SL(∞, R), Higgs bundles, and quantization

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Abstract. Nigel Hitchin recently proposed a theory of $SL(\infty,\mathbb{R})$-Higgs bundles which should parametrize a Hitchin component of representations of surface groups into $SL(\infty,\mathbb{R})$. We discuss some properties and propose a formal approximation of $SL(\infty,\mathbb{R})$ representations by $SL(n,\mathbb{R})$ representations when n goes to infinity.

In 1992 Hitchin defined the 'Hitchin component' of the moduli space of representations of a surface group into $SL(n,\mathbb{R})$. That was the beginning of 'higher Teichmüller theory', which studies higher dimensional analogues of Teichmüller space.

Recently, Hitchin [6] proposed an interpretation of the Hitchin component for $SL(\infty,\mathbb{R})$, following the same kind of ideas: since the Hitchin component is defined by Higgs bundles, the idea is to understand what a $SL(\infty,\mathbb{R})$-Higgs bundle is. It turns out that there is a natural geometric interpretation in terms of the group of Hamiltonian diffeomorphisms of the 2-sphere S^2, and this leads to interpret points in the Hitchin component for $SL(\infty,\mathbb{R})$ as geometric structures on a symplectic 2-sphere bundle M^4 over the Riemann surface Σ, namely 'folded hyperKähler structures', that is hyperKähler metrics with singularities on a circle bundle $X^3 \subset M^4$.

In [2] I showed that, as expected, this Hitchin component is locally parametrized by

$$\oplus_{i\geq 2} H^0(\Sigma, K^i)$$

(some topology has to be taken in account here), which is natural from the point of view of the limit $SL(n,\mathbb{R}) \to SL(\infty,\mathbb{R})$, since the Hitchin component for $SL(n,\mathbb{R})$ is parametrized by

$$\oplus_{i\geq 2}^{n} H^0(\Sigma, K^i).$$

The aim of this article is to study how one can approximate a point in the Hitchin component for $SL(\infty,\mathbb{R})$ by a sequence in the Hitchin

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components for $SL(n,\mathbb{R})$ when $n \to \infty$. We prove that there exists a formal solution to this problem. This is the first step to produce a genuine approximation—a problem that we will study elsewhere. Our quantization technique has some similarities with Donaldson's quantization for constant scalar curvature Kähler metrics [3], but here the quantization is done fibrewise, and the result is a solution of a geometric PDE rather than a solution of algebraic equations.

In § 1–2, we recall some basis on the Hitchin component and on the setting for $n = \infty$. In § 3 we describe our quantization procedure, and the formal solution is produced in § 4, but we avoid the most technical details coming from the machinery developed in [2].

1. Hitchin's component for $SL(n,\mathbb{R})$

Let Σ be a surface of genus $g \geq 2$. The Teichmüller space of Σ can be seen in several equivalent ways:

- the space of complex structures on Σ;
- the space of hyperbolic metrics on Σ;
- the space of discrete faithful representations $\pi_1(\Sigma) \to PSL(2,\mathbb{R})$ modulo conjugations.

It is well-known that Teichmüller space is homeomorphic to $\mathbb{R}^{6g-6}$. One construction of this isomorphism is the following. Suppose one fixes a reference hyperbolic metric g_0 on Σ (hence a complex structure), and g is another hyperbolic metric, then it follows from the Eells-Sampson theorem that there exists a unique harmonic diffeomorphism $\phi : (\Sigma, g_0) \to (\Sigma, g)$ which minimizes the energy among maps which are homotopic to the identity. Then $(\phi^* g)^{2,0} =: q$ is a quadratic differential: $q \in H^0(\Sigma, K^2)$. The map

$$g \longmapsto q$$

gives an identification of Teichmüller space with the vector space $H^0(\Sigma, K^2)$, whose complex dimension is $3g-3$. Starting from a holomorphic quadratic differential q, one can construct a corresponding g by solving some geometric PDE, which is a special case of Hitchin selfduality equations [4].

These equations enabled Hitchin [5] to prove a correspondence between:

- the component (now called Hitchin component) of the space of representations $\pi_1(\Sigma) \to SL(n,\mathbb{R})$ consisting of deformations of the representation

$$\pi_1(\Sigma) \to SL(2,\mathbb{R}) \hookrightarrow SL(n,\mathbb{R}),$$

  where the monodromy representation of a hyperbolic metric on Σ is followed by the irreducible representation of $SL(2,\mathbb{R})$ in $\mathbb{R}^n$;
- a moduli space of $SL(n,\mathbb{R})$-Higgs bundles, which we will now describe.

Let us define G-Higgs bundles for a real semisimple group G. Let $H \subset G$ be a maximal compact subgroup, and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ the Cartan decomposition, with Cartan involution θ. Then a G-Higgs bundle on the Riemann surface Σ is a pair (E,Φ), where E is a principal holomorphic $H^{\mathbb{C}}$ bundle and $\Phi \in H^0(\Sigma, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$. The Higgs bundles occurring in the Hitchin component have the following form:

- the holomorphic bundle is fixed,

$$E = K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{-\frac{n-1}{2}};$$

  this has a canonical quadratic form and is therefore a $SO(n,\mathbb{C})$ bundle;
- the Higgs field has the form

$$\Phi = \begin{pmatrix} 0 & q_2 & \cdots & q_n \\ 1 & 0 & \ddots & \vdots \\ & \ddots & \ddots & q_2 \\ & & 1 & 0 \end{pmatrix}, \quad q_i \in H^0(\Sigma, K^i).$$

  This formula defines a holomorphic 1-form with values in $E(\mathfrak{m}^{\mathbb{C}})$, the bundle of symmetric endomorphisms of E.

In particular, the Hitchin component is parametrized by the space

$$\oplus_{i=2}^{n} H^0(\Sigma, K^i), \tag{1}$$

and the q_i is calculated from the Higgs bundle by $q_i = \mathrm{Tr}(\Phi^i)$.

Finally let us recall how such Higgs bundles parametrize representations of fundamental groups. A metric h on E (that is, a reduction of the structural group of E to H) enables to define a G-connection on E by

$$D_h = A_h + \Phi - \theta_h(\Phi),$$

where A_h is the Chern connection of (E,h). One wants to solve the equation

$$F(A_h) = [\Phi, \theta_h(\Phi)],$$

which together with $\bar\partial\Phi = 0$, implies that D_h is flat, hence provides the representation $\pi_1(\Sigma) \to G$. The selfduality equations of Hitchin alluded to above are therefore the set of equations

$$\begin{aligned} \bar\partial\Phi &= 0, \\ F(A_h) &= [\Phi, \theta_h(\Phi)]. \end{aligned}$$

2. $SL(\infty,\mathbb{R})$-Higgs bundles and hyperKähler metrics

Let us see what one can do with $G = SL(\infty,\mathbb{R})$. To define a G-Higgs bundle, we actually need only an interpretation of the summands of the Cartan decomposition, that is an interpretation of $H = SO(\infty,\mathbb{R})$ and $\mathfrak{m}^{\mathbb{C}}$.

There is a well-known interpretation of $SU(\infty)$ as the group of Hamiltonian diffeomorphisms of the 2-sphere S^2. We will see in § 3 that in a precise sense $SU(\infty)$ is indeed a limit of the groups $SU(n)$ when $n \to \infty$. Therefore we can consider $\mathfrak{su}(\infty)$ as the space of functions on S^2 with zero integral. Then consider the equatorial symmetry ι on S^2: this reverses the symplectic form on S^2, so we can define a symmetric decomposition

$$\mathfrak{su}(\infty) = \mathfrak{so}(\infty,\mathbb{R}) \oplus \mathfrak{m},$$

with

$$\begin{aligned} \mathfrak{so}(\infty,\mathbb{R}) &= \{f \in \mathfrak{su}(\infty), \iota^* f = -f\}, \\ \mathfrak{m} &= \{f \in \mathfrak{su}(\infty), \iota^* f = f\}. \end{aligned}$$

In standard coordinates x_0, x_1, x_2 on S^2, we have

$$\iota(x_0,x_1,x_2) = (-x_0,x_1,x_2),$$

and their Poisson brackets satisfy the standard relations $\{x_0,x_1\} = x_2$, etc. so we have a copy of $\mathfrak{su}(2,\mathbb{R}) \hookrightarrow \mathfrak{su}(\infty,\mathbb{R})$, with x_0 generating a $\mathfrak{so}(2,\mathbb{R}) \hookrightarrow \mathfrak{su}(\infty,\mathbb{R})$.

Now, to define what a solution of the selfduality equations for the group $SL(\infty,\mathbb{R})$ is, we need:

- a H-bundle: this can be realized as a symplectic 2-sphere bundle

  $$S^2 \to M^4 \to \Sigma,$$

  with an involution ι of X which acts as the equatorial symmetry in each fibre;
- a H-connection: this is a symplectic connection on $M \to \Sigma$;

- a Higgs field: this is a (1,0)-form on Σ, with values in $E(\mathfrak{m}^{\mathbb{C}})$, that is ι-invariant complex functions on the fibre S^2.

We write down the equations in a local trivialization $\mathbb{C}\times S^2$ of the bundle X, where the 2-sphere has the standard symplectic form ω_{S^2}: let $z=x+iy$ be a holomorphic coordinate on $\mathbb{C}$, then the H-connection and the Higgs field are given by

$$A = a_0 dx + a_1 dy, \qquad \Phi = \frac{1}{2}(a_2 + ia_3)dz,$$

where $a_0, a_1 \in \mathfrak{so}(\infty,\mathbb{R})$ and $a_2, a_3 \in \mathfrak{m}$. The selfduality equations are now:

$$\begin{aligned} \partial_x a_1 - \partial_y a_0 + \{a_0, a_1\} + \{a_2, a_3\} &= 0, \\ \partial_{\bar{z}}(a_2 + ia_3) + \frac{1}{2}\{a_0 + ia_1, a_2 + ia_3\} &= 0, \end{aligned} \tag{2}$$

where the bracket is the fibrewise Poisson bracket of functions. Hitchin observed [6] that if we consider the a_i as functions defined on the total space $\mathbb{C}\times S^2$ and define

$$\begin{aligned} \omega_1 &= \omega_{S^2} - da_0 \wedge dx - da_1 \wedge dy, \\ \omega_c &= \omega_2 + i\omega_3 = d(a_2 + ia_3) \wedge dz, \end{aligned} \tag{3}$$

then the system (2) is equivalent to the fact that $(\omega_1, \omega_2, \omega_3)$ is a hyperKähler triple: the three 2-forms ω_i automatically satisfy $d\omega_i = 0$, and also the orthogonality relations

$$\omega_i \wedge \omega_j = \delta_{ij}\{a_2, a_3\}\omega_{S^2} \wedge i dz \wedge d\bar{z}. \tag{4}$$

This gives a hyperKähler metric on the locus where the volume form ω_i^2 does not degenerate, that is on the locus $\{a_2, a_3\} \neq 0$.

Here we have written the equations in local coordinates, but everything makes sense globally on the S^2 bundle $M \to \Sigma$, so we obtain a hyperKähler metric on the total space M, nondegenerate on the locus $\{a_2, a_3\} \neq 0$. Observe that $\{a_2, a_3\}$ takes values in $\mathfrak{m}$, and therefore satisfies $\iota^*\{a_2, a_3\} = -\{a_2, a_3\}$, so $\{a_2, a_3\}$ must vanish at least on the fixed point set of ι, which is a circle bundle $S^1 \to X^3 \to \Sigma$ inside the S^2 bundle $M^4 \to \Sigma$ (the equator in each sphere). The hyperKähler metric g degenerates along X in a special way and satisfies $\iota^* g = -g$, it is therefore positive on one side of X and negative on the other side. If we choose a defining function x of X, then for a certain coframe $(\theta^1, \theta^2, \theta^3)$ on X one has the behaviour near X:

$$g \sim x(dx^2 + (\theta^2)^2 + (\theta^3)^2) + x^{-1}(\theta^1)^2.$$

This kind of metric is called a folded hyperKähler metric, see [6] for details.

One important fact is that one can think of a folded hyperKähler metric in two equivalent ways:

- a ι-anti-invariant hyperKähler metric on M;
- a hyperKähler metric only on one side of X in M, say the domain $D \subset M$ where g is positive.

The domain D may be compactified in a different way. Actually, the formula $\omega_c = (a_2 + ia_3)dz = 2d\Phi$ expresses ω_c as the symplectic form of the cotangent bundle $T^*\Sigma$, with $2\Phi = (a_2 + ia_3)dz$ as the Liouville form. Therefore D can be seen as a domain in the holomorphic cotangent bundle $T^*\Sigma$.

Example. A $SL(2,\mathbb{R})$-bundle leads to a $SL(\infty,\mathbb{R})$-bundle, thanks to the embedding $\mathfrak{su}(2) \hookrightarrow \mathfrak{su}(\infty)$. In particular, the standard solution for the hyperbolic metric g_0 leads to a well-known S^1-invariant folded hyperKähler metric on the disc bundle of $T^*\Sigma$. This metric can be seen as the noncompact dual of the Eguchi-Hanson metric, and we will refer to it as the 'standard model'.

We can now define the Hitchin component for the group $SL(\infty,\mathbb{R})$ as the space of triples (D, ω_1, ω_c), where

- (D, ω_c) is a holomorphic symplectic domain of $T^*\Sigma$,
- ω_1 is a folded hyperKähler metric on (D, ω_c), with fixed cohomology class in $H^2(D, \partial D)$.

In [2] we constructed the Hitchin component near the standard model:

Theorem 1. *Near the standard model, the Hitchin's component for $SL(\infty,\mathbb{R})$ is parametrized by $\oplus_{n\geq 2} H^0(\Sigma, K^n)$.*

This indeed corresponds to the limit $n \to \infty$ in (1). The statement in [2] is more precise, and the sum is to take with respect to some topology.

There is a nice infinitesimal description of the cotangent domains which occur in the Hitchin component, near the standard model D_0 (the disc bundle). The infinitesimal deformations of D are of the form $fu\partial_u$, where f is a real function on ∂D_0 and $u\partial_u$ is the homothety vector field in the fibres. It turns out that the functions which occur for the domains D in theorem 1 are the real parts of the CR holomorphic functions on ∂D_0, and the decomposition into Fourier series on ∂D_0 gives the sum in the theorem (the functions corresponding to $H^0(\Sigma,\mathbb{C}) = \mathbb{C}$ do not preserve the cohomology class of ω_1, and those corresponding to $H^1(\Sigma, K)$ actually give trivial deformations).

3. Quantization

We now explain how one can say that the group $SU(\infty)$ is the limit of the groups $SU(n)$ when n goes to infinity. This relies on the Berezin quantization [1], see for example [7, Chapter 7]. We need an auxiliary complex structure on the symplectic 2-sphere, that is we consider S^2 as $\mathbb{C}P^1$. Then

$$E_n = H^0(\mathbb{C}P^1, \mathcal{O}(n)) \simeq \mathbb{C}^{n+1}.$$

For $f \in C^\infty(\mathbb{C}P^1, \mathbb{R})$ define the Toeplitz operator $T_{f,n} \in i\mathfrak{u}(E_n)$ by

$$T_{f,n}s = \pi_{E_n}(fs) \quad \text{for } s \in V_n,$$

where π_{E_n} is the orthogonal projection on E_n. Then, for smooth functions f, g, there is an infinite development

$$[iT_{f,n}, iT_{g,n}] \sim \frac{1}{n} iT_{\{f,g\},n} + \frac{1}{n^2} T_{P_2(f,g),n} + \frac{1}{n^3} T_{P_3(f,g),n} + \cdots \tag{5}$$

where the $P_k(f,g)$ are bi-differential operators (and $P_1(f,g) = i\{f,g\}$). The development means that for every j one has, with respect to the operator norm,

$$\big\| [iT_{f,n}, iT_{g,n}] - \sum_{k=1}^{j} \frac{1}{n^k} T_{P_k(f,g),n} \big\| = O\big(\frac{1}{n^{j+1}}\big).$$

The meaning of (5) is clear: the map $f \mapsto inT_{f,n}$ is asymptotically a Lie algebra morphism from $\mathfrak{su}(\infty)$ to $\mathfrak{su}(n+1)$. We will see in § 4 that an asymptotic inverse of this map can be constructed, making precise the statement that $\mathfrak{su}(\infty) = \lim_{n\to\infty} \mathfrak{su}(n)$.

Now come back to the Hitchin component for the group $SL(\infty,\mathbb{R})$. A point in the Hitchin component is described by a solution of the $SL(\infty,\mathbb{R})$-selfduality equations on the symplectic bundle

$$\begin{array}{ccc} S^2 & \longrightarrow & M^4 \\ & & \downarrow p \\ & & \Sigma \end{array}$$

In order to approximate the Hitchin component for $SL(\infty,\mathbb{R})$ by the Hitchin components for $SL(n,\mathbb{R})$, we want to relate such a solution with a solution of the $SL(n,\mathbb{R})$-selfduality equations. The idea is to apply the above described quantization fibrewise on M. As for usual quantization, we need an auxiliary complex structure: we describe M as the complex ruled surface

$$M = P(K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}),$$

and we denote by $\Sigma_0, \Sigma_\infty \subset M$ the zero and infinity sections. We introduce the holomorphic line bundle $L \to M$ defined by

$$L|_{\text{fibre}} = \mathcal{O}(1), \quad L|_{\Sigma_0} = K^{-\frac{1}{2}}, \quad L|_{\Sigma_\infty} = K^{\frac{1}{2}}.$$

The vector space E_n now becomes a $\mathbb{C}^{n+1}$-holomorphic bundle over Σ defined by

$$E_n = p_* L^n = K^{-\frac{n}{2}} \oplus K^{-\frac{n}{2}+1} \oplus \cdots \oplus K^{\frac{n}{2}}. \tag{6}$$

The equatorial symmetry $\iota(z) = 1/\bar{z}$ on $\mathbb{C}P^1$ lifts to an anti-holomorphic involution ι_1 of the total space of the bundle $\mathcal{O}(1)$. This gives a real structure τ on $H^0(\mathbb{C}P^1, \mathcal{O}(n))$ defined by $\tau(s) = \iota_1 \circ s \circ \iota$. Combining with the natural Hermitian metric h on $H^0(\mathbb{C}P^1, \mathcal{O}(n))$, we obtain on this space a complex quadratic form

$$q(s) = h(\tau s, s).$$

All these algebraic considerations make sense fibrewise on M, and we obtain on E_n a holomorphic quadratic form, making E_n a $SO(n+1, \mathbb{C})$-bundle. Of course, this is the standard quadratic form that one can write on E_n from the explicit form (6), but it is useful to derive q from the involution ι. In particular, if a complex function f on M satisfies $\iota^* f = \pm f$, then $T_{f,n}$ is symmetric (resp. antisymmetric). Indeed, for any $s, t \in E_n$,

$$\begin{aligned} q(T_{f,n}s, t) &= h(\tau f s, t) \\ &= \pm h(\bar{f}\tau s, t) \\ &= \pm h(\tau s, f t) \\ &= \pm q(s, f t) = \pm q(s, T_{f,n} t). \end{aligned}$$

Now let us add in the picture a solution $(\omega_1, \omega_c = 2d\Phi)$ of the $SL(\infty, \mathbb{R})$-selfduality equations. Given a vector field X on Σ, we define a horizontal lift $\tilde{X}$ of X on M by asking that

$$p_* \tilde{X} = X, \qquad \tilde{X} \perp_{\omega_1} \text{fibres}.$$

We also choose a Hermitian connection ∇ on L such that $iF(\nabla) = \omega_1$. We will require the following normalization:

(1) along each fibre S^2, the $\bar{\partial}$-operator $\nabla^{0,1}$ coincides with the holomorphic structure of L along the fibre;
(2) ∇ is compatible with the real structure τ, that is, for any section of L,

$$\nabla(\tau s) = \tau \nabla s.$$

Suppose we have another connection with the same curvature, $\nabla+i\alpha$, with α a closed 1-form on M, then the two above conditions imply respectively:

(1) α vanishes on the fibres; therefore $\alpha = p^*\alpha_0$, where $\alpha_0 \in \Omega^1(\Sigma,\mathbb{R})$ is closed;
(2) for any section s, one should have $i\alpha\tau s = \tau i\alpha s$, but since $\alpha = p^*\alpha_0$ one has $\tau i\alpha s = -i\alpha_0\tau s$ so one must have $\alpha_0 = 0$.

Our normalization therefore defines ∇ canonically. The existence is obvious in a local trivialization of (M,L) over a disk $\Delta \subset \Sigma$, such that all fibres (S^2,ω_{S^2},L) are identified (this cannot be done in natural holomorphic cotangent coordinates, since the metric on Σ is curved): in such a trivialization, if ω_1 is given by the formula (3), then

$$\nabla = \nabla_{S^2} + i(a_0dx + a_1dy).$$

We can now proceed with the quantization: we define

- a connection ∇_n on E_n by $\nabla_{X,n}s = \pi_{E_n}(\nabla_{\tilde{X}}s)$; the above compatibility $\nabla\tau = 0$ implies that the coefficients of ∇_n are antisymmetric, so ∇_n is a $SO(n+1,\mathbb{R})$-connection; this can be seen simply in a local trivialization of M as above, since we then obtain

$$\nabla_n = d + inT_{a_0+\frac{\Delta a_0}{2n},n}dx + inT_{a_1+\frac{\Delta a_1}{2n},n}dy; \tag{7}$$

  the lower order terms $\frac{\Delta a_0}{2n}$ come from the fact that the horizontal lifting of the vector field ∂_x is $\widetilde{\partial_x} = \partial_x - \xi_{a_0}$, where ξ_f is the vertical Hamiltonian vector field associated to a function f; and from the application of Tuynman's formula

$$\pi_{E_n}(\nabla_{\xi_f}s) = -iT_{\frac{\Delta f}{2},n}s$$

  for any $s \in E_n$ (the proof is a simple integration by parts);
- a Higgs field Φ_n on E_n by $\Phi_n = nT_{\Phi,n}$; since Φ is a section of p^*K, it follows that Φ_n is a section of $K \otimes \mathrm{Sym}(E_n)$, given locally by the formula $\frac{n}{2}T_{a_2+ia_3,n}dz$.

Theorem 2. *The triple (E_n,∇_n,Φ_n) is an asymptotic solution of the $SL(n+1,\mathbb{R})$-selfduality equations.*

The meaning of 'asymptotic' is made precise by the following lemma, which implies the theorem.

Lemma 1. *Suppose given (∇,Φ) on (M,L), not necessarily satisfying the selfduality equations. Then there is a full asymptotic expansion of the following quantities in powers of $\frac{1}{n}$ when $n \to \infty$, with leading terms given*

by:

$$\frac{1}{n}\nabla_{\partial_{\bar z},n}\Phi_n \sim T_{\widetilde{\partial}_{\bar z}\Phi,n} + \cdots,$$

$$\frac{1}{n}\big(F(\nabla_n) - [\Phi_n, \theta(\Phi_n)]\big)_{\partial_x,\partial_y} \sim iT_{\partial_x a_1 - \partial_y a_0 + \{a_0,a_1\}+\{a_2,a_3\},n} + \cdots.$$

Since $\widetilde{\partial}_{\bar z} = \partial_{\bar z} - \frac{1}{2}\xi_{a_0+ia_1}$, one has

$$\widetilde{\partial}_{\bar z}\Phi = \partial_{\bar z}\Phi + \frac{1}{2}\{a_0 + ia_1, \Phi\}$$

so the two leading terms of the expansions in the lemma cancel exactly when (∇, Φ) is a solution of the selfduality equations. This shows how the lemma implies the theorem.

The lemma is a consequence of the following formulas, which are contained in a forthcoming article of Xiaonan Ma and Weiping Zhang:

Lemma 2. *One has the full asymptotic expansions in powers of $\frac{1}{n}$:*

$$\nabla_{X,n}T_{f,n} \sim T_{\tilde X\cdot f,n} + \cdots$$

$$\frac{1}{n}F(\nabla_n)_{X,Y} \sim T_{F(L)_{\tilde X,\tilde Y},n} + \cdots$$

In our case the formula for $F(\nabla_n)$ is immediate from the formula (7) and the expansion (5). The first formula is more difficult and we refer the reader to Ma-Zhang.

Let us come back to lemma 1: the first formula of the lemma now follows. For the second formula, we write

$$\begin{aligned} F(L)_{\widetilde{\partial}_x,\widetilde{\partial}_y} &= -i\omega_1(\widetilde{\partial}_x, \widetilde{\partial}_y) \\ &= -i\omega_1(\partial_x - \xi_{a_0}, \partial_y - \xi_{a_1}) \\ &= i(\partial_x a_1 - \partial_y a_0 + \{a_0, a_1\}). \end{aligned}$$

On the other hand from (5) one has

$$-[\Phi_n, \theta(\Phi_n)]_{\partial_x,\partial_y} = -n^2[T_{\Phi,n}, T_{\overline{\Phi},n}] = -inT_{\{\Phi,\overline{\Phi}\},n} + \cdots$$

and

$$\{\Phi, \overline{\Phi}\} = \{a_2 + ia_3, a_2 - ia_3\}\frac{1}{4}dz\wedge d\bar z = -\{a_2, a_3\}dx\wedge dy.$$

The second formula in the lemma follows.

4. Formal approximation

We have seen that, as a consequence of the existence of the asymptotic Lie algebra morphism $\mathfrak{su}(\infty) \to \mathfrak{su}(n+1)$, a point (∇, Φ) in the $SL(\infty, \mathbb{R})$-Hitchin component gives rise to an asymptotic solution (∇_n, Φ_n) of the $SL(n+1, \mathbb{R})$ selfduality equations.

We would like to refine this to an exact solution of the selfduality equations. Here we need another ingredient from quantization which shows that $\mathfrak{su}(n+1)$ actually approximates $\mathfrak{su}(\infty)$. The Berezin symbol $\sigma_n : i\mathfrak{su}(n+1) \to \mathfrak{su}(\infty)$ is the adjoint of the map

$$T_{\cdot,n} : \mathfrak{su}(\infty) \to \mathfrak{su}(n+1). \tag{8}$$

It can be defined for $x \in S^2$ by

$$\sigma_n(A)(x) = \mathrm{Tr}(\pi_{i_n(x)} \circ A),$$

where i_n is the natural embedding $i_n : \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^n$ and $\pi_{i_n(x)}$ is the orthogonal projection on the line $i_n(x) \subset \mathbb{C}^{n+1}$. The important fact is that σ_n gives an asymptotic inverse of the Toeplitz map (8): for any $f \in C^\infty(S^2)$,

$$\sigma_n(T_{f,n}) \sim f + \sum_{k\geq 1} \frac{1}{n^k} S_k(f), \tag{9}$$

where the S_k are differential operators (in particular $S_1 = \Delta$).

Now let us emphasize the dependence of $(\nabla = d + a_0 dx + a_1 dy, \Phi = \frac{1}{2}(a_2 + ia_3)dz)$ to a by writing $(\nabla_n(a), \Phi_n(a))$, and note $(\nabla_n(a), \Phi_n(a))$ the resulting quantized $SL(n+1,\mathbb{R})$-Higgs bundle.

The Berezin symbol is injective (equivalently, the Toeplitz map is surjective). Therefore the selfduality equations for $(\nabla_n(a), \Phi_n(a))$ can be written

$$\begin{aligned} P_1(n,a) &:= \frac{1}{n}\sigma_n\big(F(\nabla_n(a)) - [\Phi_n(a), \theta(\Phi_n(a))]\big) = 0, \\ P_c(n,a) &:= \frac{1}{n}\sigma_n\big(\nabla_n(a)^{0,1}\Phi_n(a)\big) = 0. \end{aligned} \tag{10}$$

From lemma 1 and the expansion (9), it follows that there is an asymptotic development of $P(n,a) = (P_1(n,a), P_c(n,a))$ given by

$$\begin{aligned} P_1(n,a) &\sim i\big(\partial_x a_1 - \partial_y a_0 + \{a_0, a_1\} + \{a_2, a_3\}\big)dx \wedge dy + \cdots \\ P_c(n,a) &\sim \big(\partial_{\bar z}(a_2 + ia_3) + \frac{1}{2}\{a_0 + ia_1, a_2 + ia_3\}\big)d\bar z + \cdots \end{aligned} \tag{11}$$

For convenience, we have written in a local trivialization, but the leading terms in the two equations can be more intrinsically interpreted as $\omega_1(a)^2 - \frac{1}{2}\omega_c(a)\wedge\bar\omega_c(a)$ and $\omega_1(a)\wedge\omega_c(a)$ respectively, which together form the hyperKähler equations.

Theorem 3. *Given $a^{(0)}$ any solution to the $SL(\infty,\mathbb{R})$ selfduality equations, there is a formal solution $a = \sum_{k\geq 0} \frac{a^{(k)}}{n^k}$ to the equations (10).*

Let us sketch the proof of the theorem: consider the variable $\hbar = \frac{1}{n}$, we can then consider P as a function of $\hbar$ and a, with $\hbar = 0$ corresponding to $n \to \infty$. Then:

- if a is a solution of the $SL(\infty,\mathbb{R})$-selfduality equations, then $P(0,a)=0$;
- the differential $\frac{\partial P}{\partial a}(0,a)$ is the linearisation of the hyperKähler equations, which is surjective by [2].

The existence of formal developments of $P(\hbar,a)$ in powers of $\hbar$ then ensures that one can apply formally the implicit function theorem to the equation $P(\hbar,a)=0$ to find solutions for nonzero $\hbar$.

We do not give more details on the proof of theorem 3, since the technical details require the machinery developed in [2] to analyse the folded hyperKähler cotangent domains.

Conjectural considerations. The next step is to pass from a formal solution to a genuine solution of the equations. One way to do that is to truncate the formal solution at some order, and then deform to a true solution—the model of such a method being Donaldson's quantization for the problem of constant scalar curvature Kähler metrics [3]. We will deal with this problem in another article.

There are some conjectural considerations that one may deduce from such an approximation map. Let us call $\mathcal{H}_n$ the $SL(n,\mathbb{R})$ Hitchin component, and $\mathcal{H}_\infty$ that for $SL(\infty,\mathbb{R})$. We know that $\mathcal{H}_n$ is parametrized by $B_n=\oplus_{i=2}^n H^0(\Sigma,K^i)$, and the 'Hitchin map' $\mathcal{H}_n\to B_n$ is given by

$$(E,\Phi)\mapsto (p_i(\Phi)=\mathrm{Tr}(\Phi^i))_{i=2,\dots,n}.$$

On the other hand, the Hitchin map for $\mathcal{H}_\infty$ is given by

$$(\nabla,\Phi)\longmapsto \big(q_i(\Phi)=\int_{M/\Sigma}\Phi^i\omega_1\big)_{i\geq 2}.$$

Hitchin [6] showed that this defines an element of $B_\infty=\oplus_{i\geq 2}H^0(\Sigma,K^i)$. Here we do not give any precise topology on the sum.

It is almost obvious from the definitions that one has, for any function f on S^2,

$$\frac{1}{n+1}\,\mathrm{Tr}(T_{f,n})=\int_{S^2} f\omega_{S^2}.$$

There is also a full asymptotic expansion

$$T_{f,n}\circ T_{g,n}\sim T_{fg,n}+\cdots$$

It follows that

$$\begin{aligned}\frac{1}{n+1}\,\mathrm{Tr}\,\big((\frac{\Phi_n}{n})^i\big)&=\frac{1}{n+1}\,\mathrm{Tr}\,T^i_{\Phi,n}\\ &\sim \frac{1}{n+1}\,\mathrm{Tr}\,T_{\Phi^i,n}\longrightarrow\int_{M/\Sigma}\Phi^i\omega_1,\end{aligned}$$

that is,

$$\frac{1}{n^{i+1}} p_i(\Phi_n) \longrightarrow q_i(\Phi). \tag{12}$$

Now suppose that we were able to define a genuine approximation map $\mathbf{a}_n : \mathcal{H}_\infty \to \mathcal{H}_n$. Then it should follow that, with the normalization (12), we also have a convergence of the Hitchin maps:

$$\begin{array}{ccc} \mathcal{H}_\infty & \xrightarrow{\mathbf{a}_n} & \mathcal{H}_n \\ \downarrow & & \downarrow \\ B_\infty & \supset & B_n \end{array}$$

Because the Hitchin maps on $\mathcal{H}_n$ are injective, it should follow that the Hitchin map on $\mathcal{H}_\infty$ is also injective, that is, an element in $\mathcal{H}_\infty$ is determined by its polynomials q_i. Another way to state the same thing is to say that a cotangent domain of Σ carrying a folded hyper-Kähler metric is determined by its q_i's. This would be the first step in proving that $\mathcal{H}_\infty$ is parametrized by B_∞ (with the sum equipped with a suitable topology).

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