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On the construction of conditional probability densities*

Pavel V. Gapeev[†]

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Abstract

In this paper, we construct strictly positive conditional probability densities with respect to the given reference filtration in the two cases of a filtration generated by a Brownian motion and a (compound) Poisson process. Then, by means of the results contained in [23], it is possible to construct the associated random times on some extended probability space. Hence, Jacod's equivalence hypothesis, that is, the existence of strictly positive conditional densities for the random times with respect to the reference filtration, is obviously satisfied.

1 Introduction

In this paper, we construct strictly positive conditional probability densities with respect to some reference filtration. In that case, one can apply the results obtained in Jeanblanc and Song [23] to construct the associated random times on some extended probability spaces. These properties automatically lead to the satisfaction of Jacod's equivalence hypothesis, that is, to the existence of a strictly positive conditional density for the random time with respect to the reference filtration (see, e.g. Aksamit and Jeanblanc [1] for further discussions on Jacod's hypothesis). We present a general framework for this model in Section 2, and then, [under the assumption that the intensity is deterministic](#), consider the cases of Brownian reference filtrations in Section 3 as well as filtrations generated by (compound) Poisson processes in Sections 4-5. We also derive connections between our results and the models of quickest change-point (or disorder) detection (see, e.g. Shiryaev [32] and the references therein).

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2 The framework

We give a model for constructions of conditional probability densities with respect to a reference filtration. For this purpose, we assume that we work on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$, where \mathbb{F} is a given (reference) filtration. We call an \mathbb{F} -conditional density any family of strictly positive \mathbb{F} -martingales $((p_t(u))_{t \geq 0}; \forall u \geq 0)$, parameterised by $u \in [0, \infty)$, such that

$$\int_0^\infty p_t(u) g(u) du = 1, \quad \forall t \geq 0,$$

where g is a given density on \mathbb{R}^+ and $p_0(u) = 1, \forall u \geq 0$.

There are very few examples in the literature (see, e.g. [1, chapter 4]). In this paper, we will investigate the inverse problem in two cases: when \mathbb{F} is a Brownian filtration and when \mathbb{F} is a (compound) Poisson filtration, and show how to construct such families. In the both cases, we start with a deterministic bounded strictly positive function $\lambda = (\lambda(t); \forall t \geq 0)$ satisfying $\int_0^\infty \lambda(s) ds = \infty$ and a supermartingale $G = (G_t, t \geq 0)$ valued in $]0, 1]$ such that the process $(G_t e^{\int_0^t \lambda(s) ds})_{t \geq 0}$ is a (strictly) positive \mathbb{F} -local martingale. In this case, using the work of [23], on the extended space $\Omega \times [0, \infty)$, one can construct a random variable τ (in fact $\tau(\omega', \omega'') = \omega''$) and a probability \mathbb{Q} (on $\mathcal{G} \otimes \mathcal{B}([0, \infty))$) such that

$$\mathbb{Q}(\tau > t | \mathcal{F}_t) = G_t \quad \text{and} \quad \mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}, \quad \forall t \geq 0.$$

The Doob-Meyer decomposition of the \mathbb{F} -supermartingale G has a predictable part equal to $\lambda(t)G_t dt$, hence (see Proposition 2.4 in [1]), the process

$$\mathbb{1}_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \lambda(s) ds, \quad \forall t \geq 0,$$

is a martingale in the progressive enlargement of \mathbb{F} with τ , where $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function, and λ is called the intensity of τ . This implies that

$$\mathbb{Q}(\tau \leq t) = \int_0^t \lambda(s) \mathbb{Q}(\tau > s) ds, \quad \forall t \geq 0,$$

and hence,

$$\mathbb{Q}(\tau \leq t) = 1 - \exp\left(-\int_0^t \lambda(s) ds\right), \quad \forall t \geq 0,$$

Note that $\mathbb{Q}(\tau = 0) = 0$, and $G_0 = 1$, and τ admits a density g given by

$$g(u) = \lambda(u) \exp\left(-\int_0^u \lambda(v) dv\right), \quad \forall u \geq 0.$$

It follows from the deterministic character of the intensity function λ that the following Jacod's equivalence hypothesis holds, that is, the regular conditional distribution of τ given \mathcal{F}_t is equivalent to the distributional law of τ , that is expressed by

$$\mathbb{Q}(\tau \in \cdot | \mathcal{F}_t) \sim \mathbb{Q}(\tau \in \cdot) \quad (\mathbb{Q}\text{-a.s.}), \quad \forall t \geq 0. \quad (2.1)$$

We recall that Jacod's equivalence hypothesis holds if there exists a family $p(u)$ of \mathbb{F} -conditional densities $p(u) = (p_t(u); \forall t \geq 0)$, $\forall u \geq 0$, such that

$$\mathbb{Q}(\tau > u | \mathcal{F}_t) = \int_u^\infty p_t(v) g(v) dv, \forall t \geq 0, \forall u \geq 0.$$

The family of processes $(p(u); \forall u \geq 0)$ is then called the conditional density. The deterministic property of λ enables us to prove the existence of a conditional density in our model. A similar problem is studied in [23] with a different method. We refer the reader to the seminal papers of Jacod [20], Grorud and Pontier [19] and Amendinger [2], and to the book [1] for more details on Jacod's hypothesis. Note that the knowledge of the conditional density allows to give the decomposition of any martingale in the reference filtration as a semimartingale in the initial (and progressive) enlargement with τ (see [1]).

3 The case of Brownian filtrations

In this section, we consider the case where \mathbb{F} is the filtration generated by a standard Brownian motion W . We use the notation of the framework described in Section 2.

3.1 The conditional survival probability process G

Suppose that there exists a standard Brownian motion $W = (W_t)_{t \geq 0}$ generating the reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ which is complete under the probability measure \mathbb{P} . Let us consider the stochastic differential equation

$$d\Phi_t = \left((1 + \Phi_t) \lambda(t) + \frac{\Phi_t^2}{1 + \Phi_t} \right) \rho^2 dt + \Phi_t \rho dW_t \quad (3.1)$$

where $\rho > 0$ is a given constant, $\lambda = (\lambda(t); \forall t \geq 0)$ is a strictly positive deterministic function, and $\Phi_0 = 0$. Since the coefficients of the stochastic differential equation in (3.1) are Lipschitz continuous and of a linear growth, it follows from the result of [26, Chapter IV, Theorem 4.8] that the equation in (3.1) admits a pathwise unique (strong) solution process $\Phi = (\Phi_t)_{t \geq 0}$.

Let $\vartheta = \inf\{t : \Phi_t < -1/2\}$. On $[0, \vartheta]$, the drift of (3.1) is positive, hence, from [30, Theorem 3.7 chapter IX] $\Phi_t \geq \Psi_t$ where the positive process $\Psi = (\Psi_t)_{t \geq 0}$ solves the stochastic differential equation

$$d\Psi_t = \Psi_t \rho dW_t \quad (3.2)$$

and $\Psi_0 = 0$, i.e., $\Psi_t = 0$ for any $t \geq 0$ (i.e., $\vartheta = \infty$), and Φ is valued in \mathbb{R}^+ . The process $G = (G_t)_{t \geq 0}$ defined by $G_t = 1/(1 + \Phi_t)$, for all $t \geq 0$, solves the stochastic differential equation

$$dG_t = -G_t \lambda(t) dt - G_t(1 - G_t) \rho dW_t \quad (3.3)$$

and thus, it is a supermartingale valued in $(0, 1]$ with initial condition $G_0 = 1$.

Note that the equation in (3.3) has the same structure as the appropriate stochastic differential equations for the posterior probability processes $\Pi = (\Pi_t)_{t \geq 0}$ such that $\Pi_t = 1 - G_t$, $\forall t \geq 0$, in the quickest change-point detection problems for Wiener processes [?, Chapter IV, Section 4] (see also [31, Chapter IV, Section 4] and [29, Chapter VI, Section 22]).

3.2 The multiplicative decomposition for the process G

In [23], starting from an \mathbb{F} -supermartingale valued in $[0, 1]$, the authors construct a large family of \mathbb{F} -martingales $M(u)$, $\forall u \geq 0$, valued in $[0, 1]$, as solution of a stochastic differential equation such that $M_t(t) = G_t$, $\forall t \geq 0$, and $u \rightarrow M_t(u)$ is decreasing on $[0, \infty)$, $\forall t \geq 0$ fixed, and represent the \mathbb{F} -conditional law of a random time τ (under the probability measure \mathbb{Q} on the extended probability space). Here, our construction is different, however, our solution is included in [23, Section 4.2, Equation ‡]. As we explained in Section 2, our goal is to construct τ such that $\mathbb{Q}(\tau > t | \mathcal{F}_t) = G_t$, which implies that the density of τ is g and that the cumulative distribution function is

$$F(t) = \mathbb{P}(\tau \leq t) = 1 - \exp\left(-\int_0^t \lambda(s) ds\right), \forall t \geq 0.$$

In order to provide the multiplicative decomposition for the supermartingale G , let us introduce the process $Y = (Y_t)_{t \geq 0}$ by

$$Y_t = \int_0^t \frac{g(s)}{Z_s} ds + \frac{1 - F(t)}{Z_t}, \forall t \geq 0, \quad (3.4)$$

where the process $Z = (Z_t)_{t \geq 0}$ is defined by

$$Z_t = \exp\left(\rho X_t - \frac{\rho^2}{2} t\right), \forall t \geq 0, \quad (3.5)$$

and the process $X = (X_t)_{t \geq 0}$ is given by

$$X_t = \int_0^t (1 - G_s) \rho ds + W_t, \forall t \geq 0, \quad (3.6)$$

so that the inclusion $\mathcal{F}'_t \subseteq \mathcal{F}_t$ holds, $\forall t \geq 0$, for the natural filtration $\mathbb{F}' = (\mathcal{F}'_t)_{t \geq 0}$ of X defined by $\mathcal{F}'_t = \sigma(X_s | 0 \leq s \leq t)$, $\forall t \geq 0$. Then, by applying Itô's formula and the integration by parts formula, we obtain from the expressions in (3.4)-(3.6) that the processes Z , $1/Z$, and Y admit the stochastic differentials

$$dZ_t = Z_t \rho dX_t = Z_t \rho ((1 - G_t) \rho dt + dW_t) \quad (3.7)$$

$$d\left(\frac{1}{Z_t}\right) = -\frac{1}{Z_t} \rho (dX_t - \rho dt) = -\frac{1}{Z_t} \rho (dW_t - G_t \rho dt) \quad (3.8)$$

and

$$dY_t \equiv (1 - F(t)) d\left(\frac{1}{Z_t}\right) = -\frac{1 - F(t)}{Z_t} \rho (dW_t - G_t \rho dt). \quad (3.9)$$

Hence, by applying Itô's formula and the integration by parts formula, we obtain from the representations in (3.7)-(3.9) that the processes GZ , GZY , and $GZY - (1 - F)$ admit the stochastic differentials

$$\begin{aligned} d(G_t Z_t) &= G_t dZ_t + Z_t dG_t + d\langle G, Z \rangle_t \\ &= -G_t Z_t \lambda(t) dt + G_t^2 Z_t \rho dW_t \end{aligned} \quad (3.10)$$

$$\begin{aligned} d(G_t Z_t Y_t) &= (G_t Z_t) dY_t + Y_t d(G_t Z_t) + d\langle GZ, Y \rangle_t \\ &= -G_t Z_t Y_t \lambda(t) dt + (G_t Z_t Y_t - (1 - F(t))) G_t \rho dW_t \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} d(G_t Z_t Y_t - (1 - F(t))) & \\ &= (G_t Z_t Y_t - (1 - F(t))) (-\lambda(t) dt + G_t \rho dW_t) \end{aligned} \quad (3.12)$$

where we have $F(0) = 0$. Since the process $GZY - (1 - F)$ starts at $G_0 Z_0 Y_0 - (1 - F(0)) = 0$, we may conclude from the geometric structure of the stochastic differential equation (3.12) that $G_t Z_t Y_t - (1 - F(t)) = 0$, $\forall t \geq 0$, and thus, the decomposition

$$G_t = \frac{1 - F(t)}{Z_t Y_t}, \quad \forall t \geq 0, \quad (3.13)$$

so that the process G is \mathbb{F}' -adapted. In this case, we also see from the expressions in (3.13) and (3.4)-(3.6) that the inclusion $\mathcal{F}_t \subseteq \mathcal{F}'_t$ holds too, so that $\mathcal{F}'_t = \mathcal{F}_t$, $\forall t \geq 0$, and thus, we have $\mathbb{F}' \equiv \mathbb{F}$. Moreover, the expression in (3.13) gives the multiplicative decomposition with the martingale part $1/(ZY)$ and the predictable part $1 - F(t)$, $\forall t \geq 0$.

3.3 The set of strictly positive martingales $M(u)$

Let us now define a set of processes $M(u) = (M_t(u))_{t \geq 0}$ by

$$M_t(u) = \frac{1}{Y_t} \int_u^\infty \frac{g(s)}{Z_{s \wedge t}} ds \equiv \frac{1}{Y_t} \left(\int_u^{u \vee t} \frac{g(s)}{Z_s} ds + \frac{1 - F(u \vee t)}{Z_t} \right), \quad \forall t \geq 0, \quad \forall u \geq 0. \quad (3.14)$$

Then, taking into account the representations in (3.7)-(3.9), we obtain by means of standard applications of Itô's formula and the integration by parts formula that the processes $1/Y$ and $1/(ZY)$ admit the stochastic differentials

$$d\left(\frac{1}{Y_t}\right) = \frac{1 - F(t)}{Z_t Y_t^2} \rho dW_t = \frac{G_t}{Y_t} \rho dW_t \quad (3.15)$$

and

$$d\left(\frac{1}{Z_t Y_t}\right) = \frac{1}{Z_t} d\left(\frac{1}{Y_t}\right) + \frac{1}{Y_t} d\left(\frac{1}{Z_t}\right) + d\left\langle \frac{1}{Z}, \frac{1}{Y} \right\rangle_t = -\frac{1-G_t}{Z_t Y_t} \rho dW_t \quad (3.16)$$

so that $1/Y$ and $1/(ZY)$ are local martingales. Hence, by applying the integration by parts formula to the expression in (3.14), we get that the process $M(u)$ admits the stochastic differential

$$d_t M_t(u) = \left(\int_u^t \frac{g(s)}{Z_s} ds \right) d\left(\frac{1}{Y_t}\right) + (1-F(t)) d\left(\frac{1}{Z_t Y_t}\right) + \frac{1}{Z_t Y_t} d(1-F(t)) + \frac{g(t)}{Z_t Y_t} dt \quad (3.17)$$

for $0 \leq u < t$, which takes the form

$$d_t M_t(u) = \left(\frac{G_t}{Y_t} \int_u^t \frac{g(s)}{Z_s} ds - \frac{(1-F(t))(1-G_t)}{Z_t Y_t} \right) \rho dW_t \quad (3.18)$$

for $0 \leq u < t$. Thus, we conclude that $M(u)$ is a local martingale, $\forall u \geq 0$ fixed. It is also seen from the expression in (3.14) that $M(u)$ is strictly positive. Furthermore, it follows from the fact that $M_t(0) = 1$ and the property that the mapping $u \mapsto M_t(u)$ is decreasing on $[0, \infty)$, we obtain that the process $M(u)$ is a martingale valued in $(0, 1)$, $\forall u \geq 0$ fixed. Observe from the definition of the process $M(u)$ in (3.14) and the decomposition in (3.13) that $M_t(t) = G_t$, $\forall t \geq 0$, as well as $M_t(u) = (1-F(u))/(Z_t Y_t)$, $\forall 0 \leq t \leq u$, $\forall u \geq 0$ fixed. We also note that the representation

$$d_t M_t(u) = M_t(u) (1-F(t)) d\left(\frac{1}{Z_t Y_t}\right) + dK_t$$

holds, which represents the formula from [23, Section 4.2, Equation $\sharp(u)$] with $f \equiv 1$, where $K = (K_t)_{t \geq 0}$ is an \mathbb{F} -martingale which does not depend on u .

3.4 The set of conditional probability density processes $p(u)$

It therefore follows that, on an extended probability space, one can construct a random time τ and a probability measure \mathbb{Q} such that \mathbb{Q} and \mathbb{P} coincide on the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and $\mathbb{Q}(\tau > u | \mathcal{F}_t) = M_t(u)$, $\forall t \geq 0$, $\forall u \geq 0$ fixed (see [23, Section 3] and [25]). Let us finally note that the family of the conditional probability density processes $p(u) = (p_t(u))_{t \geq 0}$ defined by $p_t(u) = 1/(Z_{u \wedge t} Y_t)$, $\forall t \geq 0$, $\forall u \geq 0$ fixed, satisfies the equality

$$M_t(u) = \int_u^\infty p_t(s) g(s) ds, \quad \forall t \geq 0, \quad \forall u \geq 0. \quad (3.19)$$

Then, it follows from the expressions in (3.15) and (3.16) that the process $p(u)$ admits the stochastic differential

$$\begin{aligned} d_t p_t(u) &\equiv d_t \left(\frac{1}{Z_{u \wedge t} Y_t} \right) = \left(\mathbb{1}_{\{u < t\}} \frac{G_t}{Z_u Y_t} - \mathbb{1}_{\{u \geq t\}} \frac{1-G_t}{Z_t Y_t} \right) \rho dW_t \\ &= \left(\mathbb{1}_{\{u < t\}} G_t - \mathbb{1}_{\{u \geq t\}} (1-G_t) \right) p_t(u) \rho dW_t \end{aligned} \quad (3.20)$$

so that the representation

$$p_t(u) = \exp \left(\int_0^t \varphi_s(u) dW_s - \frac{1}{2} \int_0^t \varphi_s^2(u) ds \right), \quad \forall t \geq 0, \quad \forall u \geq 0, \quad (3.21)$$

holds with

$$\varphi_t(u) = (\mathbb{1}_{\{u < t\}} G_t - \mathbb{1}_{\{u \geq t\}} (1 - G_t)) \rho = (\mathbb{1}_{\{u < t\}} - (1 - G_t)) \rho, \quad \forall t \geq 0, \quad \forall u \geq 0. \quad (3.22)$$

4 The case of Poissonian filtrations

In this section, we consider the case, where \mathbb{F} is the filtration generated by a Poisson process N . We use the notation of the framework described in Section 2.

4.1 The conditional survival probability process G

Suppose that there exists a Poisson process $N = (N_t)_{t \geq 0}$ of the intensity $\lambda_0 > 0$ generating the reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ which is complete under the probability measure \mathbb{P} . Let us consider the stochastic differential equation

$$d\Phi_t = (1 + \Phi_{t-}) \lambda(t) dt + \Phi_{t-} \left(\frac{\lambda_1}{\lambda_0} - 1 \right) \left(dN_t - \frac{(1 + \Phi_{t-}) \lambda_0^2}{\lambda_0 + \Phi_{t-} \lambda_1} dt \right) \quad (4.1)$$

where $\lambda_1 > 0$ is a given constant, $\lambda = (\lambda(t); \forall t \geq 0)$ is a strictly positive function, and $\Phi_0 = 0$. Since the coefficients of the stochastic differential equation in (4.1) are Lipschitz continuous and of a linear growth, it follows from the result of [21, Chapter III, Theorem 2.32] that the equation in (4.1) admits a pathwise unique (strong) solution process. Moreover, between two jump times T_{n-1} and T_n , for $n \geq 1$, of the Poisson process N , the process Φ follows

$$\frac{d\Phi_t}{dt} = (1 + \Phi_t) \lambda(t) - \Phi_{t-} \left(\frac{\lambda_1}{\lambda_0} - 1 \right) \frac{(1 + \Phi_t) \lambda_0^2}{\lambda_0 + \Phi_{t-} \lambda_1} \quad \forall t \geq 0. \quad (4.2)$$

and introduce $T = \inf\{t > 0 : \Phi_t\} = 0$. Then at time T , the derivative of Φ is positive and Φ remains non negative after time T till its second hitting time of 0. Adding that the jump values of Φ are such that

$$\Phi_{T_n} = \frac{\lambda_1}{\lambda_0} \Phi_{T_n-}, \quad \forall n \geq 1,$$

we conclude that Φ is nonnegative. We may conclude that the candidate conditional survival probability process $G = (G_t)_{t \geq 0}$ defined by $G_t = 1/(1 + \Phi_t)$, for all $t \geq 0$, solves the stochastic differential equation

$$dG_t = -G_{t-} \lambda(t) dt - \frac{G_{t-}(1 - G_{t-})(\lambda_1 - \lambda_0)}{\lambda_0 + (1 - G_{t-})(\lambda_1 - \lambda_0)} (dN_t - \lambda_0 dt) \quad (4.3)$$

where $\lambda_1 > 0$ is a given constant.

Note that the equation in (4.3) has the same structure as the appropriate stochastic differential equations for the posterior probability processes $\Pi = (\Pi_t)_{t \geq 0}$ such that $\Pi_t = 1 - G_t$, $\forall t \geq 0$, in the quickest change-point detection problems for Poisson processes [28] (see also [29, Chapter VI, Section 24]).

4.2 The multiplicative decomposition for the process G

In order to provide a multiplicative decomposition for the supermartingale G , let us introduce the process $Y = (Y_t)_{t \geq 0}$ as in (3.4), where the process $Z = (Z_t)_{t \geq 0}$ is defined by

$$Z_t = \exp \left(\ln \left(\frac{\lambda_1}{\lambda_0} \right) X_t - (\lambda_1 - \lambda_0) t \right), \forall t \geq 0, \quad (4.4)$$

and the process $X = (X_t)_{t \geq 0}$ is given by

$$X_t = \left(\ln \left(\frac{\lambda_1}{\lambda_0} \right) \right)^{-1} \int_0^t \frac{(1 - G_{s-})(\lambda_1 - \lambda_0)^2}{\lambda_0 + (1 - G_{s-})(\lambda_1 - \lambda_0)} ds + N_t, \forall t \geq 0, \quad (4.5)$$

so that the inclusion $\mathcal{F}'_t \subseteq \mathcal{F}_t$ holds, $\forall t \geq 0$, for the natural filtration $\mathbb{F}' = (\mathcal{F}'_t)_{t \geq 0}$ of X defined by $\mathcal{F}'_t = \sigma(X_s | 0 \leq s \leq t)$, $\forall t \geq 0$. Then, by applying Itô's formula and the integration by parts formula for general semimartingales (see, e.g. [21, Chapter I, Theorem 4.57]) and taking into account the fact that $dN_t = \Delta N_t \equiv N_t - N_{t-}$, $\forall t \geq 0$, we obtain from the expressions in (3.4) and (4.4)-(4.5) that the processes Z , $1/Z$, and Y admit the stochastic differentials

$$dZ_t = Z_{t-} \left(\frac{\lambda_1}{\lambda_0} - 1 \right) d(X_t - \lambda_0 t) = Z_{t-} \left(\frac{\lambda_1}{\lambda_0} - 1 \right) \left(dN_t - \frac{\lambda_0^2}{\lambda_0 + (1 - G_{t-})(\lambda_1 - \lambda_0)} dt \right) \quad (4.6)$$

$$d \left(\frac{1}{Z_t} \right) = \frac{1}{Z_{t-}} \left(\frac{\lambda_0}{\lambda_1} - 1 \right) \left(dN_t - \frac{\lambda_0 \lambda_1}{\lambda_0 + \Pi_{t-}(\lambda_1 - \lambda_0)} dt \right) \quad (4.7)$$

and

$$dY_t \equiv (1 - F(t)) d \left(\frac{1}{Z_t} \right) = \frac{1 - F(t)}{Z_{t-}} \left(\frac{\lambda_0}{\lambda_1} - 1 \right) \left(dN_t - \frac{\lambda_0 \lambda_1}{\lambda_0 + (1 - G_{t-})(\lambda_1 - \lambda_0)} dt \right). \quad (4.8)$$

Hence, by applying Itô's formula and the integration by parts formula, we obtain from the representations in (4.6)-(4.8) that the processes GZ , GZY , and $GZY - (1 - F)$ admit the stochastic differentials

$$\begin{aligned} d(GZ_t) &= G_{t-} dZ_t + Z_{t-} dG_t + \Delta G_t \Delta Z_t \\ &= -G_{t-} Z_{t-} \lambda(t) dt + G_{t-} Z_{t-} \frac{G_{t-}(\lambda_1 - \lambda_0)}{\lambda_0 + (1 - G_{t-})(\lambda_1 - \lambda_0)} d(N_t - \lambda_0 t) \end{aligned} \quad (4.9)$$

$$\begin{aligned}
d(G_t Z_t Y_t) &= (G_{t-} Z_{t-}) dY_t + Y_{t-} d(G_t Z_t) + \Delta(G_t Z_t) \Delta Y_t \\
&= -G_{t-} Z_{t-} Y_{t-} \lambda(t) dt + (G_{t-} Z_{t-} Y_{t-} - (1 - F(t))) \frac{G_{t-}(\lambda_1 - \lambda_0)}{\lambda_0 + (1 - G_{t-})(\lambda_1 - \lambda_0)} d(N_t - \lambda_0 t)
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
d(G_t Z_t Y_t - (1 - F(t))) &= -(G_{t-} Z_{t-} Y_{t-} - (1 - F(t))) \lambda(t) dt \\
&+ (G_{t-} Z_{t-} Y_{t-} - (1 - F(t))) \frac{G_{t-}(\lambda_1 - \lambda_0)}{\lambda_0 + (1 - G_{t-})(\lambda_1 - \lambda_0)} d(N_t - \lambda_0 t)
\end{aligned} \tag{4.11}$$

where we have $F(0) = 0$. Since the process $GZY - (1 - F)$ starts at $G_0 Z_0 Y_0 - (1 - F(0)) = 0$, we may conclude from the geometric structure of the stochastic differential equation (4.11) that $G_t Z_t Y_t - (1 - F(t)) = 0$, for all $t \geq 0$, and thus, the decomposition as in (3.13) holds, so that the process G is \mathbb{F}' -adapted. In this case, we also see from the expressions in (3.13) and (3.4) with (4.4)-(4.5) that the inclusion $\mathcal{F}_t \subseteq \mathcal{F}'_t$ holds too, so that $\mathcal{F}'_t = \mathcal{F}_t$, for all $t \geq 0$, and thus, we have $\mathbb{F}' \equiv \mathbb{F}$. Moreover, the expression in (3.13) gives the multiplicative decomposition of G with the martingale part $1/(ZY)$ and the predictable part $1 - F(t)$, $\forall t \geq 0$.

4.3 The set of strictly positive martingales $M(u)$

Then, taking into account the representations in (4.6)-(4.8), we obtain by means of standard applications of Itô's formula and the integration by parts formula that the processes $1/Y$ and $1/(ZY)$ admit the stochastic differentials

$$d\left(\frac{1}{Y_t}\right) = \frac{1}{Y_{t-}} \frac{G_{t-}(\lambda_1 - \lambda_0)}{\lambda_0 + (1 - G_{t-})(\lambda_1 - \lambda_0)} d(N_t - \lambda_0 t) \tag{4.12}$$

and

$$\begin{aligned}
d\left(\frac{1}{Z_t Y_t}\right) &= \frac{1}{Z_{t-}} d\left(\frac{1}{Y_t}\right) + \frac{1}{Y_{t-}} d\left(\frac{1}{Z_t}\right) + \Delta\left(\frac{1}{Z_t}\right) \Delta\left(\frac{1}{Y_t}\right) \\
&= -\frac{1}{Z_{t-} Y_{t-}} \frac{(1 - G_{t-})(\lambda_1 - \lambda_0)}{\lambda_0 + (1 - G_{t-})(\lambda_1 - \lambda_0)} d(N_t - \lambda_0 t)
\end{aligned} \tag{4.13}$$

so that they are (local) martingales. Hence, by applying the integration by parts formula to the expression as in (3.14), we get that the process $M(u)$ admits the stochastic differential as in (3.17), which takes the form

$$\begin{aligned}
d_t M_t(u) &= \left(\frac{G_{t-}}{Y_{t-}} \int_u^t \frac{g(s)}{Z_s} ds - \frac{(1 - F(t))(1 - G_{t-})}{Z_{t-} Y_{t-}} \right) \\
&\quad \times \frac{\lambda_1 - \lambda_0}{\lambda_0 + (1 - G_{t-})(\lambda_1 - \lambda_0)} d(N_t - \lambda_0 t)
\end{aligned} \tag{4.14}$$

for each $0 \leq u < t$ fixed. Thus, we conclude that $M(u)$ is a local martingale, for each $u \geq 0$ fixed. It is also seen from the expression as in (3.14) that $M(u)$ is strictly positive.

Furthermore, it follows from the fact that $M_t(0) = 1$ and the property that the mapping $u \mapsto M_t(u)$ is decreasing on $(0, \infty)$, we obtain that the process $M(u)$ is a martingale valued in $(0, 1)$. Observe from the definition of the process $M(u)$ in (3.14) and the decomposition in (3.13) that $M_t(t) = G_t$, $\forall t \geq 0$, as well as $M_t(u) = (1 - F(u))/(Z_t Y_t)$, $\forall 0 \leq t \leq u$, $\forall u \geq 0$ fixed. We also note that the representation

$$d_t M_t(u) = M_{t-}(u) (1 - F(t)) d\left(\frac{1}{Z_t Y_t}\right) + dK_t$$

holds, which represents the formula from [23, Section 4.2, Equation $\sharp(u)$] with $f \equiv 1$, where $K = (K_t)_{t \geq 0}$ is an \mathbb{F} -martingale which does not depend on u .

4.4 The set of conditional probability density processes $p(u)$

It therefore follows that, on an extended probability space, one can construct a random time τ and a probability measure \mathbb{Q} such that \mathbb{Q} and \mathbb{P} coincide on the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and $\mathbb{Q}(\tau > u | \mathcal{F}_t) = M_t(u)$, for all $t \geq 0$ and each $u \geq 0$ fixed (see [21, Section 3] and [25]). Let us finally note that the family of the conditional probability density processes $p(u) = (p_t(u))_{t \geq 0}$ defined by $p_t(u) = 1/(Z_{u \wedge t} Y_t)$, for all $t \geq 0$ and each $u \geq 0$ fixed, satisfies the equality as of (3.19), for all $t \geq 0$ and each $u \geq 0$ fixed. Then, it follows from the expressions in (4.12) and (4.13) that the process $p(u)$ admits the stochastic differential

$$\begin{aligned} d_t p_t(u) &\equiv d_t \left(\frac{1}{Z_{u \wedge t} Y_t} \right) & (4.15) \\ &= \left(\mathbb{1}_{\{u < t\}} \frac{G_{t-}}{Z_u Y_{t-}} - \mathbb{1}_{\{u \geq t\}} \frac{1 - G_{t-}}{Z_{t-} Y_{t-}} \right) \frac{\lambda_1 - \lambda_0}{\lambda_0 + (1 - G_{t-})(\lambda_1 - \lambda_0)} d(N_t - \lambda_0 t) \\ &= (\mathbb{1}_{\{u < t\}} G_{t-} - \mathbb{1}_{\{u \geq t\}} (1 - G_{t-})) \frac{\lambda_1 - \lambda_0}{\lambda_0 + (1 - G_{t-})(\lambda_1 - \lambda_0)} p_{t-}(u) d(N_t - \lambda_0 t) \end{aligned}$$

so that the representation

$$p_t(u) = \exp \left(\int_0^t \ln(1 + \varphi_{s-}(u)) dN_s - \int_0^t \varphi_{s-}(u) \lambda_0 ds \right), \forall t \geq 0, \forall u \geq 0, \quad (4.16)$$

holds with

$$\begin{aligned} \varphi_{t-}(u) &= (\mathbb{1}_{\{u < t\}} G_{t-} - \mathbb{1}_{\{u \geq t\}} (1 - G_{t-})) \frac{\lambda_1 - \lambda_0}{\lambda_0 + (1 - G_{t-})(\lambda_1 - \lambda_0)} & (4.17) \\ &= (\mathbb{1}_{\{u < t\}} - (1 - G_{t-})) \frac{\lambda_1 - \lambda_0}{\lambda_0 + (1 - G_{t-})(\lambda_1 - \lambda_0)}, \forall t \geq 0, \forall u \geq 0. \end{aligned}$$

5 The case of compound Poisson filtrations

In this section, we show that the previous constructions are equivalent to the ones in [12] and [7] (see also [6] and other papers).

5.1 The conditional probability process Π

Suppose that there exists a compound Poisson process (or a pure jump Lévy process) $X = (X_t)_{t \geq 0}$ with a Lévy measure $\nu_0(dx)$ which is a positive σ -finite measure on $\mathcal{B}(\mathbb{R})$ satisfying the conditions

$$\nu_0(\{0\}) = 0, \quad \int (x^2 \wedge 1) \nu_0(dx) < \infty \quad \text{and} \quad \int |x| \nu_0(dx) < \infty. \quad (5.1)$$

Assume that the process X generates the reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ which is complete under the probability measure \mathbb{P} . Note that, in the compound Poisson case of $\nu_0(\mathbb{R}) < \infty$, the process X admits the representation $X_t = \sum_{i=1}^{N_t} \Xi_i$, where $N = (N_t)_{t \geq 0}$ is a Poisson process of intensity $\lambda_0 > 0$ and $(\Xi_i)_{i \in \mathbb{N}}$ is a sequence of independent identically distributed random variables with the distribution $\nu_0(dx)/\lambda_0$, where N and $(\Xi_i)_{i \in \mathbb{N}}$ are independent under \mathbb{P} . Let us consider the stochastic differential equation

$$d\Phi_t = (1 + \Phi_{t-}) \lambda(t) dt + \Phi_{t-} \int (\Upsilon(x) - 1) \left(\mu(ds, dx) - \frac{1 + \Phi_{t-}}{1 + \Phi_{t-} \Upsilon(x)} dt \nu_0(dx) \right) \quad (5.2)$$

where $\Upsilon(x)$ is a given positive function, and $\Phi_0 = 0$. Since the coefficients of the stochastic differential equation in (5.2) are Lipschitz continuous and of a linear growth, it follows from the result of [21, Chapter III, Theorem 2.32] that the equation in (5.2) admits a pathwise unique (strong) solution process $\Phi = (\Phi_t)_{t \geq 0}$.

The size of the jump of Φ at T_n is positive, due to the positiveness of Υ . Between two jumps, the same argument than in subsection (see (4.1) and (4.2)) allows us to establish that Φ remains non negative between two jumps. *j'ai enlevé la suite*

Hence, applying Itô's formula to the process $\Phi = (\Phi_t)_{t \geq 0}$ solving the stochastic differential equation in (5.2), we may conclude that the candidate conditional survival probability process $G = (G_t)_{t \geq 0}$ defined by $G_t = 1/(1 + \Phi_t)$, for all $t \geq 0$, solves the stochastic differential equation

$$dG_t = -G_{t-} \lambda(t) dt - \int \frac{G_{t-}(1 - G_{t-})(\Upsilon(x) - 1)}{1 + (1 - G_{t-})(\Upsilon(x) - 1)} (\mu(dt, dx) - dt \nu_0(dx)) \quad (5.3)$$

whith $G_0 = 1$. Here, $\Upsilon(x)$, for $x \in \mathbb{R}$, is a given (strictly) positive function satisfying the condition

$$\int \left(\sqrt{\Upsilon(x)} - 1 \right)^2 \nu_0(dx) < \infty \quad (5.4)$$

and $\mu(dt, dx)$ is the measure of jumps of the process X defined by

$$\mu((0, t] \times A) = \sum_{0 < s \leq t} \mathbb{1}_{\{\Delta X_s \in A\}} \quad (5.5)$$

for any Borel set $A \in \mathcal{B}(\mathbb{R})$, where we set $\Delta X_t = X_t - X_{t-}$, for all $t \geq 0$. Note that the equation in (5.3) has the same structure as the appropriate stochastic differential equations for the posterior probability processes in the quickest change-point detection problems for compound Poisson processes [12], [7], and [6].

5.2 The multiplicative decomposition for the process G

In order to provide a multiplicative decomposition for the supermartingale G , let us introduce the process $Y = (Y_t)_{t \geq 0}$ as in (3.4), where the process $Z = (Z_t)_{t \geq 0}$ is defined by

$$Z_t = \exp \left(\int_0^t \int \ln(\Upsilon(x)) \mu(ds, dx) - \int_0^t \int \frac{\Upsilon(x) - 1}{1 + (1 - G_{s-})(\Upsilon(x) - 1)} ds \nu_0(dx) \right) \quad (5.6)$$

so that the inclusion $\mathcal{F}'_t \subseteq \mathcal{F}_t$ holds, for all $t \geq 0$, for the natural filtration $\mathbb{F}' = (\mathcal{F}'_t)_{t \geq 0}$ of X defined by $\mathcal{F}'_t = \sigma(X_s | 0 \leq s \leq t)$, for all $t \geq 0$. Then, by applying Itô's formula and the integration by parts formula for general semimartingales (see, e.g. [21]), we obtain from the expressions in (3.4) and (5.6) that the processes Z , $1/Z$, and Y admit the stochastic differentials

$$dZ_t = Z_{t-} \int (\Upsilon(x) - 1) d \left(\mu(dt, dx) - \frac{1}{1 + (1 - G_{t-})(\Upsilon(x) - 1)} dt \nu_0(dx) \right) \quad (5.7)$$

$$d \left(\frac{1}{Z_t} \right) = \frac{1}{Z_{t-}} \int \left(\frac{1}{\Upsilon(x)} - 1 \right) d \left(\mu(dt, dx) - \frac{1}{1 + (1 - G_{t-})(\Upsilon(x) - 1)} dt \nu_1(dx) \right) \quad (5.8)$$

and

$$\begin{aligned} dY_t &\equiv (1 - F(t)) d \left(\frac{1}{Z_t} \right) \\ &= \frac{1 - F(t)}{Z_{t-}} \int \left(\frac{1}{\Upsilon(x)} - 1 \right) d \left(\mu(dt, dx) - \frac{1}{1 + (1 - G_{t-})(\Upsilon(x) - 1)} dt \nu_1(dx) \right) \end{aligned} \quad (5.9)$$

Hence, by applying Itô's formula and the integration by parts formula, we obtain from the representations in (5.7)-(5.9) that the processes GZ , GZY , and $GZY - (1 - F)$ admit the stochastic differentials

$$\begin{aligned} d(G_t Z_t) &= G_{t-} dZ_t + Z_{t-} dG_t + \Delta G_t \Delta Z_t \\ &= -G_{t-} Z_{t-} \lambda(t) dt + G_{t-} Z_{t-} \int \frac{G_{t-}(\Upsilon(x) - 1)}{1 + (1 - G_{t-})(\Upsilon(x) - 1)} (\mu(dt, dx) - dt \nu_0(dx)) \end{aligned} \quad (5.10)$$

$$\begin{aligned} d(G_t Z_t Y_t) &= (G_{t-} Z_{t-}) dY_t + Y_{t-} d(G_t Z_t) + \Delta(G_t Z_t) \Delta Y_t \\ &= -G_{t-} Z_{t-} Y_{t-} \lambda(t) dt \\ &\quad + (G_{t-} Z_{t-} Y_{t-} - (1 - F(t))) \int \frac{G_{t-}(\Upsilon(x) - 1)}{1 + (1 - G_{t-})(\Upsilon(x) - 1)} (\mu(dt, dx) - dt \nu_0(dx)) \end{aligned} \quad (5.11)$$

and

$$\begin{aligned} d(G_t Z_t Y_t - (1 - F(t))) &= -(G_{t-} Z_{t-} Y_{t-} - (1 - F(t))) \lambda(t) dt \\ &\quad + (G_{t-} Z_{t-} Y_{t-} - (1 - F(t))) \int \frac{G_{t-}(\Upsilon(x) - 1)}{1 + (1 - G_{t-})(\Upsilon(x) - 1)} (\mu(dt, dx) - dt \nu_0(dx)) \end{aligned} \quad (5.12)$$

where we have $F(0) = 0$. Since the process $GZY - (1 - F)$ starts at $G_0Z_0Y_0 - (1 - F(0)) = 0$, we may conclude from the geometric structure of the stochastic differential equation (5.12) that $G_tZ_tY_t - (1 - F(t)) = 0$, for all $t \geq 0$, and thus, the decomposition as in (3.13) holds, so that the process G is \mathbb{F}' -adapted. In this case, we also see from the expressions in (3.13) and (3.4) with (5.6) that the inclusion $\mathcal{F}_t \subseteq \mathcal{F}'_t$ holds too, so that $\mathcal{F}'_t = \mathcal{F}_t$, for all $t \geq 0$, and thus, we have $\mathbb{F}' \equiv \mathbb{F}$.

5.3 The set of strictly positive martingales $M(u)$

Let us now define a set of processes $M(u) = (M_t(u))_{t \geq 0}$ as in (3.14).

Then, taking into account the representations in (5.7)-(5.9), we obtain by means of standard applications of Itô's formula and the integration by parts formula that the processes $1/Y$ and $1/(ZY)$ admit the stochastic differentials

$$d\left(\frac{1}{Y_t}\right) = \frac{1}{Y_{t-}} \int \frac{G_{t-}(\Upsilon(x) - 1)}{1 + (1 - G_{t-})(\Upsilon(x) - 1)} (\mu(dt, dx) - dt \nu_0(dx)) \quad (5.13)$$

and

$$\begin{aligned} d\left(\frac{1}{Z_t Y_t}\right) &= \frac{1}{Z_{t-}} d\left(\frac{1}{Y_t}\right) + \frac{1}{Y_{t-}} d\left(\frac{1}{Z_t}\right) + \Delta\left(\frac{1}{Z_t}\right) \Delta\left(\frac{1}{Y_t}\right) \\ &= -\frac{1}{Z_{t-} Y_{t-}} \int \frac{(1 - G_{t-})(\Upsilon(x) - 1)}{1 + (1 - G_{t-})(\Upsilon(x) - 1)} (\mu(dt, dx) - dt \nu_0(dx)) \end{aligned} \quad (5.14)$$

so that they are (local) martingales. Hence, by applying the integration by parts formula to the expression as in (3.14), we get that the process $M(u)$ admits the stochastic differential as in (3.17), which takes the form

$$\begin{aligned} d_t M_t(u) &= \left(\frac{G_{t-}}{Y_{t-}} \int_u^t \frac{g(s)}{Z_s} ds - \frac{(1 - G_{t-})(1 - F(t))}{Z_{t-} Y_{t-}} \right) \\ &\quad \times \int \frac{\Upsilon(x) - 1}{1 + (1 - G_{t-})(\Upsilon(x) - 1)} (\mu(dt, dx) - dt \nu_0(dx)) \end{aligned} \quad (5.15)$$

for each $0 \leq u < t$ fixed. Thus, we conclude that $M(u)$ is a local martingale, for each $u \geq 0$ fixed. It is also seen from the expression as in (3.14) that $M(u)$ is strictly positive. Furthermore, it follows from the fact that $M_t(0) = 1$ and the property that the mapping $u \mapsto M_t(u)$ is decreasing on $(0, \infty)$, we obtain that the process $M(u)$ is a martingale valued in $(0, 1)$. We also note from the definition of the process $M(u)$ as in (3.14) and the decomposition as in (3.13) that $M_t(t) = G_t$, for all $t \geq 0$. We also note that $M_t(u) = (1 - F(u))/(Z_t Y_t)$, for all $0 \leq t \leq u$ and each $u \geq 0$ fixed.

5.4 The set of conditional probability density processes $p(u)$

It therefore follows that, on an extended probability space, one can construct a random time τ and a probability measure \mathbb{Q} such that \mathbb{Q} and \mathbb{P} coincide on the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and

$\mathbb{Q}(\tau > u | \mathcal{F}_t) = M_t(u)$, for all $t \geq 0$ and each $u \geq 0$ fixed (see [21, Section 3] and [25]). Let us finally note that the family of the conditional probability density processes $p(u) = (p_t(u))_{t \geq 0}$ defined by $p_t(u) = 1/(Z_{u \wedge t} Y_t)$, for all $t \geq 0$ and each $u \geq 0$ fixed, satisfies the equality as of (3.19), for all $t \geq 0$ and each $u \geq 0$ fixed. Then, it follows from the expressions in (4.12) and (4.13) that the process $p(u)$ admits the stochastic differential

$$\begin{aligned} d_t p_t(u) &\equiv d_t \left(\frac{1}{Z_{u \wedge t} Y_t} \right) & (5.16) \\ &= \left(\mathbb{1}_{\{u < t\}} \frac{G_{t-}}{Z_u Y_{t-}} - \mathbb{1}_{\{u \geq t\}} \frac{1 - G_{t-}}{Z_{t-} Y_{t-}} \right) \int \frac{\Upsilon(x) - 1}{1 + (1 - G_{t-})(\Upsilon(x) - 1)} (\mu(dt, dx) - dt \nu_0(dx)) \\ &= (\mathbb{1}_{\{u < t\}} G_{t-} - \mathbb{1}_{\{u \geq t\}} (1 - G_{t-})) \int \frac{\Upsilon(x) - 1}{1 + (1 - G_{t-})(\Upsilon(x) - 1)} p_{t-}(u) (\mu(dt, dx) - dt \nu_0(dx)) \end{aligned}$$

so that the representation

$$p_t(u) = \exp \left(\int_0^t \int \ln(1 + \varphi_{s-}(u, x)) \mu(ds, dx) - \int_0^t \int \varphi_{s-}(u, x) ds \nu_0(dx) \right) \quad (5.17)$$

holds with

$$\begin{aligned} \varphi_{t-}(u, x) &= (\mathbb{1}_{\{u < t\}} G_{t-} - \mathbb{1}_{\{u \geq t\}} (1 - G_{t-})) \frac{\Upsilon(x) - 1}{1 + (1 - G_{t-})(\Upsilon(x) - 1)} & (5.18) \\ &= (\mathbb{1}_{\{u < t\}} - (1 - G_{t-})) \frac{\Upsilon(x) - 1}{1 + (1 - G_{t-})(\Upsilon(x) - 1)} \end{aligned}$$

for all $t \geq 0$ and each $u \geq 0$ fixed.

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