2 \;=\; \mathcal{O}(\log n).$$

Moreover,

$$\mathbb{E}\,\Big|\zeta\big(\frac{1}{2} + iS_{n+2}\big) - \zeta\big(\frac{1}{2} + iS_n\big)\Big|^2 \;=\; 2\,\log n + o(\sqrt{\log n}), \qquad n \to \infty.$$

Proof. Put for any positive integer n

$$\begin{aligned} \zeta_{1/2,n} &= \zeta(\frac{1}{2} + iS_n), \\ \mathcal{Z}_n &= \zeta_{1/2,n} - \mathbb{E}\,\zeta_{1/2,n}. \end{aligned} \tag{4.2}$$

By Theorem 1.1 in [9], there exist explicit constants C, C_0 such that

$$\begin{aligned} &\text{(i)} \quad \mathbb{E}\,\big|\mathcal{Z}_n\big|^2 \;=\; \log n + C + o(1), \qquad n \to \infty, \\ &\text{(ii)} \quad \text{For } m > n+1, \qquad \big|\mathbb{E}\;\mathcal{Z}_n\overline{\mathcal{Z}_m}\big| \;\le\; C_0 \max\Big(\frac{1}{n}, \frac{1}{2^{m-n}}\Big). \end{aligned} \tag{4.3}$$

Therefore,

$$\begin{aligned} \mathbb{E}\,\big|\mathcal{Z}_{n+2} - \mathcal{Z}_n\big|^2 &= \mathbb{E}\,\big|\mathcal{Z}_{n+2}\big|^2 + \mathbb{E}\,\big|\mathcal{Z}_n\big|^2 - 2\mathbb{E}\,\mathcal{Z}_n\overline{\mathcal{Z}_{n+2}} \\ &= \log(n+2) + \log n + o(1) - 2\mathbb{E}\,\mathcal{Z}_n\overline{\mathcal{Z}_{n+2}} \\ &= 2\log n + \mathcal{O}(1). \end{aligned} \tag{4.4}$$

By [9, (3.3)],

$$\mathbb{E}\,\zeta_{1/2,n} = \zeta(\frac{1}{2} + n) - \frac{8n}{4n^2 - 1} \;=\; 1 + o(1). \tag{4.5}$$

By combining with (4.3)-(i) the first claim follows. We also get

$$\begin{aligned}\mathbb{E}\left|(\zeta_{1/2,n+2}-\zeta_{1/2,n})\right|^2 &= \mathbb{E}\left|\mathcal{Z}_{n+2}-\mathcal{Z}_n+(\mathbb{E}\,\zeta_{1/2,n+2}-\mathbb{E}\,\zeta_{1/2,n})\right|^2\\ &= \mathbb{E}\left|\mathcal{Z}_{n+2}-\mathcal{Z}_n+o(1)\right|^2\\ &= 2\,\log n+o(\sqrt{\log n}),\end{aligned}$$

which proves the second claim. □

For $\frac{1}{2}<\sigma<1$, the simple argument used in Lemma 4.1 no longer works since by Landau and Schnee's result, $\int_0^T|\zeta(\sigma+it)|^2\mathrm{d}t\sim\zeta(2\sigma)T$ if $\sigma>\frac{1}{2}$; in fact ([11], p.13&16, see also Conjecture (5.15) on the remainder term)

$$\int_0^T|\zeta(\sigma+it)|^2\mathrm{d}t=\zeta(2\sigma)T+(2\pi)^{2\sigma-1}\frac{\zeta\big((2(1-\sigma)\big)}{2(1-\sigma)}T^{2(1-\sigma)}+\mathcal{O}\big(T^{\frac{2(1-\sigma)}{3}}(\log T)^{\frac{2}{9}}\big),\tag{4.6}$$

if $\frac{1}{2}<\sigma<1$. Thus

$$\int_n^{n+1}\left|\zeta(\sigma+it)\right|^2\mathrm{d}t\geq A,\tag{4.7}$$

for infinitely many n, with $A<\zeta(2\sigma)$.

Remark 4.3. We don't know whether (4.7) can be obtained using elementary devices.

The question arises whether A can be arbitrary large, namely whether

Problem 4.4. *It is true that for* $1/2<\sigma\leq 1$

$$\limsup_{n\to\infty}\int_n^{n+1}\left|\zeta(\sigma+it)\right|^2\mathrm{d}t=\infty\,?$$

The answer to Problem 4.4 is positive. This follows from the strong Voronin universality Theorem, see Th. 3 in Laurinčikas [10] for instance.

We also note that

$$\|\zeta(\sigma+i.)\|_{\mathcal{S}^2}\asymp\sup_{u\in\mathbb{R}}\int_{\mathbb{R}}\frac{|\zeta(\sigma+iv)|^2}{b^2+(v-u)^2}\mathrm{d}v.$$

More precisely, for any $\sigma>0$, any $b>0$,

$$k_b\,\|\zeta(\sigma+i.)\|_{\mathcal{S}^2}\;\leq\;\sup_{u\in\mathbb{R}}\int_{\mathbb{R}}\frac{|\zeta(\sigma+iv)|^2}{b^2+(v-u)^2}\mathrm{d}v\;\leq\;K_b\,\|\zeta(\sigma+i.)\|_{\mathcal{S}^2}^2,\tag{4.8}$$

where k_b,K_b are positive finite constants depending on b only. This follows from the Lemma below.

Lemma 4.5. *Let* $b>0$. *For any* $f\in L^2_{\mathrm{loc}}(\mathbb{R})$,

$$\|f\|_{\mathcal{S}^2}<\infty\quad\textit{if and only if}\quad\sup_{u\in\mathbb{R}}\int_{\mathbb{R}}|f(t+u)|^2\frac{\mathrm{d}t}{b^2+t^2}<\infty.$$

Moreover,

$$k_b\,\|f\|_{\mathcal{S}^2}\;\leq\;\sup_{u\in\mathbb{R}}\int_{\mathbb{R}}|f(t+u)|^2\frac{\mathrm{d}t}{b^2+t^2}\;\leq\;K_b\,\|f\|_{\mathcal{S}^2}^2,$$

where $k_b=\frac{1}{b^2+1}$ *and* $K_b=2\sum_{w\geq 0}\frac{1}{b^2+w^2}+\frac{1}{b^2}$.

Proof. On the one hand, for any real u,

$$\int_{\mathbb{R}} |f(t+u)|^2 \frac{\mathrm{d}t}{b^2+t^2} \;\geq\; \frac{1}{b^2+1}\int_0^1 |f(t+u)|^2 \mathrm{d}t \;=\; \frac{1}{b^2+1}\int_u^{u+1} |f(v)|^2 \mathrm{d}v.$$

Taking supremum over u in both sides yields

$$\sup_{u\in\mathbb{R}} \int_{\mathbb{R}} |f(t+u)|^2 \frac{\mathrm{d}t}{b^2+t^2} \;\geq\; \frac{1}{b^2+1}\,\|f\|_{\mathcal{S}^2}.$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}} |f(t+u)|^2 \frac{\mathrm{d}t}{b^2+t^2} \;&=\; \sum_{x\in\mathbb{Z}} \int_x^{x+1} |f(v)|^2 \frac{\mathrm{d}v}{b^2+(v-u)^2} \\ &\leq\; \|f\|_{\mathcal{S}^2}^2 \sum_{y\in\mathbb{Z}} \max_{y\leq v\leq y+1} \Big(\frac{1}{b^2+(v-\{u\})^2}\Big).\end{aligned}$$

□

Concerning the integrals appearing in (4.8), we recall the classical formula in Titchmarsh [13, (2.1.5)],

$$\frac{\zeta(s)}{s} = -\int_0^\infty \{x\} x^{-1-s}\,\mathrm{d}x = -\int_0^\infty \{\frac{1}{x}\} x^{s-1}\,\mathrm{d}x, \qquad 0<\sigma<1, \tag{4.9}$$

where $\{x\} = x - [x]$ denotes the fractional part of x. Thus by Parseval equality for Mellin's transform,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\big|\zeta(\sigma+it)\big|^2}{\sigma^2+t^2}\mathrm{d}t \;&=\; \int_0^\infty \{\frac{1}{x}\}^2 x^{2\sigma-1}\mathrm{d}x \\ &=\; -\frac{1}{\sigma}\Big(\frac{\zeta(2\sigma)}{2} + \frac{\zeta(2\sigma-1)}{2\sigma-1}\Big),\end{aligned} \tag{4.10}$$

for $0<\sigma<1$, the quantity in brackets being negative. For the first equality, see Ivić [8, Cor. 1]. The second equality was proved by Coffey [4, Prop. 7, Cor. 8],

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\big|\zeta(\frac{1}{2}+it)\big|^2}{\frac{1}{4}+t^2}\mathrm{d}t \;=\; \log(2\pi)-\gamma, \tag{4.11}$$

and we note that by Lemma 4.1,

$$\sup_{u\in\mathbb{R}} \int_{-\infty}^{+\infty} \frac{|\zeta(\frac{1}{2}+it)|^2}{\frac{1}{4}+(t-u)^2}\mathrm{d}t \;=\; \infty. \tag{4.12}$$

Remark 4.6. It is not clear to the author whether the integrals

$$\int_{-\infty}^{+\infty} \frac{|\zeta(\sigma+iv)|^2}{\sigma^2+(v-u)^2}\mathrm{d}v, \qquad (u\in\mathbb{R}) \tag{4.13}$$

can be similarly estimated. We further could not find any reference in the literature.

Remark 4.7. We don't know any proof of formula (4.3)-(i) using complex integration. Note that this one exactly means

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \big|\zeta(\frac{1}{2}+it)\big|^2 \frac{n}{n^2+t^2}\,\mathrm{d}t \;=\; \log n + C_1 + o(1), \tag{4.14}$$

The Stepanov space $\mathcal{S}^2$ is one instance of amalgam. We recall that the weighted amalgam $\ell^q(L^p, w)$ consists of functions f on $\mathbb{R}$ such that

$$\|f\|_{p,w,q} = \Big\{ \sum_{n\in\mathbb{Z}} \Big(\int_n^{n+1} |f(x)|^p w(x) \mathrm{d}x \Big)^{\frac{q}{p}} \Big\}^{\frac{1}{q}} < \infty, \tag{4.15}$$

where $1 < p, q < \infty$ and w is a weight function. We end with the following question where Voronin's universality theorem no longer applies.

Problem 4.8. *In which weighted amalgams $\ell^q(L^2, w)$ lies the Riemann zeta function?*

Acknowledgements. The author gratefully acknowledges Antanas Laurinčikas for his warm appreciation and his interest in this work.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Appl. Math. Series **55**, U.S. Government Printing Office, Washington, DC, (1964).

[2] R. Balasubramanian and K. Ramachandra, Proof of some conjectures on the mean-value of Titchmarsh series I, *Hardy-Ramanujan J.* **13** (1990), 1–20 ; II, ibid. 14 (1991) 1-20 ; III, Proc. Indian Acad. Sci. (Math. Sci.) 102 (1992) 83-91.

[3] N. K. Bary, (1964) *A Treatise on Trigonometric Series* Vol I&II, translation by M. F. Mullins, Pergamon Press, Oxford-London-Edimburgh-New York-Paris-Frankfurt.

[4] M. W. Coffey, (2011) Evaluation of some second moment and other integrals for the Riemann, Hurwitz, and Lerch zeta functions, arXiv:1101.5722v1.

[5] C. Cuny and M. Weber, (2018) Pointwise convergence of almost periodic Fourier series and associated series of dilates, *Pacific J. Math.* **292**, No. 1, 81–101.

[6] H. B. Dwight, (1961) *Tabels of Integrals and Other Mathematical Data*, MacMillan, New York.

[7] T. H. Gronwall, (1912) *Some asymptotic expressions in the theory of numbers*, Trans. Amer. Math. Soc. **8**, 118–122.

[8] A. Ivić, *Some identities for the Riemann zeta function II*, Facta Univ. 20, 1-8 (2005); arXiv:0506214v2 (2005).

[9] M. Lifshits and M. Weber (2009) Sampling the Lindelöf conjecture with the Cauchy random walk, *Proc. London Math. Soc.* **98** (3), 241–270.

[10] A. Laurinčikas, (2010) Universality of the Riemann zeta function, *J. Number Th.* **130**, 2323–2331.

[11] K. Matsumoto, (2000) Recent Developments in the Mean Square Theory of the Riemann Zeta and Other Zeta-Functions, *Number Theory*, 241–246, *Trends Math.*, Birkhäuser, Basel.

[12] A. Sankaranarayanan and K. Srinivas, (1993) Mean-value theorem of the Riemann zeta-function over short intervals, *J. Number Theory* **45**, 320–326.

[13] E. C. Titchmarsh, (1986) *The theory of the Riemann-Zeta function*, Second Edition, revised by Heath-Brown D. R., Oxford Science Publications.

[14] M. Weber, (2022) Composed quadratic sums related to the Riemann Zeta function, *Preprint*.

[15] M. Weber, (2022) On Farey Sequence and Quadratic Farey sums, *Research in Number Theory* **8:14**.

[16] M. Weber, (2009) *Dynamical Systems and Processes*, European Mathematical Society Publishing House, IRMA Lectures in Mathematics and Theoretical Physics **14**, (2009). xiii+759p.

IRMA, UMR 7501, 10 rue du Général Zimmer, 67084 Strasbourg Cedex, France
E-mail address: michel.weber@math.unistra.fr, m.j.g.weber@mailo.com