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# How teachers address process-product dualities in mathematical argumentation processes

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*Teachers play an important role in creating environments for mathematical argumentation processes in classrooms. In the transition from arithmetic to algebra for example, students depend on how teachers address the process-product duality of mathematical argumentation and the duality of mathematical objects. Two empirical examples illustrate that eight-grade students (13-14 years old) struggle with these dualities in mathematical argumentation processes and that they need support of their teachers when creating generally valid arguments in class. This study examines how teachers' actions guide students in handling the process-product dualities and how teachers thus influence and shape the construction of mathematical arguments in classrooms.*

*Keywords: Process-product duality, mathematics teachers, mathematical argumentation, classroom research.*

## Introduction

Little research has been done on how teachers act while initiating mathematical argumentations in class discussions (e.g., Conner et al., 2014; Schwarz et al., 2006). It is still uncertain what influence teachers' specific actions have on the quality of the arguments that emerge in classrooms. In addition, there are two process-product dualities students must handle in class when it comes to mathematical argumentations. On the one hand, mathematical argumentations are processes in which participants aim to justify or refute a claim with the intent to formulate a mathematical argument – a product. On the other hand, mathematical arguments are always arguments about mathematical objects and these mathematical objects can be interpreted in a process-oriented or a product-oriented way (Sfard, 1991; Sfard & Linchevski, 1994). Whether the interpretation of the mathematical objects is more process-oriented or product-oriented becomes visible in the arguments produced by the students. How these interpretations of mathematical objects in mathematical arguments are shaped or influenced by teacher actions is still mostly unexplored and is a focus in our project. This paper presents examples of mathematical argumentations in the transition from arithmetic to algebra from two grade 8 classes. The following research questions guide our analysis: How is a process-oriented or a product-oriented interpretation of mathematical objects reflected in the mathematical argumentations and arguments? How does the teacher address these interpretation processes?

## Theoretical Framework

### Process-product dualities in mathematical argumentations

In school contexts mathematical argumentations in the transition from arithmetic to algebra include two process-product dualities: 1) In classrooms, mathematical argumentation processes are mostly oral and generally distinct from the argument (product) that emerges, which is commonly fixed in a written form (Boero, 1999). This is one duality of mathematical argumentation. 2) At the same time,

the mathematical objects in the argumentations can be understood in a dual way: operationally, this means process-oriented, or structurally, which means in a product-oriented way (Sfard, 1991; Sfard & Linchevski, 1994). The algebraic symbolic language, which is itself initially a learning object, can also be interpreted in these two ways. A process-oriented interpretation of an algebraic term means to understand the term as a “sequence of instructions” (Sfard & Linchevski, 1994, p. 191). The term, which one can calculate, represents a computational process. At the same time the term can be interpreted in a product-oriented way as a “product of a computation” (Sfard & Linchevski, 1994, p. 191). In this case, the term is treated as an object itself and mathematical structures of the term are focused. Mathematical argumentations reflect this duality.

These interplays between process-product dualities are significant in our data. For example, if one considers the sum of an even and an odd number, students calculate numerical examples on an operational level and conjecture that the sum is always odd. A process-oriented interpretation of a numerical term here means to calculate a concrete number. At the same time, it is possible to look at these numerical examples from a structural perspective and formulate them as generic examples. When arguing with their mathematical structure, a generally valid mathematical argument arises. Seeing this mathematical structure, requires a change from a process-oriented to a product-oriented interpretation, which students and teachers are often not aware of. A process-oriented interpretation of a term is not sufficient to create a valid argument for a general claim, because a generalization is needed, and this requires arguing with mathematical structures and general characteristics of the mathematical objects. How teachers and their actions contribute to this change of view and support the emergence of valid arguments is of interest here.

### **Mathematical arguments and their representation**

According to Toulmin (1958) an argument consists of different elements. A claim is a statement that participants try to justify or refute. In this study claims are marked with a C. Data (D) are accepted, undoubted facts or shared knowledge of a community, the starting point of an argument. With a warrant (W) the step from the data to the claim is legitimated. Some arguments also contain backings (B) that state why a warrant is generally applicable. To describe the certainty of an argumentation step, a modal qualifier (Q) can be integrated into the argument. Sometimes rebuttals (R) indicate “circumstances in which the general authority of the warrant would have to be set aside.” (Toulmin, 1958, p. 101). In addition to these core elements established by Toulmin (1958), Knipping and Reid (2015) introduced refutations (X) as another element of arguments. When parts of an argument are not accepted by a community or are not shared knowledge, these parts can be refuted. But a refutation is not a rebuttal. Refutations refute complete parts of arguments, whereas rebuttals only restrict the conclusion to a locally limited extent (Knipping & Reid, 2015). Therefore, refutations can apply to any element of an argument. In our data, we found statements about the applicability of the warrant in an argumentation step. We call this kind of statements backings B\*. These statements specify a characteristic of the datum and relate the warrant, the general rule, to the data. This legitimates the use of the warrant in the argumentation step. These kinds of statements aren't backings in Toulmin's sense because they don't support the authority of a warrant in general but support its applicability in the specific case. On the other hand, they have a different function than data. Toulmin (1958, p. 106) states that data must be explicit in an argument, but these statements are additional facts which

characterise the datum and don't need to be made explicit in the argument. Mathematical argumentation processes in mathematics classrooms often include more than one argumentation step. When a conclusion of a previous step is already accepted, it is possible to use this conclusion as data for another step.

Mathematical arguments are represented in different ways in mathematics classrooms. Reid and Knipping (2010) distinguish several types of arguments according to their representation. In school, in the transition from arithmetic to algebra, numeric generic, pictorial generic, narrative, and symbolic arguments are particularly important. Numeric generic arguments use concrete numbers and their inherent structures to justify a general claim. In contrast, pictorial generic arguments, also called visual proofs (Reid & Knipping, 2010, p. 136), practice transformations on visual objects to emphasize the mathematical structures and argue with them. Mathematical arguments which contain only words are called narrative arguments. Symbolic arguments are based on the algebraic symbol language. All these arguments can potentially be read and interpreted in a process- or product-oriented way (although this is already inherent in written narrative arguments). Each interpretation effects the validity of the proposed mathematical argument. Teachers are often only partly aware of these different interpretations when they guide their students in creating generally valid arguments.

### **Role of the teacher in mathematical argumentations in the transition from arithmetic to algebra**

Developing generally valid mathematical arguments in classroom discussions is challenging for students and teachers, especially in the transition from arithmetic to algebra. Teachers have a decisive influence on whether and how mathematical arguments emerge (Schwarz et al., 2006; Conner et al., 2014). In the transition from arithmetic to algebra, mathematical structures are increasingly emphasised, and mathematical properties and relations are used for generally valid arguments. Empirical research documents that symbolic arguments often dominate in mathematic classes (Brunner, 2014). These types of arguments focus mostly on process-oriented transformations of symbols. In this case, the algebraic symbol language often remains misunderstood, and learners demonstrate difficulties in grasping the underlying mathematical structures of the algebraic symbol language used (Kieran, 2020; Pedemonte, 2008). For example, students show difficulties in the interpretation of the equal sign. The equal sign can be interpreted as a “do something signal” or as a “symbol for equivalence” (Kieran, 1981, p. 317). The first, process-oriented interpretation triggers students to calculate the term. The left side of the equation is seen as a task one needs to solve, while the right side is considered as the solution of the calculation. In a product-oriented interpretation, the equal sign can be seen as a symbol between two equivalent terms. In this way, the relation between the terms is focused. Grasping mathematical structures and relations is also crucial when creating valid mathematical arguments for general claims. A process-oriented calculation of a term is not sufficient to grasp inherent mathematical structures and to capture general characteristics of the term. Often a product-oriented interpretation of mathematical objects is required. For example, when arguing with the equation “ $2 \cdot x + 2 \cdot y = 2 \cdot (x + y)$ ” that the sum of two even numbers is always even, one needs to interpret the algebraic symbols and the equal sign in a product-oriented way. Without an interpretation of the equal sign as a relational symbol (product-oriented) between two terms, one cannot grasp the relation between the terms and cannot argue that the sum of two even number is

always even. Students often struggle with these interpretations and need the support of teachers who are aware of these difficulties (Kieran, 2020).

## Methods

For researching mathematical argumentation processes in the transition from arithmetic to algebra, a learning environment with 4-5 lessons (90 min each) was designed and implemented in three eight-grade classes (13-14 years old) in Bremen (Germany). The lessons were taught by their teachers, who had different school experiences. Teacher A had been a fully trained teacher for half a year only, teacher B had been a teacher for 10 years and teacher C had been teaching about 5 years. Teacher A and C worked at an “Oberschule” (a German type of comprehensive school), teacher B at a “Gymnasium” (a selective type of school). As a starting point, different representations of mathematical arguments (Reid & Knipping, 2010) were introduced to the learners to facilitate learners’ various approaches to mathematical structures and to smooth the transition to algebra. Transcripts of the classroom discussions were made, which included drawings and descriptions of gestures and actions. Based on these transcripts the mathematical arguments that emerged in class were reconstructed with Toulmin’s (1958) functional model of arguments, which is well established in mathematics education (Knipping & Reid, 2015).

As the goal was to reconstruct and analyse actual classroom argumentations, our reconstructions include statements and arguments that are not correct from a mathematical point of view. We keep the statements in the reconstructed arguments almost the same as the original utterances of the participants during the classroom discussions. So, it is possible to use the statements in the reconstructed arguments (diagrams) to analyse the participants’ interpretations of the mathematical objects involved. For this analysis we use a distinction by Caspi and Sfard (2012), who differentiate between a processual, a granular and an objectified description of a term. A processual description of a term is a process-oriented description of the computational process. In an objectified, product-oriented description, the relationship between the objects is focused, and the actual computational process recedes into the background. Here verbs are nominalized to objects. The granular level is an intermediate step, which will not be focused in our analysis and this article. These analyses allow us to distinguish between a process-oriented and a product-oriented interpretation of mathematical objects in argumentations. For example, a student who describes the term “ $a+b$ ” as “adding  $b$  to  $a$ ” interprets the term in a process-oriented way, while “the sum of  $a$  and  $b$ ” would be considered as a product-oriented interpretation.

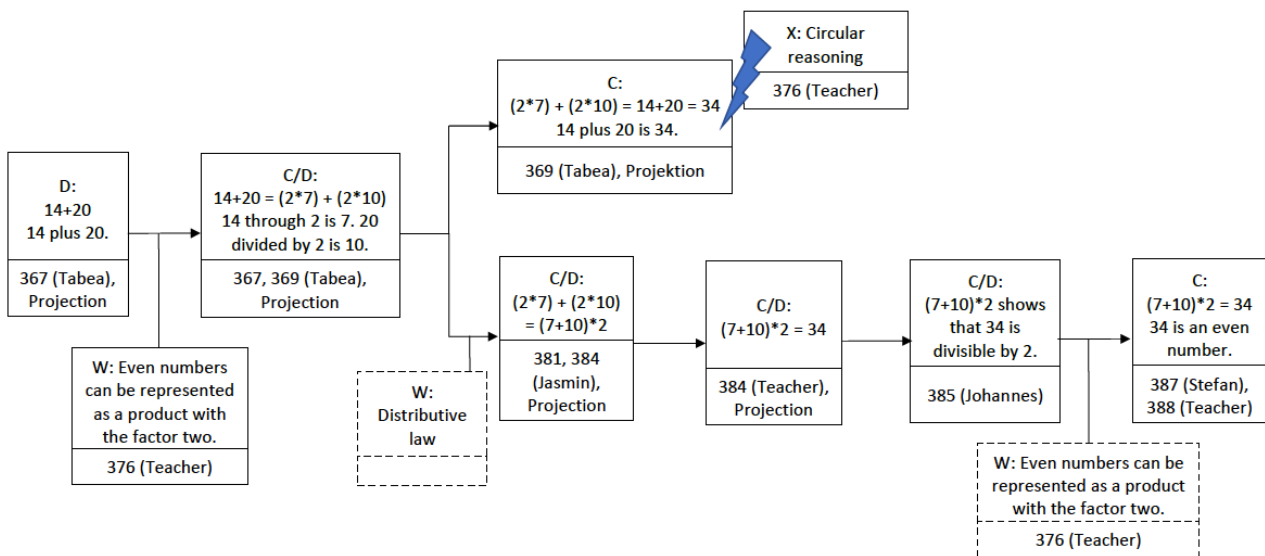
This paper focuses on an argumentation task about “The sum of two even numbers”. Students look at several numerical examples and generate a conjecture. Almost all students conjecture that the sum of two even numbers is always even. Afterwards, a reflection is requested if their conjecture is generally valid or if counterexamples exist. Finally, the students are asked to create a mathematical argument. In the following classroom discussion, students present their written arguments to the classroom community. While students’ written solutions and arguments are projected on a wall, argumentative elaborations on these products are presented, which constitute further argumentation processes. Products and processes of argumentations are intertwined here. The results of our analyses of these discussions and argumentations are focused in the next section.

## Results

In the following two examples, it becomes evident that students struggle with the process-product dualities mentioned above, when they attempt to construct a generally valid argument. The teacher shapes the students' handling of the process-product duality and thereby the creation of mathematical arguments. So, the crucial role of the teacher is apparent in these examples.

### Example 1: Numeric generic argument in teacher A's classroom

In the first example, Tabea tries to support the claim that “The sum of two even numbers is always even” with a numeric generic example (figure 1). A projection of Tabea's notes is shown to the class: “ $14+20=(2\cdot 7)+(2\cdot 10)=14+20=34$ ”. Tabea's first attempt to justify the claim that “The sum of two even numbers is even” is based on a transformation of her numerical example “ $14+20$ ”. Her first transformation “ $14+20=(2\cdot 7)+(2\cdot 10)$ ” can be seen as a step towards constructing a generic example. Tabea applies the property that even numbers are divisible by two and thus presents the numbers as a product with the factor two. In this representation a mathematical structure and a product-orientation become visible. In the second step however, Tabea switches to a calculation, “ $(2\cdot 7)+(2\cdot 10)=14+20$ ”, thus presents a process-oriented perspective. In this step, she doesn't use the represented mathematical structure for a general argument, but she comes back to her starting point. So, Tabea presents an argument that does not contain a generalisation and no connection to her general claim. Therefore, the teacher comments that Tabea “goes in a circle” and refutes Tabea's argument.



**Figure 1: Reconstruction of the numeric generic argument with the Toulmin scheme (Example 1)**

Later, the teacher repeats Tabea's first step and asks students to continue with “ $(2\cdot 7)+(2\cdot 10)$ ”: “What would be the next logical step now?”. Jasmin extends Tabea's first step to “ $(2\cdot 7)+(2\cdot 10)=(7+10)\cdot 2$ ” and uses the opportunity Tabea created. The inherent structure of the numerical example is used and represented explicitly. This requires a structural interpretation of the algebraic term and the equal sign. But now, the teacher pushes towards a calculation, saying that the product is 34. Here, the teacher switches back to a process-oriented interpretation. In the last two steps of the argument, the structure of the numerical example, inherent in the symbolic representation, is used for a conclusion by the students. Johannes finally states that the term “ $(7+10)\cdot 2$ ” shows that 34 is divisible by two,

and Stefan concludes that therefore, 34 is an even number. Both emphasise again a structural view. So, in these last two argumentation steps the structure of the generic example is used again. But like in Tabea’s attempt, a generalisation is missing - there is no explicit connection to the claim that the sum of two even numbers is even, which the class tries to legitimate.

How does the teacher shape the argument? After Jasmin’s transformation of the term, the teacher switches back to a process-oriented calculation of the result. The teacher guides the students with her statement to argue about the concrete number 34. She even asks what this term shows about 34: “What does this seven plus ten times two tell us now, what does it say about thirty-four?”. This question invites students to focus on the concrete number, which makes it difficult for the students to conduct a generalisation. The students offer a product-oriented interpretation of the term, but stick to the specific example, instead of justifying the general claim. Therefore, a final conclusion like “The sum of two even numbers is even” is missing in the whole argument. A numeric argument emerges, but its generality is not discussed and addressed by the students and the teacher.

### Example 2: Symbolic argument in teacher B’s classroom

The second example illustrates the creation of a symbolic argument about the sum of two even numbers in a classroom discussion (figure 2). As a starting point, the teacher asks the students to express an even number with variables by referring to a previous symbolic argument, which also includes even numbers. Levke introduces “ $2 \cdot n$ ” and “ $2 \cdot x$ ” and the teacher writes the sum of them on the blackboard. In these summands, a characteristic of even numbers is visible; each summand is a product with the factor two. But the students’ interpretation of the term is ambiguous. The situation allows either a process- or a product-oriented interpretation.

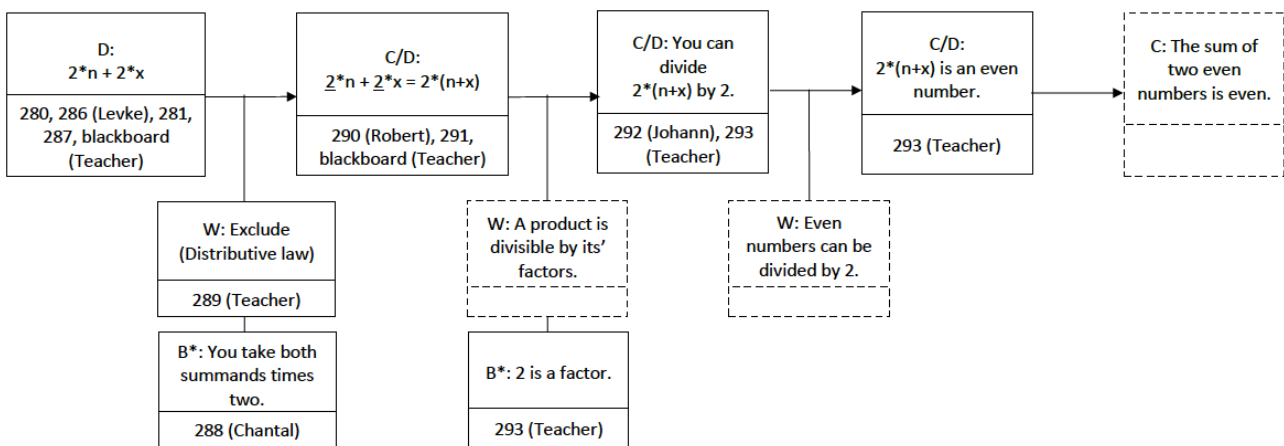


Figure 2: Reconstruction of the symbolic argument with the Toulmin scheme (Example 2)

The teacher asks what is common in both summands. Chantal points out that both are multiplied by two and thus indicates that the distributive law is applicable here. We reconstruct this as a specific kind of backing B\*, which justifies not the warrant in general, but the applicability of the warrant in this argumentation step. Chantal’s backing B\* indicates her process-oriented interpretation of the term. The associated warrant, added by the teacher, is also expressed in a process-oriented way. Consequently, the term is operationally transformed into “ $2 \cdot n + 2 \cdot x = 2 \cdot (n+x)$ ”. Now “ $2 \cdot (n+x)$ ” must be interpreted. Johann claims that one can divide this term by two. But he does not explicate a warrant

and sticks to a process-oriented interpretation. The corresponding warrant “A product is divisible by its’ factors.” is left implicit. The teacher complements Johann’s argumentation step by “2 is a factor”, which is another example of a backing of type B\*. This statement B\* relates the general warrant to the specific case. “2 is a factor” now supports a product-oriented interpretation. Finally, the teacher concludes that “ $2 \cdot (n+x)$ ” is an even number, and emphasises again a product-oriented interpretation, while the students’ interpretations of the term “ $2 \cdot (n+x)$ ” might be different. If they create a generalisation and therefore connect the argument to the implicit final conclusion is ambiguous.

All in all, the whole argumentation process is strongly guided by questions from the teacher and the students are forced to answer these questions. Students mainly suggest process-oriented statements, while the teacher sometimes switches to a structural level and supplements elements for the argument. The students seem not to grasp the structural ideas proposed by the teacher, but they keep answering and arguing in a process-oriented way.

## **Summary and Conclusion**

In the presented examples of two classrooms, both teachers try to address and shape the handling of the process-product dualities in students’ argumentations but in different ways. In both examples, a product-oriented interpretation is needed in the argumentation process to create a generalisation and produce a generally valid argument. In the first example, the teacher makes it difficult for the students to come up with such a generalisation, because she calculates the result of the generic example. The students use the structure of the created generic example for an argument, but they fail in generalising their claim; the students stick to their numeric example. The second classroom shows a teacher alternating between a process-oriented and a product-oriented interpretation, while students stick to a process-orientation. In two moments, students express a process-oriented statement and the teacher switches to a structural view. However, students do not reproduce this structural view in their contributions. In both classroom episodes a final general conclusion is implicit or even missing in the emerging argument.

All in all, the examples illustrate that a product-oriented interpretation of mathematical objects is often crucial for the creation of generally valid mathematical arguments in classrooms, while it is challenging for students and teachers. The first example also shows that a product-orientation is not sufficient. Especially for generic arguments, a generalisation is required for a complete and generally valid argument. A product-orientation does not produce a generalization by itself. In addition, students must handle the process-product duality of mathematical argumentation. The goal of these classroom discussions (example 1 and 2) is to formulate a written argument. In class, students and teachers switch back and forth between an interpretation of single steps in a written argument and the oral argumentation, where each step is justified and negotiated as reasonable.

It is evident that it matters for students how their teachers address the process-product duality of algebraic symbols. In both examples, it becomes obvious that the interplay between students’ and teachers’ process- or product-orientations is essential. Students need support in reflecting structures inherent in mathematical objects and making generalisations. The comparative analyses of the data in our project illustrate different challenges that teachers encounter in this context. More insights will be gained by further analysis in this project.



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