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Arnaud Mayeux

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# Algebraic Magnetism

Arnaud Mayeux

*With an appendix by Matthieu Romagny*

Abstract. For a diagonalizable monoid scheme $A(M)_S$ acting on an algebraic space $X$, we introduce for any submonoid $N$ of $M$ an attractor space $X^N$. We then investigate and study various aspects of attractors associated to monoids.

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*Keywords:* algebraic magnetism, magnets, attractors associated to magnets, monoids, faces of monoids, Grothendieck schemes, algebraic spaces, diagonalizable monoid schemes, diagonalizable group schemes, group schemes, reductive group schemes, parabolic groups, Levi subgroups, groups of type R, root groups, actions of group schemes on algebraic spaces, fixed-points, Z-FPR atlases

## 1. Introduction

Let $M$ be a finitely generated abelian group and let $S$ be a base scheme. Let $D(M)_S$ be the associated diagonalizable group scheme (i.e. if $M = \mathbb{Z}^r \times \prod_{i=1}^n \mathbb{Z}/n_i\mathbb{Z}$, then $D(M)_S = \mathbb{G}_{m,S}^r \times \prod_{i=1}^n \mu_{n_i,S}$ is the product of a split torus with group schemes of roots of unity.) Algebraic actions of diagonalizable group schemes appear systematically in many areas of mathematics. This article is devoted to introduce a new tool in the general context of an arbitrary algebraic action of a diagonalizable group scheme on a scheme or on an algebraic space. The style is foundational and we in fact develop the theory for any diagonalizable monoid scheme (to be defined in this article).

### 1.1 Definition of algebraic attractors

Let $S$ be a scheme. Let $M$ be an abelian monoid. Let $\mathbb{Z}[M]$ be the ring $\bigoplus_{m\in M} \mathbb{Z}X^m$ where the multiplication is induced by the structure of monoid on $M$. Let $A(M)$ be $\mathrm{Spec}(\mathbb{Z}[M])$, it is a monoid scheme over $\mathrm{Spec}(\mathbb{Z})$. We consider the base change $A(M)_S = A(M) \times_{\mathrm{Spec}(\mathbb{Z})} S$, it is called a diagonalizable monoid scheme over $S$ (if $M$ is moreover a group $A(M)_S$ is also denoted $D(M)_S$ and is a diagonalizable group scheme). For any submonoid $N$ of $M$, $A(M)_S$ acts canonically on $A(N)_S$. Let $X$ be an algebraic space over $S$ with an action $a$ of $A(M)_S$. The main idea in this paper is to introduce the following definition, for any submonoid $N$ of $M$:

Definition 1.1. Let $X^N$ be the contravariant functor from schemes over $S$ to Sets given by

$$(T \to S) \mapsto \mathrm{Hom}_T^{A(M)_T}(A(N)_T, X_T)$$

where $\mathrm{Hom}_T^{A(M)_T}(A(N)_T, X_T)$ is the set of $A(M)_T$-equivariant $T$-morphisms from $A(N)_T$ to $X_T = X \times_S T$. The functor $X^N$ is called the attractor associated to the submonoid $N$ under the action of $A(M)_S$ on $X$.

### 1.2 Magnets and attractors

We proceed with the notation from §1.1 and assume that $X \to S$ is separated. If $N \subset L$ are submonoids of $M$, then we have a canonical monomorphism $X^N \subset X^L$. Moreover $X^0$ identifies with the fixed-points functor $X^{A(M)_S}$ and $X^M$ identifies with $X$. So for any submonoid $N \subset M$, we have monomorphisms $X^0 \subset X^N \subset X$. We now use the following terminology: a magnet for the action $a$ is a submonoid $N$ of $M$. A magnet $N \subset M$ is thought as something which algebraically attracts the subspace $X^N$ of $X$. For each attractor space $X^N$ there is a minimal magnet $E$ such that $X^N = X^E$, $E$ is called a pure magnet. We have a bijection between attractor spaces and the set $\mho(a)$ of pure magnets (cf. Theorem 15.3). Assume now moreover that $X$ is finitely presented over $S$, Theorem 15.4 says that the set of pure magnets $\mho(a)$ is finite (cf. Theorem 15.4 for our precise assumptions).

### 1.3 Faces

We proceed with the notation from §1.2 and assume for simplicity that $M$ is a finitely generated abelian groups. We fix a submonoid $N \subset M$. A face of $N$ is by definition a submonoid $F$ of $N$ such that the canonical $M$-graded projection $\mathbb{Z}[N] \to \mathbb{Z}[F]$ is a morphism of rings. Equivalently, $F \subset N$ is a face if for any two elements $x, y \in N$ the following equivalence holds:

$$x + y \in F \Leftrightarrow x \in F \text{ and } y \in F.$$

Note that the only face of the group $M$ is $M$ itself, so faces matter only in the world of monoids. Each face of $N$ contains $N^*$, the face of invertible elements in $N$. Now if $F \subset N$ is a face, we get a canonical transformation of functor $\mathfrak{p}_{N,F} : X^N \to X^F$. Recall that on the other hand we have a monomorphism $X^F \to X^N$. The map $\mathfrak{p}_{N,F} : X^N \to X^F$ could be thought as a directional limit. Now if $Z \subset X^F$ is a monomorphism, we put $X^N_{F,Z} = X^N \times_{X^F} Z$ and we call it the attractor associated to $N$ with prescribed limit $Z$ relatively to the face $F$, if $F = N^*$ we also use the notation $X^N_Z$. We have a canonical monomorphism $X^N_{F,Z} \to X^N$. The concept of attractors with prescribed limits allows to reduce the fixed-points parts of attractors.

### 1.4 Summary of results

This article studies intrinsically algebraic magnetism, i.e. the formalism of attractors associated to magnets. Consequently, we prove in this paper a large number of results. We list here the most significant ones as informal slogans, with precise references.

(i) Attractors are compatible with fiber products and base changes (cf. Proposition 3.7, Proposition 3.8 and Proposition 3.9).

(ii) Attractors preserve monoid and group structures (cf. Proposition 3.14).

(iii) Attractors are compatible with equivariant actions in a natural sense (cf. Proposition 3.29).

(iv) Attractors are representable in many cases (cf. Theorems 3.19 and 8.3, Remark 8.4 and Proposition 3.29 (iii)).

(v) In the affine case, attractors are representable by explicit closed subspaces and intersections of attractors correspond to intersections of magnets (cf. Theorem 3.19 and Proposition 3.22).

(vi) Inclusions of monoids give (mono)morphisms on attractors and face inclusions provide retractions (cf. Remark 3.5, Fact 3.6, Fact 3.10, Fact 3.15, Proposition 3.27, Corollary 3.28).

(vii) Equivariant morphisms of spaces induce morphisms on attractors; moreover closed immersions give closed immersions, open immersions give open immersions, smooth morphisms give smooth morphisms, étale morphisms give étale morphisms, unramified morphisms give unramified morphisms, monomorphisms give monomorphisms, locally finitely presented morphisms give locally finitely presented morphisms (cf. Fact 3.16, Corollary 11.2, Corollary 11.6, Corollary 11.4, Fact 3.17, Lemma 3.24, Proposition 12.2, Fact 3.18). However, flat morphisms do not give flat morphisms on attractors in general (cf. Remark 11.10).

(viii) Attractors of attractors make sense and correspond to intersections of magnets (cf. Remarks 3.11 and 3.12 and Proposition 3.30).

(ix) Attractors associated to subgroups correspond to fixed-points under diagonalizable group schemes (cf. Proposition 3.32).

(x) Faces and attractors allow to obtain easily non trivial cartesian diagrams (cf. Remark 16.9 and Proposition 3.33).

(xi) Attractors preserve smoothness in two different senses (cf. Section 11, e.g. Corollaries 11.2 and 11.9).

(xii) The morphism $X^N \to X^{N^*}$ induces a bijection on the sets of connected components (cf. Proposition 12.1).

(xiii) Attractors commute with tangent spaces and Lie algebras (cf. Propositions 16.2 and 16.4).

(xiv) Attractors make sense for ind-spaces and behave as one can expect (cf. Proposition 14.4).

(xv) Attractors commute with dilatations (cf. Proposition 13.1).

### 1.5 Examples

Let us discuss some examples. We refer to §16 and Example 15.10 for some other examples.

Example 1.2. *Assume that $X = \mathbb{A}^n_S =: \mathbb{V}$ and $A(M)_S$ acts linearly so that the action induces a direct sum decomposition in weight spaces $\mathbb{V} = \bigoplus_{m\in M} \mathbb{V}_m$, then $\mathbb{V}^N = \bigoplus_{n\in N} \mathbb{V}_n \subset \mathbb{V}$.*

Example 1.3. *(Magnetic point of view on reductive groups) Let $G$ be a reductive group scheme over $S$. Let $T = D(M)_S$ be a maximal split torus of $G$. Let $a$ be the adjoint action of $T$ on $G$. The set of pure magnets is given by additively stable sets of roots. Algebraic attractors associated to the action $a$ give all the well-known classical objects of the theory of reductive groups (Levi subgroups containing $T$, parabolic subgroups containing $T$, "groupes de type R à fibres résolubles" [SGA3, Exp. XXII §5.6]). Moreover attractors with prescribed limits give unipotent radicals of such objects, namely root groups and unipotent radicals of parabolic groups. We refer to §16.4, Proposition 16.14 and [ALRR22, §6.3] for some precise statements. It is natural to elaborate further on the relation of our formalism with the study of the structure of more general algebraic groups.*

### 1.6 Relation with other works

Let us list the main sources of inspiration for our work.

(i) Our work is of a completely different nature than logarithmic algebraic geometry. However it has in common with logarithmic algebraic geometry to use schemes associated to monoids as background and [Og, Part I] were useful at some stages of the realization of our work (cf. e.g. Proposition 2.8 and Fact 2.9).

(ii) If $M = \mathbb{Z}$ and $N = \mathbb{N}$, then our attractors restrict to the well-known attractors associated to $\mathbb{G}_m$-actions. As a consequence, our work was highly inspired by the following beautiful works that include studies of $\mathbb{G}_m$-actions

$$[\mathrm{Ref}_{\mathbb{G}_m}] := \Big\{[\mathrm{Bi73}], [\mathrm{He80}], [\mathrm{Ju85}], [\mathrm{CGP10}], [\mathrm{Dr15}], [\mathrm{DG14}], [\mathrm{Mar15}], [\mathrm{Ri16}], [\mathrm{HR21}]\Big\}.$$

We now provide some examples of relations between statements in our work and in $[\mathrm{Ref}_{\mathbb{G}_m}]$:

(a) statement of Definition 3.1 was partly inspired by [Ri16, Definition 1.3], [Dr15, Definition 1.3.2], [He80, II.4.1], etc,
(b) statement of Theorem 3.19 was partly inspired by [CGP10, Lemma 2.1.4], [Dr15, §1.3.4], [Ri16, Lemma 1.9], etc,
(c) statement of Proposition 5.1 was partly inspired by [Ri16, Lemma 1.11],
(d) statements of Propositions 6.3 and 8.1 and Theorem A.1 were partly inspired by [Ri16, Lemma 1.10], [Ri16, Theorem 1.8 (i)], [Dr15, Proposition 1.2.2.], etc,
(e) statement of Theorem 8.3 was partly inspired by [Ri16, Theorem 1.8], [Dr15, Theorem 1.4.2], etc
(f) statement of Proposition 12.1 was partly inspired by [Ri16, Corollary 1.12],
(g) statement of Corollary 11.2 was partly inspired by [Mar15, Theorem 1.1], [Ri16, Theorem 1.8 (iii)], [HR21, Lemma 2.2 (ii)],
(h) statement of Corollary 11.9 was partly inspired by [Dr15, Proposition 1.4.20],
(i) statement of Proposition 12.2 was partly inspired by [HR21, Corollary 2.3].

Similarly, some of our proofs are also partly inspired by the proofs in $[\mathrm{Ref}_{\mathbb{G}_m}]$. Our proof of smoothness results (Corollaries 11.2 and 11.9) use formal smoothness (Propositions 11.1 and 11.7) and was partly inspired by [SGA3, Exp. XII Théorème 9.7 (unpublished)]. We

invite the interested reader to read [$\mathrm{Ref}_{\mathbb{G}_m}$] to form his own opinion. Of course, many statements on attractors associated to monoids in our paper do not have analogs in [$\mathrm{Ref}_{\mathbb{G}_m}$] cf. e.g. Proposition 3.24, Proposition 6.8, Fact 3.15, Proposition 3.22, Proposition 3.33, Proposition 8.4, Proposition 13.1, Corollary 13.3, Theorem 15.3, Theorem 15.4, etc. We note that Proposition 3.30 makes sense for $\mathbb{G}_m$-actions (it says that $(X^+)^- = (X^-)^+ = X^0$) but we did not come across it in [$\mathrm{Ref}_{\mathbb{G}_m}$]. Let us mention that [JS18] and [JS20] generalize attractors associated to $\mathbb{G}_m$-actions in an orthogonal direction to our.

(iii) Of course this work is written in the language of Grothendieck schemes [EGA] and Artin's algebraic spaces [Ar71], we mainly use [StP] as treatment for this theory in the present text. We use algebraic spaces instead of schemes for the same reasons than [Dr15] and [Ri16]: some actions of group schemes that we are interested in are not Zariski locally linearizable but are étale locally linearizable by deep results of Alpher-Hall-Rydh [AHR21]. This leads to use étaleness as local notion instead of openess and so to use algebraic spaces instead of schemes.

(iv) Our work was inspired by [SGA3] for many technical aspects around group schemes.

### 1.7 Organization of the paper

1.7.1 *Sections 2-16.* Section 2 introduces diagonalizable monoid schemes and often relies on [Og]. Section 3 introduces algebraic attractors associated to monoids and prove several results. Section 4 introduces attractors with prescribed limits. Sections 5-6-7-8 take care of several results used to prove the representability of attractors in non-affine cases, in particular the notion of $Z$-FPR and strongly-FPR atlases are introduced. Sections 9-10-11 deal with formal smoothness and formal étaleness results. Section 12 is about topology of attractors. Section 13 shows that attractors are compatible with dilatations. Section 14 takes care of ind-spaces. Section 15 studies pure magnets. Section 16 is mainly about examples.

1.7.2 *Appendix A.* The important Appendix A, written by M. Romagny, is devoted to prove the existence of $Z$-FPR atlases (cf. Theorem 6.8) using deep results of Alper-Hall-Rydh [AHR21]. Some important results of the paper are stated under the existence of $Z$-FPR atlases, cf. e.g Theorem 8.3. One knows that such atlases exists for Sumihiro's actions, but some actions are not Sumihiro (cf. Section 7) and in this case one needs to know the existence of $Z$-FPR atlases using other tools (this is also related to §1.6(iii)). This is why Appendix A is important. Appendix A also contains interesting generalizations of some results stated in other sections (cf. Proposition 6.3, Proposition 8.1, and Theorem A.1).

### 1.8 Applications

We expect that our formalism and results could lead to applications, simplifications and conceptual reformulations in most areas involving actions of diagonalizable group schemes. This includes the following areas of mathematics: structure of algebraic groups, Bruhat-Tits theory, representation theory, affine Grassmannians, geometric representation theory, geometric Satake equivalence, Lie theory and physics, number theory, Langlands programs, etc. For example, the reader is refered to [ALRR22, §6.3] for an application of an early version of the present work to results on the structure of algebraic groups.

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## 2. Rings associated to monoids and their spectra

### 2.1 Reference for the language of commutative monoids

We refer to [Og] for a detailed and beautiful introduction to monoids and related structures. We recall in this section some basic definitions and facts that we frequently use in our work. In this article, monoids and rings are always commutative. Readers unfamiliar with monoids should read [Og, I.1]. Let us recall some very basic notations, again we refer to [Og, I] for a more general and conceptual presentation. Let $P$ be a monoid and $N \subset P$ and $L \subset P$ be two submonoids. Let $N + L$ be $\{n + l \in P | n \in N, l \in L\}$. Then $N + L$ is a submonoid of $P$. An arbitrary intersection of submonoids is a submonoid. Let $P$ be a monoid and let $E$ be a subset of $P$, then we denote by $[E\rangle$ the smallest submonoid of $P$ containing $E$, this is the intersection of all monoids of $P$ containing $E$. Similarly if $M$ is a group and $E$ is a subset of $M$, we denote by $(E)$ the subgroup of $M$ generated by $E$. Let $M$ be an abelian group and let $N$ be a submonoid of $M$. The subgroup generated by $N$ in $M$ is denoted $N^{\mathrm{gp}}$, in fact $N^{\mathrm{gp}} = \{x - y | x, y \in N\}$. We have an obvious notion of finitely generated monoids. The monoid $(\mathbb{N}_{\geqslant 1} \times \mathbb{N}) \cup (0, 0)$ is not finitely generated and it is a submonoid of the finitely generated abelian group $\mathbb{Z} \times \mathbb{Z}$.

### 2.2 Diagonalizable monoid algebraic spaces

We fix in this section an arbitrary commutative monoid $M$. We will often work with submonoids of $M$ and will often use the symbols $N, N', L, Q$ or $F$ to denote them.

Definition 2.1. Let $N$ be a commutative monoid. Let $\mathbb{Z}[N]$ be the ring whose underlying abelian group is $\bigoplus_{n \in N} \mathbb{Z}X^n$, where for any $n$ the abelian group $\mathbb{Z}X^n$ is a formal copy of $\mathbb{Z}$, and multiplication is induced by the operation of the monoid: $X^n \times X^{n'} = X^{n+n'}$. We write $X^0 = 1$. The ring $\mathbb{Z}[N]$ is called the ring associated to the monoid $N$.

Fact 2.2. *Let $N$ be a commutative monoid. The ring $\mathbb{Z}[N]$ is a bialgebra over $\mathbb{Z}$. The augmentation is the map $\mathbb{Z}[N] \to \mathbb{Z}$ sending $X^n$ to $1$ for every $n \in N$. The comultiplication is the map $\mathbb{Z}[N] \to \mathbb{Z}[N] \otimes \mathbb{Z}[N]$ sending $X^n$ to $X^n \otimes X^n$ for $n \in N$. Moreover, if $N$ is a group, $\mathbb{Z}[N]$ is a Hopf algebra over $\mathbb{Z}$, the antipode being the map $\mathbb{Z}[N] \to \mathbb{Z}[N]$ sending $X^n$ to $X^{-n}$ for every $n \in N$.*

Definition 2.3. Let $N$ be a commutative monoid. Let $R$ be a ring. Let $B$ be an algebraic space over a scheme $S$. Let $\mathcal{O}_S$ be the structure sheaf of $S$ (cf. [StP, Tag 01IJ] and [StP, Tag 0091]). Let $\mathcal{O}_B$ be the structure sheaf of $B$ in the sense of [StP, Tag 04KD].

(i) We put $R[N] = \mathbb{Z}[N] \otimes_{\mathbb{Z}} R$. Obviously, $R[N] = \bigoplus_{n \in N} RX^n$. The ring $R[N]$ is canonically a $R$-bialgebra.

(ii) Let $\mathcal{O}_S[N]$ be the $\mathcal{O}_S$-algebra obtained by sheafification of the presheaf of algebras given by $\mathcal{O}_S(U)[N]$ for any open subset $U \subset S$, this is a sheaf of algebras (cf. [StP, Tag 00YR]). Note that the underlying $\mathcal{O}_S$-module is called the free $\mathcal{O}_S$-module with basis $N$. In particular, as $\mathcal{O}_S$-module, $\mathcal{O}_S[N]$ is isomorphic to $\bigoplus_{n \in N} \mathcal{O}_S$. If $U$ is a quasi-compact open subset of $S$, then $\mathcal{O}_S[N](U) = \mathcal{O}_S(U)[N]$ (cf. [StP, Tag 01AI]). The $\mathcal{O}_S$-algebra $\mathcal{O}_S[N]$ is canonically an

$\mathcal{O}_S$-bialgebra.

(iii) Let $\mathcal{O}_B[N]$ be the $\mathcal{O}_B$-algebra obtained by sheafification of the presheaf of algebras given by $\mathcal{O}_B(T)[N]$ for any $T \in B_{spaces,étale}$ (cf. [StP, Tag 03G0] for the notation $B_{spaces,étale}$), this is a sheaf of algebras (cf. [StP, Tag 00YR]). Note that the underlying $\mathcal{O}_B$-module is called the free $\mathcal{O}_B$-module with basis $N$ (cf. [StP, Tag 03DD]). In particular, as $\mathcal{O}_B$-module, $\mathcal{O}_B[N]$ is isomorphic to $\bigoplus_{n\in N} \mathcal{O}_B$. If $T \in B_{spaces,étale}$ is quasi-compact, then $\mathcal{O}_B[N](T) = \mathcal{O}_B(T)[N]$ by [StP, Tag 0935]. The $\mathcal{O}_B$-algebra $\mathcal{O}_B[N]$ is canonically an $\mathcal{O}_B$-bialgebra.

(iv) Assume that moreover $N$ is a group, then $R[N]$ (resp. $\mathcal{O}_S[N]$, resp. $\mathcal{O}_B[N]$) is canonically a Hopf algebra over $R$ (resp. $\mathcal{O}_S$, resp. $\mathcal{O}_B$).

Definition 2.4. (Diagonalizable monoid schemes and algebraic spaces) Let $N$ be a commutative monoid. Let $S$ be a scheme and let $B$ be an algebraic space over $S$.

(i) Let $A(N)$ be the scheme $\mathrm{Spec}(\mathbb{Z}[N])$. This is a monoid scheme over $\mathrm{Spec}(\mathbb{Z})$.

(ii) We put $A(N)_S = A(N) \times_{\mathrm{Spec}(\mathbb{Z})} S$, this is canonically a monoid scheme over $S$.

(iii) We put $A(N)_B = A(N)_S \times_S B$, this is canonically a monoid algebraic space over $B$.

(iv) The objects $A(N), A(N)_S$ and $A(N)_B$ are called diagonalizable monoid schemes and algebraic spaces. If $N = M$ is an abelian group, then $A(M)$ is denoted $D(M)$ and is a group scheme over $\mathrm{Spec}(\mathbb{Z})$ called the diagonalizable group scheme associated to $M$. We define similarly $D(M)_S$ and $D(M)_B$.

Remark 2.5. Let $M$ be an abelian group and $N$ be a monoid. The reference [SGA3] uses the notation $D_S(M)$ where we use $D(M)_S$. Similarly, the notation $A_B(N)$ can be used to denote $A(N)_B$. We like to use standard notation and see $A(N)_B$ and $D(M)_B$ as base change from $\mathrm{Spec}(\mathbb{Z})$ to $B$ of $A(N)$ and $D(M)$.

Fact 2.6. *Let $N$ be a commutative monoid. Let $B$ be an algebraic space over a scheme $S$.*

(i) *The monoid scheme $A(N)_S$ is affine and flat over $S$, moreover its quasi-coherent $\mathcal{O}_S$-bialgebra is $\mathcal{O}_S[N]$ (cf. Definition 2.3).*

(ii) *The monoid algebraic space $A(N)_B$ is affine and flat over $B$, moreover its quasi-coherent $\mathcal{O}_B$-bialgebra is $\mathcal{O}_B[N]$ (cf. Definition 2.3).*

*Proof.* Being affine and flat is stable by base change, so it is enough to prove that $A(N)$ is affine and flat over $\mathrm{Spec}(\mathbb{Z})$. It is obvious that $A(N)$ is affine and flat over $\mathrm{Spec}(\mathbb{Z})$ and this proves the first parts of both assertions. We claim that the $\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}$-bialgebra of $A(N)$ is $\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}[N]$. Let $p = A(N) \to \mathbb{Z}$. We know that $p_*\mathcal{O}_{A(N)} = \widetilde{\mathbb{Z}[N]}$. So by [StP, Tag 01ID], we have $p_*\mathcal{O}_{A(N)} = \widetilde{\bigoplus_{n\in N} \mathbb{Z}} = \bigoplus_{n\in N} \widetilde{\mathbb{Z}} = \bigoplus_{n\in N} \mathcal{O}_{\mathrm{Spec}(\mathbb{Z})} = \mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}[N]$ and this finishes to prove the claim. Now by [StP, Tag 01SA], we have that $q_*\mathcal{O}_{A(N)_S} = r^*(p_*\mathcal{O}_{A(N)})$ where $q : A(N)_S \to S$ and $r : S \to \mathbb{Z}$. So by [StP, Tag 01AJ], we have $q_*\mathcal{O}_{A(N)_S} = \mathcal{O}_S[N]$. The last assertion is proved similarly using [StP, Tag 03M1], [StP, Tag 081V] and [StP, Tag 03DC]. □

Fact 2.7. *We have a canonical identification $A(N \times L) = A(N) \times_{\mathrm{Spec}(\mathbb{Z})} A(L)$ for any pair of monoids $N, L$. If $B$ is an algebraic space over a scheme $S$, then $A(N \times L)_S = A(N)_S \times_S A(L)_S$ and $A(N \times L)_B = A(N)_B \times_B A(L)_B$.*

*Proof.* It is enough to prove the first assertion. This follows from the identity

$$\mathbb{Z}[N \times L] = \bigoplus_{(n,l)\in N\times L} \mathbb{Z}X^{(n,l)} = \big(\bigoplus_{n\in N} \mathbb{Z}X^n\big) \otimes_{\mathbb{Z}} \big(\bigoplus_{l\in L} \mathbb{Z}X^l\big) = \mathbb{Z}[N] \otimes_{\mathbb{Z}} \mathbb{Z}[L].$$

□

Recall that a monoid $N$ is cancellative if for all $x, y, z \in N$, $x + y = x + z$ implies $y = z$. In [Og], the word "integral" is used instead of "cancellative", cf. [Og, Definition 1.3.1]. A monoid is cancellative if and only if it identifies with a submonoid of a group.

Proposition 2.8. *Let $N$ be a cancellative commutative monoid. Let $B$ be an algebraic space over a scheme $S$.*

(i) *The morphism of rings $\mathbb{Z}[N] \to \mathbb{Z}[N \times N]$, $X^n \mapsto X^{(n,n)}$ is flat.*
(ii) *The multiplication $m : A(N) \times_{\mathrm{Spec}(\mathbb{Z})} A(N) \to A(N)$ of the monoid scheme $A(N)$ is flat.*
(iii) *The multiplication morphism $m : A(N)_S \times_S A(N)_S \to A(N)_S$ is flat.*
(iv) *The multiplication morphism $m : A(N)_B \times_B A(N)_B \to A(N)_B$ is flat.*

*Proof.* The monoid $N$ acts freely on $N \times N$ via $n \cdot (m, l) = (m + n, l + n)$. So the $\mathbb{Z}[N]$-module $\mathbb{Z}[N \times N]$ is flat by [Og, Prop. 4.5.12 p.134], and so (i) holds. Now (ii) follows using Facts 2.2 and 2.7. Assertions (iii) and (iv) follow since flatness is preserved by base change. □

Fact 2.9. *Let $N$ be a finitely generated monoid. Let $B$ be an algebraic space over a scheme $S$.*

(i) *The monoid $N$ is finitely presented.*
(ii) *The $\mathbb{Z}$-algebra $\mathbb{Z}[N]$ is finitely presented.*
(iii) *The structural morphism $A(N) \to \mathrm{Spec}(\mathbb{Z})$ is finitely presented.*
(iv) *The multiplication morphism $A(N) \times_{\mathrm{Spec}(\mathbb{Z})} A(N) \to A(N)$ is finitely presented.*
(v) *The morphisms $A(N)_S \to S$ and $A(N)_S \times_S A(N)_S \to A(N)_S$ are finitely presented.*
(vi) *The morphisms $A(N)_B \to B$ and $A(N)_B \times_B A(N)_B \to A(N)_B$ are finitely presented.*

*Proof.* (i) This is [Og, Theorem 2.1.7].
(ii) This follows from (i).
(iii) This follows from (ii).
(iv) Since $N$ is finitely generated, $N \times N$ is finitely generated and so $A(N) \times_{\mathrm{Spec}(\mathbb{Z})} A(N)$ is finitely presented over $\mathrm{Spec}(\mathbb{Z})$. Now the assertion follows using [StP, Tag 02FV].
(v) This follows using previous assertions and [StP, Tag 01TS].
(vi) This follows using previous assertions and [StP, Tag 049M].

□

Definition 2.10.

(i) Let $e_{\mathbb{Z}}$ be the trivial monoid scheme over $\mathrm{Spec}(\mathbb{Z})$, it is a group scheme over $\mathrm{Spec}(\mathbb{Z})$ equal to $\mathrm{Spec}(\mathbb{Z})$ as scheme. Note that $e_{\mathbb{Z}} \simeq A(0)$ where 0 is the trivial monoid.
(ii) Let $S$ be a scheme. We put $e_S = e_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} S$, this is the trivial monoid scheme over $S$.
(iii) Let $B$ be an algebraic space over a scheme $S$. We put $e_B = e_S \times_S B$, this is the trivial monoid algebraic space over $B$.

Fact 2.11. *Let $N$ be a submonoid of $M$. Let $B$ be an algebraic space over $S$. Then we have an algebraic action of $A(M)$ on $A(N)$ over $\mathrm{Spec}(\mathbb{Z})$ given by:*

$$\mathbb{Z}[N] \to \mathbb{Z}[N] \otimes \mathbb{Z}[M]$$
$$X^n \mapsto X^n \otimes X^n.$$

*Equivalently, this action comes from the general Proposition 2.21 that we discuss below in this section. Remark also that the action of $A(M)$ on $A(N)$ comes from the morphism of monoid schemes $A(M) \to A(N)$. By base change, we obtain actions of $A(M)_S$ on $A(N)_S$ and of $A(M)_B$ on $A(N)_B$.*

FACT 2.12. *Let $N' \subset N$ be submonoids of $M$, let $B$ be an algebraic space over $S$. Then we have a canonical morphism of bialgebras over $\mathbb{Z}$, $\mathbb{Z}[N'] \to \mathbb{Z}[N]$ (sending $X^{n'}$ to $X^{n'}$ for any $n' \in N'$). This induces a canonical morphism $A(N) \to A(N')$ of monoid schemes over $\mathbb{Z}$, this morphism is $A(M)$-equivariant. We obtain canonical equivariant morphisms of monoid objects $A(N)_S \to A(N')_S$ and $A(N)_B \to A(N')_B$.*

*Proof.* The inclusion morphism preserves $M$-gradings on $\mathbb{Z}[N']$ and $\mathbb{Z}[N]$. □

DEFINITION 2.13. Let $F \subset N$ be submonoids of $M$. We say that $F$ is a face of $N$ if the projection map

$$\mathbb{Z}[N] \to \mathbb{Z}[F], X^n \mapsto 0 \text{ if } n \in N \setminus F \text{ and } X^n \mapsto X^n \text{ if } n \in F$$

is a morphism of rings.

FACT 2.14. *If $F$ is a face of a submonoid $N$ of $M$, then the associated morphism of schemes $A(F) \to A(N)$ is $A(M)$-equivariant. For any algebraic space $B$ over any scheme $S$, $A(F)_S \to A(N)_S$ is $A(M)_S$-equivariant and $A(F)_B \to A(N)_B$ is $A(M)_B$-equivariant.*

*Proof.* The projection morphism preserves $M$-gradings on $\mathbb{Z}[F]$ and $\mathbb{Z}[N]$. □

PROPOSITION 2.15. *Let $F \subset N$ be submonoids of $M$. Then $F$ is a face of $N$ if and only if for all $x, y \in N$*

$$x + y \in F \Leftrightarrow x \in F \text{ and } y \in F.$$

*Proof.* Let $\phi$ denote the projection and assume it is a morphism of rings. Let $x, y \in N$. Then $x+y \in F \Leftrightarrow \phi(X^{x+y}) = X^{x+y} = \phi(X^x)\phi(X^y)$ is not zero $\Leftrightarrow$ both $x$ and $y$ are in $F$. Reciprocally assume that for all $x, y \in N$, $x + y \in F \Leftrightarrow x \in F$ and $y \in F$, then we have $\phi(X^x X^y) = \phi(X^x)\phi(X^y)$. □

PROPOSITION 2.16. *Let $N$ be a monoid. Let $N^* = \{x \in N | \ \exists y \in N \text{ such that } x + y = 0\}$, then $N^*$ is a submonoid of $N$ and a group, moreover $N^*$ is a face of $N$. The group $N^*$ is the largest subgroup of $N$, called the face of invertible elements.*

*Proof.* We apply Proposition 2.15 as follows. Take $x, y \in N$. Assume $x + y \in N^*$, then there exists $z \in N$ such that $x + y + z = 0$, this shows that $x$ and $y$ are in $N^*$. □

FACT 2.17. *Let $F$ be a face of a monoid $N$, then $N^* \subset F$.*

*Proof.* Let $n \in N^*$. Then $n + (-n) = 0$ belongs to $F$ and so $n$ belongs to $F$. □

PROPOSITION 2.18. *Let $N \subset L \subset L'$ be submonoids of $M$. Assume that $L$ is a face of $L'$. Then $N' := L' \setminus (L \setminus N)$ is a submonoid of $M$ and $N$ is a face of $N'$; moreover for any scheme $S$, $A(N')_S$ is the push-out, in the category of schemes, of*

$$\begin{tikzcd} & A(L)_S \arrow[dl] \arrow[dr] & \\ A(L')_S & & A(N)_S \end{tikzcd}.$$

*Proof.* We have $L' \setminus (L \setminus N) = (L' \setminus L) \sqcup N$ and it is clearly a submonoid of $M$. It is also clear that $N$ is a face of $N'$. Let us assume firstly that $S = \mathrm{Spec}(R)$ is affine. We have $R[L'] \times_{R[L]} R[N] \cong \{(x, y) \in R[L'] \times R[N] | f(x) = g(y)\} \cong R[N']$, indeed an element $x$ in $R[L']$ maps to an element in $R[N]$ under the projection $R[L'] \to R[L]$ if and only if $x \in R[(L' \setminus L) \sqcup N]$. The map $R[N'] \to R[N]$ is the projection morphism associated to the face inclusion $N \subset N'$. The map $R[N'] \to R[L']$ is the morphism associated to the inclusion $N' \subset L'$. So by [StP, 0ET0] the scheme $A(N')_R$ is the push-out, in the category of schemes, of the diagram

$$\begin{tikzcd} & A(L)_R \arrow[dl] \arrow[dr] & \\ A(L')_R & & A(N)_R \end{tikzcd}.$$

Now let us prove the general case. Let $S = \cup_{i \in I} U_i$ be an affine open covering and write $U_i = \mathrm{Spec}(R_i)$. Let $Y$ be a scheme and let $A(L')_S \to Y$ and $A(N)_S \to Y$ be two morphisms such that the following diagram commutes

$$\begin{tikzcd} & A(L)_S \arrow[dl] \arrow[dr] & \\ A(L')_S \arrow[dr] & & A(N)_S \arrow[dl] \\ & Y & \end{tikzcd}.$$

We then obtain, for any $i \in I$, a commutative diagram

$$\begin{tikzcd} & A(L)_{U_i} \arrow[dl] \arrow[dr] & \\ A(L')_{U_i} \arrow[dr] & & A(N)_{U_i} \arrow[dl] \\ & Y & \end{tikzcd}.$$

Now since $U_i$ is affine, we obtain a unique morphism $f_i : A(N')_{U_i} \to Y$ such that the following diagram commutes

$$\begin{tikzcd} & A(L)_{U_i} \arrow[dl] \arrow[dr] & \\ A(L')_{U_i} \arrow[dr] \arrow[ddr] & & A(N)_{U_i} \arrow[dl] \arrow[ddl] \\ & A(N')_{U_i} \arrow[d, "f_i"] & \\ & Y & \end{tikzcd}.$$

For $i, j \in I$, we have $U_i \times_S U_j = U_i \cap U_j$. Let $U_i \cap U_j = \cup_{q \in Q} V_q$ be an affine open covering. We have $f_i|_{A(N')_{V_q}} = f_j|_{A(N')_{V_q}}$ for all $q \in Q$ by the affine case done before. So we have $f_i|_{A(N')_{U_i \cap U_j}} = f_j|_{A(N')_{U_i \cap U_j}}$ by [GW, Prop. 3.5]. Thus using [GW, Prop. 3.5] again, we obtain a unique morphism

$f : A(N')_S \to Y$ such that the following diagram commutes

$$\begin{tikzcd}
 & A(L)_S \arrow[dl] \arrow[dr] & \\
A(L')_S \arrow[dr] \arrow[ddr] & & A(N)_S \arrow[dl] \arrow[ddl] \\
 & A(N')_S \arrow[d, "f"] & \\
 & Y &
\end{tikzcd}.$$

This finishes the proof. □

The category of so-called $G$-$\mathcal{O}_S$-module introduced in [SGA3, §4.7] for a group scheme $G$ over $S$ naturally extends to monoid schemes over $S$. In particular $A(M)_S$-$\mathcal{O}_S$-modules are well-defined. For the convenience of the reader we recall the definition here. Recall that if $\mathcal{F}$ is an $\mathcal{O}_S$-module, then $W(\mathcal{F})$ is defined as a functor on $Sch/S$ by the formula $W(\mathcal{F})(T) = (p^*\mathcal{F})(T)$ where $p : T \to S$ is the associated morphism. Then $W(\mathcal{F})$ is an $O_S$-module where $O_S$ is the ring functor on $Sch/S$ given by the formula $O_S(T) = \mathcal{O}_T(T)$.

Definition 2.19. An $A(M)_S$-$\mathcal{O}_S$-module is an $\mathcal{O}_S$-module $\mathcal{F}$ such that

(i) for any $S$-scheme $T$, the monoid $A(M)_S(T)$ acts on $W(\mathcal{F})(T)$ and $h\cdot(x+\lambda y) = h\cdot x+\lambda(h\cdot y)$ for all $h \in A(M)_S(T)$, $x, y \in W(\mathcal{F})(T)$ and $\lambda \in O_S(T)$,

(ii) for any $S$-morphism $T \to T'$ the actions of $A(M)_S(T)$ on $W(\mathcal{F})(T)$ and of $A(M)_S(T')$ on $W(\mathcal{F})(T')$ are compatible with the morphisms $A(M)_S(T') \to A(M)_S(T)$ and $W(\mathcal{F})(T') \to W(\mathcal{F})(T)$.

In other words, it is an $\mathcal{O}_S$-module $\mathcal{F}$ with an $O_S$-linear action of the $S$-monoid $A(M)_S$ on $W(\mathcal{F})$.

Proposition 2.20. *Let $S$ be a scheme. The category of quasi-coherent $A(M)_S$-$\mathcal{O}_S$-modules is equivalent to the category of quasi-coherent $M$-graded $\mathcal{O}_S$-modules.*

*Proof.* This is proved as [SGA3, Exp. I Proposition 4.7.3]. □

Proposition 2.21. *Let $S$ be a scheme and let $X$ be a scheme over $S$ such that $X \to S$ is affine. Actions of $A(M)_S$ on $X$ correspond to $M$-gradings of the quasi-coherent $\mathcal{O}_S$-algebra of $X$.*

*Proof.* This is proved as [SGA3, Exp. I Corollaire 4.7.3.1]. □

## 3. Algebraic attractors associated to magnets and properties

Let us fix a base scheme $S$. In this paper, an $S$-space is an algebraic space over $S$. Let us fix an $S$-space $X$. Let us fix a commutative monoid $M$ and let $A(M)_S$ be the associated diagonalizable monoid scheme over $S$ (cf. Definition 2.4). We assume that $A(M)_S$ acts on $X$, this means that we have a morphism of $S$-spaces

$$A(M)_S \times_S X \xrightarrow{action} X$$

such that the following diagrams of $S$-morphisms commute

$$\begin{tikzcd} A(M)_S \times_S A(M)_S \times_S X \arrow[r, "m\times\mathrm{Id}"] \arrow[d, "\mathrm{Id}\times action"] & A(M)_S \times_S X \arrow[d, "action"] \\ A(M)_S \times_S X \arrow[r, "action"] & X \end{tikzcd}$$

$$\begin{tikzcd} e_S \times_S X \arrow[r, "\varepsilon\times action"] \arrow[d, "\mathrm{Id}\times\mathrm{Id}"] & A(M)_S \times_S X \arrow[d, "action"] \\ S \times_S X \arrow[r, "\mathrm{Id}"] & X \end{tikzcd}$$

where $\varepsilon$ is the unit, i.e. the unique morphism of $S$-monoids from $e_S$ to $A(M)_S$. In this section, a magnet $N$ is by definition a submonoid $N \subset M$, this terminology is specific to the purpose of algebraic magnetism. For any magnet $N \subset M$, we consider $A(N)_S$ with the canonical action of $A(M)_S$ as in Fact 2.11. For any scheme $T \to S$, $\mathrm{Hom}_T^{A(M)_T}(A(N)_T, X_T)$ denotes the set of $A(M)_T$-equivariant $T$-morphisms from $A(N)_T$ to $X_T = X \times_S T$. We now introduce the attractor $X^N$ associated to a magnet $N \subset M$ under the action of $A(M)_S$ on $X$.

Definition 3.1. Let $X^N : (Sch/S) \to Set$ be the contravariant functor that associates to an object $(T \to S)$ the set

$$\mathrm{Hom}_T^{A(M)_T}(A(N)_T, X_T)$$

and that associates to the $S$-morphism $T' \to T$ the map

$$\begin{aligned} \mathrm{Hom}_T^{A(M)_T}(A(N)_T, X_T) &\to \mathrm{Hom}_{T'}^{A(M)_{T'}}(A(N)_{T'}, X_{T'}) \\ (f : A(N)_T \to X_T) &\mapsto (f_{T'} : A(N)_T \times_T T' \xrightarrow{f\times\mathrm{Id}} X_T \times_T T'). \end{aligned}$$

Proposition 3.2. *The attractor $X^N$ is an object in the topos $Sh\big((Sch/S)_{fppf}\big)$, i.e. $X^N$ is a sheaf on the site $(Sch/S)_{fppf}$.*

*Proof.* Let $PSh((Sch/S)_{fppf})$ be the category of all contravariant functors from $(Sch/S)_{fppf}$ to $Set$. The functor $X^N$ is the equalizer, in the category $PSh((Sch/S)_{fppf}, Set)$, of

$$\underline{\mathrm{Hom}}_S(A(N)_S, X) \underset{\Psi}{\overset{\Phi}{\rightrightarrows}} \underline{\mathrm{Hom}}_S(A(M)_S \times_S A(N)_S, X)$$

where $\Phi, \Psi$ are defined by: for any $T/S$ and any $f \in \mathrm{Hom}_T(A(N)_T, X)$, $\Phi(f)(h, a) = f(h \cdot a)$ and $\Psi(f)(h, a) = h \cdot f(a)$ for any $T'/T$ and any $h \in A(M)_T(T'), a \in A(N)_T(T')$. The topos $Sh\big((Sch/S)_{fppf}\big)$ is a full subcategory of $PSh((Sch/S)_{fppf}, Set)$. The functor $\underline{\mathrm{Hom}}_S(X', X) : T/S \mapsto \mathrm{Hom}_T(X'_T, X_T)$ is an fppf sheaf for any pair of $S$-schemes $X, X'$, so it belongs to the topos $Sh\big((Sch/S)_{fppf}\big)$. Equalizers in a category are unique when they exist. Equalizers exist in any topos. The forgetful functor $Sh\big((Sch/S)_{fppf}\big) \to PSh\big((Sch/S)_{fppf}\big)$ is a right adjoint (cf. [StP, Tag 00WH]) so it preserves limits and in particular equalizers. We conclude that $X^N$ is equal to the equalizer of $(\Phi, \Psi)$ in the category $Sh\big((Sch/S)_{fppf}\big)$, in particular $X^N$ belongs to $Sh\big((Sch/S)_{fppf}\big)$.

□

Remark 3.3. Definition 3.1 makes sense for an arbitrary $S$-functor endowed with an action of $A(M)_S$. We sometimes use this implicitly.

Remark 3.4. In the context of Definition 3.1, when we want to indicate that we take the attractor $X^N$ relatively to the action of the monoid scheme $A(M)_S$, we sometimes use the notation $X^{N\subset M}$.

Remark 3.5. We have an identification $X^0 = X^{A(M)_S}$ where $X^{A(M)_S}$ denotes the functor of fixed-points (cf. Proposition 3.32 for an extension of this remark in the case where $M$ is a finitely generated group). This is tautological because $A(0)_S$ identifies with $S$ endowed with the trivial action. So the concept of attractors refines the concept of fixed-points.

Fact 3.6. *We have an identification $X = X^M$ given as follows. Let $T \to S$ be a scheme. Let $\phi$ be in $X^M(T)$, we associate the morphism*

$$T = e_T \xrightarrow{\varepsilon} A(M)_T \xrightarrow{\phi} X_T \in X(T)$$

*where $\varepsilon$ is the unit. Let $\varphi$ be in $X(T)$, we associate the morphism*

$$A(M)_T = A(M)_T \times_T T \xrightarrow{\mathrm{Id}\times\varphi} A(M)_T \times X_T \xrightarrow{action} X_T.$$

Proposition 3.7. *Let $X_1, X_2, X_3$ be $S$-spaces endowed with actions of $A(M)_S$. Let $X_1 \to X_2$ and $X_3 \to X_2$ be two $A(M)_S$-equivariant morphisms of $S$-spaces. Let $N \subset M$ be a magnet.*

(i) *We have a canonical action of $A(M)_S$ on $X_1 \times_{X_2} X_3$, the canonical maps $X_1 \times_{X_2} X_3 \to X_1$ and $X_1 \times_{X_2} X_3 \to X_3$ are $A(M)_S$-equivariant and moreover the following is a cartesian square in the category of $S$-spaces endowed with action of $A(M)_S$ (morphisms in this category are $A(M)_S$-equivariant morphisms of $S$-spaces)*

$$\begin{tikzcd} X_1 \times_{X_2} X_3 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_3 \arrow[r] & X_2. \end{tikzcd}$$

(ii) *We have an isomorphism of functors*

$$(X_1 \times_{X_2} X_3)^N = {X_1}^N \times_{X_2^N} {X_3}^N.$$

*Proof.* (i) Let $T$ be a scheme. Then $A(M)_S(T)$ acts on $X_1(T) \times_{X_2(T)} X_3(T)$ via $g.(x_1, x_3) = (g.x_1, g.x_3)$ (recall that $(X_1 \times_{X_2} X_3)(T) = X_1(T) \times_{X_2(T)} X_3(T) = \{(x_1, x_3) \in X_1(T) \times X_3(T) | x_1 \equiv x_3 \text{ in } X_2(T)\}$). This proves the first assertion. The projection maps on $X_1$ and $X_3$ are $A(M)_S$-equivariant by definition. Now let $Y$ be an $S$-space with an action of $A(M)_S$. An $A(M)_S$-equivariant morphism $Y \to X_1 \times_{X_2} X_3$ gives birth by composition with projections to canonical $A(M)_S$-equivariant morphisms $Y \to X_1$ and $Y \to X_2$. Reciprocally let $Y \to X_1$ and $Y \to X_3$ be $A(M)_S$-equivariant. Then we get a morphism in the category of $S$-spaces $Y \to X_1 \times_{X_2} X_3$. This morphism is $A(M)_S$-equivariant.

(ii) This is a direct consequence of the previous assertion.

□

Proposition 3.8. *Let $N \subset M$ be a magnet. Let $Y$ and $Z$ be two algebraic spaces over $S$ endowed with actions of $A(M)_S$. Let $A(M)_S$ act componentwise on $Y \times_S Z$. We have*

$$(Y \times_S Z)^N = Y^N \times_S Z^N.$$

*Proof.* This follows from Proposition 3.7. □

Proposition 3.9. *Let $T \to S$ be an $S$-scheme. The $T$-space $X_T = X \times_S T$ is canonically endowed with an action of $A(M)_T$. Let $N \subset M$ be a magnet. We have an identification of $T$-functors*

$$X^N \times_S T = (X_T)^N.$$

*Proof.* Indeed, if $T' \to T$ is a $T$-scheme, we have identifications of sets

$$\begin{aligned}(X^N \times_S T)(T') &= \mathrm{Hom}_T(T', X^N \times_S T)\\ &= \mathrm{Hom}_S(T', X^N)\\ &= \mathrm{Hom}_{T'}^{A(M)_{T'}}(A(N)_{T'}, X_{T'}) \text{ and,}\end{aligned}$$

$$\begin{aligned}(X_T)^N(T') &= \mathrm{Hom}_{T'}^{A(M)_{T'}}(A(N)_{T'}, X_T \times_T T')\\ &= \mathrm{Hom}_{T'}^{A(M)_{T'}}(A(N)_{T'}, X_{T'}).\end{aligned}$$

□

Fact 3.10. *Let $N \subset L$ be magnets of $M$. For $T \to S$, we have a morphism*

$$\mathrm{Hom}_T^{A(M)_T}(A(N)_T, X_T) \to \mathrm{Hom}_T^{A(M)_T}(A(L)_T, X_T)$$

*obtained using the morphism $A(L)_T \to A(N)_T$ (cf. Fact 2.12). This is functorial, so we get a morphism of functors $\iota_{N,L} : X^N \to X^L$.*

Remark 3.11. Let $N \subset M$ be a magnet. We have an action of the monoid scheme $A(M)_S$ on $X^N$ given as follows. For any $S$-scheme $T$, we have actions of $A(M)(T)$ on $X_T$ and $A(N)_T$, in particular for any $t \in A(M)(T)$, we have arrows $X_T \xrightarrow{t\cdot} X_T$ and $A(N)_T \xrightarrow{t\cdot} A(N)_T$. Now let $f \in X^N(T) = \mathrm{Hom}_T^{A(M)_T}(A(N)_T, X_T)$ and $t \in A(M)(T)$. The composition $A(N)_T \xrightarrow{f} X_T \xrightarrow{t\cdot} X_T$ equals the composition $A(N)_T \xrightarrow{t\cdot} A(N)_T \xrightarrow{f} X_T$ and is denoted $t \cdot f$. This defines an action of $A(M)_S$ on $X^N$.

Remark 3.12. We proceed with the notation from Remark 3.11. We have an action of the monoid scheme $A(N)_S$ on $X^N$ given as follows. Let $f \in X^N(T) = \mathrm{Hom}_T^{A(M)_T}(A(N)_T, X_T)$ and $t \in A(N)(T)$. We define $t \cdot f$ to be $A(N)_T \xrightarrow{t\cdot} A(N)_T \xrightarrow{f} X_T$.

Remark 3.13. The action of Remark 3.11 can be obtained from the action of Remark 3.12 via the canonical morphism of monoid schemes $A(M)_S \to A(N)_S$. This follows from definitions and the fact that the action of $A(M)_S$ on $A(N)_S$ comes from the canonical morphism of monoid schemes $A(M)_S \to A(N)_S$ induced by the inclusion $N \subset M$. Let us mention a sanitary check. If $L \subset M$ is an other magnet, then under some working assumptions $(X^{N\subset M})^{L\subset M} = (X^{N\subset M})^{(N\cap L)\subset N}$ (e.g. use directly 3.29, 3.30, 3.19 or 8.3).

Proposition 3.14. *Let $N \subset M$ be a magnet.*

(i) *If $X$ is a monoid algebraic space over $S$ and the action of $A(M)_S$ is by monoid endomorphisms, then $X^N$ is a monoid functor.*

(ii) *If $X$ is a group algebraic space over $S$ and the action of $A(M)_S$ is by group automorphisms, then $X^N$ is a group functor.*

*Proof.* (i) Let $T$ be a scheme over $S$. Let $A(N)_T \xrightarrow{g} X_T$ and $A(N)_T \xrightarrow{h} X_T$ be two elements in $X^N(T)$. Then we define $gh$ as the composition

$$A(N)_T \to A(N)_T \times_T A(N)_T \to X_T \times_T X_T \to X_T$$

where the first morphism is the diagonal morphism, the second is $g \times h$, and the third is the multiplication morphism coming from the group structure on $X$. The two firsts are

equivariant by definitions and the third is equivariant because $A(M)_S$ acts on $X_S$ by monoid endomorphisms. This defines a monoid law on $X^N$.

(ii) By (i) we have a monoid law on $X^N$. This law is a group law.

□

FACT 3.15. *Let $F$ and $N$ be magnets of $M$ and assume that $F$ is a face of $N$. Then for all $T \to S$ the $A(M)_T$-equivariant morphism $A(F)_T \to A(N)_T$ (cf. Fact 2.14) induces a morphism*

$$\mathrm{Hom}_T^{A(M)_T}(A(N)_T, X_T) \to \mathrm{Hom}_T^{A(M)_T}(A(F)_T, X_T).$$

*So we obtain a morphism of functors $X^N \to X^F$, that we denote $\mathfrak{p}_{N,F}$. The morphism $\mathfrak{p}_{N,F}$ satisfies $\mathfrak{p}_{N,F} \circ \iota_{F,N} = \mathrm{Id}_{X^F}$.*

*Proof.* Clear. □

FACT 3.16. *If $f : X \to Y$ is an $A(M)_S$-equivariant morphism of algebraic spaces, then for any magnet $N$ of $M$, we have a morphism of functors $f^N : X^N \to Y^N$.*

*Proof.* For any $T \to S$, send an equivariant arrow $A(N)_T \to X_T$ to $A(N)_T \to X_T \to Y_T$. □

FACT 3.17. *Let $N \subset M$ be a magnet. Assume $X \to Y$ is an $A(M)_S$-equivariant monomorphism of $S$-spaces, then $X^N \to Y^N$ is a monomorphism.*

*Proof.* Let $T \to S$ be a scheme. Let $A(N)_T \underset{g}{\overset{f}{\rightrightarrows}} X_T$ be two equivariant $S$-morphisms and assume the compositions $A(N)_T \underset{g}{\overset{f}{\rightrightarrows}} X_T \longrightarrow Y_T$ are equal. Since $X_T \to Y_T$ is a monomorphism, we get $f = g$. □

FACT 3.18. *Let $N \subset M$ be a magnet. Assume $f : X \to Y$ is an $A(M)_S$-equivariant morphism of $S$-spaces. If $f$ is locally of finite presentation, then $f^N : X^N \to Y^N$ is locally of finite presentation.*

*Proof.* This follows from the definitions and [StP, Tag 04AK]. □

THEOREM 3.19. *Assume that $X$ is affine over $S$. Let $p : X \to S$ be the structural morphism and $\mathcal{A} = p_*\mathcal{O}_X$ so that $X = \mathrm{Spec}_S(\mathcal{A})$. Let $N$ be a magnet of $M$. The functor $X^N$ is representable by a closed subscheme of $X$ whose quasi-coherent ideal sheaf $\mathcal{J}_N$ is the ideal sheaf generated $\{\mathcal{A}_m | m \in M \setminus N\}$ where $\mathcal{A}_m$ is the component appearing in the direct sum decomposition $\mathcal{A} = \bigoplus_{m \in M} \mathcal{A}_m$ coming from the action of $A(M)_S$ on $X$ (cf. Proposition 2.21).*

*Proof.* Since $X^N$ is a fppf sheaf by Proposition 3.2 and in particular a Zariski sheaf, the statement is local on $S$ using e.g. [StP, Tag 01JJ]. We assume $S = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(A)$, let $A = \bigoplus_{m \in M} A_m$ be the decomposition coming from the action. Let $J$ be the ideal of $A$ generated by $\{A_m | m \in M \setminus N\}$. Let $B'$ be a $B$-algebra. It is enough to define functorial maps $\Theta$ and $\Psi$

$$\mathrm{Hom}_B(A/J, B') \underset{\Psi}{\overset{\Theta}{\rightleftarrows}} \mathrm{Hom}_B^{M\text{-graded}}(A, B'[N])$$

such that $\Theta \circ \Psi = \mathrm{Id}$ and $\Psi \circ \Theta = \mathrm{Id}$. Take $A/J \xrightarrow{F} B'$ on the left-hand-side and define a map $f = \Theta(F)$ on the right-hand-side as

$$\begin{aligned} A = \oplus_{m \in M} A_m &\xrightarrow{f} B'[N] \\ a_m \in A_m &\mapsto F([a_m])X^m \end{aligned}$$

Let us check that this map $\Theta$ is well-defined. Since $F([a_m]) = 0$ if $m \in M \setminus N$, the element $F([a_m])X^m$ belongs to $B'[N]$. We have to explain that $f$ is a morphism of $B$-algebras. This is a consequence of the identity

$$f(a_m a_{m'}) = F([a_m a_{m'}])X^{m+m'} = F([a_m][a_{m'}])X^m X^{m'} = f(a_m)f(a_{m'}).$$

So $\Theta$ is well-defined. Now take $A \xrightarrow{f} B'[N]$ on the right-hand-side. Then $f(a_m) = 0$ for all $a_m \in A_m$ for all $m \in M \setminus N$, so $f$ vanishes on $J$, i.e $f$ factors through $A \to A/J \xrightarrow{f} B'[N]$. Now we define $F = \Psi(f)$ as the composition $A/J \xrightarrow{f} B'[N] \xrightarrow{X^n \mapsto 1} B'$, this is a morphism of $B$-algebras. Let us prove that $\Theta \circ \Psi = \mathrm{Id}$. Let $f$ be a morphism on the right-hand-side. Let $a_n \in A_n$ for $n \in N$, we have $f(a_n) = \lambda_n X^n$. We have

$$((\Theta \circ \Psi)(f))(a_n) = (\Theta(\Psi(f)))(a_n) = (\Psi(f))([a_n]) \cdot X^n = (f(a_n))|_{X^n=1} \cdot X^n = \lambda_n X^n = f(a_n).$$

Now let $a_m \in M \setminus N$, then

$$((\Theta \circ \Psi)(f))(a_m) = (\Theta(\Psi(f)))(a_m) = (\Psi(f))([a_m]) \cdot X^m = 0 = f(a_m).$$

This proves that $\Theta \circ \Psi = \mathrm{Id}$. Let us prove that $\Psi \circ \Theta = \mathrm{Id}$. Let $F$ be a morphism on the left-hand-side, and let us look at the image of $[a_n]$ for some $a_n \in A_n$ with $n \in N$ under $(\Psi \circ \Theta)(F) = \Psi(\Theta(F))$ :

$$\begin{aligned} A/J &\xrightarrow{\Theta(F)} B'[N] \to B' \\ [a_n] &\mapsto F([a_n])X^n \mapsto F([a_n]). \end{aligned}$$

This finishes the proof of Theorem 3.19. □

Corollary 3.20. *Assume $X$ is affine over $S$. If $N \subset L$ are magnets of $M$, then $X^N \to X^L$ is a closed immersion.*

*Proof.* With the notation of Theorem 3.19, we have $\mathcal{J}_L \subset \mathcal{J}_N$. □

Remark 3.21. Corollary 3.20 does not generalize outside the $S$-affine case.

Proposition 3.22. *Assume $X$ is affine over $S$. Let $\{N_i\}_{i \in I}$ be magnets of $M$. Then*

$$\bigcap_{i \in I} X^{N_i} = X^{\cap_{i \in I} N_i} = \prod_{i \in I}{}_X X^{N_i}.$$

*Here $\cap$ means the scheme theoretic intersection as in [StP, Tag 0C4H] and infinite products of affine morphisms make sense by [StP, Tag 0CNI].*

*Proof.* This follows from the description of affine attractors given in Theorem 3.19 and the identity $\sum_{i \in I} \mathcal{J}_{N_i} = \mathcal{J}_{\cap_{i \in I} N_i}$ that follows from the identity $\cup_{i \in I}(M \setminus N_i) = M \setminus (\cap_{i \in I} N_i)$. □

Remark 3.23. The first equality in Proposition 3.22 does not make sense outside the $S$-affine case in general. This is because $X^N$ is not a closed subspace of $X$ in general if $X$ is not $S$-affine. Moreover in many non-affine cases $X^{N \cap L} \neq X^N \times_X X^L$ (e.g. cf. [Dr15, Remark 1.6.3]).

Lemma 3.24. *Let $f : Z \to Y$ be an $A(M)_S$-equivariant closed immersion of $S$-affine schemes. Let $N$ be a magnet of $M$. Then the morphism $Z^N \to Y^N$ is a closed immersion, and more precisely $Z^N = Z \times_Y Y^N$.*

*Proof.* This is local on $S$ so we assume $S = \mathrm{Spec}(R)$ is affine, moreover we identify $Z$ with a closed subscheme of $Y$. Let $A$ be the $R$-algebra of $Y$ and $I$ be the ideal of $A$ defining $Z$. Let $J$

be the ideal of $A$ defining $Y^N$, cf. Proof of Theorem 3.19 . Then the ideal of $A/I$ defining $Z^N$ is $I + J/I$ using Theorem 3.19. So the ideal of $A$ defining $Z^N$ is $I + J$. Now the isomorphism $A/I \otimes_A A/J \cong A/(I + J)$ finishes the proof. □

Remark 3.25. Proposition 12.2 extends Lemma 3.24 outside the $S$-affine case. Note that the proof of Proposition 12.2 uses Lemma 3.24.

Proposition 3.26. *Let $\mathcal{E}$ be a quasi-coherent $A(M)_S$-$\mathcal{O}_S$-module and let $\mathbb{V}(\mathcal{E}) = \mathrm{Spec}_S(\mathrm{Sym}\ \mathcal{E})$ be the associated quasi-coherent bundle defined by $\mathcal{E}$. Then $A(M)_S$ acts linearly on $\mathbb{V}(\mathcal{E})$. Let $N$ be a magnet of $M$. Then the attractor $\big(\mathbb{V}(\mathcal{E})\big)^N$ associated to $N$ is canonically isomorphic to $\mathbb{V}(\mathcal{E}^N)$ where $\mathcal{E}^N$ is the $N$-graded component of $\mathcal{E}$ relatively to the $A(M)_S$-action on $\mathcal{E}$ (cf. Proposition 2.20).*

*Proof.* Let $p : T \to S$ be a scheme over $S$. We have $\mathbb{V}(\mathcal{E}) \times_S T = \mathbb{V}(p^*\mathcal{E})$. The quasi-coherent $\mathcal{O}_T$-module inherits a $M$-grading and we have $(p^*\mathcal{E})^N = p^*(\mathcal{E}^N)$. The following identifications finish the proof

$$\begin{aligned}
\mathrm{Hom}_T^{A(M)_T}(A(N)_T, \mathbb{V}(p^*\mathcal{E})) &= \mathrm{Hom}^{M\text{-graded}}_{\mathcal{O}_T\text{-alg}}\big(\mathrm{Sym}\ p^*\mathcal{E}, \mathcal{O}_T[N]\big) \\
&= \mathrm{Hom}^{M\text{-graded}}_{\mathcal{O}_T\text{-mod}}\big(p^*\mathcal{E}, \mathcal{O}_T[N]\big) \\
&= \mathrm{Hom}^{M\text{-graded}}_{\mathcal{O}_T\text{-mod}}\big(p^*\mathcal{E}^N, \mathcal{O}_T[N]\big) \\
&= \mathrm{Hom}_{\mathcal{O}_T\text{-mod}}\big(p^*\mathcal{E}^N, \mathcal{O}_T\big) \\
&= \mathrm{Hom}_{\mathcal{O}_T\text{-alg}}\big(\mathrm{Sym}\ p^*\mathcal{E}^N, \mathcal{O}_T\big) \\
&= \mathbb{V}(\mathcal{E}^N)(T).
\end{aligned}$$

□

Proposition 3.27. *Assume that $X$ is separated and let $N \subset N'$ be two magnets of $M$. Then the map of functors $X^N \to X^{N'}$ is a monomorphism, i.e. for any scheme $T$ over $S$ we have a canonical inclusion $X^N(T) \subset X^{N'}(T)$.*

*Proof.* Since $N \subset N'$, by [StP, Tag 01R8], the scheme theoretic image of $A(N')_T \to A(N)_T$ is $A(N)_T$ for any scheme $T$ over $S$. Now let $f_1, f_2$ be two elements in $X^N(T)$ such that their images in $X^{N'}(T)$ coincide. Consider the schematic kernel of the maps $(f_1, f_2) : A(N)_T \to X_T$. Since $X_T$ is separated, $\ker(f_1, f_2)$ is a closed subscheme of $A(N)_T$ (the proof of [GW, Def./Prop. 9.7] works in this context). So since the scheme theoretic image of $A(N')_T \to A(N)_T$ is $A(N)_T$, we have $\ker(f_1, f_2) = A(N)_T$ and so $f_1 = f_2$. □

Corollary 3.28. *Assume that $X$ is separated and let $N$ be a magnet of $M$. Then the natural map of functors $X^N \to X$ is a monomorphism.*

*Proof.* Combine Proposition 3.27 and Remark 3.6. □

Proposition 3.29. *Let $Z$ be a monoid and let $f : M \to Z$ be a morphism of monoids. Let $Y$ be a magnet of $Z$ and let $N$ be $f^{-1}(Y)$. Then $N$ is a magnet of $M$. Assume that one of the following conditions holds*

(i) *$X$ is affine over $S$,*

(ii) *$M$ and $Z$ are finitely generated groups,*

(iii) *$M$ is cancellative, $Z = M^{\mathrm{gp}}$, $N = Y$ and $X$ is separated,*

(iv) *add more as needed,*

*then in each case we have an identification of functors $X^N = X^Y$, where on the left-hand-side $X$ is seen as an $A(M)_S$-space and on the right-hand-side as an $A(Z)_S$-space (via $A(Z)_S \to A(M)_S$ dual to $f : M \to Z$). In other words, with the notation of Remark 3.4, we have $X^{N\subset M} = X^{Y\subset Z}$.*

*Proof.* (i) We reduce to the case where $S$ and $X = \mathrm{Spec}(A)$ are affine. We have two compatible gradings on $A$, one given by $Z$ and one given by $M$. For any $y \in Y$, we have $A_y = \oplus_{n\in f^{-1}(y)} A_n$. So $\oplus_{z\in Z\setminus Y} A_z = \oplus_{n\in M\setminus N} A_n$. Then the ideal defining $X^N$ equals the ideal defining $X^Y$, cf. Theorem 3.19.

(ii) Let $T \to S$ be a scheme. We have a canonical map

$$\theta : \mathrm{Hom}_T^{D(M)_T}(A(N)_T, X_T) \to \mathrm{Hom}^{D(Z)_T}(A(Y)_T, X_T)$$

obtained by precomposition with $A(Y)_T \to A(N)_T$. Let us show that $\theta$ is an isomorphism. We construct the reciprocal map. Let $f : A(Y)_T \to X_T$ be a $D(Z)_T$-equivariant map. We get a map $f' : D(M)_T \times_T A(Y)_T \to X_T$ given on points by $(g, x) \mapsto g \cdot f(x)$. Let $K$ be $f(M) + Y^{\mathrm{gp}}$, this is a subgroup of $Z$. We have a morphism of groups $\phi : f(M) \times Y^{\mathrm{gp}} \to f(M) + Y^{\mathrm{gp}}$ given by $(f(m), y) \mapsto f(m) + y$. The morphism $\phi$ induces a closed immersion of group schemes $D(f(M) + Y^{\mathrm{gp}})_T \to D(f(M) \times Y^{\mathrm{gp}})_T = D(f(M))_T \times_T D(Y^{\mathrm{gp}})_T$. Let us consider the action of $D(f(M))_T \times_T D(Y^{\mathrm{gp}})_T$ on $D(M)_T \times_T A(Y)_T$ given by $(g, h)\cdot(g', x) = (g \cdot g', h^{-1} \cdot x)$, remark that this action is free. So we obtain by composition a free action $\star$ of $D(K)_T$ on $D(M)_T \times_T A(Y)_T$. Let us consider the morphisms

$$\xi_\star, \xi : D(K)_T \times_T D(M)_T \times_T A(Y)_T \to X_T$$

where $\xi_\star$ is given by $(k, g, y) \mapsto f'(k \star (g, y))$ and $\xi$ is given by $(k, g, y) \mapsto f'(g, y)$. Let us prove that $\xi_\star = \xi$. Let us consider the canonical morphism

$$p : D(Z)_T \times_T D(M)_T \times_T A(Y)_T \to D(K)_T \times_T D(M)_T \times_T A(Y)_T$$

induced by the inclusion $K \subset Z$. Then $p \circ \xi = p \circ \xi_\star$, and so $\ker(\xi, \xi_\star) = D(K)_T \times_T D(M)_T \times_T A(Y)_T$ because the schematic image of $p$ is $D(K)_T \times_T D(M)_T \times_T A(Y)_T$. This finishes the proof of the claim $\xi = \xi_\star$. So we have $f'(k \star (g, x)) = f'((g, x))$ for all $T' \to T$, $k \in D(K)_T(T')$, $g \in D(M)_T(T')$ and $x \in A(Y)_T(T')$. So the map $f'$ induces by factorization a map $(D(M)_T \times_T A(Y)_T)/D(K)_T \to X_T$ where $(D(M)_T \times_T A(Y)_T)/D(K)_T$ is the fpqc quotient as in [SGA3, VIII Th. 5.1]. Now let $\mathcal{A}$ be the quasi-coherent algebra of the $T$-affine scheme $(D(M)_T \times_T A(Y)_T)$. Consider the $K$-grading on $\mathcal{A}$ associated to the action of $D(K)_T$ on $(D(M)_T \times_T A(Y)_T)$. By [SGA3, VIII Th. 5.1], $(D(M)_T \times_T A(Y)_T)/D(K)_T$ is affine with quasi-coherent algebra the degree zero part $\mathcal{A}_0$ in $\mathcal{A}$. The $K$-grading on $\mathcal{A}$ is given locally by $\deg(X^{(m,y)}) = f(m) - y \in K$. This implies that $\mathcal{A}_0 \simeq \mathcal{O}_T[f^{-1}(Y)]$. This identifies $(D(M)_T \times_T A(Y)_T)/D(K)_T$ with $A(N)_T$. So the map $f$ induces a map $A(N)_T \to X_T$. This map is $D(M)_T$-equivariant. The obtained map $\mathrm{Hom}^{D(Z)_T}(A(Y)_T, X_T) \to \mathrm{Hom}^{D(M)_T}(A(N)_T, X_T)$ is the reciprocal map of $\theta$.

(iii) We have to show that for any $T/S$, $\mathrm{Hom}_T^{D(M^{\mathrm{gp}})_T}(A(N)_T, X_T) = \mathrm{Hom}_T^{A(M)_T}(A(N)_T, X_T)$. The inclusion $\supset$ is clear. Reciprocally let $\phi : A(N)_T \to X_T$ be $D(M^{\mathrm{gp}})_T$-equivariant. We use that $X$ is separated and that the schematic image of

$$A(N)_T \times_T D(M^{\mathrm{gp}})_T \to A(N)_T \times_T A(M)_T$$

is $A(N)_T \times_T A(M)_T$ to prove that $\phi$ is $A(M)_T$-equivariant (cf. the proof of Proposition 3.27 for similar arguments).

□

PROPOSITION 3.30. *Let $L$ and $N$ be arbitrary submonoids of $M$. Assume that one of the following conditions holds*

(i) *$X$ is affine over $S$,*

(ii) *$M$ is a finitely generated abelian group,*

(iii) *$M$ is cancellative, $M^{\mathrm{gp}}$ is finitely generated as abelian group, and $X$ and $X^N$ are separated algebraic spaces,*

(iv) *add more as needed,*

*then we have canonical identifications*

$$(X^N)^L = X^{N\cap L},$$

*cf. 3.11 for the actions of $A(M)_S$ on $X^N$ and $X^L$ that we used implicitely on the left-hand sides.*

*Proof.* (i) We use the explicit description given in Theorem 3.19.

(ii) Let us use [SGA3, VIII Th. 5.1] as follows. We remark that

$$(X^N)^L(T) = \mathrm{Hom}_T^{D(M)_T\times_T D(M)_T}(A(N)_T \times_T A(L)_T, X_T)$$

where $D(M)_T \times_T D(M)_T$ acts on $X_T$ via $(g,h)\cdot x = g\cdot(h\cdot x)$ and on $A(N)_T \times_T A(L)_T$ via $(g,h)\cdot(a,b) = (g\cdot a, h\cdot b)$. Let us consider the action $\star$ of $D(N^{\mathrm{gp}}+L^{\mathrm{gp}})_T$ on $A(N)_T\times_T A(L)_T$ given by $g\star(a,b) = (g^{-1}\cdot a, g\cdot b)$. The action $\star$ is free because firstly the action of $D(N^{\mathrm{gp}}\times L^{\mathrm{gp}})_T$ on $A(N\times L)_T$ given by $(\lambda,\beta)\cdot(a,b) = (\lambda\cdot a, \beta^{-1}\cdot b)$ is free and secondly because the morphism of groups $N^{\mathrm{gp}}\times L^{\mathrm{gp}} \to N^{\mathrm{gp}}+L^{\mathrm{gp}}$ given by $(n,l)\mapsto n+l$ is surjective and so $D(N^{\mathrm{gp}}+L^{\mathrm{gp}})_T$ is a closed subgroup scheme of $D(N^{\mathrm{gp}}\times L^{\mathrm{gp}})_T$. Now let $F \in \mathrm{Hom}_T^{D(M)_T\times_T D(M)_T}(A(N)_T\times_T A(L)_T, X_T)$, $F : A(N)_T\times_T A(L)_T \to X_T$. Then $F(g\star(a,b)) = F((a,b))$ for any $T' \to T$, $g\in D(N^{\mathrm{gp}}+L^{\mathrm{gp}})_T(T')$ and $(a,b)\in (A(N)_T\times_T A(L)_T)(T')$. So $F$ induces a morphism $f$ from the fpqc quotient, $f : \big(A(N)_T\times_T A(L)_T\big)/D(N^{\mathrm{gp}}+L^{\mathrm{gp}})_T \to X_T$. We have $A(N)_T\times_T A(L)_T = \mathrm{Spec}(\mathbb{Z}[N\times L])\times_{\mathbb{Z}} T$ and the degree zero part of $\mathbb{Z}[N\times L]$ (relatively to the $N^{\mathrm{gp}}+L^{\mathrm{gp}}$-grading induced by the action $\star$ of $D(N^{\mathrm{gp}}+L^{\mathrm{gp}})$) is $\mathbb{Z}[N\cap L]$. So we have an identification $\big(A(N)_T\times_T A(L)_T\big)/D(N^{\mathrm{gp}}+L^{\mathrm{gp}})_T = A(N\cap L)_T$. So $F$ induces a natural morphism $f : A(N\cap L)_T \to X_T$, that is $D(M)_T$-equivariant. Now let $f\in X^{N\cap L}(T)$, $f : A(N\cap L)_T \to X_T$ and consider the composition

$$F : A(N)_T\times_T A(L)_T \to A(N\cap L)_T\times_T A(N\cap L)_T \to A(N\cap L)_T \xrightarrow{f} X_T$$

where the second morphism is the multiplication of $A(N\cap L)_T$. Then $F$ is $D(M)_T\times_T D(M)_T$-equivariant. The previous maps $F\mapsto f$ and $f\mapsto F$ induce a canonical bijection between

$$\mathrm{Hom}_T^{D(M)_T\times_T D(M)_T}(A(N)_T\times_T A(L)_T, X_T)$$

and $\mathrm{Hom}_T^{D(M)_T}(A(N\cap L)_T, X_T)$.

(iii) Using three times Proposition 3.29 and one time (ii), we have

$$(X^{N\subset M})^{(L\subset M)} = (X^{N\subset M^{\mathrm{gp}}})^{L\subset M^{\mathrm{gp}}} = X^{(L\cap N)\subset M^{\mathrm{gp}}} = X^{(N\cap L)\subset M}.$$

□

FACT 3.31. *Let $\{N_i\}_{i\in I}\subset M$ be magnets such that $X^{N_i} = X^{N_j}$ for all $i,j\in I$. Assume that $X$ is separated and that one of the following conditions holds*

(i) *$X$ is affine over $S$,*

(ii) *$M$ is a finitely generated abelian group,*

(iii) *$M$ is cancellative, $M^{\mathrm{gp}}$ is finitely generated as abelian group, and $X^{\cap_{l\in J}N_l}$ is a separated algebraic spaces for any finite subset $J \subset I$,*

(iv) *add more as needed,*

*then, for all $i \in I$,*

$$X^{N_i} = X^{\cap_{l\in I}N_l}.$$

*Proof.* If $X$ is affine, we use Proposition 3.22. Now we prove (ii) and (iii). The inclusion $\cap_{l\in I}N_l \subset N_i$ induces a canonical morphism $X^{\cap_{l\in I}N_l} \to X^{N_i}$. Since $X$ is separated, this canonical morphism is a monomorphism by Proposition 3.27. So for any scheme $T \to S$, $X^{\cap_{l\in I}N_l}(T) \subset X^{N_i}(T)$. We now prove the reverse inclusion. We write $I$ as a directed colimit of finite sets $J$. Then, in the category of sets, $\cap_{l\in I}N_l = \lim_{J\subset I} \cap_{l\in J}N_l$. Furthermore, in the category of rings, $\mathbb{Z}[\cap_{l\in I}N_l] = \lim_{J\subset I} \mathbb{Z}[\cap_{l\in J}N_l]$. So $A(\cap_{l\in I}N_l)_S = \mathrm{colim}_{J\subset I} A(\cap_{l\in J}N_l)_S$. By Proposition 3.30, $X^{\cap_{l\in J}N_l} = X^{N_i}$ for any finite subset $J$ of $I$. We use these observations to obtain

$$\begin{aligned}
X^{\cap_{l\in I}N_l}(T) &= \mathrm{Hom}_T^{A(M)_S}(A(\cap_{l\in I}N_l)_T, X_T) \\
&= \mathrm{Hom}_T^{A(M)_S}(\mathrm{colim}_{J\subset I} A(\cap_{l\in J}N_l)_T, X_T) \\
\text{[StP, Tag 002H] and equivariance is clear} &\supset \lim_{J\subset I} \mathrm{Hom}_T^{A(M)_S}(A(\cap_{l\in J}N_l)_T, X_T) \\
&= \lim_{J\subset I} X^{\cap_{l\in J}N_l}(T) \\
&= \lim_{J\subset I} X^{N_i}(T) \\
&= X^{N_i}(T).
\end{aligned}$$

This finishes the proof. □

Proposition 3.32. *Assume $M$ is a finitely generated group. Let $Z \subset M$ be a subgroup. Then the attractor space $X^Z$ for the action of $D(M)_S$ on $X$ is identified with the fixed-points space $X^{D(M/Z)_S}$ of $X$ under the action of $D(M/Z)_S$.*

*Proof.* We have an exact sequence of abelian groups

$$0 \to Z \to M \to M/Z \to 0.$$

By [SGA3, Exp. VIII] we obtain an exact sequence of diagonalizable group schemes

$$1 \to D(M/Z)_S \to D(M)_S \to D(Z)_S \to 1.$$

Let $T \to S$ be a scheme and let us prove that $X^Z(T) = X^{D(M/Z)_S}(T)$. Note that we have a $D(M)_S$-equivariant identification of $S$-scheme $A(Z)_S = D(Z)_S$. We have

$$\begin{aligned}
X^Z(T) &= \mathrm{Hom}_T^{D(M)_T}(A(Z)_T, X_T) \\
&= \mathrm{Hom}_T^{D(M)_T}(D(Z)_T, X_T) \\
&= \mathrm{Hom}_T^{D(M)_T}(D(M)_T/D(M/Z)_T, X_T) \\
&= \mathrm{Hom}_T^{D(M)_T}(D(M)_T, (X_T)^{D(M/Z)_T}).
\end{aligned}$$

Now Remark 3.6 finishes the proof. □

Proposition 3.33. *Assume that $M$ is an abelian group and that $X$ is a scheme. Let $N, N', L$ and $L'$ be magnets in $M$ such that $L \subset L'$, $N' \subset L', N = L \cap N'$ and $L' = L + N'$. Assume that $X^E$ is representable by a scheme for all $E \in \{N, N', L, L'\}$. Assume that $L$ is a face of $L'$. Then $N$ is a face of $N'$. Assume that one of the following conditions holds*

(i) *we have an equality $N' = L' \setminus (L \setminus N)$*

(ii) *$S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$ are affine, $A_l A_{n'} = A_{l+n'}$ for all $l \in L \setminus N$ and $n' \in N'$ (as usual $A_m$ denote the $m$-graded part of $A$)*

(iii) *add more as needed,*

*then the following diagram is a cartesian square in the category of schemes*

$$\begin{tikzcd}
 & X^{N'} \arrow[dl, "\iota_{N',L'}"'] \arrow[dr, "\mathfrak{p}_{N,N'}"] & \\
X^{L'} \arrow[dr, "\mathfrak{p}_{L,L'}"] & & X^N \arrow[dl, "\iota_{N,L}"] \\
 & X^L &
\end{tikzcd} .$$

*Proof.* The monoid $N$ is clearly a face of $N'$.

(i) Let $T$ be a scheme and let $T \to X^{L'}$, $T \to X^N$ be two morphisms of schemes such that the following diagram commutes

$$\begin{tikzcd}
 & T \arrow[dl] \arrow[dr] & \\
X^{L'} \arrow[dr, "\mathfrak{p}_{L,L'}"'] & & X^N \arrow[dl, "\iota_{N,L}"] \\
 & X^L &
\end{tikzcd} .$$

This corresponds to a diagram

$$\begin{tikzcd}
 & A(L)_T \arrow[dl] \arrow[dr] & \\
A(L')_T \arrow[dr] & & A(N)_T \arrow[dl] \\
 & X_T &
\end{tikzcd}$$

where all arrows are $D(M)_T$-equivariant. By Lemma 2.18, we obtain a unique arrow $A(N')_T \to$

$X_T$ such that the following diagram commutes

$$\begin{tikzcd}
 & A(L)_T \arrow[dl] \arrow[dr] & \\
A(L')_T \arrow[dr] \arrow[ddr] & & A(N)_T \arrow[dl] \arrow[ddl] \\
 & A(N')_T \arrow[d] & \\
 & X_T &
\end{tikzcd}.$$

It is enough to show that the arrow $A(N')_T \to X_T$ is $D(M)_T$-equivariant. Consider the diagram

$$\begin{tikzcd}
 & A(L)_T \times_T D(M)_T \arrow[rr] \arrow[dl] \arrow[dr] & & A(L)_T \arrow[dl] \arrow[dr] & \\
A(L')_T \times_T D(M)_T \arrow[dr] \arrow[ddr] & & A(N)_T \times_T D(M)_T \arrow[dl] \arrow[ddl] & A(L')_T \arrow[dr] \arrow[ddr] & A(N)_T \arrow[dl] \arrow[ddl] \\
 & A(N')_T \times_T D(M)_T \arrow[rr] \arrow[d] & & A(N')_T \arrow[d] & \\
 & X_T \times_T D(M)_T \arrow[rr] & & X_T &
\end{tikzcd}$$

obtained by fiber product with $D(M)_T$. We have

$$A(E)_T \times_T D(M)_T = (A(E) \times_{\mathrm{Spec}(\mathbb{Z})} T) \times_T D(M)_T = A(E)_{D(M)_T}$$

for $E \in \{N, N', L, L'\}$, so the left diamond is a push-out by Proposition 2.18. Now we want to show that the lower rectangle is commutative. Consider the upper right composition in this rectangle and precompose it with the right part of the left diamond, denote this arrow by $a_1$. Consider the lower left composition in the rectangle and precompose it with the left part of the left diamond, denote this arrow by $a_2$. Now using the commutative diagrams coming from the $D(M)_T$-equivariant morphisms on the right, we see that $a_1$ and $a_2$ are both equal to the composition

$$A(L)_T \times_T D(M)_T \to A(L)_T \to A(N')_T \to X_T.$$

Using the left push-out diamond, this now implies that the lower rectangle is commutative. So the arrow $A(N')_T \to X_T$ is $D(M)_T$-equivariant. This finishes the proof.

(ii) Let $x \in X^{N'}(R)$. Then $x$ is a morphism $A \to R[N']$. Now we have an equality of compositions

$$(A \to R[N'] \to R[L'] \to R[L]) = (A \to R[N'] \to R[N' \cap L] \to R[L]).$$

This shows that the diagram is commutative. Now let us prove that it is cartesian. Let $Y = \mathrm{Spec}(B)$ be an affine $R$-scheme and let $f : Y \to X^{L'}$ and $g : Y \to X^N$ be two morphisms such that $\mathfrak{p}_{L,L'} \circ f = \iota_{L,N} \circ g$. So $f$ is a morphism of graded algebras $A \to B[L']$ and $g$ is a morphism of graded algebras $A \to B[N]$. Let $m \in L' = L + N'$ and let $A_m$ be the $m$-graded part of $A$. Let $x \in A_m$ and let $\lambda_m$ such that $f(x) = \lambda_m X^m$. Then since $\mathfrak{p}_{L,L'} \circ f = \iota_{L,N} \circ g$, we obtain that $\lambda_m = 0$ for all $l \in L \setminus (N' \cap L)$. So we get $f(x) = 0$ for all $x \in A_m$ for all $m \in L' \setminus N'$ (we use that $A_l A_{n'} = A_{l+n'}$ for all $l \in L \setminus (N' \cap L)$ and $n' \in N'$). So we obtain a unique morphism $h$ from $Y$ to $X^{N'}$ with the cartesian property.

□

Proposition 3.34. *Assume that $X$ is separated and locally of finite presentation over $S$. Let $N$ be a magnet of $M$. Write $N = \mathrm{colim}_{i\in I} N_i$ as a directed colimit of submonoids of $N$. Let $T \to S$ be a scheme, then*

$$X^N(T) = \mathrm{colim}_{i\in I} X^{N_i}(T).$$

*Proof.* For any scheme $T$, we have $A(N)_T = \lim_{i\in I} A(N_i)_T$ by [StP, Tag 01YW]. So $X(A(N)_T) = \mathrm{colim} X(A(N_i)_T)$ by [StP, Tag 049J]. Let $A(N)_T \to X_T$ be an $A(M)_T$-equivariant morphism. By the previous assertion this factorizes through a morphism $A(N_i)_T \xrightarrow{f} X_T$. We want to show that $f$ is $A(M)_T$-equivariant. Consider the diagram

$$\begin{tikzcd} A(M)_T \times A(N_i)_T \arrow[r, shift left] \arrow[r, shift right] & X_T \\ A(M)_T \times A(N)_T \arrow[u] \arrow[ur] & \end{tikzcd}$$

where the horizontal arrows $(f_1, f_2)$ correspond to $(g, x) \mapsto f(gx)$ and $(g, x) \mapsto gf(x)$. Since $X$ is separated, the kernel of $(f_1, f_2)$ is a closed subscheme of $A(M)_T \times A(N_i)_T$ (the proof of [GW, Def./Prop. 9.7] works in this context). Since the schematic image of the vertical morphism $\phi$ is $A(M)_T \times_T A(N_i)_T$ and because $f_1 \circ \phi = f_2 \circ \phi$, we have an equality of schemes $\ker(f_1, f_2) = A(M)_T \times_T A(N_i)_T$. So we have an equality of morphisms of schemes $f_1 = f_2$, and so $f$ is equivariant. We deduce that $X^N(T) = \mathrm{colim}_{i\in I} X^{N_i}(T)$. □

## 4. Attractors with prescribed limits

We introduce in this section another functor. Let $X$ be an algebraic space over a base scheme $S$. Let $M$ be a commutative monoid and let $A(M)_S$ be the associated diagonalizable monoid scheme over $S$. Assume that $A(M)_S$ acts on $X$. Let $N \subset M$ be a magnet. Let $F$ be a face of $N$. Let $Z$ be an other $S$-functor with a monomorphism $Z \to X^F$. We now introduce the attractor $X^N_{F,Z}$ associated to the magnet $N$ under the action of $A(M)_S$ on $X$ with prescribed limit in $Z$ relatively to the face $F$. Recall that we have a canonical morphism $X^N \to X^F$ (cf. Fact 3.15). Since $Z(T) \subset X^F(T)$ for any $S$-scheme $T$, the following definition makes sense.

Definition 4.1. Let $X^N_{F,Z}$ be the contravariant functor

$$(Sch/S) \to Set, (T \to S) \mapsto \{f \in X^N(T) \mid \text{ the image of } f \text{ in } X^F(T) \text{ belongs to } Z(T)\}.$$

If $F = N^*$, we omit $F$ in the notation, i.e. we put $X^N_{N^*,Z} =: X^N_Z$.

Proposition 4.2. *We have a canonical isomorphism $X^N_{F,Z} \simeq X^N \times_{X^F} Z$.*

*Proof.* Clear since

$$\begin{aligned} X^N_{F,Z}(T) &= \{f \in X^N(T) \mid \text{ the image of } f \text{ in } X^F(T) \text{ belongs in } Z(T)\} \\ &= \{(f, g) \in X^N(T) \times Z(T) \mid \ f = g \text{ in } X^F(T)\} \\ &= X^N(T) \times_{X^F(T)} Z(T). \end{aligned}$$

□

Fact 4.3. *Assume that $X$ is separated. Then we have a canonical monomorphism $X^N_{F,Z} \to X^N$ of $S$-functors. In particular we have a canonical monomorphism $X^N_{F,Z} \to X$ of $S$-functors.*

*Proof.* Clear by Corollary 3.28. □

## 5. Base change of affine étale morphisms along faces

Let $S$ be a scheme and let $X$ be an $S$-algebraic space with an action of $D(M)_S$ for some finitely generated abelian group $M$. Let $N$ be a magnet of $M$.

Proposition 5.1. *Let $Q$ be a face of $N$ and let $X^N \to X^Q$ be the associated morphism on attractors. Let $U$ be an $S$-affine scheme. Let $U \to X$ be an étale, $D(M)_S$-equivariant morphism of algebraic spaces. Assume that one of the following conditions hold*

(i) *$N$ is finitely generated as monoid,*

(ii) *$X$ is separated,*

*then the natural map $U^N \to X^N \times_{X^Q} U^Q$ is an isomorphism.*

*Proof.* We need some terminologies on monoids. Recall that a monoid is fine if it is cancellative and finitely generated. Recall also that a monoid $L$ is sharp if $L^* = 0$. See [Og, §I] for more details on monoids.

(i) Firstly, we remark that it is enough to treat the case $Q = N^*$. Indeed assume that Proposition 5.1 is true for the face of invertible elements. Since $Q^* = N^*$, we have

$$X^N \times_{X^Q} U^Q = X^N \times_{X^Q} (X^Q \times_{X^{N^*}} U^{N^*}) = X^N \times_{X^{N^*}} U^{N^*} = U^N.$$

So we assume $Q = N^*$. We now remark that we can assume $N^* = 0$ and that $N$ is fine and sharp using the map $M \to M/N^*$ and Proposition 3.29. So we now assume that $Q = 0$ and $N$ is fine and sharp and we adapt [Ri16, Lemma 1.11]. It is enough to construct the inverse morphism $X^N \times_{X^0} U^0 \to U^N$. For this let $p : T \to S$ be an $S$-scheme and let

$$\begin{tikzcd} T = A(0)_T \arrow[r, "f_0"] \arrow[d] & U_T \arrow[d] \\ A(N)_T \arrow[r] & X_T \end{tikzcd}$$

be a diagram corresponding to a $T$-point of $X^N \times_{X^0} U^0$. We want to find a diagonal filling $A(N)_T \to U_T$. Let $I$ be the kernel of $\mathcal{O}_T[N] \to \mathcal{O}_T[0]$, i.e the ideal associated to $N \setminus 0$ in $\mathcal{O}_T[N]$. Let $V(I^k) = \mathrm{Spec}_T(\mathcal{O}_T[N]/I^k)$ be the infinitesimal neighbourhoods of $A(0)_T$ inside $A(N)_T$. Since $U_T \to X_T$ is étale, it is smooth by [StP, Tag 04XX], and formally smooth by [StP, Tag 02H6], so by the infinitesimal lifting property the morphism $f_0 : T \to U_T$ lifts uniquely to a compatible family of morphisms $f_k : V(I^k) \to U_T$. These liftings are equivariant because the two maps $D(M)_T \times V(I^k) \to U_T$, $(g, x) \mapsto f_k(gx)$ and $(g, x) \mapsto gf_k(x)$ are common liftings of the map $D(M)_T \times A(0)_T \to U_T$, $(g, x) \mapsto f_0(gx) = gf_0(x)$ hence by uniqueness they are equal. Writing $U = \mathrm{Spec}_S(\mathcal{A})$ for a $M$-graded quasi-coherent $\mathcal{O}_S$-algebra $\mathcal{A}$, we have and $U \times_S T = \mathrm{Spec}_T(\mathcal{A}_T)$ where $\mathcal{A}_T = p^*\mathcal{A}$. Moreover we have a family of $M$-graded morphisms $\mathcal{A}_T \to \mathcal{O}_T[N]/I^k$. This means that $(\mathcal{A}_T)_m$ goes to 0 when $m \notin N$ and to $(\mathcal{O}_T[N]/I^k)_m$ when $m \in N$. The induced morphism to the completion $\mathcal{A}_T \to \mathcal{O}_T[[N]]$ has image in $\mathcal{O}_T[N]$, yielding the desired lifting $A(N)_T \to U_T$ (cf. [Og, Chap. I,§3 Prop. 3.6.1] for the local description of the completion $\mathcal{O}_T[[N]]$).

(ii) Let $T \to S$ be a scheme. Write $N = \mathrm{colim}_{i \in I} N_i$ as a colimit of finitely generated monoids.

Using (i) and Proposition 3.34 we have

$$\begin{aligned} U^N(T) &= \mathrm{colim}_{i\in I} U^{N_i}(T) \\ &= \mathrm{colim}_{i\in I}\big(X^{N_i}(T) \times_{X^Q(T)} U^Q(T)\big) \\ &= \mathrm{colim}_{i\in I}\big(X^{N_i}(T)\big) \times_{X^Q(T)} U^Q(T) \\ &= X^N(T) \times_{X^Q(T)} U^Q(T) \\ &= (X^N \times_{X^Q} U^Q)(T). \end{aligned}$$

□

## 6. Fixed-point-reflecting atlases

Let $M$ be an abelian group. Let $S$ be a scheme. Let $X$ be a quasi-separated $S$-space endowed with an action of $D(M)_S$.

We discuss fixed-point-reflecting atlases in this section. Roughly, fixed-point-reflecting étale atlases are atlases such that we can learn from the charts the fixed points of our space. This concept makes sense for an action of an arbitrary group scheme and is well-studied. In this section we introduce several related concepts and definitions, $Z$-FPR morphisms and strongly-FPR morphisms, useful and motivated by algebraic magnetism ($Z$ is a subgroup of $M$ here). A strongly-FPR atlas is $Z$-FPR for all subgroups $Z \subset M$. We formulate a theorem (Theorem 6.8) that provides the existence of $Z$-FPR atlases and a conjecture (Conjecture 6.9) about the existence of strongly-FPR atlases. In this work, $Z$-FPR atlases are used to prove representability results (Proposition 8.1 and Theorem 8.3) and strongly-FPR atlases are needed to study magnets (Theorem 15.4). In the next section (Section 7), we will notice that Conjecture 6.9 is true in many cases (namely for Zariski locally linearizable actions, e.g. Sumihiro's actions 7.4) in particular one has $Z$-FPR atlases in this case. Romagny's appendix offers a proof of Theorem 6.8.

Definition 6.1. Let $f : U \to X$ be a $D(M)_S$-equivariant morphism of $S$-spaces.

(i) We say that $f$ is fixed-point reflecting (FPR) if the canonical morphism of functors $U^{D(M)_S} \to U \times_X X^{D(M)_S}$ is an isomorphim.
(ii) We say that $f$ is $Z$-FPR if it is fixed-point reflecting for the induced action of $D(M/Z)_S$, i.e. if the canonical morphism of functors $U^{D(M/Z)_S} \to U \times_X X^{D(M/Z)_S}$ is an isomorphim.
(iii) We say that $f$ is strongly-FPR if it is $Z$-FPR for all subgroups $Z \subset M$.
(iv) We say that $f$ is an equivariant atlas if $f$ is étale and surjective and $U = \coprod_{\tau\in\mathscr{A}} U_\tau$ is the disjoint union of $D(M)_S$-stable $S$-affine schemes; moreover in this case we say that the equivariant atlas $f$ is $S$-affine if $\mathscr{A}$ may be chosen finite.
(v) We say that $f$ is a $Z$-FPR atlas if $f$ is a $Z$-FPR equivariant atlas.
(vi) We say that $f$ is a strongly-FPR atlas if $f$ is a strongly-FPR equivariant atlas.

Remark 6.2. Since a finite disjoint union of $S$-affine schemes is $S$-affine, if an equivariant atlas is $S$-affine then we can assume that $\mathscr{A}$ is a singleton. In other words an $S$-affine equivariant atlas for $X$ is a $D(M)_S$-equivariant étale surjective morphism $U \to S$ where $U$ is an $S$-affine scheme.

The following is an immediate generalization of [Ri16, Lemma 1.10].

Proposition 6.3. *Assume that $M \cong \mathbb{Z}^r$ for some integer $r$ and let $U \to X$ be an étale $D(M)_S$-equivariant morphism, then $U^{D(M)_S} \to X^{D(M)_S} \times_X U$ is an isomorphism. In other words, if $M$ is torsion-free, then every étale morphism is FPR.*

*Proof.* The proof of [Ri16, Lemma 1.10] works replacing "$\mathbb{G}_m$" by "$D(\mathbb{Z}^r)$". □

Remark 6.4. In his appendix, Romagny provides a generalization of Proposition 6.3 to arbitrary flat group schemes with connected fibers and in particular provides a detailled proof of Proposition 6.3, cf. Theorem A.1 .

Remark 6.5. If $M$ is not torsion-free, and $U \to X$ is an étale $D(M)_S$-equivariant morphism, then $U^{D(M)_S} \to X^{D(M)_S} \times_X U$ is obviously not an isomorphism in general. For example, take $M = \mathbb{Z}/2\mathbb{Z}$, $S = \mathrm{Spec}(\mathbb{Z})$, $X = S$ endowed with the trivial action of $D(M)$ and $U = D(M)$ endowed with the non-trivial action of itself by multiplication. Then the $D(M)$-equivariant morphism $U \to S$ is smooth of relative dimension zero and so it is étale; but $U^{D(M)} = \emptyset$, $X^{D(M)} = X$ and therefore $U^{D(M)} \ncong X^{D(M)} \times_X U$.

Remark 6.6. Let us provide an example of a space $X$ endowed with an action of a diagonalizable group $D(M)$ and an FPR equivariant atlas $U \to X$ that is not strongly-FPR. Choose an exact sequence of abelian groups $0 \to Z \to M \to M/Z \to 0$ with $Z \neq M$. Then $f : D(M) \to D(Z)$ is étale, surjective and $D(M)$-equivariant. Put $U = D(M)$ and $X = D(Z)$. Since $U^{D(M)} = X^{D(M)} = \emptyset$, $f$ is FPR. Now $f$ is not $Z$-FPR because $U^{D(M/Z)} = \emptyset$ and $X^{D(M/Z)} = X$.

Lemma 6.7. *Let $X, X'$ be $S$-algebraic spaces endowed with actions of $D(M)_S$. Let $Z$ be a subgroup of $M$. Let $X' \to X$ be a $D(M)_S$-equivariant affine morphism of $S$-algebraic spaces. Let $U \to X$ be a $Z$-FPR atlas of $X$. Then*

(i) *The canonical morphism $U \times_X X' \to X'$ is a $Z$-FPR atlas of $X'$.*
(ii) *If $U \to X$ is an $S$-affine $Z$-FPR atlas of $X$, then $U \times_X X' \to X'$ is an $S$-affine $Z$-FPR atlas of $X'$.*

*Proof.* (i) Since $X' \to X$ is an affine morphism, $U \times_X X' \to U$ is affine and so the composition $U \times_X X' \to S$ is a disjoint union of $S$-affine schemes. The morphism $U \times_X X' \to X'$ is étale and surjective because $U \to X$ is so. By Proposition 3.7, we have canonical identifications

$$(X' \times_X U)^Z = X'^Z \times_{X^Z} U^Z = X'^Z \times_{X^Z} (U \times_X X^Z) = X'^Z \times_X U = X'^Z \times_{X'} (X' \times_X U).$$

Assertion (ii) is now immediate. □

Theorem 6.8 was motivated by finding a generalization of Proposition 6.3 for general diagonalizable group schemes. Theorem 6.8 is essentially a theorem of Alper, Hall and Rydh [AHR21]. The present version of Theorem 6.8 was formulated by Mayeux. The proof of Theorem 6.8 could be seen as a corollary of [AHR21]. Establishing Theorem 6.8 as a corollary of [AHR21] is due to Romagny and is the topic of the appendix. Note that Theorem 6.8 is numbered as Theorem A.2 in the appendix.

Theorem 6.8. *Assume that $M$ is finitely generated as abelian group. Let $X$ be a quasi-separated $S$-algebraic space locally of finite presentation endowed with an action of $D(M)_S$. Let $Z \subset M$ be a subgroup and assume that one of the following assertions holds:*

(i) *$X$ is separated over $S$,*
(ii) *$M/Z$ is torsion-free.*

*Then there exists a $Z$-FPR atlas $U \to X$, which may be chosen quasi-compact (in particular $S$-affine here) if $X \to S$ is quasi-compact.*

*Proof.* The proof is given in the Appendix of this article, cf. Theorem A.2. □

In fact the author expects that the following stronger assertions should be true. We will study the correct generality in which this holds. We think that this is an interesting problem.

Conjecture 6.9. *Assume that $M$ is finitely generated as abelian group. Let $S$ be an arbitrary scheme. Let $X$ be a separated $S$-algebraic space locally of finite presentation endowed with an action of $G = D(M)_S$. The following assertions hold.*

(i) *There exists a strongly-FPR atlas $U \to X$.*
(ii) *If $X/S$ is quasi-compact, there exists an $S$-affine strongly-FPR atlas $U \to X$.*

Note that §7 already ensures the existence of strongly-FPR atlases for Sumihiro actions.

## 7. Zariski locally linearizable actions

We proceed with the notation from §6. We recall the definition of Zariski locally linearizable actions and explain that they have strongly-FPR atlases. We recall that some actions are far from being locally linearizable; this gives a reason to use étale atlases (cf. § 6) and the theory of algebraic spaces.

Fact 7.1. *Let $U \to X$ be a $D(M)_S$-equivariant open immersion, then for any subgroup $Z \subset M$,*

$$U^{D(M/Z)_S} \to X^{D(M/Z)_S} \times_X U$$

*is an isomorphism. In other words, every $D(M)_S$-equivariant open immersion is $Z$-FPR for all subgroups.*

*Proof.* This works for any equivariant monomorphism. For any scheme $T$ over $S$, $U(T) \times_{X(T)} X^{D(M/Z)_S}(T) = U^{D(M/Z)_S}(T)$. □

Definition 7.2. The action of $D(M)_S$ on $X$ is Zariski locally linearizable if there are $S$-affine open $D(M)_S$-stable subspaces of $X$ covering $X$. We also refer to these actions as Sumihiro actions or Sumihiro spaces.

Fact 7.3. *Let $X$ be an $S$-space endowed with a Zariski locally linearizable action of $D(M)_S$, then there exists a strongly-FPR equivariant atlas.*

*Proof.* Let $U$ be the disjoint union of $S$-affine open $D(M)_S$-stable subspaces of $X$ covering $X$, then $U \to X$ is as required by Fact 7.1. □

Remark 7.4. Assume that $M$ is finitely generated. The result [Su75, Corollary 3.11] shows that if $D(M)_S$ is smooth over $S$ and if $X$ is a scheme and satisfies the condition (N) (e.g. $X$ is normal, cf. [Su75, 3.4, 3.5]), then the action of $D(M)_S$ on $X$ is Zariski locally linearizable.

Remark 7.5. There exists a simple example of an action of $D(\mathbb{Z})_S$ on a quasi-separated locally finitely presented scheme $X/S$ that is not Zariski locally linearizable (e.g. cf. [Ri16, §0.2]).

## 8. Representability and properties

Let $M$ be an abelian group. Let $S$ be an arbitrary scheme. Let $X$ be an $S$-algebraic space endowed with an action of $D(M)_S$. We prove that attractors are representable once we know the existence of FPR atlases. Recall that Theorem 6.8 or Section 7 provide FPR atlases in great generality, so that our representabilty results hold in great generality.

Proposition 8.1. *Let $Z \subset M$ be a subgroup. Assume that there exists a $Z$-FPR atlas for $X$ under the action of $D(M)_S$ (cf. Definition 6.1). Then the attractor space $X^Z$ (which identifies with the fixed space $X^{D(M/Z)_S}$ by Proposition 3.32) is representable by a closed subspace of $X$.*

*Proof.* Let $U$ be a $Z$-FPR atlas. We have a cartesian square

$$\begin{tikzcd} U^Z \arrow[r] \arrow[d] & X^Z \arrow[d] \\ U \arrow[r] & X \end{tikzcd}.$$

By Proposition 3.20 the left vertical arrow is a closed immersion of schemes. The lower arrow is surjective étale. So by [StP, Tag 03I2], $X^Z$ is an algebraic space and the right vertical arrow is a closed immersion. We used that being a closed immersion is stable under base change, fppf-local on the base and closed immersions satisfy fppf-descent. □

Remark 8.2. Theorem A.1 of the appendix provides a result similar to Proposition 8.1 about closedness.

Theorem 8.3. *Let $N \subset M$ be a magnet, i.e. an arbitrary submonoid. Let $N^* \subset N$ be the face of intertible elements of $N$. Assume that there exists a $N^*$-FPR atlas for $X$ under the action of $D(M)_S$ (cf. Definition 6.1). Assume that one of the following conditions hold*

(a) *$X$ is separated over $S$, or*
(b) *$N$ is finitely generated as monoid,*

*then the attractor $X^N$ is representable by an algebraic space over $S$. Moreover*

(i) *The morphism of algebraic spaces $X^N \to X^{N^*}$ is affine.*
(ii) *If $X/S$ is quasi-separated, then $X^N/S$ is quasi-separated.*
(iii) *If $X/S$ is separated, then $X^N/S$ is separated.*
(iv) *If $X/S$ is locally of finite presentation, then $X^N/S$ is locally of finite presentation.*
(v) *If $X/S$ is quasi-compact then $X^N/S$ is quasi-compact.*
(vi) *If $X/S$ if of finite presentation, then $X^N/S$ is of finite presentation.*
(vii) *If $X$ is a scheme then $X^N$ is a scheme.*

*Proof.* Let $U \to X$ be a $N^*$-FPR atlas. Using Proposition 5.1, we obtain a diagram with cartesian squares:

$$\begin{tikzcd} U^N \arrow[r] \arrow[d] & U^{N^*} \arrow[r] \arrow[d] & U \arrow[d] \\ X^N \arrow[r] & X^{N^*} \arrow[r] & X. \end{tikzcd}$$

The vertical maps are étale and surjective, and we know from Theorem 3.19 that $U^N$ is representable by a disjoint sum of $S$-affine schemes. By Proposition 8.1, $X^{N^*}$ is an $S$-algebraic space. It follows from [StP, Tag 03I2] that $X^N$ is representable.

Now let us prove the listed properties. We have a cartesian square

$$\begin{tikzcd} U^N \arrow[r] \arrow[d] & X^N \arrow[d] \\ U^{N^*} \arrow[r] & X^{N^*}. \end{tikzcd}$$

As explained before, the lower horizontal arrow is étale and surjective. The left vertical arrow is affine because $U$ is $S$-affine and $U^N$ and $U^{N^*}$ are $S$-affine schemes by Theorem 3.19. Now, we apply [StP, Tag 03I2] to conclude that the right vertical arrow is affine. We used that being affine is preserved under base change, fppf local on the base and satisfies descent for fppf coverings. Assume that $X$ is locally of finite presentation, i.e. commutes with colimits of affine schemes in $(Sch/S)_{\text{fppf}}$, then the functor $X^N$ commutes with colimits of affine schemes in $(Sch/S)_{\text{fppf}}$, i.e. $X^N$ is locally of finite presentation. Assume that $X$ is separated (resp. quasi-separated, quasi-compact, resp. is a scheme). Then $X^{N^*}$ is separated (resp. quasi-separated, quasi-compact, resp. is a scheme) because it is closed in $X$ (cf. Proposition 8.1). Since $X^N \to X^{N^*}$ is affine, in particular representable, quasi-compact and separated it follows that $X^N$ is separated (resp. quasi-compact, resp. is a scheme). Note that by definition $X$ is of finite presentation if it is locally of finite presentation, quasi-compact and quasi-separated. □

Remark 8.4. Note that Proposition 3.29 (iii) and Theorem 8.3 together show representability of many attractors under actions of diagonalizable monoid schemes outside affine cases.

Proposition 8.5. *Assume that $M$ is finitely generated as abelian group. Assume that $X \to S$ is quasi-compact and separated. Then there exists a finitely generated submonoid $N_c$ of $N$ such that $X^N = X^{N_c}$ and $N^* = N_c^*$.*

*Proof.* Let $\kappa : \mathbb{N} \to N$ be a bijection. For any $i \in \mathbb{N}$, let $N_i$ be the submonoid of $N$ generated by $\{N^* \cup \kappa(\mathbb{N}_{\leqslant i})\}$. Then $N_i^* = N^*$ and $N_i$ is finitely generated for any $i \in \mathbb{N}$, moreover $\bigcup_{i \geqslant 0} N_i = N$. By Proposition 3.34 $X^N(T) = \text{colim} X^{N_i}(T)$. Let $U \to X$ be an $S$-affine $N^*$-FPR atlas as in Theorem 6.8. Using that $U$ is an $S$-affine scheme of finite presentation over $S$, there exists an integer $c$ such that $U^{N_c} = U^{N_i}$ for all $i \geqslant c$. To see this, write $U$ as the spectrum of a graded $\mathcal{O}_S$-algebra, consider degrees of a finite set of generators of this $\mathcal{O}_S$-algebra and use that gradings of $\mathcal{O}_S$-algebras are preserved under equivariant morphisms of $S$-affine schemes. Let $i \geqslant c$, and consider the diagram

$$\begin{tikzcd} U^{N_c} \arrow[r] \arrow[d] & X^{N_c} \arrow[d] \\ U^{N_i} \arrow[r] & X^{N_i} \end{tikzcd} .$$

The horizontal arrows are étale and surjective using Proposition 5.1. Moreover, the left vertical arrow is an isomorphism since $i \geqslant c$. Furthermore, the diagram is a cartesian square because by Proposition 5.1

$$\begin{aligned} U^{N_i} \times_{X^{N_i}} X^{N_c} &\simeq (U^{N^*} \times_{X^{N^*}} X^{N_i}) \times_{X^{N_i}} X^{N_c} \\ &\simeq U^{N^*} \times_{X^{N^*}} X^{N_c} \\ &\simeq U^{N_c}. \end{aligned}$$

Using [StP, Tag 03I2] for the property "isomorphism", we deduce that for all $i \geqslant c$, $X^{N_i} = X^{N_c}$. This implies that $X^N = X^{N_c}$. □

## 9. Hochschild cohomology for diagonalizable monoids via formulas

Let $S$ be a scheme and $B$ be an algebraic space over $S$. Let $B_{spaces,fppf}$ be the small fppf site of $B$ (defined as in the case of schemes, e.g. cf. [Sc17]). In this section $\mathcal{O}_B$ is the canonical sheaf of rings on $B_{spaces,fppf}$.

### 9.1 Affine morphisms and sheaves

PROPOSITION 9.1. *Let $f : X \to B$ be an affine morphism of algebraic spaces. Let $\mathcal{A}$ be the quasi-coherent $\mathcal{O}_B$-algebra of $X$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_B$-module. Then*

$$f_* f^* \mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_B} \mathcal{A}.$$

*Proof.* Let $f^* f_* \mathcal{O}_X \to \mathcal{O}_X$ be the canonical morphism coming from adjunction. It induces a canonical morphism

$$f^*\mathcal{F} \otimes_{\mathcal{O}_X} f^* f_* \mathcal{O}_X \to f^*\mathcal{F}.$$

Since $\mathcal{A} = f_*\mathcal{O}_X$, we get obtain a canonical morphism $f^*\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{A} \to f^*\mathcal{F}$. Now using [StP, Tag 03EL], we obtain a canonical morphism

$$f^*(\mathcal{F} \otimes_{\mathcal{O}_B} \mathcal{A}) \to f^*\mathcal{F}.$$

By adjunction, we finally get a canonical morphism $\mathcal{F} \otimes_{\mathcal{O}_B} \mathcal{A} \to f_* f^* \mathcal{F}$. It remains to prove that it is an isomorphism. For this, we reduce to the affine case and apply [StP, Tag 01I8]. □

### 9.2 Fppf Hochschild monoid cohomology over algebraic spaces

Let $H$ be a monoid algebraic space over $B$. Let $\mathcal{O}_B$ be the sheaf of rings of $B$ on $B_{spaces,fppf}$.

DEFINITION 9.2. A $H$-$\mathcal{O}_B$-module over $B_{spaces,fppf}$ is an $\mathcal{O}_B$-module $\mathcal{F}$ in the sense of [StP, Tag 03CW] such that $H$ acts $\mathcal{O}_B$-linearly on $\mathcal{F}$, i.e:

(i) For any $T \to B \in B_{spaces,fppf}$, $H(T)$ acts on $\mathcal{F}(T)$. Moreover for any $g \in H(T), x, y \in \mathcal{F}(T)$ and $a \in \mathcal{O}_B(T)$, we have $g \cdot (x + ay) = g \cdot x + a(g \cdot y)$.

(ii) If $T' \to T$ is a morphism in $B_{spaces,fppf}$, the action of $H(T')$ on $\mathcal{F}(T')$ is compatible with the action of $H(T)$ on $\mathcal{F}(T)$.

REMARK 9.3. Note that Definition 9.2 is similar to Definition 2.19, but is different.

DEFINITION 9.4. Let $\mathcal{F}$ be a $H$-$\mathcal{O}_B$-module over $B_{spaces,fppf}$. Let $n \geqslant 0$ be an integer. We put

$$C^n(H, \mathcal{F}) = Mor_{B_{spaces,fppf}}(H^n, \mathcal{F}) = \text{The set of natural transformations from } H^n \text{ to } \mathcal{F},$$

where $H^n$ and $\mathcal{F}$ are seen as functors from $B_{spaces,fppf}$ to Sets. The set $C^n(H, \mathcal{F})$ is canonically endowed with a structure of $\mathcal{O}_B(B)$-module via

$$\text{for any } \Theta, \Psi \in C^n(H, \mathcal{F}), a \in \mathcal{O}_B(B), T \in B_{spaces,fppf}; (\Theta + a\Psi)(T) = \Theta(T) + a\,|_T \Psi(T).$$

We have a morphism of $\mathcal{O}_B(B)$-modules

$$\partial : C^n(H, \mathcal{F}) \to C^{n+1}(H, \mathcal{F})$$

sending $\Theta$ to $\partial\Theta$ where $\partial\Theta$ is the transformation from $H^{n+1}$ to $\mathcal{F}$ such that for any $T \in B_{spaces,fppf}$, $\partial\Theta(T)$ is the map sending $g_1, \ldots, g_{n+1} \in H^{n+1}(T)$ to

$$g_1 \cdot \Big(\Theta(T)\big(g_2, \ldots, g_{n+1}\big)\Big) + \sum_{i=1}^{n} (-1)^i \Theta(T)\big(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}\big) + (-1)^{n+1} f(g_1, \ldots, g_n) \in \mathcal{F}(T).$$

We also have an internal version of $C^n(H, \mathcal{F})$ as follows.

DEFINITION 9.5. Let $\mathcal{F}$ be a $H$-$\mathcal{O}_B$-module. Let $n \geqslant 0$ be an integer. We put for any $V \in B_{spaces,fppf}$

$$\underline{C^n(H, \mathcal{F})}(V) = C^n(H|_V, \mathcal{F}|_V) = Mor_{V_{spaces,fppf}}(H^n|_V, \mathcal{F}|_V).$$

The functor $\underline{C^n(H,\mathcal{F})}$ is canonically endowed with a structure of $\mathcal{O}_B$-module. We have a morphism of $\mathcal{O}_B$-modules $\underline{\partial} : \underline{C^n(H,\mathcal{F})} \to \underline{C^{n+1}(H,\mathcal{F})}$ induced by $\partial$.

Remark 9.6. For any positive integer $n$ the composition $\underline{\partial} \circ \underline{\partial}$ is the zero map.

Definition 9.7. Let $\mathcal{F}$ be a $H$-$\mathcal{O}_B$-module over $B_{spaces,fppf}$. We put

$$\begin{aligned}
&Z^n_{monoid}(H,\mathcal{F}) = \ker(C^n(H,\mathcal{F}) \xrightarrow{\partial} C^{n+1}(H,\mathcal{F})) \text{ for any } n \geqslant 0\\
&B^0_{monoid}(H,\mathcal{F}) = 0\\
&B^n_{monoid}(H,\mathcal{F}) = \mathrm{im}(C^{n-1}(H,\mathcal{F}) \xrightarrow{\partial} C^n(H,\mathcal{F}) \text{ for any } n \geqslant 1\\
&H^n_{monoid}(H,\mathcal{F}) = \frac{Z^n(H,\mathcal{F})}{B^n(H,\mathcal{F})} \text{ for any } n \geqslant 0.
\end{aligned}$$

$$\begin{aligned}
&\underline{Z^n_{monoid}(H,\mathcal{F})} = \ker(\underline{C^n(H,\mathcal{F})} \xrightarrow{\partial} \underline{C^{n+1}(H,\mathcal{F})}) \text{ for any } n \geqslant 0\\
&\underline{B^0_{monoid}(H,\mathcal{F})} = 0\\
&\underline{B^n_{monoid}(H,\mathcal{F})} = \underline{\mathrm{im}(C^{n-1}(H,\mathcal{F})} \xrightarrow{\partial} \underline{C^n(H,\mathcal{F})} \text{ for any } n \geqslant 1\\
&\underline{H^n_{monoid}(H,\mathcal{F})} = \frac{\underline{Z^n(H,\mathcal{F})}}{\underline{B^n(H,\mathcal{F})}} \text{ for any } n \geqslant 0.
\end{aligned}$$

We finish this subsection with the description of $Z^n_{monoid}$ and $B^n_{monoid}$ for $n = 1$.

Remark 9.8. We have

$$Z^1_{monoid}(H,\mathcal{F}) = \{c \in C^1(H,\mathcal{F}) \mid \forall T \in B_{spaces,fppf}, \forall g, g' \in H(T),\ c(gg') = c(g) + g \cdot c(g')\}.$$

$$B^1_{monoid}(H,\mathcal{F}) = \{b \in C^1(H,\mathcal{F}) \mid \exists v \in \mathcal{F}(B); \forall T \in B_{sp.,fppf}, \forall g \in G(T), b(g) = g \cdot v_T - v_T\}.$$

### 9.3 Modules under affine monoid algebraic spaces

Let $M$ be a finitely generated cancellative monoid. The structural morphism $A(M)_B \to B$ and the multiplication morphism $A(M)_B \times_B A(M)_B \to A(M)_B$ are surjective, finitely presented and flat by Proposition 2.8 and Fact 2.9.

Proposition 9.9. *Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_B$-modules (over $B_{spaces,fppf}$). Assume $\mathcal{G}$ is quasi-coherent. Let*

$$X' = \mathrm{Spec}_B(\mathcal{A}') \xrightarrow{m} X = \mathrm{Spec}_B(\mathcal{A}) \xrightarrow{f} B.$$

*be affine morphisms of $S$-spaces. Assume $m$ and $f$ are flat and finitely presented. Then we have a canonical identification*

$$\mathcal{H}om_{\mathcal{O}_B}(\mathcal{F},\mathcal{G})(X) = \mathrm{Hom}_{\mathcal{O}_B}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_B} \mathcal{A}).$$

*Moreover the map*

$$\mathcal{H}om_{\mathcal{O}_B}(\mathcal{F},\mathcal{G})(X) \to \mathcal{H}om_{\mathcal{O}_B}(\mathcal{F},\mathcal{G})(X')$$

*associated to $m$ corresponds to the map*

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_B}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_B} \mathcal{A}) &\to \mathrm{Hom}_{\mathcal{O}_B}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_B} \mathcal{A}')\\
\varphi &\mapsto (\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \otimes_{\mathcal{O}_B} \mathcal{A} \xrightarrow{\mathrm{Id}\otimes\phi_m} \mathcal{G} \otimes_{\mathcal{O}_B} \mathcal{A}')
\end{aligned}$$

*where $\phi_m : \mathcal{A} \to \mathcal{A}'$ is dual to $m$ via [StP, Tag 081V].*

*Proof.* Note that $m$ and $f$ are flat and finitely presented. The following identifications prove the first assertion

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_B}(\mathcal{F},\mathcal{G})(X) &= \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{F}, f^*\mathcal{G})\\ &= \mathrm{Hom}_{\mathcal{O}_B}(\mathcal{F}, f_*f^*\mathcal{G})\\ &= \mathrm{Hom}_{\mathcal{O}_B}(\mathcal{F}, \mathcal{G}\otimes_{\mathcal{O}_B}\mathcal{A}).\end{aligned}$$

The second assertion also follows from these intermediate identifications. □

Proposition 9.10. *Let $H = \mathrm{Spec}_B(\mathcal{A})$ be a monoid algebraic space over $B$ such that $H \to B$ is affine, flat and finitely presented. An $H$-$\mathcal{O}_B$-module structure (over $B_{spaces,fppf}$) on a quasi-coherent $\mathcal{O}_B$-module $\mathcal{F}$ corresponds to an $\mathcal{O}_B$-comodule structure on $\mathcal{F}$, i.e a morphism of $\mathcal{O}_B$-modules $\mathcal{F} \xrightarrow{\delta} \mathcal{F}\otimes_{\mathcal{O}_B}\mathcal{A}$ such that $(\delta\otimes\mathrm{Id}_{\mathcal{A}})\circ\delta = (\mathrm{Id}_{\mathcal{F}}\otimes\Delta_{\mathcal{A}})\circ\delta$ and $(\mathrm{Id}_{\mathcal{F}}\otimes\varepsilon_{\mathcal{A}})\circ\delta = \mathrm{Id}_{\mathcal{F}}$.*

*Proof.* Let us first remark that the set of $H$-$\mathcal{O}_B$-module structures on an $\mathcal{O}_B$-module $\mathcal{F}$ is a subset of the set $Mor_{B_{spaces,fppf}}(H, \mathcal{H}om_{\mathcal{O}_{B,fppf}}(\mathcal{F},\mathcal{F}))$. Using Proposition 9.9 we get

$$\begin{aligned}Mor_{B_{spaces,fppf}}(H, \mathcal{H}om_{\mathcal{O}_B}(\mathcal{F},\mathcal{F})) &= \mathcal{H}om_{\mathcal{O}_B}(\mathcal{F},\mathcal{F})(H)\\ &= \mathrm{Hom}_{\mathcal{O}_B}(\mathcal{F}, \mathcal{F}\otimes_{\mathcal{O}_B}\mathcal{A}).\end{aligned}$$

The condition that $\mathcal{F}$ is an $H$-module means that the transformations of functors

$$T_1 : H\times H \xrightarrow{m} H \xrightarrow{\Psi} \mathcal{H}om_{\mathcal{O}_B}(\mathcal{F},\mathcal{F})$$

and

$$T_2 : H\times H \xrightarrow{\Psi\times\Psi} \mathcal{H}om_{\mathcal{O}_B}(\mathcal{F},\mathcal{F})\times\mathcal{H}om_{\mathcal{O}_B}(\mathcal{F},\mathcal{F}) \xrightarrow{\circ} \mathcal{H}om_{\mathcal{O}_B}(\mathcal{F},\mathcal{F})$$

are equal and that the composition

$$e_B \to H \xrightarrow{\Psi} \mathcal{H}om_{\mathcal{O}_B}(\mathcal{F},\mathcal{F})$$

is the identity. The transformations $T_1$ and $T_2$ correspond to elements in

$$\mathrm{Hom}_{\mathcal{O}_B}(\mathcal{F}, \mathcal{F}\otimes_{\mathcal{O}_B}\mathcal{A}\otimes_{\mathcal{O}_B}\mathcal{A}),$$

namely $T_1$ corresponds to $(\mathrm{Id}_{\mathcal{F}}\otimes\Delta_{\mathcal{A}})\circ\delta$ (cf. Proposition 9.9) and $T_2$ corresponds to $(\delta\otimes\mathrm{Id}_{\mathcal{A}})\circ\delta$. The composition

$$e_B \to H \xrightarrow{\Psi} \mathcal{H}om_{\mathcal{O}_B}(\mathcal{F},\mathcal{F})$$

corresponds to an element in $\mathrm{Hom}_{\mathcal{O}_B}(\mathcal{F}, \mathcal{F}\otimes_{\mathcal{O}_B}\mathcal{O}_B)$, namely $(\mathrm{Id}_{\mathcal{F}}\otimes\varepsilon_{\mathcal{A}})\circ\delta$. This finishes the proof. □

Proposition 9.11. *An $A(M)_B$-$\mathcal{O}_B$-module (over $B_{spaces,fppf}$) structure on a quasi-coherent $\mathcal{O}_B$-module $\mathcal{F}$ corresponds to a collection $(\mu_m)_{m\in M}\in\mathrm{Hom}_{\mathcal{O}_B}(\mathcal{F},\mathcal{F})$ such that for all $m,k\in M$,*

$$\mu_m\circ\mu_k = \mu_k \text{ if } k=m,\ \ \mu_m\circ\mu_k = 0 \text{ if } k\neq m \text{ and } \sum_{m\in M}\mu_m = \mathrm{Id}_{\mathcal{F}}.$$

*Proof.* Proposition 9.10 shows that an $A(M)_B$-$\mathcal{O}_B$-module corresponds to a morphism of $\mathcal{O}_B$-comodule

$$\mu : \mathcal{F} \to \mathcal{F}\otimes_{\mathcal{O}_B}\mathcal{O}_B[M] = \bigoplus_{m\in M}\mathcal{F}.$$

Now $\mu$ gives a collection $(\mu_m)_{m\in M}$ with $\mu_m\in\mathrm{Hom}_{\mathcal{O}_B}(\mathcal{F},\mathcal{F})$. Saying that $\mu$ is a comodule is equivalent to the conditions in the statement (cf. Fact. 2.2 and Def. 2.3 for the $\mathcal{O}_B$-coalgebra structure on the quasi-coherent $\mathcal{O}_B$-algebra of the affine monoid algebraic space $A(M)_B$, namely $\mathcal{O}_B[M]$). □

Proposition 9.12. *Let $\mathcal{F}$ be a quasi-coherent $A(M)_B$-$\mathcal{O}_B$-module over $B_{spaces,fppf}$. Let $V \in B_{spaces,fppf}$. We have*

$$\underline{C^n(A(M)_B,\mathcal{F})}(V) = \Big(\bigoplus_{(m_1,\dots,m_n)\in M^n} \mathcal{F}X^{m_1}\cdots X^{m_n}\Big)(V)$$

*where for every $m = (m_1,\dots,m_n) \in M^n$, $\mathcal{F}X^{m_1}\cdots X^{m_n} = \mathcal{F}^m$ is a formal copy of $\mathcal{F}$. In particular if $V$ is quasi-compact we have*

$$\underline{C^n(A(M)_B,\mathcal{F})}(V) = \bigoplus_{m\in M^n} \mathcal{F}^m(V)$$

*Moreover for any $n \geqslant 0$, the linear map $\partial : C^n \to C^{n+1}$ is given on the component $\mathcal{F}X^{m_1}\cdots X^{m_n}$ and on quasi-compact objects by the formula*

$$\begin{aligned} f^{m_1,\dots,m_n}X^{m_1}\cdots X^{m_n} \mapsto & \mu(f^{m_1,\dots,m_n})X^{m_1}\cdots X^{m_n} \\ & + \sum_{i=1}^{n}(-1)^i f^{m_1,\dots,m_n}X^{m_1}\cdots \Delta(X^{m_i})\cdots X^{m_n} \\ & +(-1)^{n+1}f^{m_1,\cdots,m_n}X^{m_1}\dots X^{m_n}X^0 \end{aligned}$$

*where $\Delta(X^{m_i}) = X^{m_i}X^{m_i}$.*

*Proof.* Put $p : A(M)^n_V \to V$ and $\mathcal{F}_V = \mathcal{F}|_V$. The identifications

$$\begin{aligned} \underline{C^n(A(M)_B,\mathcal{F})}(V) &= Mor_{V_{spaces,fppf}}(A(M)^n_V,\mathcal{F}_V) \\ &= Mor_{V_{spaces,fppf}}(A(M)^n_V,\mathcal{H}om_{\mathcal{O}_V}(\mathcal{O}_V,\mathcal{F}_V)) \\ &= \mathcal{H}om_{\mathcal{O}_V}(\mathcal{O}_V,\mathcal{F}_V)(A(M)^n_V) \\ &= \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{O}_V,\mathcal{F}_V\otimes_{\mathcal{O}_V}\mathcal{O}_V[M]\otimes_{\mathcal{O}_V}\cdots\otimes_{\mathcal{O}_V}\mathcal{O}_V[M]) \\ &= \Big(\mathcal{F}_V\otimes_{\mathcal{O}_V}\mathcal{O}_V[M]\otimes_{\mathcal{O}_V}\cdots\otimes_{\mathcal{O}_V}\mathcal{O}_V[M]\Big)(V) \\ &= \Big(\mathcal{F}\otimes_{\mathcal{O}_B}\mathcal{O}_B[M]\otimes_{\mathcal{O}_B}\cdots\otimes_{\mathcal{O}_B}\mathcal{O}_B[M]\Big)(V) \\ &= \Big(\bigoplus_{(m_1,\dots,m_n)\in M^n}\mathcal{F}X^{m_1}\cdots X^{m_n}\Big)(V) \end{aligned}$$

prove the first assertion. The second assertion is now a consequence of Definition 9.5. □

Proposition 9.13. *Let $\mathcal{F}$ be a quasi-coherent $A(M)_B$-$\mathcal{O}_B$-module over $B_{spaces,fppf}$, then*

$$H^1_{monoid}(A(M)_B,\mathcal{F}) = 0.$$

*Proof.* It is enough to prove that $Z^1_{monoid}(A(M)_B,\mathcal{F}) \subset B^1(A(M)_B,\mathcal{F})$. Let $U$ be a quasi-compact object in $B_{spaces,fppf}$. Put $F = \Gamma(U,\mathcal{F})$ and $R = \Gamma(U,\mathcal{O}_B)$. Let $\xi_U = \sum_{m\in M} f_m X^m$ be an element in $\Gamma(U,\mathcal{F}\otimes\mathcal{O}_S[M]) = \bigoplus_{m\in M} FX^m$. Assume that $\partial(\xi_U) = 0$, then

$$0 = \Big(\sum_{(k,m)\in M^2}\mu_k(f_m)X^{(k,m)}\Big) - \Big(\sum_{l\in M} f_l X^{(l,l)}\Big) + \Big(\sum_{n\in M} f_n X^{(n,0)}\Big).$$

So $\mu_k(f_0) = -f_k$ for all $k \in M$ with $k \neq 0$ and $\mu_0(f_0) = 0$. Put $e_U := -f_0 \in F = \Gamma(U,\mathcal{F})$. We have

$$\partial(e_U) = \Big(\sum_{k\in M}\mu_k(e_U)X^k\Big) - e_U X^0 = \sum_{0\neq k\in M} f_k X^k + f_0 X^0 = \xi_U.$$

So $\xi_U$ belongs to the image of $\Gamma(U,\mathcal{F}) \xrightarrow{\partial} \Gamma(U,\mathcal{F}\otimes\mathcal{O}_B[M])$. Now let $\xi \in \Gamma(B,\mathcal{F}\otimes\mathcal{O}_B[M])$ and assume $\partial(\xi)=0$, i.e. $\xi \in Z^1_{monoid}(A(M)_B,\mathcal{F})$. Let us consider the set $\{U_i\}_{i\in I}$ of all quasi-compact objects in $B_{spaces,fppf}$, of course it provides an fppf covering of $B$. For each $i\in I$, $\partial(\xi|_{U_i})=0$, so the above computation gives us elements $e_i \in \Gamma(U_i,\mathcal{F})$ such that $\partial(e_i)=\xi|_{U_i}$. These $e_i$ are compatible and there exists an element $e\in\Gamma(B,\mathcal{F})$ such that $e|_{U_i}=e_i$ for all $i\in I$. We have $\partial(e)=\xi$. So $\xi \in B^1_{monoid}(A(M)_B,\mathcal{F})$. This finishes the proof.

□

Remark 9.14. We conjecture that our method will lead to a proof that $H^n_{monoid}(A(M)_B,\mathcal{F})=0$ also for $n\geqslant 2$. Note that [SGA3, Exp I. Théorème 5.3.3] proves that $H^n(D(M)_S,\mathcal{F})=0$ $(n\geqslant 1)$ under the assumption that $S$ is affine and that $\mathcal{F}$ is quasi-coherent, using derived functors. Our computational method, highly inspired by formulas given –and unexploited– in [SGA3], shows that the assumption that $S$ is affine is unnecessary in [SGA3, Exp. I Théorème 5.3.3] (at least in the case $n=1$).

## 10. Equivariant infinitesimal deformations

Let $S$ be a scheme. Let $G$ be a group algebraic space over $S$.

Proposition 10.1. *Let $X, X', Y$ and $Z$ be $S$-algebraic spaces endowed with $S$-actions of $G$. Let*

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "i"] & Y \arrow[d, "f"] \\ X' \arrow[r, "\varphi"] & Z \end{tikzcd}$$

*be a commutative diagram of $G$-equivariant $S$-morphisms where the left vertical morphism $i$ is a first order thickening, cf. [StP, Tag 05ZK]. Assume that $Y\to Z$ is formally smooth and*

$$H^1_{monoid}(G, p_*\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z},\mathcal{C}_{X/X'}))=0,$$

*where $p_*: Sh(X'_{spaces,fppf})\to Sh(S_{spaces,fppf})$ (cf. [StP, Tag 00X6] for $p_*$, § 9.2 for $H^1_{monoid}(-,-)$ and [StP, Tag 04CN] for $\mathcal{C}_{X/X'}$). Then there exists a $G$-equivariant morphism $\phi: X'\to Y$ filling the diagram, i.e. such that $\phi\circ i = a$ and $f\circ\phi=\varphi$.*

*Proof.* For an object $U'$ of $(X')_{spaces,fppf}$ with $U = X\times_{X'} U'$, consider morphisms $a': U'\to Y$ such that

(1) $a'$ is a morphism over $Z$, and
(2) $a'|_U = a|_U$.

Then the arguments of [StP, Tag 061A] show that the rule $U'\mapsto\{a': U'\to Y'$ such that (1) and (2) hold.$\}$ defines a sheaf of sets $\mathcal{P}$ on $(X')_{spaces,fppf}$. Note that the condition (3) in [StP, 061A] is empty because $Y=Y'$ in our situation. By [StP, Tag 061C], there is an action of the sheaf of abelian groups $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z},\mathcal{C}_{X/X'})$ on the sheaf $\mathcal{P}$. Moreover, the action of $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z},\mathcal{C}_{X/X'})$ on $\mathcal{P}$ is simply transitive for any object $U'$ of $(X')_{spaces,fppf}$ over which the sheaf $\mathcal{P}$ has a section. Let $p: X'\to S$ be the structural morphism. By [StP, Tag 00X6], $p$ induces a morphism of topoi $p_*$

$$\begin{aligned} p_*: Sh(X'_{spaces,fppf}) &\to Sh(S_{spaces,fppf}) \\ \mathcal{F} &\mapsto \big((T\to S)\mapsto \mathcal{F}(X'\times_S T)\big). \end{aligned}$$

The action of $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z},\mathcal{C}_{X/X'}))$ on $\mathcal{P}$ induces an action of the sheaf of abelian groups $p_*\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z},\mathcal{C}_{X/X'})$ on the sheaf $p_*\mathcal{P}$. We want to show that $G$ acts on the sheaf of

abelian groups $p_*\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z}, \mathcal{C}_{X/X'})$ and on the sheaf $p_*\mathcal{P}$, in a compatible way. Let $T$ be an algebraic space over $S$ and let $g \in G(T)$. An element $a' \in (p_*\mathcal{P})(T)$ is by definition an $S$-morphism $X' \times_S T \to Y$ such that $a'$ is an $S$-morphism over $Z$ and $a' \mid_{X\times_S T} = a \mid_{X\times_S T}$. In other words, an element $a' \in (p_*\mathcal{P})(T)$ is by definition a $T$-morphism $X' \times_S T \to Y \times_S T$ such that $a'$ is a $T$-morphism over $Z \times_S T$ and $a' \mid_{X\times_S T} = a \mid_{X\times_S T}$. Let $g^{-1}$ be the inverse of $g$ in $G(T)$. As usual, $X'_T, X_T, Y_T$ and $Z_T$ denote $X' \times_S T, X \times_S T, Y \times_S T$ and $Z \times_S T$. The element $g^{-1}$ gives us elements $g^{-1}_{X'_T} \in \mathrm{Aut}_T(X'_T)$ and $g^{-1}_{X_T} \in \mathrm{Aut}_T(X_T)$. The element $g$ gives us an element $g_{Y_T} \in \mathrm{Aut}_T(Y_T)$ and an element $g_{Z_T} \in \mathrm{Aut}_T(Z_T)$. For any $a' \in (p_*\mathcal{P})(T)$, we put $g \cdot a' := g_{Y_T} \circ a' \circ g^{-1}_{X'_T}$, this is a $T$-morphism $X'_T \to Y_T$. Let us check that $g \cdot a'$ belongs to $(p_*\mathcal{P})(T)$. Since $a$ is $G$ equivariant we have $a = g_{Y_T} \circ a \circ g^{-1}_{X_T}$, this shows that $g \cdot a' \mid_{X_T} = a \mid_{X_T}$. Let us now show that $g \cdot a'$ is a $T$-morphism over $Z \times_S T$. Let $f_T$ and $\phi_T$ denote the base change from $S$ to $T$ of $f$ and $\phi$. We have

$$\begin{aligned} f_T \circ g_{Y_T} \circ a' \circ g^{-1}_{X'_T} &= \\ \text{because } f_T \text{ is equivariant} &= g_{Z_T} \circ f_T \circ a' \circ g^{-1}_{X'_T} \\ \text{because } a' \in (p_*\mathcal{P})(T) &= g_{Z_T} \circ \varphi_T \circ g^{-1}_{X'_T} \\ \text{because } \varphi_T \text{ is equivariant} &= \varphi_T. \end{aligned}$$

This finishes the verification that $g \cdot a' \in (p_*\mathcal{P})(T)$. We now define the action of $G$ on

$$p_*\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z}, \mathcal{C}_{X/X'})$$

giving the action of $g \in G(T)$ on $v \in (p_*\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z}, \mathcal{C}_{X/X'}))(T)$. The diagram

$$\begin{tikzcd} Y_T \arrow[r, "g_{Y_T}"] \arrow[d, "f_T"] & Y_T \arrow[d, "f_T"] \\ Z_T \arrow[r, "g_{Z_T}"] & Z_T \end{tikzcd},$$

whose horizontal arrows are isomorphisms, is commutative since $f_T$ is equivariant. So we obtain an automorphism $g_{\Omega_T}$ of $\Omega_{Y_T/Z_T}$, e.g. by [StP, Tag 04CX]. Note that $g_{\Omega_T} \circ h_{\Omega_T} = (hg)_{\Omega_T}$, e.g. by [StP, Tag 05Z7]. That induces similar automorphisms of $a_T^*\Omega_{Y_T/Z_T}$. Similarly, using [StP, Tag 04CP] and [StP, Tag 04G2], we get for any $g, h \in G(T)$ automorphisms $g_{\mathcal{C}_T}$ of $\mathcal{C}_{X_T/X'_T}$ such that $g_{\mathcal{C}_T} \circ h_{\mathcal{C}_T} = (hg)_{\mathcal{C}_T}$. Now let $v \in \mathrm{Hom}_{\mathcal{O}_X}(a_T^*\Omega_{Y_T/Z_T}, \mathcal{C}_{X_T/X'_T}) = (p_*\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z}, \mathcal{C}_{X/X'}))(T)$ and $g \in G(T)$. We put $g \cdot v = g^{-1}_{\mathcal{C}_T} \circ v \circ g_{\Omega_T}$. The element $g \cdot v$ belongs to $(p_*\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z}, \mathcal{C}_{X/X'}))(T)$. This defines a left action because

$$h \cdot (g \cdot v) = h^{-1}_{\mathcal{C}_T} \circ g^{-1}_{\mathcal{C}_T} \circ v \circ g_{\Omega_T} \circ h_{\Omega_T} = (g_{\mathcal{C}_T} \circ h_{\mathcal{C}_T})^{-1} \circ v \circ g_{\Omega_T} \circ h_{\Omega_T} = (hg)^{-1}_{\mathcal{C}_T} \circ v \circ hg_{\Omega_T} = hg \cdot v.$$

Using [StP, Tag 0618], we obtain that the actions are compatible in the sense that

$$g \cdot a' + g \cdot v = g \cdot (a' + v) \text{ for any } g \in G(T), a' \in (p_*\mathcal{P})(T), v \in (p_*\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z}, \mathcal{C}_{X/X'}))(T).$$

We now define an element $c$ in $Z^1_{monoid}(G, (p_*\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z}, \mathcal{C}_{X/X'})))$. Let us fix $a' \in (p_*\mathcal{P})(S)$, this is possible because $Y \to Z$ is formally smooth. For any $T \to S \in S_{spaces,fppf}$, put $c(T) : G(T) \to (p_*\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z}, \mathcal{C}_{X/X'}))(T)$, $g \mapsto g \cdot a'_T - a'_T$. Then $c$ is indeed a cocycle because for any $T \in S_{spaces,fppf}$ and $g, g' \in G(T)$ we have

$$c(g) + g \cdot c(g') = g \cdot a'_T - a'_T + g \cdot (g' \cdot a'_T - a'_T) = gg' \cdot a'_T - a'_T = c(gg').$$

By assumption, $H^1_{monoid}(G, (p_*\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z}, \mathcal{C}_{X/X'}))) = 0$, so there is an element

$$v \in (p_*\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z}, \mathcal{C}_{X/X'}))(S)$$

such that $c(g) = g \cdot v_T - v_T$ for all $T \in S_{spaces,fppf}$ and all $g \in G(T)$. So for any $T \in S_{spaces,fppf}$ and any $g \in G(T)$, we have $g \cdot a'_T - a'_T = g \cdot v_T - v_T$, so we have $g \cdot (a'_T - v_T) = a'_T - v_T$. We now put $\phi := a' - v \in (p_*\mathcal{P})(S) = \mathcal{P}(X')$. Then $\phi$ is a morphism from $X'$ to $Y$ satisfying all the required properties, this finishes the proof.

□

Remark 10.2. The structure of the proof of Proposition 10.1 is partly similar to the argument sketched in [SGA3, Exp. XII proof of Lemma 9.4] (unpublished), though frameworks are different.

## 11. Formal étaleness, formal smoothness and formal unramifiedness

Let $S$ be a scheme and let $M$ be a finitely generated abelian group. Let $X \to Y$ be a $D(M)_S$-equivariant morphism of algebraic spaces over $S$. Let $N \subset M$ be a submonoid.

Proposition 11.1. *Assume that $f : X \to Y$ is formally smooth and locally of finite presentation (i.e. smooth by [StP, Tag 04AM]), then $f^N : X^N \to Y^N$ is formally smooth as transformation of functors (cf. e.g. [StP, Tag 049S]).*

*Proof.* Let $\iota_T : \overline{T} \to T$ be a first order thickening of affine schemes, over $S$. Let $D$:

$$\begin{tikzcd} \overline{T} \arrow[r, "\overline{\phi}"] \arrow[d, "\iota_T"] & X^N \arrow[d, "f^N"] \\ T \arrow[r, "\varphi"] & Y^N \end{tikzcd}$$

be a commutative diagram of $S$-functors. We have to prove that there exists an $S$-morphism $\phi : T \to X^N$ such that the diagram

$$\begin{tikzcd} \overline{T} \arrow[r, "\overline{\phi}"] \arrow[d, "\iota_T"] & X^N \arrow[d, "f^N"] \\ T \arrow[r, "\varphi"] \arrow[ur, "\phi"] & Y^N \end{tikzcd}$$

commutes. The diagram $D$ corresponds to the following data (i), (ii)

(i) $\overline{\phi}$ is a $D(M)_{\overline{T}}$-equivariant $\overline{T}$-morphism $A(N)_{\overline{T}} \to X_{\overline{T}}$

(ii) $\varphi$ is a $D(M)_T$-equivariant $T$-morphism $A(N)_T \to Y_T$

such that $\varphi|_{\overline{T}} : A(N)_{\overline{T}} \to Y_{\overline{T}}$ equals the map $A(N)_{\overline{T}} \xrightarrow{\overline{\phi}} X_{\overline{T}} \xrightarrow{f_{\overline{T}}} Y_{\overline{T}}$. Let us now consider the commutative diagram

$$\begin{tikzcd} A(N)_{\overline{T}} \arrow[r, "\overline{\phi}"] \arrow[d] & X_{\overline{T}} \arrow[r] & X_T \arrow[d, "f_T"] \\ A(N)_T \arrow[rr, "\varphi"] & & Y_T \end{tikzcd} .$$

The spaces $A(N)_{\overline{T}}$ and $X_{\overline{T}}$ are canonically endowed with actions of $D(M)_T$, moreover all arrows in the diagram are $D(M)_T$-equivariant. By Propositions 10.1 and 9.13, we get a $D(M)_T$-

equivariant map $\phi : A(N)_T \to X_T$ such that the diagram

$$\begin{tikzcd} A(N)_{\overline{T}} \arrow[r, "\overline{\phi}"] \arrow[d] & X_{\overline{T}} \arrow[r] & X_T \arrow[d, "f_T"] \\ A(N)_T \arrow[urr, "\phi"] \arrow[rr, "\varphi"] & & Y_T \end{tikzcd}$$

is commutative. The map $\phi$ corresponds to a morphism $T \to X^N$ and satisfies all the required properties to make the diagram

$$\begin{tikzcd} \overline{T} \arrow[r, "\overline{\phi}"] \arrow[d, "\iota_T"] & X^N \arrow[d, "f^N"] \\ T \arrow[r, "\varphi"] \arrow[ur, "\phi"] & Y^N \end{tikzcd}$$

commutative. □

Corollary 11.2. *Assume that $X \to Y$ is smooth and that $X^N$ and $Y^N$ are representable by algebraic spaces. Then $X^N \to Y^N$ is smooth as algebraic spaces.*

*Proof.* Trivial by Propositions 3.18 and 11.1, and [StP, Tag 060G]. □

Proposition 11.3. *Assume that $f : X \to Y$ is formally unramified, then $f^N : X^N \to Y^N$ is formally unramified as transformation of functors (cf. e.g. [StP, Tag 049S]).*

*Proof.* Let $\iota_T : \overline{T} \to T$ be a first order thickening of affine schemes, over $S$. Let $D$:

$$\begin{tikzcd} \overline{T} \arrow[r, "\overline{\phi}"] \arrow[d, "\iota_T"] & X^N \arrow[d, "f^N"] \\ T \arrow[r, "\varphi"] & Y^N \end{tikzcd}$$

be a commutative diagram of $S$-functors. We have to prove that there exists a most one $S$-morphism $\phi : T \to X^N$ such that the diagram

$$\begin{tikzcd} \overline{T} \arrow[r, "\overline{\phi}"] \arrow[d, "\iota_T"] & X^N \arrow[d, "f^N"] \\ T \arrow[r, "\varphi"] \arrow[ur, "\phi"] & Y^N \end{tikzcd}$$

commutes. Let $\phi_a, \phi_b$ be two such morphisms, we have to prove that $\phi_a = \phi_b$. The diagram $D$ corresponds to the following data (i), (ii)

(i) $\overline{\phi}$ is a $D(M)_{\overline{T}}$-equivariant $\overline{T}$-morphism $A(N)_{\overline{T}} \to X_{\overline{T}}$
(ii) $\varphi$ is a $D(M)_T$-equivariant $T$-morphism $A(N)_T \to Y_T$

such that $\varphi|_{\overline{T}} : A(N)_{\overline{T}} \to Y_{\overline{T}}$ equals the map $A(N)_{\overline{T}} \xrightarrow{\overline{\phi}} X_{\overline{T}} \xrightarrow{f_{\overline{T}}} Y_{\overline{T}}$. Let us now consider the commutative diagram

$$\begin{tikzcd} A(N)_{\overline{T}} \arrow[r, "\overline{\phi}"] \arrow[d] & X_{\overline{T}} \arrow[r] & X_T \arrow[d, "f_T"] \\ A(N)_T \arrow[rr, "\varphi"] & & Y_T \end{tikzcd} .$$

The morphisms $\phi_a, \phi_b$ provide two diagonals of the diagram above. Now using that $f_T$ if formally unramified, we have $\phi_a = \phi_b$. □

Corollary 11.4. *Assume that $X \to Y$ is formally unramified and that $X^N$ and $Y^N$ are representable by algebraic spaces. Then $X^N \to Y^N$ is formally unramified as morphism of algebraic spaces.*

*Proof.* Trivial by Proposition 11.3 and [StP, Tag 04G7]. □

Proposition 11.5. *Assume that $f : X \to Y$ is formally étale and locally of finite presentation (i.e étale by [StP, Tag 0616]), then $f^N : X^N \to Y^N$ is formally étale as transformation of functors (cf. e.g. [StP, Tag 049S]).*

*Proof.* This follows from [StP, Tag 049S] and Propositions 11.3 and 11.1. □

Corollary 11.6. *Assume that $X \to Y$ is étale and that $X^N$ and $Y^N$ are representable by algebraic spaces. Then $X^N \to Y^N$ is étale as morphism of algebraic spaces.*

*Proof.* Trivial by Proposition 11.5, [StP, Tag 04GC] and [StP, Tag 0616]. □

Proposition 11.7. *Assume that $f : X \to S$ is smooth, let $F$ be a face of $N$, then $f^{N,F} : X^N \to X^F$ is formally smooth as transformation of functors (cf. e.g. [StP, Tag 049S]).*

*Proof.* Let $\iota_T : \overline{T} \to T$ be a first order thickening of affine schemes, over $S$. Let $R$ and $I \subset R$ such that $T = \mathrm{Spec}(R)$ and $\overline{T} = \mathrm{Spec}(R/I)$. Let $D$:

$$\begin{tikzcd} \overline{T} \arrow[r, "\overline{\phi}"] \arrow[d, "\iota_T"] & X^N \arrow[d, "f^{N,F}"] \\ T \arrow[r, "\varphi"] & X^F \end{tikzcd}$$

be a commutative diagram of $S$-functors. We have to prove that there exists an $S$-morphism $\phi : T \to X^N$ such that the diagram

$$\begin{tikzcd} \overline{T} \arrow[r, "\overline{\phi}"] \arrow[d, "\iota_T"] & X^N \arrow[d, "f^{N,F}"] \\ T \arrow[r, "\varphi"] \arrow[ur, "\phi"] & X^F \end{tikzcd}$$

commutes. The diagram $D$ corresponds to the following data (i), (ii)

(i) $\overline{\phi}$ is a $D(M)_{\overline{T}}$-equivariant $\overline{T}$-morphism $A(N)_{\overline{T}} \to X_{\overline{T}}$
(ii) $\varphi$ is a $D(M)_T$-equivariant $T$-morphism $A(F)_T \to X_T$

such that $\varphi|_{\overline{T}} : A(F)_{\overline{T}} \to Y_{\overline{T}}$ equals the map $A(F)_{\overline{T}} \to A(N)_{\overline{T}} \xrightarrow{\overline{\phi}} X_{\overline{T}}$. We need to prove the following fact.

Fact 11.8. *The push-out of the diagram $A(F)_T \leftarrow A(F)_{\overline{T}} \to A(N)_{\overline{T}}$, in the category of algebraic spaces over $T$, exists and is denoted*

$$A(F)_T \coprod_{A(F)_{\overline{T}}} A(N)_{\overline{T}}.$$

*Moreover $A(F)_T \coprod_{A(F)_{\overline{T}}} A(N)_{\overline{T}}$ is an affine scheme over $T$ and is in fact a push-out of*

$$A(F)_T \leftarrow A(F)_{\overline{T}} \to A(N)_{\overline{T}}$$

*in the category whose objects are algebraic spaces over $T$ and whose morphisms are $D(M)_T$-equivariant morphisms of algebraic spaces over $T$.*

*Proof.* By [StP, Tag 0ET0], the push-out of $A(F)_T \leftarrow A(F)_{\overline{T}} \rightarrow A(N)_{\overline{T}}$ exists in the category of schemes and is given by

$$A(F)_T \coprod_{A(F)_{\overline{T}}} A(N)_{\overline{T}} := \mathrm{Spec}\big(\bigoplus_{n\in F} R.X^n \oplus \bigoplus_{n\in N\setminus F} (R/I).X^n\big).$$

Since $D(M)_T$-equivariant morphisms of affine schemes correspond to $M$-graded morphisms, we obtain that $A(F)_T \coprod_{A(F)_{\overline{T}}} A(N)_{\overline{T}}$ is endowed with an action of $D(M)_T$ and that

$$\nu_1 : A(F)_T \rightarrow A(F)_T \coprod_{A(F)_{\overline{T}}} A(N)_{\overline{T}}$$

and

$$\nu_2 : A(N)_{\overline{T}} \rightarrow A(F)_T \coprod_{A(F)_{\overline{T}}} A(N)_{\overline{T}}$$

are $D(M)_T$-equivariant. By [StP, Tag 0ET0], the push-out of

$$D(M)_T \times_T A(F)_T \leftarrow D(M)_T \times_T A(F)_{\overline{T}} \rightarrow D(M)_T \times_T A(N)_{\overline{T}}$$

exists in the category of schemes and the explicit formula allows us to identify it canonically with

$$D(M)_T \times_T \big(A(F)_T \coprod_{A(F)_{\overline{T}}} A(N)_{\overline{T}}\big).$$

We now remark that push-outs in the category of schemes over $T$ give push-outs in the full category of algebraic spaces over $T$ by [StP, Tag 07SY]. Now let $X$ be an algebraic space over $T$ with $D(M)_T$-equivariant morphisms $f_1 : A(F)_T \rightarrow X$ and $f_2 : A(N)_{\overline{T}} \rightarrow X$ such that the diagram

$$\begin{tikzcd} A(F)_{\overline{T}} \arrow[r, "f_1"] \arrow[d] & A(N)_{\overline{T}} \arrow[d, "f_2"] \\ A(F)_T \arrow[r] & X \end{tikzcd}$$

commutes; we have to prove that the obtained morphism $f_1 \coprod f_2 : A(F)_T \coprod_{A(F)_{\overline{T}}} A(N)_{\overline{T}} \rightarrow X$

is $D(M)_T$-equivariant. For this consider the diagram

$$\begin{tikzcd}
D(M)_T \times_T A(F)_{\overline{T}} \arrow[r, "Id\times i"] \arrow[d, "Id\times\iota"] \arrow[dd, bend right=60, "m_{A(F)_{\overline{T}}}"'] & D(M)_T \times_T A(N)_{\overline{T}} \arrow[d, "Id\times a_2"] \arrow[dd, bend right=60, "m_{A(N)_{\overline{T}}}"'] \arrow[dr, "Id\times f_2"] & \\
D(M)_T \times A(F)_T \arrow[r, "Id\times a_1"] \arrow[rr, bend right, "Id\times f_1"'] \arrow[d, bend right=60, "m_{A(F)_T}"'] & D(M)_T \times_T \left(A(F)_T \coprod_{A(F)_{\overline{T}}} A(N)_{\overline{T}}\right) \arrow[r, "Id\times(f_1\coprod f_2)"] & D(M)_T \times_T X \arrow[dd, "m_X"] \\
A(F)_{\overline{T}} \arrow[r, "i"] \arrow[d, "\iota"] & A(N)_{\overline{T}} \arrow[d, "a_2"] \arrow[dr, "f_2"] & \\
A(F)_T \arrow[r, "a_1"] \arrow[rr, bend right, "f_1"'] \arrow[urr, "m_{A(F)_T \coprod_{A(F)_{\overline{T}}} A(N)_{\overline{T}}}"] & A(F)_T \coprod_{A(F)_{\overline{T}}} A(N)_{\overline{T}} \arrow[r, "f_1\coprod f_2"] & X
\end{tikzcd}$$

where $m_{\mathcal{S}}$ denote the multiplication morphism for any $D(M)_T$-space $\mathcal{S}$. We have to show that $m_X \circ \big(Id \times (f_1 \coprod f_2)\big) = (f_1 \coprod f_2) \circ m_{A(F)_T \coprod_{A(F)_{\overline{T}}} A(N)_{\overline{T}}}$. Since, as we noted before, $D(M)_T \times_T (A(F)_T \coprod_{A(F)_{\overline{T}}} A(N)_{\overline{T}})$ is the push-out of

$$D(M)_T \times_T A(F)_T \xleftarrow{Id\times\iota} D(M)_T \times_T A(F)_{\overline{T}} \xrightarrow{Id\times i} D(M)_T \times_T A(N)_{\overline{T}},$$

it is enough to prove that

(i) $m_X \circ \big(Id \times (f_1 \coprod f_2)\big) \circ (Id \times a_1) = (f_1 \coprod f_2) \circ m_{A(F)_T \coprod_{A(F)_{\overline{T}}} A(N)_{\overline{T}}} \circ (Id \times a_1)$

(ii) $m_X \circ \big(Id \times (f_1 \coprod f_2)\big) \circ (Id \times a_2) = (f_1 \coprod f_2) \circ m_{A(F)_T \coprod_{A(F)_{\overline{T}}} A(N)_{\overline{T}}} \circ (Id \times a_2)$.

These identities are easy to check on the diagram using that $a_1, a_2, f_1, f_2$ are equivariant and that $f_1 = (f_1 \coprod f_2) \circ a_1$, $f_2 = (f_1 \coprod f_2) \circ a_2$. $\square$

We now continue the proof of Proposition 11.7. Let us consider the diagram

$$\begin{tikzcd}
A(F)_{\overline{T}} \arrow[r, "i"] \arrow[d, "\iota"] & A(N)_{\overline{T}} \arrow[d] \arrow[dr, "\overline{\phi}"] \arrow[ddd, bend left] & & \\
A(F)_T \arrow[r] \arrow[dr] \arrow[ddrr, bend right, "\varphi"'] & A(F)_T \coprod_{A(F)_{\overline{T}}} A(N)_{\overline{T}} \arrow[d, "\Psi"] \arrow[ddr, "\overline{\Phi}"] & X_{\overline{T}} \arrow[dd] & \\
& A(N)_T \arrow[ddr] & & \\
& & X_T \arrow[d] & \\
& & T &
\end{tikzcd}$$

where

(i) $\overline{\Phi}$ is the map obtained from the universal property of pushout from $\overline{\phi}$ and $\varphi$.

(ii) $\Psi$ is the map obtained from the universal property of pushout from the canonical morphisms $A(N)_{\overline{T}} \to A(N)_T$ and $A(F)_T \to A(N)_T$.

By Fact 11.8, $i$ and $\iota$ are $D(M)_T$-equivariant and so $\Psi$ and $\overline{\Phi}$ are also $D(M)_T$-equivariant. Now we apply Propositions 10.1 and 9.13 to get a $D(M)_T$-equivariant morphism $\phi : A(N)_T \to X_T$ such that the right lower quadrigone commutes. The morphism $\phi$ satisfies all required properties to make the diagram

$$\begin{tikzcd}
\overline{T} \arrow[r, "\overline{\phi}"] \arrow[d, "\iota_T"] & X^N \arrow[d, "f^{N,F}"] \\
T \arrow[r, "\varphi"] \arrow[ur, "\phi"] & X^F
\end{tikzcd}$$

commutative. □

Corollary 11.9. *Let $F$ be a face of a magnet $N$. Assume that $X \to S$ is smooth and that $X^N$ and $X^F$ are representable by algebraic spaces. Then $X^N \to X^F$ is smooth as algebraic spaces.*

*Proof.* Clear by Proposition 11.7, and [StP, Tag 060G]. □

Remark 11.10. If $X \to Y$ is flat, then $X^N \to Y^N$ is not flat in general (as we know that flatness is not preserved by taking fixed-points, e.g. cf. Vistoli's answer in Mathoverflow "Under what hypotheses are schematic fixed points of a flat deformation themselves flat?").

Remark 11.11. If $X$ is Cohen-Macaulay, then $X^N$ is not Cohen-Macaulay in general (as we know that being Cohen-Macaulay is not preserved by taking fixed-points, e.g. cf. Vistoli's answer in Mathoverflow "Are schematic fixed-points of a Cohen-Macaulay scheme Cohen-Macaulay?").

## 12. Topology of attractors

Let $M$ be a finitely generated abelian group. Let $X$ be an $S$-algebraic space locally of finite presentation endowed with an action of $D(M)_S$.

Proposition 12.1. *Let $N \subset M$ be a submonoid. Assume that $N$ is finitely generated and that there exists a $N^*$-FPR atlas of $X$. Then the map $X^N \to X^{N^*}$ (which is affine by Proposition 8.3) has geometrically connected fibers and induces a bijection on the sets of connected components $\pi_0(X^N) \simeq \pi_0(X^{N^*})$ of the underlying spaces.*

*Proof.* We adapt [Ri16, Cor. 1.12]. Using Proposition 3.29 we can assume that $N^* = 0$ and that $N$ is fine and sharp. Let $K$ be a field, and let $x : \mathrm{Spec}(K) \to X^0$ be a point. Let $X^N_x = X^N \times_{X^0,x} \mathrm{Spec}(K)$. We claim that its underlying topological space $|X^N_x|$ is connected. Let $L$ be a field and let $y : \mathrm{Spec}(L) \to X^N_x$ be a point, and denote by $x_L$ the composition $\mathrm{Spec}(L) \to \mathrm{Spec}(K) \xrightarrow{x} X^0$. Then $x_L$ and $x$ define the same point of $|X^N_x|$. Recall that we have a natural action of the monoid scheme $A(N)_S$ on the attractor $X^N$. The $A(N)_L$ orbit of $y$ defines a map $h : A(N)_L \to X^N_x$ with $h(1) = y$ and $h(0) = x_L$. Since $N$ is sharp, $L[N]$ is integral. So $A(N)_L$ is connected. So $x$ and $y$ lie in the connected set $|h|(|A(N)_L|)$. Since $y$ was arbitrary, this shows that $|X^N_x|$ is connected. So the continous map $|X^N| \to |X^0|$ has connected fibers, and the assertion on connected components follows from the existence of a continous section $|X^0| \subset |X^N|$. □

Proposition 12.2. *Let $N \subset M$ be a magnet. Let $f : X \to Y$ be a $D(M)_S$-equivariant morphism of $S$-algebraic spaces such that $X^N$ and $Y^N$ are representable by $S$-algebraic spaces. Let $f^N : X^N \to Y^N$ be the canonical morphism obtained on attractors. The following assertions hold.*

(i) *If $f$ is an open immersion, then $f^N$ is an open immersion.*

(ii) *If the following conditions hold*

  (a) *$f$ is a closed immersion,*
  (b) *there exists a $N^*$-FPR atlas of $Y$,*
  (c) *$N$ is finitely generated or $X$ is separated,*

  *then $f^N$ is a closed immersion, moreover $X^N \cong Y^N \times_Y X$.*

*Proof.* (i) Since $f$ is an open immersion, it is smooth and a monomorphism. By Corollary 11.2, $f^N$ is smooth and in particular locally finitely presented and flat (cf. [StP, Tag 04TA]). By Fact 3.17, $f^N$ is a monomorphism. Therefore, by [StP, Tag 05VH], $f^N$ is universally injective and unramified. Consequently, by [StP, Tag 06LU], $f^N$ is étale. Finally, by [StP, Tag 05W5], $f^N$ is an open immersion.

(ii) Assume $f$ is a closed immersion. The canonical morphisms $X^N \to Y^N$ and $X^N \to X$ induces a canonical morphism $i : X^N \to Y^N \times_Y X$. We are going to prove that $i$ is an isomorphism. Let $U$ be a $N^*$-FPR atlas of $Y$. By Proposition 6.7, the map $U \times_Y X \to X$ is a $N^*$-FPR atlas of $X$. So by Proposition 5.1 the canonical map $(U \times_Y X)^N \to X^N$ is étale and surjective. The canonical map $U^N \to Y^N$ (étale and surjective) induces an étale and

surjective map $(U \times_Y X) \times_U U^N = X \times_Y U^N \to X \times_Y Y^N$. So we get a diagram

$$\begin{tikzcd} (U \times_Y X)^N \arrow[r] \arrow[d] & X^N \arrow[d] \\ (U \times_Y X) \times_U U^N = X \times_Y U^N \arrow[r] & X \times_Y Y^N. \end{tikzcd}$$

The left arrow is an isomorphism by Lemma 3.24 (write $U$ as a disjoint union of stable $S$-affine schemes $U_i$ and use that $U_i \times_Y X \to U_i \times_Y Y = U_i$ is affine for all $i$). The horizontal arrows are étale and surjective. The diagram is a cartesian square because

$$(X \times_Y U^N) \times_{X \times_Y Y^N} X^N = U^N \times_{Y^N} X^N = (U \times_Y X)^N$$

by Proposition 3.7. So by [StP, Tag 03I2] we obtain $X^N \cong X \times_Y Y^N$.

□

## 13. Attractors and dilatations

Let $S$ be a scheme and let $S'$ be a closed locally principal subscheme of $S$. Let $X$ be an algebraic space over $S$ with a $D(M)_S$-action where $M$ is an abelian group. Put $D = X_{S'}$. Let $Y$ be a closed subspace of $X_{S'}$. Then by [Ma23] (or [MRR20] if $X$ is a scheme), we get a space $\mathrm{Bl}_Y^D X$ called the dilatation of $X$ with center $Y$ along $S'$, and an affine morphism of spaces $\mathrm{Bl}_Y^D X \to X$. Let $N$ be a submonoid in $M$. Assume that $Y$ is stable under the action of $D(M)_{S'}$ on $X_{S'}$. Then by Proposition 12.2 $Y^N \to D^N$ is a closed immersion. Moreover $D^N = (X_{S'})^N = (X^N)_{S'}$ (cf. Proposition 3.9) is a locally principal closed subscheme of $X^N$. So $\mathrm{Bl}_{Y^N}^{D^N} X^N$ is well-defined.

Proposition 13.1. *Assume moreover that $X \to S$, $(\mathrm{Bl}_Y^D X)^N \to S$ and $\mathrm{Bl}_{Y^N}^{D^N} X^N \to S$ are flat. Then $D(M)_S$ acts naturally on $\mathrm{Bl}_Y^D X$, moreover we get a canonical isomorphism*

$$\Theta : (\mathrm{Bl}_Y^D X)^N \cong \mathrm{Bl}_{Y^N}^{D^N} X^N.$$

Remark 13.2. We refer to [MRR20, Prop. 2.16] for conditions ensuring flatness of dilatations.

*Proof.* Remark first that since $D(M)_S \to S$, $X \to S$, $\mathrm{Bl}_Y^D X \to S$ are flat, $D(M)_S$, $X$ and $\mathrm{Bl}_Y^D X$ belong to $\mathrm{Spaces}_S^{S'\text{-}reg}$ (cf. [Ma23] for the definition of $\mathrm{Spaces}_S^{S'\text{-}reg}$), moreover by flatness any products of these objects in the category $\mathrm{Spaces}/S$ or $\mathrm{Spaces}_S^{S'\text{-}reg}$ coincide. So to check that we have an action of $D(M)_S$ on $\mathrm{Bl}_Y^D X$ it is enough to show that $D(M)_S(T)$ acts on $\mathrm{Bl}_Y^D X(T)$ functorially for any $T \in \mathrm{Spaces}_S^{S'\text{-}reg}$. So let $T \in \mathrm{Spaces}_S^{S'\text{-}reg}$. Let $(g, x) \in D(M)_S(T) \times \mathrm{Bl}_Z^D(X)(T)$, note that $x$ corresponds to a morphism $T \xrightarrow{x} X$ such that $T|_{S'} \to X_{S'}$ factors through $Y$. We define $g.x$ as the composition

$$T \xrightarrow{(g,x)} D(M)_S \times X \xrightarrow{\text{action}} X.$$

Then $g.x$ restricted to $S'$ factors through $Y$ and so $g.x \in \mathrm{Bl}_Z^D X(T)$. So $D(M)_S$ acts on $\mathrm{Bl}_Y^D(X)$. We obtain the following two diagrams

$$(\mathrm{Bl}_Y^D X)^N \to X^N,$$

$$\begin{tikzcd} (\mathrm{Bl}_Y^D X)^N|_{S'} = (\mathrm{Bl}_Y^D X|_{S'})^N \arrow[rr] \arrow[dr] & & X^N|_{S'} = (X|_{S'})^N \\ & Y^N \arrow[ur] & \end{tikzcd}.$$

By the universal property of dilatations, we obtain a morphism

$$\Theta : (\mathrm{Bl}_Y^D X)^N \to \mathrm{Bl}_{Y^N}^{D^N} X^N.$$

We now prove that it is an isomorphism. Again, let $T \in \mathrm{Spaces}_S^{S'-reg}$ and let $T'$ be $T \times_S S'$. Then $T'$ is a closed and locally principal subspace of $T$. Moreover, since $A(N)_T$ is flat over $T$, $A(N)_T$ belongs to $\mathrm{Spaces}_T^{T'-reg}$. Since $\mathrm{Bl}_Y^D X \to S$ is flat, by [Ma23, §3.6], we have $(\mathrm{Bl}_Y^D X)_T = \mathrm{Bl}_{Y_{T'}}^{D_{T'}} X_T$. So, on the one hand

$$\begin{aligned}(\mathrm{Bl}_Y^D X)^N(T) &= \mathrm{Hom}^{D(M)_T}(A(N)_T, (\mathrm{Bl}_Y^D X)_T) \\ &= \mathrm{Hom}^{D(M)_T}(A(N)_T, \mathrm{Bl}_{Y_{T'}}^{D_{T'}} X_T) \\ &= \{A(N)_T \xrightarrow{f} X_T |\ f \text{ is } D(M)_T\text{-equiv. and } A(N)_T|_{T'} \xrightarrow{f_{T'}} X|_{T'} \text{ factors through } Y_{T'}\},\end{aligned}$$

moreover, on the other hand,

$$\begin{aligned}(\mathrm{Bl}_{Y^N}^{D^N} X^N)(T) &= \{T \to X^N |\ \ T|_{S'} \to X^N|_{S'} \text{ factors through } Y^N\} \\ &= \{A(N)_T \xrightarrow{f} X |\ f \text{ is } D(M)_T\text{-equiv. and } A(N)_T|_{T'} \xrightarrow{f_{T'}} X|_{T'} \text{ factors through } Y_{T'}\}.\end{aligned}$$

Now since $\mathrm{Bl}_{Y^N}^{D^N} X^N$ and $(\mathrm{Bl}_Y^D X)^N$ are flat over $S$, they belong to $\mathrm{Spaces}_S^{S'\mathrm{reg}}$. So by Yoneda $\mathrm{Bl}_{Y^N}^{D^N} X^N = (\mathrm{Bl}_Y^D X)^N$. This finishes the proof. □

Corollary 13.3. *Let $S$ be a scheme and let $S'$ be an effective Cartier divisor on $S$. Let $X$ be a smooth scheme over $S$ with a $D(M)_S$-action where $M$ is an abelian group. Put $D = X_{S'}$. Let $Y$ be a closed subscheme of $X_{S'}$ such that $Y \to S'$ is smooth. Let $\mathrm{Bl}_Y^D X$ be the dilatation of $X$ with center $Y$ along $S'$. Let $N$ be a submonoid in $M$. Assume that $Y$ is stable under the action of $D(M)_{S'}$ on $X_{S'}$. Then $D(M)_{S'}$ acts on $\mathrm{Bl}_Y^D X$ and we get a canonical isomorphism*

$$\Theta : (\mathrm{Bl}_Y^D X)^N \cong \mathrm{Bl}_{Y^N}^{D^N} X^N.$$

*Proof.* Since $X \to S, D \to S'$ and $Y \to S$ are smooth, by Corollary 11.2 $X^N \to S, D^N \to S$ and $Y^N \to S$ are smooth. So by [MRR20, Proposition 2.16], $\mathrm{Bl}_Y^D X \to S$ and $\mathrm{Bl}_{Y^N}^{D^N} X^N \to S$ are smooth. So using Corollary 11.2 again, $(\mathrm{Bl}_Y^D X)^N \to S$ is smooth. Now since smooth implies flat, Corollary 13.3 follows from Proposition 13.1. □

Remark 13.4. We note that the fact that dilatations commute with attractors may be used to study valued root data as in Bruhat-Tits theory. Indeed by Section 16.5 root groups of reductive groups are examples of attractors (cf. also Section 16.2) and dilatations allow to define filtrations.

## 14. Ind-algebraic spaces

Let $S$ be a scheme and let $(Aff/S)$ be the category of affine schemes over $S$. Its objects are morphisms $\mathrm{Spec}(R) \to S$ from affine schemes to $S$. We use [HR21] for the definition of ind-algebraic spaces.

Definition 14.1. An ind-algebraic space (resp. ind-scheme) over $S$ is a functor $(Aff/S) \to Set$ which admits a presentation $X \cong \mathrm{colim}_{i \in I} X_i$ as a filtered colimit of $S$-algebraic spaces (resp. $S$-schemes) where all transition maps $\phi_{ij} : X_i \to X_j$, $i \leqslant j$ are closed immersions. The category of ind-algebraic spaces (resp. ind-schemes) over $S$ is the full subcategory of functors $(Aff/S) \to Set$ whose objects are ind-algebraic spaces (resp. ind-schemes) over $S$.

Any algebraic space (resp. scheme) over $S$ is naturally an ind-algebraic space (resp an ind-scheme) over $S$. Any ind-scheme over $S$ is naturally an ind-algebraic space over $S$.

Remark 14.2. [HR21, §1.5] If $X = \mathrm{colim}_i X_i$ and $Y = \mathrm{colim}_j Y_j$ are presentations of ind-algebraic spaces (resp. ind-schemes) over $S$, and if each $X_i$ is quasi-compact, then as sets $\mathrm{Hom}(X, Y) = \lim_i \mathrm{colim}_j \mathrm{Hom}(X_i, Y_j)$, because every map $X_i \to Y$ factors over some $Y_j$ by quasi-compactness of $X_i$. The categories of ind-algebraic spaces and ind-schemes are closed under fiber product. If $\mathbf{P}$ is a property of algebraic spaces (resp. schemes), then an $S$-ind-algebraic space (resp. $S$-ind-scheme) $X$ is said to have ind-$\mathbf{P}$ if there exists a presentation $X = \mathrm{colim}_i X_i$ where each $X_i$ has property $\mathbf{P}$. A map $f : X \to Y$ of $S$-ind-algebraic spaces (resp. $S$-ind-schemes) is said to have property $\mathbf{P}$ if $f$ is representable and for all schemes $T \to Y$, the pullback $f \times_Y T$ has property $\mathbf{P}$. Note that every representable quasi-compact map of $S$-ind-schemes is schematic.

Definition 14.3. Let $G$ be a group scheme over $S$ and let $X$ be an ind-algebraic space over $S$.

(i) A categorical action of $G$ on $X$ is an action of $G$ on $X$ seen as a functor. Equivalently by Yoneda, a categorical action is a morphism in the category of ind-algebraic spaces over $S$ $\sigma : G \times_S X \to X$ satisfying the usual axioms, i.e.

  (a) $\sigma \circ (\mathrm{Id}_G \times \sigma) = \sigma \circ (m \times \mathrm{Id}_X)$ where $m : G \times_S G \to G$ is the group law of $G$,
  (b) $\sigma \circ (e \times \mathrm{Id}_X) = \mathrm{Id}_X$ where $e : S \to G$ is the identity section of $G$.

(ii) A collection of actions $\sigma_i : G \times_S X_i \to X_i$ such that for $i \leqslant j$ we have $\phi_{ij} \circ \sigma_i = \sigma_j \circ (\mathrm{Id}_G \times \phi_{ij})$ gives birth to a categorical action of $G$ on $X$. Such categorical actions are called ind-actions.

Let $M$ be a finitely generated abelian group.

Proposition 14.4. *Let $X$ be an ind-algebraic space (resp. ind-scheme) over $S$. Assume that we have a presentation $X = \mathrm{colim}_i X_i$ with $X_i$ quasi-separated and locally finitely presented over $S$. Assume that $D(M)_S$ acts on $X$ via an ind-action on the presentation $X = \mathrm{colim}_i X_i$. Let $N \subset M$ be a monoid. Assume that $N$ is finitely generated or that $X$ is separated. Then the attractor functor*

$$(T \to S) \mapsto \mathrm{Hom}^{D(M)_T}(A(N)_T, X_T)$$

*is representable by the ind-algebraic space (resp. ind-scheme) $\mathrm{colim}_i X_i^N$. Moreover the natural morphism $X^N \to X$ is representable by algebraic spaces (resp. schemes). If $N = Z$ is a group the natural morphism $X^Z \to X$ is representable by a closed immersion.*

*Proof.* The first assertion is a direct corollary of Proposition 12.2. To show that $X^N \to X$, is representable by algebraic spaces, we notice that if $T$ is an affine scheme and $T \to X$ is a morphism, then there exists $i$ such that this morphism is induced by a morphism $T \to X_i$, and then we have $T \times_X X^N = T \times_{X_i} {X_i}^N$. If $N = Z$ is a group, the last assertion follows from Corollary 8.1. □

## 15. Pure magnets

Let $M$ be an abelian monoid. Let $X$ be a separated algebraic space over $S$ endowed with an action $a$ of $A(M)_S$. Let $m(a)$ be the set of magnets of $a$, i.e. the set of all submonoids of $M$. We also use the notation $m(M)$ to denote $m(a)$.

Proposition 15.1. *Let $N \in m(a)$, the following conditions are equivalent.*

(i) *For any $L \in m(a)$, $L \subsetneq N \Rightarrow X^L \subsetneq X^N$.*
(ii) *For any $L \in m(a)$, $X^N = X^L \Rightarrow N \subset L$.*

*Proof.* Let us prove (i) $\Rightarrow$ (ii). So let $L \in m(M)$ and assume $X^N = X^L$. Then $X^N = X^{N \cap L}$ by Proposition 3.30. Since $N \cap L \subset N$, (i) implies that $N \cap L = N$. So $N \subset L$. Reciprocally assume (ii) holds. Let $L \in m(a)$ such that $L \subsetneq N$. By Proposition 3.27, we have $X^L \subset X^N$. It remains to prove that $X^L \neq X^N$. This is clear, indeed otherwise (ii) implies that $N \subset L$ and so $N \neq N$ which is absurd. □

Definition 15.2. We use the following terminology.

(i) A pure magnet for the action $a$ is a magnet $N \in m(a)$ satisfying the equivalent properties of Proposition 15.1.
(ii) The set of all pure magnets of the action $a$ is denoted $\mho(a)$.

Theorem 15.3. *We have a canonical bijection between $\mho(a)$ and the set*

$$\{X^N \subset X | N \in m(M)\}$$

*with the convention that we identify $X^N$ and $X^L$ if and only if $X^N(T) = X^L(T) \subset X(T)$ for all $T/S$ (cf. Proposition 3.28). The bijection sends a pure magnet $N \in \mho(a)$ to the attractor $X^N$. The reciprocal bijection sends $Y \in \{X^N \subset X | N \in m(M)\}$ to the pure magnet*

$$E(Y) := \bigcap \big\{N \in m(M) | X^N = Y\big\}.$$

*Proof.* Let $Y \in \{X^N \subset X | N \in m(M)\}$ and let $E(Y)$ be the monoid defined in the statement, i.e. $E(Y) = \bigcap \big\{N \in m(M) | X^N = Y\big\}$. By Proposition 3.31, we have $Y = X^{E(Y)}$. Let us prove that $E(Y)$ is a pure magnet. Let $L \in m(M)$ such that $X^L = X^{E(Y)} = Y$, by definition of $E(Y)$, we have $E(Y) \subset L$ and so $E(Y)$ is a pure magnet. It remains to prove that for all $N \in \mho(a)$ and all $Y \in \{X^N \subset X | N \in m(M)\}$, we have $E(X^N) = N$ and $Y = X^{E(Y)}$. So let us first take $N \in \mho(a)$. It is obvious that $E(X^N) \subset N$. On an other hand, by Proposition 3.31, we have $X^N = X^{E(X^N)}$ and so $N \subset E(X^N)$ because $N$ is a pure magnet. Now let us take $Y \in \{X^N \subset X | N \in m(M)\}$, Proposition 3.31 implies that $Y = X^{E(Y)}$. □

Theorem 15.4. *Assume moreover that $X$ is finitely presented over $S$. Assume additionally that one of the following conditions is satisfied*

(i) *$X$ is affine over $S$,*
(ii) *$M$ is a group and there exists an $S$-affine strongly-FPR atlas for $X$ (e.g. the action is Zariski locally linearizable),*
(iii) *add more as needded,*

*then $\mho(a)$ is a finite poset.*

*Proof.* The set $\mho(a)$ is a poset for the inclusion. It remains to show that $\mho(a)$ is finite. Assume (i) holds. Write $X = \mathrm{Spec}_S(\mathcal{A})$, then we have an $M$-grading $\mathcal{A} = \bigoplus_{m \in M} \mathcal{A}_m$. Because of $X$ is

finitely presented over $S$, there exists a finite subset $E \subset M$ such that $\mathcal{A}$ is generated locally by homogeneous elements of degree in $E$. Now for any monoid $N$, we have $X^N = X^{[N \cap E\rangle}$. This implies that $\mathfrak{V}(a)$ is finite. Case (ii) follows from case (i) and the following result:

Proposition 15.5. *Let $U \to X$ be an $S$-affine strongly-FPR atlas. Let $a_U$ denote the action of $D(M)_S$ on $U$. Then for all submonoids $N, L$ of $M$, we have $U^N = U^L \Rightarrow X^N = X^L$; moreover*

$$\mathfrak{V}(a_X) \subset \mathfrak{V}(a_U).$$

*Proof.* Note that $X^P$ is representable for any magnet $P$ by Proposition 8.3. Let $N, L$ be submonoids of $M$ and assume $U^N = U^L$. By Proposition 3.30 or Proposition 3.22, we have $U^N = U^{N \cap L} = U^L$. So it is enough to prove that $X^{N \cap L} = X^L$, in other words we can change notation and assume $N \subset L$. Let us remark that $U \to S$ is of finite presentation, indeed $U \to S$ being affine, it is quasi-compact and quasi-separated and $U \to X \to S$ is locally of finite presentation as compositions of two such morphisms. The map $U^N \to X^N$ is étale and surjective. Indeed since $u$ is strongly-FPR, by Proposition 5.1 we get a diagram

$$\begin{tikzcd} U^N \arrow[r] \arrow[d] & U^{N^*} \arrow[r] \arrow[d] & U \arrow[d] \\ X^N \arrow[r] & X^{N^*} \arrow[r] & X \end{tikzcd}$$

with Cartesian squares and $U \to X$ is étale and surjective. We get a commutative triangle

$$\begin{tikzcd} U^N \arrow[rr, "f"] \arrow[dr, "p"'] & & X^N \arrow[dl, "q"] \\ & X^L & \end{tikzcd}$$

where $f$ is surjective and étale, $p$ is étale by Corollary 11.6 and $q$ is locally of finite presentation (e.g. by [StP, Tag 05WT] and Theorem 8.3). So $q$ is étale by [StP, Tag 0AHE] and in particular unramified. Now since $X$ is separated, $q$ is a monomorphism by Proposition 3.27. Being an unramified monomorphism, $q$ is universally injective by [StP, Tag 05W6]. So $q$ being étale and universally injective, by [StP, Tag 05W5], it is an open immersion. Now since $p$ is surjective, we get that $q$ is surjective. So $q$ being a surjective open immersion, it is an isomorphism. So we proved that $X^N = X^L$. Now let $N \in \mathfrak{V}(a_X)$ and let us prove that $N \in \mathfrak{V}(a_U)$. So let $L \in m(M)$ such that $U^N = U^L$. We proved that $X^N = X^L$ and so $L \subset N$ because $N \in \mathfrak{V}(a_X)$. This finishes the proof. □

□

Remark 15.6. Theorem 15.4 shows in particular that, under the same assumptions, the set $\{X^{D(M/Z)_S} \subset X \mid Z \subset M \text{ is a subgroup of } M\}$ is finite. We did not know a reference for this fact.

Conjecture 15.7. *We conjecture that the conclusion of Theorem 15.4 remains true without the additional assumption (i),(ii) or (iii).*

Remark 15.8. Conjecture 6.9 implies Conjecture 15.7.

If $N$ is a monoid, we denote by $gen(N)$ the set of all subsets of $N$ that generate $N$ as monoid. The emptyset generates the zero monoid.

Definition 15.9. The rank $mk(N)$ (possibly infinite) of a monoid $N$ is the cardinal defined by

$$mk(N) = \min\{\#E | E \in gen(N)\}.$$

Understanding combinatorial aspects of $\mho(a)$, including ranks of pure magnets, is a fundamental invariant of the action $a$.

Example 15.10.

(i) *Let $a$ be the trivial action of $A(M)$ on a space $X$, then $\mho(a) = \{0\}$.*

(ii) *Let $a$ be the action of $D(\mathbb{Z}^n)$ on itself by multiplication. Then $\mho(a) = \{0, \mathbb{Z}^n\}$.*

Fact 15.11. *We have $mk(\mathbb{Z}^n) = n + 1$.*

*Proof.* Indeed, as monoid, $\mathbb{Z}^n$ is generated by $\{\{e_i\}_{i\in\{1,\ldots,n\}}, -\sum_{i=1}^n e_i\}$. Assume that $\mathbb{Z}^n$ is generated by $n$ elements $f_1, \ldots, f_n$, then there exists $p_1, \ldots, p_n > 0$ such that $-(f_1 + \ldots + f_n) = p_1 f_1 + \ldots + p_n f_n$, this implies $(p_1 + 1)f_1 + \ldots + (p_n + 1)f_n = 0$. So $f_1, \ldots, f_n$ are linked and can not generate $\mathbb{Z}^n$ as group. This is absurd. So $\mathbb{Z}^n$ can not be generated by only $n$ elements. □

(iii) *Let $a$ be the action of $A(\mathbb{N})$ on itself by multiplication. Then $\mho(a) = \{0, \mathbb{N}\}$, note that $mk(\mathbb{N}) = 1$.*

(iv) *Let $a$ be the action of $D(\mathbb{Z})$ on $A(\mathbb{N})$ by multiplication. Then $\mho(a) = \{0, \mathbb{N}\}$.*

(v) *Let $a$ be the action of $D(\mathbb{Z})$ on $A(\mathbb{N})$ given by $\lambda \cdot x = \lambda^2 x$. Then $\mho(a) = \{0, 2\mathbb{N}\}$, note that $mk(2\mathbb{N}) = 1$.*

(vi) *Let $a$ be the action of $D(\mathbb{Z})$ on $\mathbb{P}^1$, then $\mho(a) = \{0, \mathbb{N}, -\mathbb{N}, \mathbb{Z}\}$.*

(vii) *Let $a$ be the action of $D(\mathbb{Z}^2)$ on $A(\mathbb{N}^2)$ by multiplication, then $\mho(a) = \{0 \times 0, \mathbb{N} \times 0, 0 \times \mathbb{N}, \mathbb{N} \times \mathbb{N}\}$.*

(viii) *Let $a$ be the action of $D(\mathbb{Z}/6\mathbb{Z})$ on $D(\mathbb{Z}/6\mathbb{Z}) \times D(\mathbb{Z}/6\mathbb{Z}) \times D(\mathbb{Z}/6\mathbb{Z})$ given by*

$$\lambda \cdot (x, y, z) = (\lambda x, \lambda^2 y, \lambda^3 z).$$

*Then $\mho(a) = \{\mathbb{Z}/6\mathbb{Z}, 2\mathbb{Z}/6\mathbb{Z}, 3\mathbb{Z}/6\mathbb{Z}, 6\mathbb{Z}/6\mathbb{Z}\}$.*

(ix) *Let $a$ be the adjoint action of a maximal split torus $T = D(M)$ on a given reductive group scheme $G$. Then $N \mapsto N \cap \Phi$ provides a bijection between $\mho(a)$ and the set of subsets of $\Phi(G, T)$ that are closed under addition. Pure magnets of rank 1 correspond to roots (cf. §16.4 for more details about the example of reductive groups).*

## 16. Complements and examples

### 16.1 Tangent spaces and attractors

Let $S$ be a scheme. We denote by $I_S$ the scheme of dual numbers over $S$ as in [SGA3, Exp. II Définition 2.1]. For any scheme $T$ over $S$, we have $I_T = I_S \times_S T$. Explicitly, $I_S = \mathrm{Spec}(\mathbb{Z}[\varepsilon]) \times_{\mathbb{Z}} S$ where $\mathbb{Z}[\varepsilon] \overset{\varepsilon=[T]}{=} \mathbb{Z}[T]/(T^2)$. Let $X$ be a functor over $S$. Let $T_X$ be the tangent space of $X$ as in [SGA3, Exp. II Définition 3.1]. This is a functor from $\mathrm{Sch}_S$ to $Set$ sending a scheme $R$ over $S$ to

$$\mathrm{Hom}_R(I_R, X_R).$$

Remark 16.1.

(i) For $S$-morphisms $X \to Y$ and $Z \to Y$, we have $T_{X \times_Y Z} \cong T_X \times_{T_Y} T_Z$.

(ii) If $X \to Y$ is a monomorphism, then $T_X \to T_Y$ is a monomorphism.

(iii) We have a canonical identification $T_S = S$.

Let $M$ be an abelian group and assume that $D(M)_S$ acts on $X$. Then $D(M)_S$ acts naturally on $T_X$ using the definition of $T_X$. Let $N$ be a submonoid in $M$.

Proposition 16.2. *We have a canonical isomorphism*

$$(T_X)^N \cong T_{X^N}.$$

*Proof.* It is enough to show that $(T_X)^N(S') \cong T_{X^N}(S')$ for any $S$-scheme $S'$. We have

$$\begin{aligned}
(T_X)^N(S') &= \mathrm{Hom}_{S'}^{D(M)_{S'}}(A(N)_{S'}, T_{X_{S'}}) \\
&= \mathrm{Hom}_{S'}^{D(M)_{S'}}(A(N)_{S'} \times_{S'} I_{S'}, X_{S'}) \text{ where } D(M)_{S'} \text{ acts trivially on } I_{S'} \\
&= \mathrm{Hom}_{I_{S'}}^{D(M)_{I_{S'}}}(A(N)_{I_{S'}}, X_{I_{S'}}) \\
&= \mathrm{Hom}_{S'}(I_{S'}, X_{S'}^N) \\
&= (T_{X^N})(S').
\end{aligned}$$

This shows that we have a canonical isomorphism. □

### 16.2 Lie algebras and attractors

Let $G/S$ be a group functor over a scheme $S$. Recall that in this case $T_G$ is a group functor over $S$ and we have two canonical morphisms of group functors $G \to T_G$ and $T_G \to G$ by [SGA3, Exp. II]. Let $e_S$ be the trivial group over $S$, as $S$-scheme we have $e_S = S$. The Lie algebra of $G$ is defined as the fiber product

$$\begin{tikzcd}
\mathrm{Lie}(G) = e_S \times_G T_G \arrow[r] \arrow[d] & T_G \arrow[d] \\
e_S \arrow[r] & G
\end{tikzcd}$$

where $e_S \to G$ is the canonical morphism of group functors from $e_S$ to $G$. As in the previous section, let $M$ be an abelian group and assume that $D(M)_S$ acts on $G$. We assume moreover that this action is compatible with the group structure on $G$, i.e $D(M)_S$ acts by automorphisms on $G$. Then the induced action of $D(M)_S$ on $T_G$ is by group automorphisms. We thus obtain an action of $D(M)_S$ on $\mathrm{Lie}(G)$ by group automorphisms.

Remark 16.3.

(i) For $S$-group functors $G, K, H$ with morphisms $G \to K, H \to K$, we have a canonical isomorphism $\mathrm{Lie}(G \times_K H) \cong \mathrm{Lie}(G) \times_{\mathrm{Lie}(K)} \mathrm{Lie}(H)$.

(ii) If $G \to H$ is a monomorphism, then $\mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is a monomorphism.

(iii) We have a canonical identification $\mathrm{Lie}(e_S) = e_S$.

Proposition 16.4. *We have a canonical isomorphism of group functors over $S$*

$$\big(\mathrm{Lie}(G)\big)^N \cong \mathrm{Lie}(G^N)$$

*Proof.* Using Proposition 3.7 and Proposition 16.2, we have

$$\big(\mathrm{Lie}(G)\big)^N = \big(S \times_G T_G\big)^N \cong S^N \times_{G^N} {T_G}^N \cong S \times_{G^N} T_{G^N} = \mathrm{Lie}(G^N).$$

□

Let us fix now a group functor $H$ over $S$ and a monomorphism $H \to G^{N^*}$ preserving the group structures. Recall that $G_H^N$ is the attractor with prescribed limit $H$ as in Definition 4.1.

Proposition 16.5. *We have a canonical isomorphism of group functors over $S$*

$$\big(\mathrm{Lie}(G)\big)^N_{\mathrm{Lie}(H)} \cong \mathrm{Lie}(G^N_H).$$

*Proof.* Using Proposition 16.4 and Remark 16.3 we have

$$\begin{aligned}\mathrm{Lie}(G)^N_{\mathrm{Lie}(H)} &= \mathrm{Lie}(G)^N \times_{\mathrm{Lie}(G)^{N^*}} \mathrm{Lie}(H)\\ &\cong \mathrm{Lie}(G^N) \times_{\mathrm{Lie}(G^{N^*})} \mathrm{Lie}(H)\\ &\cong \mathrm{Lie}(G^N \times_{G^{N^*}} H)\\ &= \mathrm{Lie}(G^N_H)\end{aligned}$$

This shows that we have the desired canonical isomorphism. □

### 16.3 Relation to $\mathbb{G}_m$-attractors

Let $X$ be an $S$-algebraic space with and action of $D(M)_S$. Let $N$ be a submonoid of the group $M$. Then in some cases the attractor space $X^N$ can be obtained as a succession of attractors under $\mathbb{G}_m$ (cf. the introduction and [Ri16]) and fixed-points. Let us gives two examples.

(i) Let $\alpha \in M$ and let us consider $X^{[\alpha\rangle}$. Under mild assumptions, by Proposition 3.30 we have $X^{[\alpha\rangle} = \big(X^{(\alpha)}\big)^{[\alpha\rangle}$. By Proposition 3.32 $X^{(\alpha)}$ identifies with the fixed-points space $X^{D(M/(\alpha))_S}$. So the operation $X \rightsquigarrow X^{(\alpha)}$ can be realized as taking fixed-points. By Remarks 3.12 and 3.11, $D((\alpha))_S$ and $D(M)_S$ act on $X^{(\alpha)}$, and using Proposition 3.29, we have $\big(X^{(\alpha)}\big)^{[\alpha\rangle\subset(\alpha)} = \big(X^{(\alpha)}\big)^{[\alpha\rangle\subset M}$ (cf. Remark 3.4 for the notation $X^{N\subset M}$). But $\big([\alpha\rangle \subset (\alpha)\big) \simeq \big(\mathbb{N} \subset \mathbb{Z}\big)$, so the operation $X^{(\alpha)} \rightsquigarrow \big(X^{(\alpha)}\big)^{[\alpha\rangle}$ can be realized as taking attractor under $\mathbb{G}_m = D(\mathbb{Z})_S$. So $X \rightsquigarrow X^{[\alpha\rangle}$ can be realized as fixed-points followed by taking the attractor under an action of $\mathbb{G}_m$.

(ii) Assume that $M = \mathbb{Z} \times \mathbb{Z}$ and that $N = \mathbb{N} \times \mathbb{N}$. Using Propositions 3.30 and 3.29, we have

$$X^N = \big(X^{\mathbb{N}\times\mathbb{Z}\subset\mathbb{Z}\times\mathbb{Z}}\big)^{\mathbb{Z}\times\mathbb{N}\subset\mathbb{Z}\times\mathbb{Z}} = \big(X^{\mathbb{N}\times 0\subset\mathbb{Z}\times 0}\big)^{0\times\mathbb{N}\subset 0\times\mathbb{Z}}.$$

This shows that $X \rightsquigarrow X^N$ can be realized as two stages of $\mathbb{G}_m$-attractors.

(iii) If $M = \mathbb{Z}^r$ and $N = \mathbb{N}^r$ then $X \rightsquigarrow X^N$ can be realized as $r$ stages of $\mathbb{G}_m$-attractors.

### 16.4 Magnetic point of view on reductive groups

This section is devoted to the observation that parabolic and Levi subgroups of reductive group schemes are easily described using attractors. We work with a split reductive group over a field for simplicity and accessibility but similar results hold more generally (e.g. cf. Proposition 16.14 and [ALRR22, §6.3]). In fact, we expect attractors theory provide a natural framework to study some aspects of the advanced theory of group schemes from the beginning, but this is not the purpose of the present work. So let $G$ be a split connected reductive group scheme over a field $R$. Let $T$ be a maximal split torus and choose a Borel $B$ containing $T$. Let $\Phi = \Phi(G,T) \subset X^*(T)$ denote the set of roots associated to $(G,T)$ and $\Phi^+ = \Phi(B,T)$ the roots in $B$. Let $\mathcal{B}$ be the basis of $\Phi$ determined by $\Phi^+$. For $\alpha \in \Phi$, let $U_\alpha$ be the associated unipotent root group and $\mathfrak{u}_\alpha$ be the root group associated to $\alpha$ in the Lie algebra of $G$. We refer to [SGA3, Exp. XXII] for the definition of $U_\alpha$ and $\mathfrak{u}_\alpha$. Let $U$ be the unipotent radical of $B$ and let $\mathfrak{u} \subset \mathfrak{b} \subset \mathfrak{g}$ be the Lie algebras of $U, B$ and $G$. Consider the adjoint action of $T$ on $G, U$ and $\mathfrak{u}$.

Proposition 16.6. *There exists a $T$-equivariant isomorphism of $R$-schemes $\mathfrak{u} \simeq U$.*

*Proof.* This is a direct consequence of [SGA3, Exp. XXII Th. 1.1, Exp. XXVI Prop. 1.12], indeed these results imply the following assertions. For each root $\alpha \in \Phi^+$, we have a $T$-equivariant isomorphism $U_\alpha \simeq \mathfrak{u}_\alpha$. We have $T$-equivariant isomorphisms of schemes $\mathfrak{u} = \Pi_{\alpha\in\Phi^+}\mathfrak{u}_\alpha$ and $U = \Pi_{\alpha\in\Phi^+}U_\alpha$. This finishes the proof. □

Let us now fix $\alpha \in \Phi^+$. Recall that $[\alpha\rangle \subset X^*(T)$ is the submonoid generated by $\alpha$.

Proposition 16.7. *We have a canonical isomorphism $\mathfrak{u}_\alpha \simeq \mathfrak{u}^{[\alpha\rangle}$ (resp. $U_\alpha \simeq U^{[\alpha\rangle}$), between root group and $[\alpha\rangle$-attractor for the action of $T = \mathrm{Spec}(R[X^*(T)])$ on $\mathfrak{u}$ (resp. $U$).*

*Proof.* By Proposition 16.6, we have a $T$-equivariant isomorphism $\mathfrak{u} \simeq U$, so it is enough to prove the statement for $\mathfrak{u}$. Since $G$ is split, $\Phi$ is reduced and Proposition 3.26 finishes the proof. □

Recall that we have a bijection between parabolic subgroups of $G$ containing $B$ and subsets of $\mathcal{B}$, cf e.g. [Co14, Page 35, lines 4-5] or [CGP10, 2.2.8]. If $\Sigma$ is a subset of $X^*(T)$, we also use the notation $N_\Sigma$ to denote $[\Sigma\rangle$, the monoid generated by $\Sigma$ in $X^*(T)$.

Proposition 16.8. *Let $\zeta \subset \mathcal{B}$. Let $\Theta$ be $\zeta \cup -\zeta$. Let $\Sigma$ be $\mathcal{B} \cup -\zeta$.*

(i) *The attractor $G^{N_\Theta}$ is the Levi subgroup $L_\Theta$ such that $\Phi(L_\Theta, T) = N_\Theta \cap \Phi$.*

(ii) *The attractor $G^{N_\Sigma}$ is the associated parabolic subgroup, moreover $L_\Theta$ is a Levi component of $P$.*

(iii) *Let $\xi \subset \zeta$. Let $\Gamma = \mathcal{B} \cup -\xi$. Let $N$ be a submonoid of $N_\Sigma$ such that $N \cap \Sigma = \Gamma$. Then $G^N = G^{N_\Gamma}$.*

*Proof.* Let $P_\Sigma$ be the parabolic corresponding to $\zeta$. By [CGP10, 2.2.8, 2.2.9], there exists a $\lambda \in X_*(T)$ such that $P_\Sigma$ is the attractor associated to the monoid $\mathbb{N}$ relatively to the action of $\mathbb{G}_m = D(\mathbb{Z})_S$ on $G$ via $x.g = \mathrm{ad}_{\lambda(x)}g$ and such that $\lambda(\beta) \geqslant 0$ for all $\beta \in \Sigma$ and $\lambda(\beta) = 0$ for all $\beta \in \Theta$. The Levi subgroup $L_\Theta$ corresponding to $\Theta$ is the fixed space in $G$ under the action of $\lambda$ by conjuguation, i.e $L_\Theta = G^0$. Now we prove the Proposition.

(i) Assume first that $\zeta = \mathcal{B}$. Then $L_\Theta = G$ and $N_\Theta = N_{\Phi(G,T)}$. Using Prop. 16.7, we deduce that the big cell $\Omega = \Pi_{\alpha\in\Phi^-}U_\alpha \times T \times \Pi_{\alpha\in\Phi^+}U_\alpha$ is in $G^{N_\Theta}$. Now we have inclusions $\Omega \subset G^{N_\Theta} \subset G$ with $\Omega$ dense in $G$ and $G^{N_\Theta}$ closed in $G$. This implies $G^{N_\Theta} = G$. Let us now prove the general case, let $\zeta \subset \mathcal{B}$. By Proposition 3.29, we have $G^{f^{-1}(0)} = G^0 = L_\Theta$. We have $\Theta \subset f^{-1}(0)$, and so $N_\Theta \subset f^{-1}(0)$, consequently $G^{N_\Theta} \subset G^{f^{-1}(0)}$. So we have proved that $G^{N_\Theta} \subset L$ and let us now prove that this is an equality. We remark that $N_\Theta = N_{\Phi(L_\Theta,T)}$. Now since $L \subset G$ and using the first case done before, we have $L = L^{N_{\Phi(L_\Theta,T)}} \subset G^N_\Theta$. This finishes the proof.

(ii) Recall that $\lambda : \mathbb{G}_m \to T$ corresponds to the morphism of abelian groups $f : X^*(T) \to \mathbb{Z}, \chi \mapsto (\lambda, \chi)$. Now we see $G$ as a $\mathbb{G}_m$-scheme and as a $T$-scheme. By Proposition 3.29, we have $G^{\mathbb{N}} = G^{f^{-1}(\mathbb{N})}$. Since, for all $\beta \in \Sigma$, $f(\beta) = \lambda(\beta) \geqslant 0$, we have $f(\beta) \in \mathbb{N}$ and so $\Sigma \subset f^{-1}(\mathbb{N})$, and so $N_\Sigma \subset f^{-1}(\mathbb{N})$. Consequently, $P_\Sigma = G^{\mathbb{N}} = G^{f^{-1}(\mathbb{N})} \supset G^{N_\Sigma}$. Let us prove that $P_\Sigma \subset G^{N_\Sigma}$. We have $P_\Sigma = L_\Theta \times \mathrm{R_u}(P_\Sigma)$ where $\mathrm{R_u}(P_\Sigma)$ is the unipotent radical of $P_\Sigma$. Using (i), we have $L_\Theta = L_\Theta^{N_\Theta} \subset L_\Theta^{N_\Sigma} \subset L_\Theta$, and so $L_\Theta^{N_\Sigma} = L_\Theta$. Using Proposition 16.7, one has $(\mathrm{R_u}(P_\Sigma))^{N_\Sigma} = \mathrm{R_u}(P_\Sigma)$. So $P_\Sigma^{N_\Sigma} = P_\Sigma$, and so $P_\Sigma \subset G^{N_\Sigma}$.

(iii) We have a canonical closed immersion $G^{N_\Gamma} \subset G^N$. We have $N \cap N_\Theta = N_{\zeta\cup-\xi}$ and $N \cap N_\Theta$ it is thus included in $N_\Gamma$. By Proposition 3.22 we have $G^N = (G^{N_\Sigma})^N$. By the previous assertions we have $G^{N_\Sigma} = G^{N_\Theta} \times \mathrm{R_u}(G^{N_\Sigma})$. Now by 3.22 we have

$$\left(G^{N_\Theta} \times \mathrm{R_u}(G^{N_\Sigma})\right)^N = (G^{N_\Theta})^N \times (\mathrm{R_u}(G^{N_\Sigma}))^N = G^{N_{\zeta\cup-\xi}} \times \mathrm{R_u}(G^{N_\Sigma}).$$

This implies that $G^{N_\Gamma} \supset G^N$ and finishes the proof.

□

Remark 16.9. Proposition 16.8 implies that any parabolic or Levi subgroup of $G$ containing $T$ can be obtained as an attractor under the conjuguation action of $T$ on $G$. Moreover, assume that $\mathbf{B}$ is a parabolic subgroup in a parabolic $\mathbf{P}$ and $\mathbf{L}$ is a Levi component of $\mathbf{P}$. We assume that $\mathbf{B}, \mathbf{P}$ and $\mathbf{L}$ contain $T$. Let $\mathbf{M}$ be $\mathbf{B} \cap \mathbf{L}$, this is a parabolic subgroup in $L$. Then one has a cartesian square

$$\begin{tikzcd} & \mathbf{B} \arrow[dl] \arrow[dr] & \\ \mathbf{P} \arrow[dr] & & \mathbf{M} \arrow[dl] \\ & \mathbf{L} & \end{tikzcd} .$$

This square can be obtained using Proposition 3.33. Indeed let $L'$ be the submonoid generated by $\Phi(\mathbf{P}, T)$, let $L$ be the submonoid generated by $\Phi(\mathbf{L}, T)$ and $N$ be the submonoid generated by $\Phi(\mathbf{M}, T)$. Using Proposition 2.16, we deduce that $L$ is a face of $L'$. Now let $N'$ be $L' \setminus (L \setminus N)$. We have $G^{L'} = \mathbf{P}, G^N = \mathbf{M}$ and $G^L = \mathbf{L}$. Using Proposition 16.8 we have $G^{N'} = \mathbf{B}$.

For any root $\alpha \in \Phi^+$ we denote by $H_\alpha \subset G$ the semidirect product $T \ltimes U_\alpha$, this is a group scheme whose unipotent radical equals $U_\alpha$.

Proposition 16.10. *We have a canonical isomorphism $H_\alpha \simeq G^{[\alpha\rangle}$.*

*Proof.* Since $[\alpha\rangle \subset \Sigma_{\Phi^+}$ and by Proposition 16.8, we have a closed immersion $G^{[\alpha\rangle} \subset B = G^{\Sigma_{\Phi^+}}$ where $B$ is the Borel subgroup. So by Lemma 3.24, we get a closed immersion $G^{[\alpha\rangle} \subset B^{[\alpha\rangle}$ and thus an equality $G^{[\alpha\rangle} = B^{[\alpha\rangle}$. Now we have a $T$-equivariant isomorphism of schemes $B \simeq T \times U$. Using Proposition 16.7, we get

$$U_\alpha = U^{[\alpha\rangle}. \tag{16.1}$$

It is obvious that

$$T = T^{[\alpha\rangle}. \tag{16.2}$$

Now equations 16.1 and 16.2 and Proposition 3.7 (ii) imply that $B^{[\alpha\rangle} = H_\alpha$. This finishes the proof. □

Corollary 16.11.

(i) *Let $e_G$ be the closed subscheme of $G$ corresponding to the unit section. Then the attractors $G^{[\alpha\rangle}_{e_G}$ with prescribed limit $e_G$ equals $U_\alpha$.*

(ii) *We have a canonical isomorphism $U_\alpha = \mathrm{R_u}(G^{[\alpha\rangle})$ where $\mathrm{R_u}$ means the unipotent radical.*

*Proof.* Clear from Proposition 16.10 and Section 4. □

### 16.5 Root groups

Let $S$ be a scheme, let $G$ be a group scheme over $S$. Let $M$ be an abelian group. Assume that $D(M)_S$ acts on $G$ by group automorphisms. Then we get an action of $D(M)_S$ on $\mathrm{Lie}(G)_S$ by group automorphisms. Let $N$ be a submonoid of $M$. Let $G^N_{e_S}$ be the attractor associated to $N$ with prescribed limit the trivial subgroup $e_S/S$ relatively to the face $N^*$.

Definition 16.12. Let $\alpha \in M$. We call $G^{[\alpha\rangle}_{e_S}$ the root group associated to $\alpha$ under the action of $D(M)_S$ on $G$. We call $G^{[\alpha\rangle}$ the non prescribed root group associated to $\alpha$.

Proposition 16.13. *Let $\alpha \in M$.*

(i) *If $G$ is affine over $S$, we have closed immersions of group schemes over $S$*

$$G^{[\alpha\rangle}_{e_S} \subset G^{[\alpha\rangle} \subset G.$$

(ii) *If $G/S$ is smooth, then $G^{[\alpha\rangle}_{e_S}$ and $G^{[\alpha\rangle}$ are smooth over $S$.*

(iii) *We have canonical isomorphisms $\mathrm{Lie}(G^{[\alpha\rangle}_{e_S}) = \big(\mathrm{Lie}(G)\big)^{[\alpha\rangle}_{e_S}$ and $\mathrm{Lie}(G^{[\alpha\rangle}\rangle = \mathrm{Lie}(G)^{[\alpha\rangle}$.*

*Proof.* (i) We know that $G^{[\alpha\rangle}$ is a closed subgroup of $G$ by Theorem 3.19. We have $G^{[\alpha\rangle}_{e_S} = G^{[\alpha\rangle} \times_{G^0} e_S$ and so $G^{[\alpha\rangle}_{e_S}$ is a closed subgroup of $G^{[\alpha\rangle}$.

(ii) By Corollary 11.9, $G^{[\alpha\rangle} \to S$ and $G^{[\alpha\rangle} \to G^0$ are smooth, this implies the claim.

(iii) These are special cases of Propositions 16.5 and 16.4.

□

We observe that our definition is compatible with the definition given by Conrad and SGA3.

Proposition 16.14. *Let $G \to S$ be a reductive group scheme over a non-empty scheme $S$, $T \cong D_S(M)$ a split maximal torus, and $\alpha \in M$ a root. Let $\exp_\alpha(\mathbf{W}(\mathfrak{g}_\alpha)) \subset G$ be the $\alpha$-root group for $(G, T, M)$ considered by B. Conrad in [Co14, Theorem 4.1.4], then*

$$\exp_\alpha(\mathbf{W}(\mathfrak{g}_\alpha)) = G^{[\alpha\rangle}_{e_S}$$

$$\exp_\alpha(\mathbf{W}(\mathfrak{g}_\alpha)) \times_S T = G^{[\alpha\rangle}.$$

*Proof.* We remark that we have $(\mathbf{W}(\mathfrak{g}_\alpha) \times_S T)^{[\alpha\rangle} = \mathbf{W}(\mathfrak{g}_\alpha) \times_S T$. By [Co14, Theorem 4.1.4] we have a closed immersion $\mathbf{W}(\mathfrak{g}_\alpha) \times_S T \to G$. So Lemma 3.24 gives us a closed immersion $f : \mathbf{W}(\mathfrak{g}_\alpha) \times_S T \to G^{[\alpha\rangle}$. We now prove that $f$ is an open immersion. Since $f$ is of finite presentation by [StP, Tag 02FV], using [GW, Proposition 14.18] it is enough to prove that $f$ is flat. Since $\mathbf{W}(\mathfrak{g}_\alpha) \times_S T \to S$ is flat, it is enough to prove that for any $s \in S$, the morphism $f_s : \big(\mathbf{W}(\mathfrak{g}_\alpha) \times_S T\big)_s \to \big(G^{[\alpha\rangle}\big)_s$ is flat (cf. [GW, Proposition 14.25]). Since all involved constructions are compatible by base change, $f_s$ is an isomorphism (in particular flat) for any $s \in S$ by Corollary 16.11. Now $f$ is an open and closed immersion that gives isomorphisms on fibers. This implies that $f$ is an isomorphism and proves the second equality. Now the first equality is clear because $[\alpha\rangle^* = 0$, $G^0 = T$ and so

$$G^{[\alpha\rangle}_{e_S} = G^{[\alpha\rangle} \times_{G^0} e_S = \big(\exp_\alpha(\mathbf{W}(\mathfrak{g}_\alpha)) \times_S T\big) \times_T e_S = \exp_\alpha(\mathbf{W}(\mathfrak{g}_\alpha)).$$

□

Remark 16.15. Let us remark that, $G^{[\alpha\rangle} = \big(G^{D(M/(\alpha))_S}\big)^{[\alpha\rangle\subset(\alpha)}$ (e.g. by §16.3). So $G^{[\alpha\rangle}$ can be obtained as a first stage of fixed-points followed by a stage of $\mathbb{G}_m$-attractor. Similarly $G^{[\alpha\rangle}_{e_S} = \big(G^{(\alpha)}_{T_\alpha}\big)^{[\alpha\rangle}_{e_S}$ where $T_\alpha$ is defined in [Co14, Lemma 4.1.3] or [SGA3], this explains why [Co14] works with small semisimple groups of rank one as a first stage in order to build root groups using $\mathbb{G}_m$-attractors. Note that [SGA3] also uses semisimple groups of rank one as a first stage [SGA3, Exp. XX] before defining root groups in the general case [SGA3, Exp. XXII].

### 16.6 Monoschemes and toric schemes

Let $M$ be an abelian group and let $N$ and $L$ be submonoids of $M$. Let us consider the attractors $A(N)^L$ associated to the monoid $L$ under the action of $D(M)$ on $A(N)$. By Theorem 3.19 $A(N)^L$ equals $\mathrm{Spec}\big(\mathbb{Z}[N]/(X^n|n \in N \setminus (N \cap L))\big)$. The ideal $\mathcal{I} := (X^n|n \in N \setminus (N \cap L))$ of $\mathbb{Z}[N]$ equals $\bigoplus_{i\in I} \mathbb{Z}X^i$ where $I$ is the ideal of $N$ generated by $N \setminus (N \cap L)$. If $I$ is a prime ideal of $N$ (cf. [Og]) then $N \setminus I$ is a submonoid of $N$ (and necessarily a face) and $\mathbb{Z}[N]/\mathcal{I} = \mathbb{Z}[N \setminus I]$. In this case, $A(N)^L$ is also a scheme associated to a submonoid of $M$. For example if $N \cap L$ is a face of $N$, then $N \setminus I = N \cap L$ and $A(N)^L = A(N \cap L)$. In general $I$ is not a prime ideal and so $N \setminus I$ is not a submonoid of $N$ (e.g. take $M = \mathbb{Z}^2, N = [(1,1),(1,-1),(1,0)\rangle$ and $L = [(1,0)\rangle$, then $N \setminus I = \{(0,0),(1,0)\}$).

More generally, let $\mathcal{N}$ be a toric monoscheme whose associated finitely generated abelian group $\Gamma$ is $M$ (cf. [Og, II §1.9]). Let $A(\mathcal{N})$ be the scheme asssociated to $\mathcal{N}$ (cf. [Og, II Prop. 1.9.1]), this is a toric scheme. Let $\{\mathrm{spec}(N_\tau)\}_{\tau\in\mathcal{A}}$ be an open affine covering of $\mathcal{N}$ ($N_\tau \subset M$ for all $\tau$). Then $\{A(N_\tau)\}_{\tau\in\mathcal{A}}$ is an open affine covering of $A(\mathcal{N})$. Then $\coprod_\tau A(N_\tau) \to A(\mathcal{N})$ gives an FPR atlas of $A(\mathcal{N})$ (recall that a $D(M)$-equivariant open immersion is $Z$-FPR for any subgroup $Z \subset M$). Let $L$ be a submonoid of $M$, then we obtain that $\{A(N_\tau)^L\}$ is an affine open covering of $A(\mathcal{N})^L$.

## Appendix A. Fixed-point-reflecting morphisms, after Drinfeld and Alper-Hall-Rydh.
## by Matthieu Romagny

Let $U \to X$ be an equivariant morphism of $S$-algebraic spaces endowed with actions of an affine $S$-group scheme $G$ whose function algebra is free over $\mathcal{O}_S$ (e.g. a diagonalizable $S$-group scheme). It is classical that if $U/S$ is separated then the fixed point functor is a closed subspace $U^G \to U$ ([SGA3, Exp. VIII, § 6]). By étale descent, if $U \to X$ is étale, surjective and reflects the fixed points, then the same conclusion follows for $X^G \to X$. This need not always hold: if $X$ is the affine line with doubled origin over the field of complex numbers, and $V, W \subset X$ are the two glued copies of the affine line, then the action of $\mathbb{Z}/2\mathbb{Z}$ that permutes $V$ and $W$ induces an action on $X$ whose fixed point scheme is the complement of the two origins.

Thus there is a close relationship between the existence of FPR atlases and the closedness of fixed points. In this appendix we present two situations where these properties occur. The first is a useful generalization of an argument of Drinfeld, who in [Dr15, Prop. 1.2.2] considers the case where $S$ is the spectrum of a field and $G$ is the multiplicative group $\mathbb{G}_m$ (cf. also [Ri16, Lemma 1.10] and Propositions 8.1 and 6.3).

Theorem A.1. *Let $G$ be a flat, finitely presented $S$-group scheme with* connected fibres. *Let $X, Y$ be locally finitely presented $S$-algebraic spaces endowed with $G$-actions. Assume either*

(i) *$S$ is locally noetherian, or*

(ii) *$X, Y$ are quasi-separated.*

*Then $X^G \to X$, $Y^G \to Y$ are closed immersions of finite presentation and all étale equivariant morphisms $X \to Y$ are fixed point reflecting.*

Interestingly this gives a case where closedness of $X^G \to X$ is ensured by an assumption on

the group (connectedness) rather than on the space (separation). The second result, Theorem A.2, is essentially a corollary of results of Alper, Hall and Rydh [AHR21] and the present version was formulated by Mayeux (cf. Theorem 6.8).

THEOREM A.2. *Let $X$ be a quasi-separated $S$-algebraic space locally of finite presentation endowed with an action of a finitely presented diagonalizable group scheme $G = D(M)_S$. Let $H = D(M/Z)$ be a subgroup scheme and assume that one of the following assertions holds:*

(i) *$X$ is separated over $S$,*

(ii) *$H$ has connected fibres.*

*Then there exists a $Z$-FPR atlas $U \to X$ (cf. Definition 6.1), which may be chosen quasi-compact if $X \to S$ is.*

Let us get the proofs started. We start with Drinfeld's result on the closedness of fixed points. We shall prepare the discussion with three preliminary lemmas. Let $F = (F, e : S \to F)$ be a pointed $S$-algebraic space. An *action* of $F$ on $X$ is a morphism of $S$-algebraic spaces $F \times X \to X$, $(f, x) \mapsto fx$ through which the section $e$ acts trivially. Examples are the projection $\mathrm{pr}_2 : F \times X \to X$, called the *trivial action*, and the action induced when $F$ is a subscheme of an $S$-group scheme $G$ acting on $X$ in the usual sense. To any action is associated its *stabilizer*, the pointed sub-$X$-algebraic space of $F \times X$ defined by pulling back the map $F \times X \to X \times X$, $(f, x) \mapsto (x, fx)$ along the diagonal:

$$\begin{tikzcd} \mathrm{Stab} \arrow[r, "\varphi"] \arrow[d] & F \times X \arrow[d] \\ X \arrow[r, "\Delta"] & X \times X. \end{tikzcd}$$

LEMMA A.3. *Let $F = (F, e)$ be a finite locally free, infinitesimal pointed $S$-algebraic space acting on the $S$-algebraic space $X$. Then the functor of $F$-fixed points, whose values over an $S$-algebraic space $T$ are the $F$-equivariant maps $u : T \to X$ where $T$ is endowed with the trivial action, is representable by a closed subspace $X^F \subset X$.*

*Proof.* Note that $X^F$ is the Weil restriction of $\varphi : \mathrm{Stab} \to F \times X$ along the projection $\mathrm{pr}_2 : F \times X \to X$. In particular, its representability by an algebraic space is standard, for all $F$ finite locally free (see e.g. [Ry11, Th. 3.7]). To prove the closed immersion property in the infinitesimal case, we argue as follows. Because the stabilizer is sandwiched as

$$X \hookrightarrow \mathrm{Stab} \hookrightarrow F \times X$$

between two finite infinitesimal $X$-spaces, then also $\mathrm{Stab} \to X$ is finite infinitesimal and $\varphi$ is a closed immersion. Let $\mathcal{A}$ be the $\mathcal{O}_X$-algebra of functions of $F \times X$ and $\mathcal{I} \subset \mathcal{A}$ the defining ideal sheaf of $\varphi$. Let $u : T \to X$ be a morphism. Saying that $F$ acts trivially on $T$ means that $\varphi$ restricts to an isomorphism above $F \times T$. The latter assertion means that after the base change $u^\sharp : \mathcal{O}_X \to u_*\mathcal{O}_T$, the map $\mathcal{I} \to \mathcal{A}$ becomes the zero map. The $\mathcal{O}_X$-algebra $\mathcal{A}$ being finite locally free, the formation of its linear dual commutes with base change; hence it is equivalent to say that $\mathcal{I} \otimes \mathcal{A}^\vee \to \mathcal{O}_X$ becomes the zero map. Equivalently the $\mathcal{O}_X$-ideal $\mathcal{J} := \mathrm{im}(\mathcal{I} \otimes \mathcal{A}^\vee \to \mathcal{O}_X)$ is contained in $\ker(u^\sharp)$. This shows that the functor of fixed points is representable by the closed subscheme $V(\mathcal{J}) \subset X$. □

LEMMA A.4. *Let $R \to A$ be a ring map and $x_1, \ldots, x_d$ elements of $A$. Let $I_n$ be the ideal of $A$ generated by the powers $(x_1)^n, \ldots, (x_d)^n$, for each $n \geqslant 1$. If $A/I_1$ is finite over $R$, then $A/I_n$ is finite over $R$ for all $n \geqslant 1$.*

*Proof.* Let $(I_1)^n$ be the powers of the ideal $I_1$. By induction, using that $(I_1)^n/(I_1)^{n+1}$ is a finite $A/I_1$-module hence finite over $R$, we see that $A/(I_1)^n$ is finite over $R$ for all $n$. The containments $(I_1)^{d(n-1)+1} \subset I_n \subset (I_1)^n$ show that the sequences of ideals $\{I_n\}_{n\geqslant 1}$ and $\{(I_1)^n\}_{n\geqslant 1}$ are cofinal, and the claim follows. □

Lemma A.5. *Let $G$ be a flat, finitely presented $S$-group scheme with connected fibres. Then for each point $s \in S$ there is an étale neighbourhood $S' \to S$ and a sequence of finite locally free infinitesimal neighbourhoods of the unit section of $G \times_S S'$ which is cofinal to the canonical sequence of $n$-th order thickenings.*

*Proof.* Let $e = e(s)$ be the unit section of the fibre $G_s$. Since the local ring $\mathcal{O}_{G_s,e}$ is Cohen-Macaulay ([SGA3, Exp. VII$_B$, Corollaire 5.5.1]), it admits a regular sequence $\bar{x} = (\bar{x}_1, \dots, \bar{x}_d)$ of length $d = \dim(\mathcal{O}_{G_s,e})$. Let $U$ be an open subscheme of $G$ over which the germs $\bar{x}_i$ extend to local functions $x_i$ belonging to the augmentation ideal $\ker(e^\sharp : \mathcal{O}_U \to \mathcal{O}_S)$. Because $G \to S$ is flat and the sequence $\bar{x}$ is regular, the closed subscheme $F \subset U$ cut out by $x_1, \dots, x_d$ is *flat* and quasi-finite over $S$ at the point $e$. Thus shrinking $U$ is necessary, we may assume that $F \to S$ is flat and quasi-finite; it is furthermore finitely presented. By [EGA, IV, Th. 18.12.1] there is an étale extension $S' \to S$ and an open neighbourhood $V'$ of $e' = e(s')$ in $G \times_S S'$ such that $F' := (F \times_S S') \cap V'$ is *finite* over $S'$, and therefore locally free. For each $n \geqslant 1$, the sequence $(x_1)^n, \dots, (x_d)^n$ is again regular and it now follows from Lemma A.4 that its vanishing locus $F'_n$ in $V'$ is also finite locally free over $S'$. □

Remark A.6. If $G \to S$ is smooth, the $n$-th order thickenings of the unit section are finite locally free, hence they fit the bill.

We are ready to prove that "$X^G \to X$ is closed when $G$ is connected".

*Proof of Theorem A.1.* We start with the statement about $X^G \to X$. The assumptions and conclusion being local over $S$, we may assume that $S$ is affine. Because $G \to S$ is flat and finitely presented, the orbit in $X$ of an open quasi-compact subspace $W \subset X$ is open and quasi-compact. We may replace $X$ by one such orbit and hence assume that $X$ is quasi-compact. In the case when $X$ is assumed quasi-separated, it is then of finite presentation. By standard results on limits (as in [StP, Tag 07SJ]), the space $X$ and the $G$-action then come from a finitely presented algebraic space $X_0 \to S_0$ with $G_0$-action by a base change $S \to S_0$ where $S_0$ is of finite type over $\mathbb{Z}$. Thus in all cases we may assume that $S$ is noetherian and $X$ is locally noetherian.

The claims about $X^G \to X$ are étale-local over $S$ so using Lemma A.5 we can assume that there exists a sequence $\{F_n\}$ of finite locally free infinitesimal neighbourhoods of the unit section of $G$ which is cofinal to the canonical sequence of $n$-th order thickenings. By Lemma A.3 the fixed points $X^{F_n}$ are closed subspaces of $X$. Let $X_0$ be the closed subspace of $X$ equal to their intersection. From the obvious inclusions $X^G \subset X^{F_n}$ follows that $X^G \subset X_0$. In order to prove the opposite inclusion and conclude, it is enough to prove that the top map $\varphi_0$ in the pullback diagram

$$\begin{tikzcd} \mathrm{Stab}_0 \arrow[r, "\varphi_0"] \arrow[d] & G \times X_0 \arrow[d] \\ \mathrm{Stab} \arrow[r, "\varphi"] & G \times X \end{tikzcd}$$

is an isomorphism. We view $\varphi_0$ as a map of $X_0$-group schemes. Note that $\varphi_0$ is of finite presentation. We claim that $\varphi_0$ is formally étale along the unit section. Since the target is locally

noetherian, argueing as in [EGA, IV, Prop. 17.14.2] it is enough to prove that each diagram

$$\begin{tikzcd} \mathrm{Stab}_0 \arrow[r, "\varphi_0"] & G \times X_0 \\ \mathrm{Spec}(A) \arrow[u] \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}(A') \arrow[u] \end{tikzcd}$$

can be filled as indicated, where $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$ is a square-zero thickening of artinian local schemes. Base-changing along $\mathrm{Spec}(A') \to X_0$ we reduce to the case where $X_0$ is local artinian, in which case the group schemes involved are separated ([SGA3, Exp. $\mathrm{VI}_A$, § 0.3]) and $\mathrm{Spec}(A') \to G \times X_0$ is a closed immersion. Since $\mathrm{Spec}(A')$ is artinian, it is included in one of the $F_n \times X_0$, and then the lifting result follows from the definition of $X_0$.

This proves that the maximal open subscheme $U \subset \mathrm{Stab}_0$ where $\varphi_0$ is étale contains the unit section. Clearly $U$ is stable by multiplication by local sections of $U$, and also by inversion. Hence $U$ is an open subgroup scheme of $\mathrm{Stab}_0$ and also (because $\varphi_{0|U}$ is étale) of $G \times X_0$. Since $G \to S$ has connected fibres, its only open subgroup scheme is itself, whence $U = G = \mathrm{Stab}_0$ and the conclusion.

We now consider the statement about étale equivariant morphisms $X \to Y$. Proceeding as before we reduce to the situation where we have a family $\{F_n\}$ with equalities $X^G = \cap_{n \geqslant 0} X^{F_n}$ and $Y^G = \cap_{n \geqslant 0} Y^{F_n}$. Since the intersections commute with pullback along $X \to Y$, it is enough to prove that the natural map $X^{F_n} \to X \times_Y Y^{F_n}$ is an isomorphism. For this it is enough to prove that the action of $F_n$ on the subspace $X_0 := X \times_Y Y^{F_n}$ of $X$ is trivial. Consider the diagram:

$$\begin{tikzcd} X_0 \arrow[r, "e \times \mathrm{id}"] \arrow[d] & F_n \times X_0 \arrow[d] \arrow[dl, dashed] \\ X \arrow[r, "\Delta"'] & X \times_Y X. \end{tikzcd}$$

Here the right vertical map is $(f, x) \mapsto (x, fx)$ and the bottom arrow is the diagonal. The diagram is commutative by the very definition of $X_0$. The top arrow is a homeomorphism and the diagonal is an open immersion, since $X \to Y$ is étale. Therefore we obtain a diagonal filling as indicated. This proves our claim. □

Remark A.7. If $G \to S$ is infinitesimal (understood, finite locally free) then as in the proof of Lemma A.3 we see that the stabilizer of the action is finite infinitesimal over $X$. Then there is a quotient algebraic space $q : X \to Y = X/G$ such that $q$ is affine ([Ry13, Th. 5.3]). Let $V \to Y$ be an étale surjective map whose source is a disjoint sum of affine schemes; then $U = V \times_Y X \to X$ has the same properties. Moreover $U$ is endowed with a $G$-action such that $U \to X$ is equivariant, and from the known affine case we see that $U^G \hookrightarrow U$ is closed. Since $U^G \simeq X^G \times_X U$, it follows that $X^G \hookrightarrow X$ is closed. Note that this proof may seem simpler than the one given above, but it uses the existence of quotients.

Finally we provide the proof of the existence of FPR-atlases.

*Proof of Theorem A.2.* Proceeding as in the beginning of the proof of Theorem A.1, we reduce to the case where $S$ is affine and $S, X$ are of finite type over $\mathbb{Z}$. In particular, the set of closed points $|X^{\mathrm{cl}}| \subset |X|$ is dense. It will thus be enough to find an affine, $G$-equivariant étale neighbourhood $U(x) \to X$ of each closed point $x \in |X^{\mathrm{cl}}|$ and to eventually consider $U := \amalg_{x \in |X^{\mathrm{cl}}|} U(x)$.

For such a closed point $x$ with image $s \in S$, the residue field extension $\kappa(x)/\kappa(s)$ is finite. By [AHR21], Corollary 20.2 there exists an affine pointed scheme $(U_0 = \mathrm{Spec}(A), u_0)$ and an

étale, $G$-equivariant morphism $(U_0, u_0) \to (X, x)$ which induces an isomorphism of residue fields $\kappa(x) \simeq \kappa(u_0)$ and an isomorphism of stabilizers $G_{u_0} \simeq G_x$. Write

$$H = D(M/Z) = T \times F$$

as the product of a split torus with a finite diagonalizable group scheme. In case (ii) where $H$ has connected fibres (which means that $F$ is infinitesimal), it follows from Theorem A.1 that $U_0^H \to X^H \times_X U_0$ is an isomorphism, in other words $U(x) := U_0$ is the desired $Z$-FPR atlas at $x$. It remains to consider the case (i) where $X$ is separated. In this case the stabilizer of the action of $F$ is finite over $X$ hence there is a quotient $X \to X/F$ which is a finite morphism (see [Ry13, Th. 5.3 and Prop. 4.7]). Of course, the same is true for the stabilizer of the action of $F$ on $U_0$ so there exists also a quotient $U_0 \to U_0/F$. Let $U_1 := U_0/F \times_{X/F} X$ and $u_1 := (u_0, x) \in U_1$. The following properties are seen to hold:

- the scheme $U_1$ is a disjoint sum of affine $S$-schemes (because this is true for $U_0/F$ and its pullback along the affine map $X \to X/F$),
- the group $G/F$ acts on $U_0/F$ and on $X/F$, so $G$ acts on the fibred product $U_1$ diagonally on the factors,
- the map of point stabilizers $G_{u_1} \to G_x$ is an isomorphism,
- the map $U_1^F \to X^F \times_X U_1$ is an isomorphism.

Moreover, by Theorem A.1 the morphism $U_1^T \to X^T \times_X U_1$ is an isomorphism. Since $U_1^H = U_1^T \cap U_1^F$ and $X^H = X^T \cap X^F$, we conclude that $U(x) := U_1$ is a $Z$-FPR atlas at $x$.

The final claim that $U \to X$ can be chosen quasi-compact if $X \to S$ is quasi-compact is obvious. □

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