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Arnaud Mayeux

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Actions of diagonalizable groups and algebraic attractors associated to monoids

Arnaud Mayeux

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ABSTRACT. For a diagonalizable group scheme $D(M)_S$ acting on an algebraic space X over a scheme S , we introduce for any submonoid N of M an attractor space X^N . We then extend and refine the study of \mathbb{G}_m -actions on algebraic spaces to actions of diagonalizable group schemes and general monoids. We expect that this formalism will shed light on some primordial objects, as an evidence we observe that root groups are natural from the view point of attractors associated to monoids.

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1. Introduction

Contents. Let S be a scheme. Let M be an abelian group and let $D(M)_S$ be the associated diagonalizable group scheme over S . Let $N \subset M$ be a submonoid and let N^* be the face of invertible elements of N . Let $A(N)_S$ be $\text{Spec}(\mathbb{Z}[N]) \times_{\text{Spec}(\mathbb{Z})} S$, this is a monoid scheme over S with a canonical action of $D(M)_S$. Let X be an algebraic space over S with an action of $D(M)_S$. Our main idea is to introduce the following definition.

DEFINITION 1.1. Let X^N be the functor from schemes over S to *Sets* given by

$$(T \rightarrow S) \mapsto \text{Hom}_T^{D(M)_T}(A(N)_T, X_T)$$

where $\text{Hom}_T^{D(M)_T}(A(N)_T, X_T)$ is the set of $D(M)_T$ -equivariant T -morphisms from $A(N)_T$ to $X_T = X \times_S T$. The functor X^N is called the attractor associated to the monoid N under the action of $D(M)_S$ on X .

If Z is a subgroup of M seen as a monoid, we prove that X^Z equals the space of fixed points $X^{D(M/Z)_S}$ under the action of $D(M/Z)_S$ on X . The following Theorems hold.

THEOREM 1.2. *Assume that X is S -affine. Then the attractor X^N is representable by a closed subscheme of X .*

THEOREM 1.3. *Assume that M is finitely generated. Assume that X is quasi-compact and separated over S , then there exists a finitely generated submonoid N' of N such that $X^N = X^{N'}$. In particular X^N is representable by the following Theorem.*

THEOREM 1.4. *Assume that M is finitely generated. Assume that X is locally of finite presentation and quasi-separated over S . Assume that N is finitely generated as monoid, then the attractor X^N is representable by a quasi-separated and locally finitely presented S -algebraic space. Moreover the following assertions hold.*

- (i) *If X/S is separated, then X^N/S is separated.*
- (ii) *If X/S is quasi-compact then X^N/S is quasi-compact.*
- (iii) *If X/S is of finite presentation, then X^N/S is of finite presentation.*
- (iv) *If X/S is smooth, then X^N/S is smooth.*
- (v) *If X is a scheme then X^N is a scheme.*
- (vi) *Let $\pi^{N^*} : X^N \rightarrow X^{N^*}$ be the canonical morphism coming from the fact that N^* is a face of N , then π^{N^*} is affine and induces a bijection on the sets of connected components of underlying topological spaces. Note that X^{N^*} is the space of fixed points $X^{D(M/N^*)_S}$.*
- (vii) *Let Y be an other separated and locally finitely presented algebraic space over S . Let $f : X \rightarrow Y$ be a morphism of algebraic spaces and let $f^N : X^N \rightarrow Y^N$ be the canonical morphism obtained on attractors. Then*
 - (1) *if f is an open immersion, then f^N is an open immersion,*

(2) if f is a closed immersion, then f^N is a closed immersion.

In our paper, we also state several elementary Propositions that we do not recall in this introduction. We show that attractors are highly functorial. We show that attractors are compatible with tangent spaces and Lie algebras. We show that root groups of reductive groups are well interpreted in the language of attractors (cf. Section 14 and Proposition 14.3). We show that attractors are compatible with dilatations.

Relation with other works. To prove our results, we adapt the strategy used by Drinfeld and Richarz in [Dr13] and [Ri16] in the case of \mathbb{G}_m . Let us recall briefly the situation for $\mathbb{G}_m (= D(\mathbb{Z}))$. Richarz [Ri16] works with an algebraic space X endowed with an étale locally linearizable action of $D(\mathbb{Z})_S$. This means that there exists an equivariant étale atlas $\{U_i \rightarrow X\}_{i \in I}$ such that each U_i is S -affine. Then Richarz proves that for any equivariant étale morphism $U \rightarrow X$, there is a canonical isomorphism $U^{\mathbb{N}} = X^{\mathbb{N}} \times_{X^0} U^0$ (cf. [Ri16, Lemma 1.11]). Richarz also proves that for any equivariant étale morphism $U \rightarrow X$ from an S -affine scheme to X , we have a canonical isomorphism $U^0 = X^0 \times_X U$ (cf. [Ri16, Lemma 1.10]). Using that being étale is stable by base change, Richarz obtains that $\{U_i^{\mathbb{N}} \rightarrow X^{\mathbb{N}}\}_{i \in I}$ is an étale atlas of $X^{\mathbb{N}}$ and that $X^{\mathbb{N}}$ is representable (cf. [Ri16, Th. 1.8]) by an algebraic space. Note that Richarz and Drinfeld use the notations

$$X^+ \quad X^- \quad X^0 \quad \mathbb{G}_{m,S} \quad (\mathbb{A}_S^1)^+ \quad (\mathbb{A}_S^1)^-$$

where in our formalism we write

$$X^{\mathbb{N}} \quad X^{-\mathbb{N}} \quad X^0 \quad D(\mathbb{Z})_S \quad A(\mathbb{N})_S \quad A(-\mathbb{N})_S.$$

Richarz uses in his proof that there exists a canonical map $X^{\mathbb{N}} \rightarrow X^0$. However in our general situation where M is a finitely generated abelian group and N is an arbitrary submonoid, we do not have a canonical map $X^N \rightarrow X^0$. Moreover it is not true that $U^0 = X^0 \times_X U$ for any $D(M)_S$ -equivariant étale morphism from an S -affine scheme U to X . In our proof, instead of using the map $X^{\mathbb{N}} \rightarrow X^0$, we use the canonical map $\pi^{N^*} : X^N \rightarrow X^{N^*}$. Note that X^{N^*} is a fixed points space of X , namely the fixed points under $D(M/Z)_S$. In our paper we say that a $D(M)_S$ -equivariant morphism $U \rightarrow X$ is fixed point-reflecting (FPR) relatively to a subgroup $Z \subset M$ if the identity $U^Z = X^Z \times_X U$ holds. We prove in this paper the existence of étale FPR atlas for any subgroup Z using [AHR21]. We show that for any equivariant étale morphism $U \rightarrow X$, we have $U^N = X^N \times_{X^{N^*}} U$. Altogether and using the representability in the S -affine case, we deduce the representability of attractors. To prove the listed properties, we often adapt arguments of [Ri16] or [Dr13].

We are pleased to indicate in this introduction general and powerful results of Alper-Hall-Rydh and Halpern-Leistner-Preygel. The content of [AHR21, Corollary 14.10] gives a powerful representability result and can be used to deduce the representability by algebraic spaces of our attractors in the case where S is excellent. The content of [HP19, Introduction and Theorem 5.1.1] gives also a powerful representability result over specific bases. Our method (in the spirit of [Dr13] and [Ri16]), taking into account the explicit description of attractors in the S -affine case and using methods and results of [AHR21] (for the existence of FPR-atlas), allows to prove all assertions contained in Theorem 1.4 over arbitrary bases. We expect that our attractors make sense for stacks and could lead to a generalization of [AHR20, §5.4 §5.5].

In [JS18], over fields, the authors generalize in some sense [Dr13] to actions of much more general groups than \mathbb{G}_m and prove the existence of Bialynicki-Birula decompositions [Bi73] in

this context. Our construction goes in a different generality than [JS18] for two main reasons. Firstly, in the case of \mathbb{G}_m (i.e $M = \mathbb{Z}$), monoids that are reminiscent in the construction of [JS18] are the same than those in [Dr13] or [Ri16] (i.e. $0, \mathbb{N}$ or $-\mathbb{N}$), whereas we can also consider monoids like $0 \cup \mathbb{N}_{\geq k}$, $a\mathbb{N}$, $a\mathbb{N} + b\mathbb{N}$ or $\mathbb{N} \setminus \{1, 2, 4, 5, 7, 10, 13\}$. Secondly, we allow bases that are not fields. To our knowledge, in the \mathbb{G}_m -case, the definition of attractors as functors using equivariant morphisms appeared first in the works of Hesselink [He80, §4] and Jurciewick [Ju85, §1.2.10].

2. Rings associated to monoids and their spectra

We refer to [Og] for a detailed and beautiful introduction to monoids and related structures. We recall in this section some basic definitions and facts that we frequently use in our work. In this article, monoids and rings are always commutative. Readers unfamiliar with monoids should read [Og, I.1]. Let us recall some very basic notations, again we refer to [Og, I] for a more general and conceptual presentation. Let P be a monoid and $N \subset P$ and $L \subset P$ be two submonoids. Then $N \cap L$ is a submonoid of N, L and P . Let $N + L$ be $\{n + l \in P | n \in N, l \in L\}$. Then $N + L$ is a submonoid of P . Let P be a monoid and let Σ be a subset of P , then we denote by N_Σ or by $[\Sigma]$ the smallest submonoid of P containing Σ , this is the intersection of all monoids of P containing Σ . Similarly if M is a group and Σ is a subset of M , we denote by $\langle \Sigma \rangle$ the subgroup of M generated by Σ . Let M be an abelian group and let N be a submonoid of M . The subgroup generated by N in M is denoted N^{gp} , in fact $N^{\text{gp}} = N - N$. We have an obvious notion of finitely generated monoids. The monoid $(\mathbb{N}_{\geq 1} \times \mathbb{N}) \cup (0, 0)$ is not finitely generated and it is a submonoid of the finitely generated abelian group $\mathbb{Z} \times \mathbb{Z}$.

DEFINITION 2.1. Let R be a ring and let N be a monoid, then we denote by $R[N]$ the associated R -algebra whose underlying R -module is $\bigoplus_{n \in N} R \cdot X^n$ and multiplication is induced by the operation of the monoid: $X^n \times X^{n'} = X^{n+n'}$. By convention we have $X^0 = 1$.

DEFINITION 2.2. Let N be a monoid, then we define $A(N)$ to be the scheme $\text{Spec}(\mathbb{Z}[N])$. If S is a scheme, then we put $A(N)_S = A(N) \times_{\text{Spec}(\mathbb{Z})} S$. This is a monoid scheme over S . If M is an abelian group, then $A(M)$ is denoted $D(M)$ and is a group scheme called the diagonalizable group scheme associated to M [SGA3, Exp 1, §4.4].

If $N' \subset N$, then we obtain morphisms of monoid schemes $A(N)_S \rightarrow A(N')_S$, for every scheme S .

REMARK 2.3. Let us recall the augmentation and the comultiplication of the monoid scheme $A(N)$ for any monoid N . The augmentation is the map $\mathbb{Z}[N] \rightarrow \mathbb{Z}$ sending X^n to 1 for every $n \in N$. The comultiplication is the map $\mathbb{Z}[N] \rightarrow \mathbb{Z}[N] \otimes \mathbb{Z}[N]$ that sends X^n to $X^n \otimes X^n$ for $n \in N$. Moreover, if $N = M$ is a group the antipode is the map $\mathbb{Z}[M] \rightarrow \mathbb{Z}[M]$ that sends X^m to X^{-m} for every $m \in M$.

Let e_S be the trivial monoid scheme over S , it is a group scheme over S , isomorphic to S as scheme. So for any monoid N , we have a canonical closed immersion $e_S \rightarrow A(N)_S$ called the neutral section. We also denote by $e_{A(N)_S}$ the corresponding closed submonoid of $e_{A(N)_S}$.

Let M be an abelian group and N be a submonoid of M . Then we have an algebraic action

of $D(M)$ on $A(N)$ over \mathbb{Z} given by:

$$\begin{aligned}\mathbb{Z}[N] &\rightarrow \mathbb{Z}[N] \otimes \mathbb{Z}[M] \\ X^n &\mapsto X^n \otimes X^n.\end{aligned}$$

By base change, we obtain an action of $D(M)_S$ on $A(N)_S$ for every scheme S . Remark that the action of $D(M)_S$ on $A(N)_S$ comes from the morphism of monoid schemes $D(M)_S \rightarrow A(N)_S$.

REMARK 2.4. We have $A(N \times L)_S = A(N)_S \times_S A(L)_S$ for any pair of monoids and any scheme.

DEFINITION 2.5. Let N be a submonoid of a monoid L . We say that N is a face of L if the projection map

$$\mathbb{Z}[L] \rightarrow \mathbb{Z}[N], X^l \mapsto 0 \text{ if } l \in L \setminus N \text{ and } X^l \mapsto X^l \text{ if } l \in N$$

is a morphism of rings.

Remark that if N is a face of L , then the associated morphism of schemes $A(N)_S \rightarrow A(L)_S$ is $D(M)_S$ -equivariant for any scheme S .

PROPOSITION 2.6. Let $N \subset L$ be monoids. Then N is a face of L if and only if for all $x, y \in L$

$$x + y \in N \Leftrightarrow x \in N \text{ and } y \in N.$$

Proof. Let ϕ denote the projection and assume it is a morphism of rings. Let $x, y \in L$. Then $x + y \in N \Leftrightarrow \phi(X^{x+y}) = X^{x+y} = \phi(X^x)\phi(X^y)$ is not zero \Leftrightarrow both x and y are in N . Reciprocally assume that for all $x, y \in L$, $x + y \in N \Leftrightarrow x \in N$ and $y \in N$, then we have $\phi(X^x X^y) = \phi(X^x)\phi(X^y)$. \square

PROPOSITION 2.7. Let N be a monoid in M . Let $N^* = \{x \in N \mid \exists y \in N \text{ such that } x + y = 0\}$, then N^* is a submonoid of N and a group, moreover N^* is a face of N .

Proof. Take $x, y \in N$. Assume $x + y \in N^*$, then there exists $z \in N$ such that $x + y + z = 0$, this shows that x and y are in N^* . Now apply Proposition 2.6. \square

3. Attractors associated to monoids and properties

Let X be an algebraic space over a base scheme S . Let M be an abelian group and let $D(M)_S$ be the associated diagonalizable S -group scheme. Assume that $D(M)_S$ acts on X , this means that we have a morphism of algebraic spaces $D(M)_S \times_S X \rightarrow X$ satisfying the usual axioms. Let N be a submonoid of M and consider $A(N)_S$ with the canonical action of $D(M)_S$ as in the previous section. We now introduce the attractor X^N associated to the monoid N under the action of $D(M)_S$ on X .

DEFINITION 3.1. Let X^N be the functor

$$(Sch/S) \rightarrow Sets, (T \rightarrow S) \mapsto \text{Hom}_T^{D(M)_T}(A(N)_T, X_T)$$

where $\text{Hom}_T^{D(M)_T}(A(N)_T, X_T)$ is the set of $D(M)_T$ -equivariant T -morphisms from $A(N)_T$ to $X_T = X \times_S T$.

REMARK 3.2. There are several ways to generalize this definition. First of all we can consider an algebraic space with the action of $A(M)_S$ where M is just a monoid instead of an abelian group and consider X^N for submonoids $N \subset M$, this definition leads to reasonable statements in the

affine case and some of our results remain true in this generality. Secondly we can remove the assumption that $N \subset M$ and consider not necessarily injective morphisms of monoids $N \rightarrow M$, however this second generalization leads to a degenerate theory because in the case $M = 0$ and $N = \mathbb{N}$, this leads to call for any scheme X (endowed with the trivial action of the trivial group) the functor $T \mapsto \text{Hom}(\mathbb{A}_T^1, X_T)$ an attractor space. This is not representable in general. So this second naive generalization is not interesting to us. A third possible generalization is to try to find a formalism covering both [JS18], [JS20] and our formalism. We would like to mention [AHR21, Corollary 14.10] as a possible ingredient for generalization. However we think that the case of diagonalizable groups works well and is very convenient for many applications and interpretations of many objects. Note that algebraic actions of diagonalizable groups appear very often in many parts of mathematics.

REMARK 3.3. In the context of Definition 4.1, when we want to indicate that we take the attractor X^N relatively to the diagonalizable group $D(M)_S$, we also use the notation $X^{N \subset M}$.

REMARK 3.4. We have an identification $X \simeq X^M$. For all scheme $T \rightarrow S$ and $x \in X^M(T)$, we associate the morphism $T = e_T \xrightarrow{\varepsilon} D(M)_T \xrightarrow{x} X_T \in X(T)$ where ε is the neutral section.

REMARK 3.5. Let $N \subset L$ be submonoids of M . For $T \rightarrow S$, we have a morphism

$$\text{Hom}_T^{D(M)_T}(A(N)_T, X_T) \rightarrow \text{Hom}_T^{D(M)_T}(A(L)_T, X_T)$$

obtained using the morphism $A(L)_T \rightarrow A(N)_T$ corresponding to the morphisms of rings $\mathbb{Z}[N] \subset \mathbb{Z}[L]$. So we get a morphism of functors $\iota_{N,L} : X^N \rightarrow X^L$.

REMARK 3.6. We have an action of the group scheme $D(M)_S$ on X^N given as follows. For any S -scheme T , we have an action of $D(M)(T)$ on X_T (for any $t \in D(M)(T)$, we have an arrow $X_T \xrightarrow{t} X_T$). Now let $f \in X^N(T) = \text{Hom}_T^{D(M)_T}(A(N)_T, X_T)$ and $t \in D(M)(T)$. We define $t \cdot f$ to be $A(N)_T \xrightarrow{f} X_T \xrightarrow{t} X_T$. The morphism $X^N \rightarrow X$ is $D(M)_S$ -equivariant.

REMARK 3.7. We have an action of the monoid scheme $A(N)_S$ on X^N given as follows. For any S -scheme T , we have an action of $A(N)(T)$ on itself by multiplication (for any $t \in A(N)(T)$, we have an arrow $A(N)_T \xrightarrow{t} A(N)_T$). Now let $f \in X^N(T) = \text{Hom}_T^{D(M)_T}(A(N)_T, X_T)$ and $t \in A(N)(T)$. We define $t \cdot f$ to be $A(N)_T \xrightarrow{t} A(N)_T \xrightarrow{f} X_T$. The morphism of monoid schemes $A(N^{\text{gp}})_S \rightarrow A(N)_S$ implies that we also have a natural action of $D(N^{\text{gp}})_S$ on X^N .

PROPOSITION 3.8. *If X is a group algebraic space over S and the action of $D(M)_S$ is by group automorphisms, then X^N is a group functor.*

Proof. Let T be a scheme over S . Let $A(N)_T \xrightarrow{g} X_T$ and $A(N)_T \xrightarrow{h} X_T$ be two elements in $X^N(T)$. Then we define gh as the composition

$$A(N)_T \rightarrow A(N)_T \times_T A(N)_T \rightarrow X_T \times_T X_T \rightarrow X_T$$

where the first morphism is the diagonal morphism, the second is $g \times h$, and the third is the multiplication morphism coming from the group structure on X . The two first are equivariant by definitions and the third is equivariant because $D(M)_S$ acts on X_S by group automorphisms. This defines a group law on X^N . \square

DEFINITION 3.9. Let N and L be submonoids of an abelian group M and assume that N is a face of L . Let X be an S -algebraic space with an action of $D(M)_S$. Then for all $T \rightarrow S$ the

morphism $A(N)_T \rightarrow A(L)_T$ induces a morphism

$$\mathrm{Hom}_T^{D(M)_T}(A(L)_T, X_T) \rightarrow \mathrm{Hom}_T^{D(M)_T}(A(N)_T, X_T).$$

So we obtain a morphism of functors $X^L \rightarrow X^N$, that we denote $\mathfrak{p}_{N,L}$.

When N is a face of L , the morphism \mathfrak{p} satisfies $\mathfrak{p} \circ \iota = \mathrm{Id}$.

REMARK 3.10. If $X \rightarrow Y$ is a $D(M)_S$ -equivariant morphism of algebraic spaces, then for any submonoid N of M , we have a morphism of functors $X^N \rightarrow Y^N$.

PROPOSITION 3.11. *Let M be a finitely generated abelian group. Let S be a scheme and let X be an algebraic space with a $D(M)_S$ -action. Let $f : M \rightarrow Z$ be a morphism of abelian groups. Let $Y \subset Z$ be a submonoid and let N be $f^{-1}(Y)$. Then we have an isomorphism of functors $X^N \simeq X^Y$, where on the left-hand-side X is seen as a $D(M)_S$ -space and on the right-hand-side as a $D(Z)_S$ -space. In other words, with notation of Remark 3.3, we have $X^{N \subset M} = X^{Y \subset Z}$.*

Proof. Let $T \rightarrow S$ be a scheme. We have a canonical map $\theta : \mathrm{Hom}_T^{D(M)_T}(A(N)_T, X_T) \rightarrow \mathrm{Hom}_T^{D(Z)_T}(A(Y)_T, X_T)$ obtained by precomposition with $A(Y)_T \rightarrow A(N)_T$. Let us show that θ is an isomorphism. We construct the reciprocal map. Let $f : A(Y)_T \rightarrow X_T$ be a $D(Z)_T$ -equivariant map. We get a map $f' : D(M)_T \times_T A(Y)_T \rightarrow X_T$ given on points by $(g, x) \mapsto g \cdot f(x)$. Let K be $f(M) + Y^{\mathrm{gp}}$, this is a subgroup of Z . We have a morphism of groups $\phi : f(M) \times Y^{\mathrm{gp}} \rightarrow f(M) + Y^{\mathrm{gp}}$ given by $(f(m), y) \mapsto f(m) + y$. The morphism ϕ induces a closed immersion of group schemes $D(f(M) + Y^{\mathrm{gp}})_T \rightarrow D(f(M) \times Y^{\mathrm{gp}})_T = D(f(M))_T \times_T D(Y^{\mathrm{gp}})_T$. Let us consider the action of $D(f(M))_T \times_T D(Y^{\mathrm{gp}})_T$ on $D(M)_T \times_T A(Y)_T$ given by $(g, h) \cdot (g', x) = (g \cdot g', h^{-1} \cdot x)$, remark that this action is free. So we obtain by composition a free action \star of $D(K)_T$ on $D(M)_T \times_T A(Y)_T$. Let us consider the morphisms

$$\xi_\star, \xi : D(K)_T \times D(M)_T \times_T A(Y)_T \rightarrow X_T$$

where ξ_\star is given by $(k, g, y) \mapsto f'(k \star (g, y))$ and ξ is given by $(k, g, y) \mapsto f'(g, y)$. Let us prove that $\xi_\star = \xi$. Let us consider the canonical morphism

$$p : D(Z)_T \times D(M)_T \times_T A(Y)_T \rightarrow D(K)_T \times D(M)_T \times_T A(Y)_T$$

induced by the inclusion $K \subset Z$. Then $p \circ \xi = p \circ \xi_\star$, and so $\ker(\xi, \xi_\star) = D(K)_T \times D(M)_T \times_T A(Y)_T$ because the schematic image of p is $D(K)_T \times D(M)_T \times_T A(Y)_T$. This finishes the proof of the claim $\xi = \xi_\star$. So we have $f'(k \star (g, x)) = f'((g, x))$ for all $T' \rightarrow T$, $k \in D(K)_T(T')$, $g \in D(M)_T(T')$ and $x \in A(Y)_T(T')$. So the map f' induces by factorization a map $(D(M)_T \times_T A(Y)_T) / D(K)_T \rightarrow X_T$ where $(D(M)_T \times_T A(Y)_T) / D(K)_T$ is the $fpqc$ quotient as in [SGA3, VIII Th. 5.1]. Now let \mathcal{A} be the quasi-coherent algebra of the T -affine scheme $(D(M)_T \times_T A(Y)_T)$. Consider the K -grading on \mathcal{A} associated to the action of $D(K)_T$ on $(D(M)_T \times_T A(Y)_T)$. Now by [SGA3, VIII Th. 5.1], $(D(M)_T \times_T A(Y)_T) / D(K)_T$ is affine with quasi-coherent algebra the degree zero part \mathcal{A}_0 in \mathcal{A} . Now the K -graduation on \mathcal{A} is given locally by $\deg(X^{(m,y)}) = f(m) - y \in K$. This implies that $\mathcal{A}_0 \simeq \mathcal{O}_T[f^{-1}(Y)]$. This identifies $(D(M)_T \times_T A(Y)_T) / D(K)_T$ with $A(N)_T$. So the map f induces a map $A(N)_T \rightarrow X_T$. This map is $D(M)_T$ -equivariant. The obtained map $\mathrm{Hom}_T^{D(Z)_T}(A(Y)_T, X_T) \rightarrow \mathrm{Hom}_T^{D(M)_T}(A(N)_T, X_T)$ is the reciprocal map of θ . \square

REMARK 3.12. If X is S -affine, Proposition 3.11 can be deduced easily from Theorem 5.1, cf. [Ma22].

The next proposition shows that $X^N \rightarrow X^{N'}$ is a monomorphism if X is separated and $N \subset N'$.

PROPOSITION 3.13. Assume that X is separated and let $N \subset N'$ be two submonoids of M . Then the map of functors $X^N \rightarrow X^{N'}$ is a monomorphism, i.e for any scheme T over S we have an inclusion $X^N(T) \subset X^{N'}(T)$.

Proof. Since $N \subset N'$, by [StP, Tag 01R8], the scheme theoretic image of $A(N')_T \rightarrow A(N)_T$ is $A(N)_T$ for any scheme T over S . Now let f_1, f_2 be two elements in $X^N(T)$ such that their images in $X^{N'}(T)$ coincide. Consider the schematic kernel of the maps $(f_1, f_2) : A(N)_T \rightarrow X_T$. Since X_T is separated, $\ker(f_1, f_2)$ is a closed subscheme of $A(N)_T$ (the proof of [GW, Def./Prop. 9.7] works in this context). So since the scheme theoretic image of $A(N')_T \rightarrow A(N)_T$ is $A(N)_T$, we have $\ker(f_1, f_2) = A(N)_T$ and so $f_1 = f_2$. \square

COROLLARY 3.14. Assume that X is separated and let N be a submonoid of M . Then the natural map of functors $X^N \rightarrow X$ is a monomorphism.

Proof. Combine Proposition 3.13 and Remark 3.4. \square

REMARK 3.15. Assume $X \rightarrow Y$ is a $D(M)_S$ -equivariant monomorphism of S -algebraic spaces, then $X^N \rightarrow Y^N$ is a monomorphism.

PROPOSITION 3.16. Let L and N be submonoids of M , then we have canonical identifications

$$(X^N)^L = (X^L)^N = X^{N \cap L}.$$

Proof. Let us use [SGA3, VIII Th. 5.1] as follows. We remark that

$$(X^N)^L(T) = \mathrm{Hom}_T^{D(M)_T \times_T D(M)_T}(A(N)_T \times_T A(L)_T, X_T)$$

where $D(M)_T \times_T D(M)_T$ acts on X_T via $(g, h) \cdot x = g \cdot (h \cdot x)$ and on $A(N)_T \times_T A(L)_T$ via $(g, h) \cdot (a, b) = (g \cdot a, h \cdot b)$. Let us consider the action \star of $D(N^{\mathrm{gp}} + L^{\mathrm{gp}})_T$ on $A(N)_T \times_T A(L)_T$ given by $g \star (a, b) = (g^{-1} \cdot a, g \cdot b)$. The action \star is free because firstly the action of $D(N^{\mathrm{gp}} \times L^{\mathrm{gp}})_T$ on $A(N \times L)_T$ given by $(\lambda, \beta) \cdot (a, b) = (\lambda \cdot a, \beta^{-1} \cdot b)$ is free and secondly because the morphism of groups $N^{\mathrm{gp}} \times L^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}} + L^{\mathrm{gp}}$ given by $(n, l) \mapsto n + l$ is surjective and so $D(N^{\mathrm{gp}} + L^{\mathrm{gp}})_T$ is a closed subgroup scheme of $D(N^{\mathrm{gp}} \times L^{\mathrm{gp}})_T$. Now let $F \in \mathrm{Hom}_T^{D(M)_T \times_T D(M)_T}(A(N)_T \times_T A(L)_T, X_T)$, $F : A(N)_T \times_T A(L)_T \rightarrow X_T$. Then $F(g \star (a, b)) = F((a, b))$ for any $T' \rightarrow T$, $g \in D(N^{\mathrm{gp}} + L^{\mathrm{gp}})_T(T')$ and $(a, b) \in (A(N)_T \times_T A(L)_T)(T')$. So F induces a morphism f from the *fpqc* quotient, $f : (A(N)_T \times_T A(L)_T) / D(N^{\mathrm{gp}} + L^{\mathrm{gp}})_T \rightarrow X_T$. We have $A(N)_T \times_T A(L)_T = \mathrm{Spec}(\mathbb{Z}[N \times L]) \times_{\mathbb{Z}} T$ and the degree zero part of $\mathbb{Z}[N \times L]$ (relatively to the $N^{\mathrm{gp}} + L^{\mathrm{gp}}$ -grading induced by the action \star of $D(N^{\mathrm{gp}} + L^{\mathrm{gp}})$) is $\mathbb{Z}[N \cap L]$. So we have an identification $(A(N)_T \times_T A(L)_T) / D(N^{\mathrm{gp}} + L^{\mathrm{gp}})_T = A(N \cap L)_T$. So F induces a natural morphism $f : A(N \cap L)_T \rightarrow X_T$, that is $D(M)_T$ -equivariant. Now let $f \in X^{N \cap L}(T)$, $f : A(N \cap L)_T \rightarrow X_T$ and consider the composition

$$F : A(N)_T \times_T A(L)_T \rightarrow A(N \cap L)_T \times_T A(N \cap L)_T \rightarrow A(N \cap L)_T \xrightarrow{f} X_T$$

where the second morphism is the multiplication of $A(N \cap L)_T$. Then F is $D(M)_T \times_T D(M)_T$ -equivariant. The previous maps $F \mapsto f$ and $f \mapsto F$ induce a canonical bijection between

$$\mathrm{Hom}_T^{D(M)_T \times_T D(M)_T}(A(N)_T \times_T A(L)_T, X_T)$$

and $\mathrm{Hom}_T^{D(M)_T}(A(N \cap L)_T, X_T)$. \square

REMARK 3.17. If X is S -affine, Proposition 3.16 can be deduced easily from Theorem 5.1, cf. [Ma22].

Let $Z \subset M$ be a subgroup, taking fixed points under the closed subgroup scheme $D(M/Z)_S$ of $D(M)_S$ corresponds to taking the attractor associated to the monoid Z , as the following proposition shows. So, in this context, we view attractors as a formalism generalizing fixed points.

PROPOSITION 3.18. *Let $Z \subset M$ be a subgroup. Then the attractor space X^Z for the action of $D(M)_S$ on X is identified with the fixed points $X^{D(M/Z)_S}$ of X under the action of $D(M/Z)_S$.*

Proof. We have an exact sequence of abelian groups

$$0 \rightarrow Z \rightarrow M \rightarrow M/Z \rightarrow 0.$$

By [SGA3, Exp. VIII] we obtain an exact sequence of diagonalizable group schemes

$$1 \rightarrow D(M/Z)_S \rightarrow D(M)_S \rightarrow D(Z)_S \rightarrow 1.$$

Let $T \rightarrow S$ be a scheme and let us prove that $X^Z(T) = X^{D(M/Z)}(T)$. Note that we have a $D(M)_S$ -equivariant identification of S -scheme $A(Z)_S = D(Z)_S$. We have

$$\begin{aligned} X^Z(T) &= \mathrm{Hom}_T^{D(M)_T}(A(Z)_T, X_T) \\ &= \mathrm{Hom}_T^{D(M)_T}(D(Z)_T, X_T) \\ &= \mathrm{Hom}_T^{D(M)_T}(D(M)_T/D(M/Z)_T, X_T) \\ &= \mathrm{Hom}_T^{D(M)_T}(D(M)_T, (X_T)^{D(M/Z)_T}). \end{aligned}$$

Now Remark 3.4 finishes the proof. □

PROPOSITION 3.19. *Let S be a scheme. Let M be an abelian group and let X/S be a $D(M)_S$ -scheme. Let N, N', L and L' be submonoids in M such that $L \subset L'$, $N' \subset L'$, $N = L \cap N'$ and $L' = L + N'$. We assume that X^E is representable by a scheme for all $E \in \{N, N', L, L'\}$. We assume that L is a face of L' . Then N is a face of N' . Assume that one of the following conditions holds*

- (i) we have an equality $N' = L' \setminus (L \setminus N)$
- (ii) $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$ are affine, $A_l A_{n'} = A_{l+n'}$ for all $l \in L \setminus N$ and $n' \in N'$ (as usual A_m denote the m -graded part of A)

then the following diagram is a cartesian square in the category of schemes

$$\begin{array}{ccc} & X^{N'} & \\ \iota_{N',L'} \swarrow & & \searrow \mathfrak{p}_{N,N'} \\ X^{L'} & & X^N \\ \mathfrak{p}_{L,L'} \searrow & & \swarrow \iota_{N,L} \\ & X^L & \end{array}$$

Proof. We refer to [Ma22, § Cartesian squares]. □

We conclude this section with a useful elementary Proposition.

PROPOSITION 3.20. *Let X_1, X_2, X_3 be S -algebraic spaces endowed with actions of $D(M)_S$. Let $X_1 \rightarrow X_2$ and $X_3 \rightarrow X_2$ be two $D(M)_S$ -equivariant morphisms of S -algebraic spaces. Let $N \subset M$ be a monoid.*

- (i) We have a canonical action of $D(M)_S$ on $X_1 \times_{X_2} X_3$, the canonical maps $X_1 \times_{X_2} X_3 \rightarrow X_1$ and $X_1 \times_{X_2} X_3 \rightarrow X_3$ are $D(M)_S$ -equivariant and moreover the following is a cartesian square in the category of S -algebraic space endowed with action of $D(M)_S$ (morphisms in this category are $D(M)_S$ -equivariant morphisms of S -algebraic spaces)

$$\begin{array}{ccc} X_1 \times_{X_2} X_3 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_3 & \longrightarrow & X_2. \end{array}$$

- (ii) We have an isomorphism of functors

$$(X_1 \times_{X_2} X_3)^N = X_1^N \times_{X_2^N} X_3^N.$$

Proof. (i) Let T be a scheme. Then $D(M)(T)$ acts on $X_1(T) \times_{X_2(T)} X_3(T)$ via $g.(x_1, x_3) = (g.x_1, g.x_3)$ (recall that $(X_1 \times_{X_2} X_3)(T) = X_1(T) \times_{X_2(T)} X_3(T) = \{(x_1, x_3) \in X_1(T) \times X_3(T) \mid x_1 \equiv x_3 \text{ in } X_2(T)\}$). This proves the first assertion. The projection maps on X_1 and X_3 are $D(M)_S$ -equivariant by definition. Now let Y be a $D(M)_S$ -algebraic space. A $D(M)_S$ -equivariant morphism $Y \rightarrow X_1 \times_{X_2} X_3$ gives birth by composition with projections to canonical $D(M)_S$ -equivariant morphisms $Y \rightarrow X_1$ and $Y \rightarrow X_2$. Reciprocally let $Y \rightarrow X_1$ and $Y \rightarrow X_3$ be $D(M)_S$ -equivariant. Then we get a morphism in the category of S -algebraic spaces $Y \rightarrow X_1 \times_{X_2} X_3$. This morphism is $D(M)_S$ -equivariant by definitions.

- (ii) This is a direct consequence of the previous assertion. □

4. Attractors with prescribed limits

We introduce in this section an other functor. Let X be an algebraic space over a base scheme S . Let M be an abelian group and let $D(M)_S$ be the associated diagonalizable S -group scheme. Assume that $D(M)_S$ acts on X . Let F be a face of N . Let Z be an other S -algebraic space with a monomorphism $Z \rightarrow X^F$. We now introduce the attractor $X_{F,Z}^N$ associated to the monoid N under the action of $D(M)_S$ on X with prescribed limit in Z relatively to the face F . Since $Z(T) \subset X^F(T)$ for any S -scheme T , the following definition makes sense.

DEFINITION 4.1. Let $X_{F,Z}^N$ be the functor

$$(Sch/S) \rightarrow Sets, (T \rightarrow S) \mapsto \{f \in X^N(T) \mid \text{the image of } f \text{ in } X^F(T) \text{ belongs to } Z(T)\}.$$

If $F = N^*$, we omit F from the notation.

PROPOSITION 4.2. We have a canonical isomorphism $X_{F,Z}^N \simeq X^N \times_{X^F} Z$.

Proof. Clear since

$$\begin{aligned} X_{F,Z}^N(T) &= \{f \in X^N(T) \mid \text{the image of } f \text{ in } X^F(T) \text{ belongs in } Z(T)\} \\ &= \{(f, g) \in X^N(T) \times Z(T) \mid f = g \text{ in } X^F(T)\} \\ &= X^N(T) \times_{X^F(T)} Z(T). \end{aligned}$$

□

5. Attractors in the affine case

Let us explain that attractors are representable in the affine case and state some useful results about them. We follow [Ma22]. Let M be an abelian group and N be a submonoid of M . Let S be a scheme and X an S -affine scheme endowed with an action of $D(M)_S$. Let \mathcal{A} be the \mathcal{O}_S -quasi-coherent algebra of X . By [SGA3, SGA3.1 Corollaire 4.7.3.1], \mathcal{A} is M -graded. So we have a decomposition $\mathcal{A} = \bigoplus_{m \in M} \mathcal{A}_m$ with each \mathcal{A}_m quasi-coherent.

THEOREM 5.1. *The functor X^N is representable by a closed subscheme of X whose quasi-coherent ideal sheaf \mathcal{J} is the ideal sheaf generated locally by homogeneous element in \mathcal{A}_m for $m \in M \setminus N$.*

Proof. This is [Ma22, Theorem 5.1]. □

PROPOSITION 5.2. *If $N \subset L$ are submonoids in M , then $\iota : X^N \rightarrow X^L$ is a closed immersion.*

Proof. This is [Ma22, Proposition 5.3]. □

REMARK 5.3. Proposition 5.2 does not generalize outside the S -affine case.

PROPOSITION 5.4. [Ma22] *Let N, L be two submonoids of M . Then $X^{N \cap L} = X^N \cap X^L$. Here \cap means the scheme theoretic intersection as in [StP, Tag 0C4H].*

Proof. This is [Ma22, Proposition 5.4]. □

REMARK 5.5. Proposition 5.4 does not make sense outside the S -affine case in general. This is because X^N is not a closed subspace of X in general if X is not S -affine.

LEMMA 5.6. *Let S be a scheme and let $f : X \rightarrow Y$ be a $D(M)_S$ -equivariant closed immersion of S -affine $D(M)_S$ -schemes. Let N be a submonoid of M . Then the morphism $X^N \rightarrow Y^N$ is a closed immersion, and more precisely $X^N = X \times_Y Y^N$.*

Proof. This is [Ma22, Lemma 5.5] □

REMARK 5.7. Proposition 9.3 generalizes Lemma 5.6 outside the S -affine case. Note that the proof of Proposition 9.3 uses Lemma 5.6.

PROPOSITION 5.8. *Let \mathcal{E} be a quasi-coherent \mathcal{O}_S -module and let $\mathbb{V}(\mathcal{E}) = \text{Spec}_S(\text{Sym } \mathcal{E})$ be the associated quasi-coherent bundle defined by \mathcal{E} . Assume that $D(M)_S$ acts linearly on $\mathbb{V}(\mathcal{E})$. Let N be a submonoid of M . Then the attractor $(\mathbb{V}(\mathcal{E}))^N$ associated to N is canonically isomorphic to $\mathbb{V}(\mathcal{E}^N)$ where \mathcal{E}^N is the N -graded component of \mathcal{E} relatively to the $D(M)_S$ -action on \mathcal{E} .*

Proof. This is [Ma22, Proposition 5.7] □

6. Base change of affine étale morphisms along faces

Let S be a scheme and let X be an S -algebraic space with an action of $D(M)_S$ for some finitely generated abelian group M . Let N be a finitely generated submonoid of M .

PROPOSITION 6.1. *Let Q be a face of N and let $X^N \rightarrow X^Q$ be the associated morphism on attractors. Let U be an S -affine scheme. Let $U \rightarrow X$ be an étale, $D(M)_S$ -equivariant morphism of algebraic spaces. Then the natural map $U^N \rightarrow X^N \times_{X^Q} U^Q$ is an isomorphism.*

Proof. Firstly, we remark that it is enough to treat the case $Q = N^*$. Indeed assume the Proposition is true for the face of invertible elements. Since $Q^* = N^*$, we have

$$X^N \times_{X^Q} U^Q = X^N \times_{X^Q} (X^Q \times_{X^{N^*}} U^{N^*}) = X^N \times_{X^{N^*}} U^{N^*} = U^N.$$

So we assume $Q = N^*$. We now remark that we can assume $N^* = 0$ and that N is fine and sharp using the map $M \rightarrow M/N^*$ and Proposition 3.11. So we now assume that $Q = 0$ and N is fine and sharp. It is enough to construct the inverse morphism $X^N \times_{X^0} U^0 \rightarrow U^N$. For this let T be an S -scheme and let

$$\begin{array}{ccc} T = A(0)_T & \xrightarrow{f_0} & U_T \\ \downarrow & & \downarrow \\ A(N)_T & \longrightarrow & X_T \end{array}$$

be a diagram corresponding to a T -point of $X^N \times_{X^0} U^0$. We want to find a diagonal filling $A(N)_T \rightarrow U_T$. Let I be the kernel of $\mathcal{O}_T[N] \rightarrow \mathcal{O}_T[0]$, i.e the ideal associated to $N \setminus 0$ in $\mathcal{O}_T[N]$. Let $V(I^k) = \text{Spec} \mathcal{O}_T[N]/I^k$ be the infinitesimal neighbourhoods of $A(0)_T$ inside $A(N)_T$. Since $U_T \rightarrow X_T$ is étale, it is smooth by [StP, Tag 04XX], and formally smooth by [StP, Tag 02H6], so by the infinitesimal lifting property the morphism $f_0 : T \rightarrow U_T$ lifts uniquely to a compatible family of morphisms $f_k : V(I^k) \rightarrow U_T$. These liftings are equivariant because the two maps $D(M)_T \times V(I^k) \rightarrow U_T$, $(g, x) \mapsto f_k(gx)$ and $(g, x) \mapsto gf_k(x)$ are common liftings of the map $D(M)_T \times A(0)_T \rightarrow U_T$, $(g, x) \mapsto f_0(gx) = gf_0(x)$ hence by uniqueness they are equal. Writing $U = \text{Spec}(\mathcal{A})$ for a graded quasi-coherent \mathcal{O}_S -algebra \mathcal{A} , we have a family of M -graded morphisms $\mathcal{A}_T \rightarrow \mathcal{O}_T[N]/I^k$. This means that $(\mathcal{A}_T)_m$ goes to 0 when $m \notin N$ and to $(\mathcal{O}_T[N]/I^k)_m$ when $m \in N$. The induced morphism to the completion $\mathcal{A}_T \rightarrow \mathcal{O}_T[[N]]$ has image in $\mathcal{O}_T[N]$, yielding the desired lifting $A(N)_T \rightarrow U_T$ (cf. [Og, Chap. I, §3 Prop. 3.6.1] for the definition of the completion $\mathcal{O}_T[[N]]$). \square

7. Fixed-point reflecting atlases

We introduce and study fixed-point reflecting atlases in this section. We thank a lot Matthieu Romagny for determining remarks, help and leading ideas giving birth to this section. Let M be a finitely generated abelian group.

DEFINITION 7.1. Let S be a scheme. Let X be an S -algebraic space endowed with an action of $D(M)_S$.

- (i) Let $f : U \rightarrow X$ be a morphism of $D(M)_S$ -algebraic spaces. We say that f is fixed-point reflecting (FPR) if the canonical morphism of functors $U^{D(M)_S} \rightarrow U \times_X X^{D(M)_S}$ is an isomorphism.
- (ii) An FPR atlas for X under the action of $D(M)_S$ is a $D(M)_S$ -equivariant étale surjective morphism $f : U \rightarrow X$ such that $U = \coprod_{\tau \in \mathcal{A}} U_\tau$ is the disjoint union of S -affine schemes with $D(M)_S$ -actions and such that f is FPR.
- (iii) If Z is a subgroup of M , then a Z -FPR atlas for X under the action of $D(M)_S$ is an FPR atlas for the action of $D(M/Z)_S$ on X . In particular, by Proposition 3.18, a Z -FPR atlas for $U \rightarrow X$ satisfies $U^Z = X^Z \times_X U$.
- (iv) An FPR or Z -FPR atlas is finite if \mathcal{A} may be chosen finite in (ii).

REMARK 7.2. Since a finite disjoint union of S -affine schemes is S -affine, if an FPR atlas or a Z -FPR atlas is finite then we can assume that \mathcal{A} is a singleton. In other words a finite Z -FPR atlas for X is a $D(M)_S$ -equivariant étale surjective morphism $U \rightarrow S$ where U is an S -affine scheme such that $U^Z = X^Z \times_X U$.

LEMMA 7.3. *Let X, X' be S -algebraic spaces endowed with actions of $D(M)_S$. Let Z be a subgroup of M . Let $X' \rightarrow X$ be a $D(M)_S$ -equivariant affine morphism of S -algebraic spaces. Let $U \rightarrow X$ be a Z -FPR atlas of X . Then*

- (i) *The canonical morphism $U \times_X X' \rightarrow X'$ is a Z -FPR atlas of X' .*
- (ii) *If $U \rightarrow X$ is a finite Z -FPR atlas of X , then $U \times_X X' \rightarrow X'$ is a finite Z -FPR atlas of X' .*

Proof. (i) Since $X' \rightarrow X$ is an affine morphism, $U \times_X X' \rightarrow U$ is affine and so the composition $U \times_X X' \rightarrow S$ is a disjoint union of S -affine schemes. The morphism $U \times_X X' \rightarrow X'$ is étale and surjective because $U \rightarrow X$ is so. By Proposition 3.20, we have canonical identifications

$$(X' \times_X U)^Z = X'^Z \times_{X^Z} U^Z = X'^Z \times_{X^Z} (U \times_X X^Z) = X'^Z \times_X U = X'^Z \times_{X'} (X' \times_X U).$$

Assertion (ii) is now immediate. \square

Because we will assume that X is locally of finite presentation, we will be able to construct FPR atlases. In order to build FPR atlases we rely on a result of Alper, Hall and Rydh [AHR21] together with some results on actions of finite locally free group schemes which are the subject of the following two lemmas.

LEMMA 7.4. *Let S be a scheme. Let X be an S -algebraic space with an action of a finite locally free S -group scheme Ω and let $x \in X$ be a point fixed by Ω . Then any étale neighbourhood $(U, u) \rightarrow (X, x)$ has an Ω -equivariant étale refinement $(V, v) \rightarrow (U, u) \rightarrow (X, x)$. Moreover if S is separated, one may choose for V an S -affine scheme.*

For basics on points of algebraic spaces, we refer the reader to [StP, Tag 03BT].

Proof. Write $f : U \rightarrow X$ for the given morphism. Let V be the Weil restriction of the étale map $\text{id} \times f : \Omega \times_S U \rightarrow \Omega \times_S X$ along the action morphism $\Omega \times_S X \rightarrow X$:

$$V := \text{Res}_{\Omega \times_S X/X}(\Omega \times_S U).$$

This is representable by an étale X -algebraic space, see e.g. [Ry11, Proposition 3.5]. The points of V over some $a : T \rightarrow X$ are the morphisms $\alpha : \Omega \times_S T \rightarrow U$ that lift the map $\Omega \times_S T \rightarrow X$, $(g, t) \mapsto g^{-1}a(t)$. Otherwise said, as an S -functor, its T -points are the pairs (a, α) with $a : T \rightarrow X$ and α a lifting as above. The space V is endowed with an action of Ω defined by $h \cdot \alpha : \Omega \times_S T \rightarrow U$, $(g, t) \mapsto \alpha(h^{-1}g, t)$ for all sections $h \in \Omega$, $\alpha \in V$. Moreover, there is a morphism $V \rightarrow U$ defined by $\alpha \mapsto \alpha|_{\{1\} \times T}$ and the composition $V \rightarrow U \rightarrow X$ is Ω -equivariant. To make V pointed we choose representatives of x and u defined over a field K :

$$\begin{array}{ccc} & & U \\ & \nearrow u & \downarrow f \\ \text{Spec}(K) & \xrightarrow{x} & X \end{array}$$

Since x is fixed by Ω , the composition

$$\Omega \times \text{Spec}(K) \xrightarrow{\text{pr}_2} \text{Spec}(K) \xrightarrow{u} U$$

defines a point $v \in V$. Obviously v lifts u and this finishes the construction of an étale, equivariant neighbourhood $(V, v) \rightarrow (U, u) \rightarrow (X, x)$. Finally, since S is separated, shrinking the initial scheme U if necessary we may assume that it is S -affine, in which case the Weil restriction V is S -affine, see [Ry11, Proposition 3.5]. \square

A consequence of Lemma 7.4 is that equivariant étale (affine) neighbourhoods of $x \in X$ form a cofinal system among all étale (affine) neighbourhoods. In particular, the étale local ring

$$\mathcal{O}_{X,x}^{\text{ét}} = \text{colim}_{(U,u) \rightarrow (X,x)} \mathcal{O}_U(U)$$

can be computed using only equivariant affine pointed neighbourhoods. Thus the local scheme $X_x = \text{Spec}(\mathcal{O}_{X,x}^{\text{ét}})$, called the *local space of X at x* , is canonically endowed with an Ω -action.

We will now take a look at the local space of the Ω -fixed points X^Ω . That X^Ω is an algebraic space follows from general results on Weil restrictions as in [Ry11], because if $\varphi : \Omega \times_S X \rightarrow X \times_S X$ is the map given by the action and the second projection, then X^Ω can be seen as the Weil restriction of $\varphi^{-1}(\Delta_X) \rightarrow \Omega \times_S X$ along the projection $\text{pr}_2 : \Omega \times_S X \rightarrow X$. Thus $X^\Omega \rightarrow X$ is a closed immersion, or an immersion, or etc, if X is separated, or locally separated, or satisfies other separation axioms (as in [StP, Tag 03BR]).

LEMMA 7.5. *Let S be a separated scheme. Let X be an S -algebraic space with an action of a finite locally free S -group scheme Ω , and $x \in X$ a point fixed by Ω . Then Ω acts on the local scheme X_x and there are isomorphisms*

$$(X_x)^\Omega \xrightarrow[\text{(i)}]{\simeq} (X^\Omega)_x \xrightarrow[\text{(ii)}]{\simeq} \lim U^\Omega,$$

the limit being indexed by equivariant affine étale neighbourhoods $(U, u) \rightarrow (X, x)$.

Proof. The fact that Ω acts on X_x was proven before the lemma. To prove the first isomorphism, observe that like any closed subscheme of a henselian local scheme, $(X_x)^\Omega$ is a henselian local scheme. Once this is noticed, it is easy to check that $(X_x)^\Omega \rightarrow X^\Omega$ is universal among morphisms from a henselian local scheme to X^Ω , that is, it satisfies the universal property of $(X^\Omega)_x \rightarrow X^\Omega$. To prove the second isomorphism, observe that $(X_x)^\Omega$ satisfies the universal property of the limit of the system $\{U^\Omega\}_{(U,u) \rightarrow (X,x)}$ in the category of schemes. \square

Recall that M is a finitely generated abelian group.

PROPOSITION 7.6. *Let S be an arbitrary scheme. Let X be a quasi-separated S -algebraic space locally of finite presentation endowed with an action of $G = D(M)_S$. Let $Z \subset M$ be a subgroup. The following assertions hold.*

- (i) *There exists a Z -FPR atlas $U \rightarrow X$.*
- (ii) *If X/S is quasi-compact, there exists a finite Z -FPR atlas $U \rightarrow X$.*

Proof. (i) Since the assumptions and conclusion are local over S , we may assume that S is affine. Because $G \rightarrow S$ is flat and finitely presented, the orbit in X of an open quasi-compact subspace $W \subset X$ is open and quasi-compact. Since the result to be proved is local over X , we may replace X by one such orbit and hence assume that X is of finite presentation. By standard results on limits (as in [StP, Tag 07SJ]), the space X and the G -action then come from a finitely presented algebraic space $X_0 \rightarrow S_0$ with G -action by a base change $S \rightarrow S_0$ where S_0 is of finite type over \mathbb{Z} . Changing notation, we may therefore assume that S and X are of finite type over \mathbb{Z} . In particular, the set of closed points $|X^{\text{cl}}| \subset |X|$ is dense. It will thus be enough to find an affine, G -equivariant étale neighbourhood $U(x) \rightarrow X$ of each closed point $x \in |X^{\text{cl}}|$ and to eventually consider $U := \coprod_{x \in |X^{\text{cl}}|} U(x)$ (or in fact, a finite subcover, by quasi-compactity).

For such a closed point x with image $s \in S$, the residue field extension $\kappa(x)/\kappa(s)$ is finite. By [AHR21, Corollary 20.2], there exists an affine pointed scheme $(U = \text{Spec}(A), u)$ and

an étale, G -equivariant morphism $(U, u) \rightarrow (X, x)$ which induces an isomorphism of residue fields $\kappa(x) \simeq \kappa(u)$ and an isomorphism of stabilizers $G_u \simeq G_x$. Hence since x is fixed by $D(M/Z)_S$, then so is u . Write $D(M/Z) = \mathbf{T} \times \Omega$ as the product of a split torus with a finite diagonalizable group scheme. We have an isomorphism $X^H = (X^{\mathbf{T}})^\Omega$. We need the following Lemma

LEMMA 7.7. *Let $\mathbf{T} \cong D(\mathbb{Z}^r)_S$ be a split torus over S and let $U \rightarrow X$ be an étale morphism, then $U^{\mathbf{T}} \rightarrow X^{\mathbf{T}} \times_X U$ is an isomorphism.*

Proof. The proof of [Ri16, Lemma 1.10] works replacing " \mathbb{G}_m " by " \mathbf{T} ". □

According to 7.7, the morphism

$$U^{\mathbf{T}} \rightarrow X^{\mathbf{T}} \times_X U$$

is an isomorphism. Replacing X, U, H by $X^{\mathbf{T}}, U^{\mathbf{T}}, \Omega$ we reduce to the case where $H = \Omega$ is finite locally free. Since $(U, u) \rightarrow (X, x)$ is étale with trivial residue field extension, the map of local spaces $U_u \rightarrow X_x$ is an equivariant isomorphism. It follows from Lemma 7.5(i) that the closed immersion $U^\Omega \rightarrow X^\Omega \times_X U$ is an isomorphism $(U^\Omega)_u \xrightarrow{\simeq} (X^\Omega)_x$ on the germs. By 7.5(ii) we know that there is an étale equivariant refinement $(U', u') \rightarrow (U, u) \rightarrow (X, x)$ such that the previous isomorphism extends to an isomorphism $U'^\Omega \xrightarrow{\simeq} X^\Omega \times_X U'$. Choosing $U(x) = U'$ provides the sought-for atlas.

- (ii) If X/S is quasi-compact, we extract a finite subcovering of a covering obtained with the first assertion. □

COROLLARY 7.8. *Let X be a quasi-separated S -algebraic space locally of finite presentation endowed with an action of $D(M)_S$. Let $Z \subset M$ be a subgroup. Then the attractor space X^Z (which identifies with the fixed space $X^{D(M/Z)_S}$ by Proposition 3.18) is representable by a closed subspace of X .*

Proof. Let U be a Z -FPR atlas as in Proposition 7.6, in particular we have a cartesian square

$$\begin{array}{ccc} U^Z & \longrightarrow & X^Z \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array} .$$

By Proposition 5.2 the left vertical arrow is a closed immersion of schemes. The lower arrow is surjective étale. So by [StP, Tag 03I2], X^Z is an algebraic space and the right vertical arrow is a closed immersion. We used that being a closed immersion is stable under base change, fppf-local on the base and closed immersions satisfy fppf-descent. □

8. Representability and properties

Let M be a finitely generated abelian group. Let S be an arbitrary scheme. Let X be a quasi-separated S -algebraic space locally of finite presentation endowed with an action of $D(M)_S$.

THEOREM 8.1. *Let $N \subset M$ be an arbitrary submonoid. The following assertions hold.*

- (i) *Assume that N is finitely generated. Then the attractor X^N is representable by an algebraic space over S .*

(ii) Assume that $X \rightarrow S$ is quasi-compact and separated. Then there exists a finitely generated submonoid N_c of N such that $X^N = X^{N_c}$ and $N^* = N_c^*$.

Proof. Assume first that N is finitely generated. Choosing $U \rightarrow X$ as provided by Proposition 7.6 (with $Z = N^*$) and using Proposition 6.1, we obtain a diagram with cartesian squares:

$$\begin{array}{ccccc} U^N & \longrightarrow & U^{N^*} & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ X^N & \longrightarrow & X^{N^*} & \longrightarrow & X. \end{array}$$

The vertical maps are étale and surjective, and we know from Theorem 5.1 that U^N is representable by a disjoint sum of S -affine schemes. By Proposition 7.8, X^{N^*} is an S -algebraic space. It follows from [StP, Tag 03I2] that X^N is representable.

Now let us assume that N is arbitrary and that X is separated and quasi-compact. It is enough to find a finitely generated monoid $N_c \subset N$ such that $X^N = X^{N_c}$ and $N^* = N_c^*$. Let $\kappa : \mathbb{N} \rightarrow N$ be a bijection. For any $i \in \mathbb{N}$, let N_i be the submonoid of N generated by $\{N^* \cup \kappa(\mathbb{N}_{\leq i})\}$. Then $N_i^* = N^*$ and N_i is finitely generated for any $i \in \mathbb{N}$, moreover $\bigcup_{i \geq 0} N_i = N$. For any scheme T , we have $A(N)_T = \lim A(N_i)_T$ by [StP, Tag 01YW]. So $X(A(N)_T) = \operatorname{colim} X(A(N_i)_T)$ by [StP, Tag 049I]. Let $A(N)_T \rightarrow X_T$ be a $D(M)_T$ -equivariant morphism, by the previous assertion this factorizes through a morphism $A(N_i)_T \xrightarrow{f} X_T$. We want to show that f is $D(M)_T$ -equivariant. Consider the diagram

$$\begin{array}{ccc} D(M)_T \times A(N_i)_T & \rightrightarrows & X_T \\ \uparrow & \searrow & \\ D(M)_T \times A(N)_T & & \end{array}$$

where the horizontal arrows (f_1, f_2) correspond to $(g, x) \mapsto f(gx)$ and $(g, x) \mapsto gf(x)$. Since X is separated, the kernel of (f_1, f_2) is a closed subscheme of $D(M)_T \times A(N_i)_T$ (the proof of [GW, Def./Prop. 9.7] works in this context). Since the schematic image of the vertical morphism ϕ is $D(M)_T \times_T A(N_i)_T$ and because $f_1 \circ \phi = f_2 \circ \phi$, we have an equality of schemes $\ker(f_1, f_2) = D(M)_T \times_T A(N_i)_T$. So we have an equality of morphisms of schemes $f_1 = f_2$, and so f is equivariant. We deduce that

$$X^N(T) = \operatorname{colim} X^{N_i}(T). \quad (8.1)$$

Let $U \rightarrow X$ be a finite N^* -FPR atlas as in Proposition 7.6. Using that U is an S -affine scheme of finite presentation over S , there exists an integer c such that $U^{N_c} = U^{N_i}$ for all $i \geq c$.

Let $i \geq c$, and consider the diagram

$$\begin{array}{ccc} U^{N_c} & \longrightarrow & X^{N_c} \\ \downarrow & & \downarrow \\ U^{N_i} & \longrightarrow & X^{N_i} \end{array}$$

The horizontal arrows are étale and surjective. The left vertical arrow is an isomorphism since $i \geq c$. Moreover the diagram is a cartesian square because by Proposition 6.1

$$\begin{aligned}
U^{N_i} \times_{X^{N_i}} X^{N_c} &\simeq (U^{N^*} \times_{X^{N^*}} X^{N_i}) \times_{X^{N_i}} X^{N_c} \\
&\simeq U^{N^*} \times_{X^{N^*}} X^{N_c} \\
&\simeq U^{N_c}.
\end{aligned}$$

Using [StP, Tag 03I2] for the property "isomorphism", we deduce that for all $i \geq c$

$$X^{N_i} = X^{N_c}. \quad (8.2)$$

Now 8.1 and 8.2 imply that $X^N = X^{N_c}$. \square

PROPOSITION 8.2. *Let $N \subset M$ be a finitely generated submonoid. Recall that X is a quasi-separated and locally finitely presented algebraic space over S . The following assertions hold.*

- (i) *The morphism of algebraic spaces $X^N \rightarrow X^{N^*}$ is affine.*
- (ii) *The attractor X^N/S is quasi-separated and locally of finite presentation.*
- (iii) *If X/S is separated, then X^N/S is separated.*
- (iv) *If X/S is quasi-compact then X^N/S is quasi-compact.*
- (v) *If X/S is of finite presentation, then X^N/S is of finite presentation.*
- (vi) *If X is a scheme then X^N is a scheme.*

Proof. Let U be a N^* -FPR atlas as in Proposition 7.6. We have a cartesian square

$$\begin{array}{ccc}
U^N & \longrightarrow & X^N \\
\downarrow & & \downarrow \\
U^{N^*} & \longrightarrow & X^{N^*}
\end{array}$$

The lower horizontal arrow is étale and surjective. The left vertical arrow is affine because U is S -affine and U^N and U^{N^*} are S -affine schemes by Theorem 5.1. Now, we apply [StP, Tag 03I2] to conclude that the right vertical arrow is affine. We used that being affine is preserved under base change, fppf local on the base and satisfy descent for fppf coverings.

Since, as functor, X commutes with colimits of affine schemes in $(\text{Sch}/S)_{\text{fppf}}$, the functor X^N commutes with colimits of affine schemes in $(\text{Sch}/S)_{\text{fppf}}$, i.e. X^N is locally of finite presentation.

Assume that X is separated (resp. quasi-separated, quasi-compact, resp. is a scheme). Then X^{N^*} is separated (resp. quasi-separated, quasi-compact, resp. is a scheme) because it is closed in X . Since $X^N \rightarrow X^{N^*}$ is affine, in particular representable, quasi-compact and separated it follows that X^N is separated (resp. quasi-compact, resp. is a scheme). Note that by definition X is of finite presentation if it is locally of finite presentation, quasi-compact and quasi-separated. \square

PROPOSITION 8.3. *Let $N \subset M$ be a finitely generated submonoid. Recall that X is a quasi-separated and locally finitely presented over S . Assume $X \rightarrow S$ is smooth. Then $X^N \rightarrow S$ is smooth moreover $X^N \rightarrow X^{N^*}$ is smooth.*

Proof. Using [SGA3, Exp I Th 5.3.3], one adapts Drinfeld's argument [Dr13, Proof of Proposition 1.4.20] (cf. also [SGA3, Exp XII §9]). \square

REMARK 8.4. We expect that if $X \rightarrow Y$ is a smooth $D(M)_S$ -equivariant morphism of S -algebraic spaces, then $X^N \rightarrow Y^N$ is also smooth. We will study this soon.

REMARK 8.5. If $X \rightarrow S$ is flat, then $X^N \rightarrow S$ is not flat in general.

REMARK 8.6. Let X be an S -algebraic space with an action of $D(M)_S$. Let N be a submonoid of the group M . Then in some cases the attractor space X^N can be obtained as a succession of attractors under \mathbb{G}_m (cf. the introduction and [Ri16]) and fixed points. Let us give two examples.

- (i) Let $\alpha \in M$ and let us consider $X^{[\alpha]}$, then by Proposition 3.16 we have $X^{[\alpha]} = (X^{(\alpha)})^{[\alpha]}$. By Proposition 3.18 $X^{(\alpha)}$ identifies with the fixed point space $X^{D(M/(\alpha))_S}$. So the operation $X \rightsquigarrow X^{(\alpha)}$ can be realized as taking fixed points. By Remarks 3.7 and 3.6, $D((\alpha))_S$ and $D(M)_S$ act on $X^{(\alpha)}$, and using Proposition 3.11, we have $(X^{(\alpha)})^{[\alpha] \subset (\alpha)} = (X^{(\alpha)})^{[\alpha] \subset M}$ (cf. Remark 3.3 for the notation $X^{N \subset M}$). But $([\alpha] \subset (\alpha)) \simeq (\mathbb{N} \simeq \mathbb{Z})$, so the operation $X^{(\alpha)} \rightsquigarrow (X^{(\alpha)})^{[\alpha]}$ can be realized as taking attractor under $\mathbb{G}_m = D(\mathbb{Z})_S$. So $X \rightsquigarrow X^{[\alpha]}$ can be realized as fixed points followed by taking an attractor under \mathbb{G}_m .
- (ii) Assume that $M = \mathbb{Z} \times \mathbb{Z}$ and that $N = \mathbb{N} \times \mathbb{N}$. Using Proposition 3.3 and Proposition 3.11, we have

$$X^N = (X^{\mathbb{N} \times \mathbb{Z} \subset \mathbb{Z} \times \mathbb{Z}})^{\mathbb{Z} \times \mathbb{N} \subset \mathbb{Z} \times \mathbb{Z}} = (X^{\mathbb{N} \times 0 \subset \mathbb{Z} \times 0})^{0 \times \mathbb{N} \subset 0 \times \mathbb{Z}}.$$

This shows that $X \rightsquigarrow X^N$ can be realized as two stages of \mathbb{G}_m -attractors.

- (iii) If $M = \mathbb{Z}^r$ and $N = \mathbb{N}^r$ then $X \rightsquigarrow X^N$ can be realized as r stages of \mathbb{G}_m -attractors.

REMARK 8.7. (**Monoschemes and toric schemes**) Let M be an abelian group and let N, L be submonoids of M . Let us consider the attractors $A(N)^L$ associated to the monoid L under the action of $D(M)_S$ on $A(N)$. By Theorem 5.1 $A(N)^L$ equals $\text{Spec}(\mathbb{Z}[N]/(X^n | n \in N \setminus (N \cap L)))$. The ideal $\mathcal{I} := (X^n | n \in N \setminus (N \cap L))$ of $\mathbb{Z}[N]$ equals $\bigoplus_{i \in I} \mathbb{Z}X^i$ where I is the ideal of N generated by $N \setminus (N \cap L)$. If I is a prime ideal of N (cf. [Og]) then $N \setminus I$ is a submonoid of N (and necessarily a face) and $\mathbb{Z}[N]/\mathcal{I} = \mathbb{Z}[N \setminus I]$. In this case, $A(N)^L$ is also a scheme associated to a submonoid of M . For example if $N \cap L$ is a face of N , then $N \setminus I = N \cap L$ and $A(N)^L = A(N \cap L)$. In general I is not a prime ideal and so $N \setminus I$ is not a submonoid of N (e.g. take $M = \mathbb{Z}^2$, $N = [(1, 1), (1, -1), (1, 0)]$ and $L = [(1, 0)]$, then $N \setminus I = \{(0, 0), (1, 0)\}$).

More generally, let \mathcal{N} be a toric monoscheme whose associated finitely generated abelian group Γ is M (cf. [Og, II §1.9]). Let $A(\mathcal{N})$ be the scheme associated to \mathcal{N} (cf. [Og, II Prop. 1.9.1]), this is a toric scheme. Let $\{\text{spec}(N_\tau)\}_{\tau \in \mathcal{A}}$ be an open affine covering of \mathcal{N} ($N_\tau \subset M$ for all τ). Then $\{A(N_\tau)\}_{\tau \in \mathcal{A}}$ is an open affine covering of $A(\mathcal{N})$. Then $\coprod_{\tau} A(N_\tau) \rightarrow A(\mathcal{N})$ gives an FPR atlas of $A(\mathcal{N})$ (note that a $D(M)$ -equivariant open immersion is Z -FPR for any subgroup $Z \subset M$). Let L be a submonoid of M , then we obtain that $\{A(N_\tau)^L\}$ is an affine open covering of $A(\mathcal{N})^L$.

9. Topology of attractors

Let M be a finitely generated abelian group. Let X be a quasi-separated S -algebraic space locally of finite presentation endowed with an action of $D(M)_S$.

PROPOSITION 9.1. *Let $N \subset M$ be a submonoid. Assume that N is finitely generated. Then the map $X^N \rightarrow X^{N^*}$ (which is affine by Proposition 8.2) has geometrically connected fibers and induces a bijection on the sets of connected components $\pi_0(X^N) \simeq \pi_0(X^{N^*})$ of the underlying spaces.*

Proof. Using Proposition 3.11 we can assume that $N^* = 0$ and that N is fine and sharp. Let K be a field, and let $x : \text{Spec}(K) \rightarrow X^0$ be a point. Let $X_x^N = X^N \times_{X^0, x} \text{Spec}(K)$. We claim that its underlying topological space $|X_x^N|$ is connected. Let L be a field and let $y : \text{Spec}(L) \rightarrow X_x^N$

be a point, and denote by x_L the composition $\text{Spec}(L) \rightarrow \text{Spec}(K) \xrightarrow{x} X^0$. Then x_L and x define the same point of $|X_x^N|$. Recall that we have a natural action of the monoid scheme $A(N)_S$ on the attractor X^N . The $A(N)_L$ orbit of y defines a map $h : A(N)_L \rightarrow X_x^N$ with $h(1) = y$ and $h(0) = x_L$. Since N is sharp, $L[N]$ is integral. So $A(N)_L$ is connected. So x and y lie in the connected set $|h|(|A(N)_L|)$. Since y was arbitrary, this shows that $|X_x^N|$ is connected. So the continuous map $|X^N| \rightarrow |X^0|$ has connected fibers, and the assertion on connected components follows from the existence of a continuous section $|X^0| \subset |X^N|$. \square

PROPOSITION 9.2. *Let N be a finitely generated submonoid of M . Then the canonical morphism $X^{N^*} \rightarrow X^N$ is a closed immersion.*

Proof. By Proposition 3.16, we have $X^{N^*} = (X^N)^{N^*}$, now Corollary 7.8 finishes the proof. \square

PROPOSITION 9.3. *Let $N \subset M$ be a finitely generated submonoid. Let $f : X \rightarrow Y$ be a $D(M)$ -equivariant morphism of quasi-separated S -algebraic spaces locally of finite presentations. Let $f^N : X^N \rightarrow Y^N$ be the canonical morphism obtained on attractors. The following assertions hold.*

- (i) *If f is an open immersion, then f^N is an open immersion.*
- (ii) *If f is a closed immersion, then f^N is a closed immersion, moreover $X^N \cong Y^N \times_Y X$.*

Proof. Assume f is an open immersion. Recall that $X^{N^*} = X^{D(M/N^*)_S}$. Since f is an open immersion, we have $X^{N^*} = X \times_Y Y^{N^*}$, so $X^{N^*} \rightarrow Y^{N^*}$ is an open immersion. Now by Proposition 6.1, we get $X^N = X^{N^*} \times_{Y^{N^*}} Y^N$, and so $f^N : X^N \rightarrow Y^N$ is an open immersion.

Assume f is a closed immersion. The canonical morphisms $X^N \rightarrow Y^N$ and $X^N \rightarrow X$ induces a canonical morphism $i : X^N \rightarrow Y^N \times_Y X$. We are going to prove that i is an isomorphism. Let U be a N^* -FPR atlas of Y . By Proposition 7.3, the map $U \times_Y X \rightarrow X$ is a N^* -FPR atlas of X . So by Proposition 6.1 the canonical map $(U \times_Y X)^N \rightarrow X^N$ is étale and surjective. The canonical map $U^N \rightarrow Y^N$ (étale and surjective) induces an étale and surjective map $(U \times_Y X) \times_U U^N = X \times_Y U^N \rightarrow X \times_Y Y^N$. So we get a diagram

$$\begin{array}{ccc} (U \times_Y X)^N & \longrightarrow & X^N \\ \downarrow & & \downarrow \\ (U \times_Y X) \times_U U^N = X \times_Y U^N & \longrightarrow & X \times_Y Y^N \end{array}$$

The left arrow is an isomorphism by Lemma 5.6. The horizontal arrows are étale and surjective. The diagram is a cartesian square because $(X \times_Y U^N) \times_{X \times_Y Y^N} X^N = U^N \times_{Y^N} X^N = (U \times_Y X)^N$ by Lemma 3.20. So by [StP, Tag 03I2] we obtain $X^N \cong X \times_Y Y^N$. \square

10. Tangent spaces and attractors

Let S be a scheme we denote by I_S the scheme of dual numbers over S as in [SGA3, Exp. II Définition 2.1]. For any scheme T over S , we have $I_T = I_S \times_S T$. Explicitly, $I_S = \text{Spec}(\mathbb{Z}[\varepsilon]) \times_{\mathbb{Z}} \mathbb{Z}$ where $\mathbb{Z}[\varepsilon] \stackrel{\varepsilon=[X]}{=} \mathbb{Z}[X]/(X^2)$. Let X be a scheme over S . Let T_X be the tangent space of X as in [SGA3, Exp. II Définition 3.1]. This is a scheme over S representing the functor from Sch_S to Sets sending a scheme R over S to

$$\text{Hom}_R(I_R, X_R).$$

REMARK 10.1. (i) For S -morphisms $X \rightarrow Y$ and $Z \rightarrow Y$, we have $T_{X \times_Y Z} \cong T_X \times_{T_Y} T_Z$.

- (ii) If $X \rightarrow Y$ is a monomorphism, then $T_X \rightarrow T_Y$ is a monomorphism.
- (iii) We have a canonical identification $T_S = S$.

Let M be an abelian group and assume that $D(M)_S$ acts on X . Then $D(M)_S$ acts naturally on T_X using the definition of T_X . Let N be a submonoid in M .

PROPOSITION 10.2. *We have a canonical isomorphism*

$$(T_X)^N \cong T_{X^N}.$$

Proof. It is enough to show that $(T_X)^N(R) \cong T_{X^N}(R)$ for any S -scheme R . We can change notation and assume $R = S$. We have

$$\begin{aligned} (T_X)^N(S) &= \text{Hom}_S^{D(M)_S}(A(N)_S, T_X) \\ &= \text{Hom}_S^{D(M)_S}(A(N)_S \times_S I_S, X) \text{ where } D(M)_S \text{ acts trivially on } I_S \end{aligned}$$

$$\begin{aligned} (T_{X^N})(S) &= \text{Hom}_S(I_S, X^N) \\ &= \text{Hom}_{I_S}^{D(M)_S \times_S I_S}(A(N)_S \times_S I_S, X \times_S I_S). \end{aligned}$$

This shows that we have a canonical isomorphism. □

11. Lie algebras and attractors

Let G/S be a group scheme over a scheme S . Recall that in this case T_G is a group scheme over S and we have two canonical morphisms of group schemes $G \rightarrow T_G$ and $T_G \rightarrow G$ by [SGA3, Exp. II]. Let e_S be the trivial group over S , as S -scheme we have $e_S = S$. The Lie algebra of G is defined as the fiber product

$$\begin{array}{ccc} \text{Lie}(G) = e_S \times_G T_G & \longrightarrow & T_G \\ \downarrow & & \downarrow \\ e_S & \longrightarrow & G \end{array}$$

where $e_S \rightarrow G$ is the canonical morphism of group schemes from e_S to G . As in the previous section, let M be an abelian group and assume that $D(M)_S$ acts on G . We assume moreover that this action is compatible with the group structure on G , i.e $D(M)_S$ acts by automorphisms on G . Then the induced action of $D(M)_S$ on T_G is by group automorphisms. We thus obtain an action of $D(M)_S$ on $\text{Lie}(G)$ by group automorphisms.

REMARK 11.1. (i) For S -group schemes G, K, H with morphisms $G \rightarrow K, H \rightarrow K$, we have a canonical isomorphism $\text{Lie}(G \times_K H) \cong \text{Lie}(G) \times_{\text{Lie}(K)} \text{Lie}(H)$.

(ii) If $G \rightarrow H$ is a monomorphism, then $\text{Lie}(G) \rightarrow \text{Lie}(H)$ is a monomorphism.

(iii) We have a canonical identification $\text{Lie}(e_S) = e_S$.

PROPOSITION 11.2. *We have a canonical isomorphism of group schemes over S*

$$(\text{Lie}(G))^N \cong \text{Lie}(G^N)$$

Proof. Using Proposition 3.20 and Proposition 10.2, we have

$$(\text{Lie}(G))^N = (S \times_G T_G)^N \cong S^N \times_{G^N} T_G^N \cong S \times_{G^N} T_{G^N} = \text{Lie}(G^N).$$

□

Let us fix now a group scheme H over S and a monomorphism $H \rightarrow G^{N^*}$ of schemes preserving the group scheme structures. Recall that G_H^N is the attractor with prescribed limit H as in Definition 4.1.

PROPOSITION 11.3. *We have a canonical isomorphism of group schemes over S*

$$(\mathrm{Lie}(G))_{\mathrm{Lie}(H)}^N \cong \mathrm{Lie}(G_H^N).$$

Proof. Using Proposition 11.2 and Remark 11.1 we have

$$\begin{aligned} \mathrm{Lie}(G)_{\mathrm{Lie}(H)}^N &= \mathrm{Lie}(G)^N \times_{\mathrm{Lie}(G)^{N^*}} \mathrm{Lie}(H) \\ &\cong \mathrm{Lie}(G^N) \times_{\mathrm{Lie}(G^{N^*})} \mathrm{Lie}(H) \\ &\cong \mathrm{Lie}(G^N \times_{G^{N^*}} H) \\ &= \mathrm{Lie}(G_H^N) \end{aligned}$$

This shows that we have the desired canonical isomorphism. \square

12. Ind-algebraic spaces

Let S be a scheme and let AffSch_S be the category of affine schemes over S . Its objects are morphisms $\mathrm{Spec}(R) \rightarrow S$ from affine schemes to S . We use [HR21] for the definition of ind-algebraic spaces.

DEFINITION 12.1. An ind-algebraic space (resp. ind-scheme) over S is a functor $\mathrm{AffSch}_S \rightarrow \mathit{Sets}$ which admits a presentation $X \cong \mathrm{colim}_{i \in I} X_i$ as a filtered colimit of S -algebraic spaces where (resp. S -schemes) all transition maps $\phi_{ij} : X_i \rightarrow X_j$, $i \leq j$ are closed immersions. The category of ind-algebraic spaces (resp. ind-schemes) over S is the full subcategory of functors $\mathrm{AffSch}_S \rightarrow \mathit{Sets}$ whose objects are ind-algebraic spaces (resp. ind-schemes) over S .

Any algebraic space (resp. scheme) over S , is naturally an ind-algebraic space (resp an ind-scheme) over S . Any ind-scheme over S is naturally an ind-algebraic space over S .

REMARK 12.2. [HR21, §1.5] If $X = \mathrm{colim}_i X_i$ and $Y = \mathrm{colim}_j Y_j$ are presentations of ind-algebraic spaces (resp. ind-schemes) over S , and if each X_i is quasi-compact, then as sets $\mathrm{Hom}(X, Y) = \lim_i \mathrm{colim}_j \mathrm{Hom}(X_i, Y_j)$, because every map $X_i \rightarrow Y$ factors over some Y_j by quasi-compactness of X_i . The category ind-algebraic spaces and ind-schemes are closed under fiber product. If \mathbf{P} is a property of algebraic spaces (resp. schemes), then an S -ind-algebraic space (resp. S -ind-scheme) X is said to have ind- \mathbf{P} if there exists a presentation $X = \mathrm{colim}_i X_i$ where each X_i has property \mathbf{P} . A map $f : X \rightarrow Y$ of S -ind-algebraic spaces (resp. S -ind-schemes) is said to have property \mathbf{P} if f is representable and for all schemes $T \rightarrow Y$, the pullback $f \times_Y T$ has property \mathbf{P} . Note that every representable quasi-compact map of S -ind-schemes is schematic.

DEFINITION 12.3. Let G be a group scheme over S and let X be an ind-algebraic space over S .

- (i) A categorical action of G on X is an action of G on X seen as a functor. Equivalently by Yoneda, a categorical action is a morphism in the category of ind-algebraic spaces over S $\sigma : G \times_S X \rightarrow X$ satisfying the usual axioms, i.e.
 - (a) $\sigma \circ (\mathrm{Id}_G \times \sigma) = \sigma \circ (m \times \mathrm{Id}_X)$ where $m : G \times_S G \rightarrow G$ is the group law of G ,
 - (b) $\sigma \circ (e \times \mathrm{Id}_X) = \mathrm{Id}_X$ where $e : S \rightarrow G$ is the identity section of G .

- (ii) A collection of actions $\sigma_i : G \times_S X_i \rightarrow X_i$ such that for $i \leq j$ we have $\phi_{ij} \circ \sigma_i = \sigma_j \circ (\text{Id}_G \times \phi_{ij})$ gives birth to a categorical action of G on X . Such categorical actions are called ind-actions.

Let M be a finitely generated abelian group.

PROPOSITION 12.4. *Let X be an ind-algebraic space (resp. ind-scheme) over S . Assume that we have a presentation $X = \text{colim}_i X_i$ with X_i quasi-separated and locally finitely presented over S . Assume that $D(M)_S$ acts on X via an ind-action on the presentation $X = \text{colim}_i X_i$. Let $N \subset M$ be a monoid. Then the attractor functor*

$$(T \rightarrow S) \mapsto \text{Hom}^{D(M)_T}(A(N)_T, X_T)$$

is representable by the ind-algebraic space (resp. ind-scheme) $\text{colim}_i X_i^N$. Moreover the natural morphism $X^N \rightarrow X$ is representable by algebraic spaces (resp. schemes). If $N = Z$ is a group the natural morphism $X^Z \rightarrow X$ is representable by a closed immersion.

Proof. The first assertion is a direct corollary of Proposition 9.3. To show that $X^N \rightarrow X$, is representable by algebraic spaces, we notice that if T is an affine scheme and $T \rightarrow X$ is a morphism, then there exists i such that this morphism is induced by a morphism $T \rightarrow X_i$, and then we have $T \times_X X^N = T \times_{X_i} X_i^N$. If $N = Z$ is a group, the last assertion follows from Corollary 7.8. \square

13. Weights

Let M be a finitely generated abelian group. Let X be an algebraic space separated over S with an action of $D(M)_S$. We generalize in this section the notion of weights of an action of a torus on a linear space to a more general setting using attractor functors. If m is an element in M , we denote by $[m]$ the submonoid generated by m in M .

DEFINITION 13.1. An element $m \in M$ is said to be

- (i) a magnet if $X^{[m]}$ is distinct of X^0 (where X^0 is the attractor associated to 0, i.e the fixed points),
- (ii) a weight if it is a magnet and moreover for any $m' \in M$ such that $[m'] \subsetneq [m]$, we have

$$X^{[m']} \subsetneq X^{[m]}.$$

We denote respectively by $m(X)$ and $w(X)$ the sets of magnets and weights.

DEFINITION 13.2. Let w, w' be two distinct weights in $w(X)$. We say that w is multiple of w' if $[w] \subset [w']$. If w is a multiple of w' , we also say that w' is a divisor of w . We say that w is indivisible if w does not have any divisor in $w(X)$. We say that w is a tip if w as no multiple in $w(X)$.

DEFINITION 13.3. Let w be a weight and assume it is a tip, then we define the associated weight space as $X^{[w]}$.

DEFINITION 13.4. A weight $w \in w(X)$ is said to be reduced if it is indivisible and a tip.

DEFINITION 13.5. The action on the space X is said to be reduced if all weights are reduced.

DEFINITION 13.6. A chain of weights is a sequence w_1, \dots, w_i, \dots of distinct weights in $w(X)$ such that $[w_1] \supset \dots \supset [w_i] \supset \dots$. A maximal chain of weight is a chain of weights that can not be refined.

DEFINITION 13.7. Let w_1, \dots, w_i, \dots be a maximal chain of weights, then we define the associated chain weight spaces as $X^{[w_1]} \supset \dots \supset X^{[w_i]} \supset \dots$

DEFINITION 13.8. If N is a monoid we denote by N_{\boxplus} the monoid generated by $N \cap m(X)$ and N_{\square} the monoid generated by $N \cap w(X)$. Obviously, $N_{\square} \subset N_{\boxplus} \subset N$.

REMARK 13.9. (i) The inclusions, in the sequence $[w_1] \supset \dots \supset [w_i] \supset \dots$ associated to a chain of weights, are strict by definitions.

(ii) The space X is reduced if and only if all chains of weights have lengths at most 1.

REMARK 13.10. Assume X is finitely presented over S . Then we expect that $m(X)$ and $w(X)$ are finite subsets of M . In particular, all chain of weights should be finite length. We plan to study this expectation and the following more difficult question. Let $N \subset M$ be a monoid. We say that an algebraic space with a $D(M)_S$ -action is N -proper if $X^N = X^{N_{\boxplus}} = X^{N_{\square}}$. What can we say about this notion? Note that the multiplication of $D(\mathbb{Z})$ is not \mathbb{N} -proper.

REMARK 13.11. Our notion of reduced actions is compatible with the notion of reduced root systems associated to reductive groups.

14. General root groups

Let S be a scheme, let G be a group scheme over S . Let M be an abelian group. Assume that $D(M)_S$ acts on G by group automorphisms. Then we get an action of $D(M)_S$ on $\text{Lie}(G)_S$ by group automorphisms. Let N be a submonoid of M . Let $G_{e_S}^N$ be the attractor associated to N with prescribed limit the trivial subgroup e_S/S relatively to the face N^* .

DEFINITION 14.1. Let $\alpha \in M$. We call $G_{e_S}^{[\alpha]}$ the root group associated to α under the action of $D(M)_S$ on G . We call $G^{[\alpha]}$ the non prescribed root group associated to α .

PROPOSITION 14.2. *Let $\alpha \in M$.*

(i) *If G is affine over S , we have closed immersions of group schemes over S*

$$G_{e_S}^{[\alpha]} \subset G^{[\alpha]} \subset G.$$

(ii) *If G/S is smooth, then $G_{e_S}^{[\alpha]}$ and $G^{[\alpha]}$ are smooth over S .*

(iii) *We have canonical isomorphisms $\text{Lie}(G_{e_S}^{[\alpha]}) = (\text{Lie}(G))_{e_S}^{[\alpha]}$ and $\text{Lie}(G^{[\alpha]}) = \text{Lie}(G)^{[\alpha]}$.*

Proof. (i) We know that $G^{[\alpha]}$ is a closed subgroup of G by Theorem 5.1. We have $G_{e_S}^{[\alpha]} = G^{[\alpha]} \times_{G^0} e_S$ and so $G_{e_S}^{[\alpha]}$ is a closed subgroup of $G^{[\alpha]}$.

(ii) By Proposition 8.3, $G^{[\alpha]} \rightarrow S$ and $G^{[\alpha]} \rightarrow G^0$ are smooth, this implies the claim.

(iii) These are special cases of Propositions 11.3 and 11.2.

□

We show that our definition is compatible with the definition given by Conrad and SGA3.

PROPOSITION 14.3. *Let $G \rightarrow S$ be a reductive group scheme over a non-empty scheme S , $T \cong D_S(M)$ a split maximal torus, and $\alpha \in M$ a root. Let $\exp_{\alpha}(\mathbf{W}(\mathfrak{g}_{\alpha})) \subset G$ be the α -root group for (G, T, M) considered by B. Conrad in [Co14, Theorem 4.1.4], then*

$$\begin{aligned} \exp_{\alpha}(\mathbf{W}(\mathfrak{g}_{\alpha})) &= G_{e_S}^{[\alpha]} \\ \exp_{\alpha}(\mathbf{W}(\mathfrak{g}_{\alpha})) \times_S T &= G^{[\alpha]}. \end{aligned}$$

Proof. We remark that we have $(\mathbf{W}(\mathfrak{g}_\alpha) \times_S T)^{[\alpha]} = \mathbf{W}(\mathfrak{g}_\alpha) \times_S T$. By [Co14, Theorem 4.1.4] we have a closed immersion $\mathbf{W}(\mathfrak{g}_\alpha) \times_S T \rightarrow G$. So Lemma 5.6 gives us a closed immersion $f : \mathbf{W}(\mathfrak{g}_\alpha) \times_S T \rightarrow G^{[\alpha]}$. We now prove that f is an open immersion. Since f is of finite presentation by [StP, Tag 02FV], using [GW, Proposition 14.18] it is enough to prove that f is flat. Since $\mathbf{W}(\mathfrak{g}_\alpha) \times_S T \rightarrow S$ is flat, it is enough to prove that for any $s \in S$, the morphism $f_s : (\mathbf{W}(\mathfrak{g}_\alpha) \times_S T)_s \rightarrow (G^{[\alpha]})_s$ is flat (cf. [GW, Proposition 14.25]). Since all involved constructions are compatible by base change, f_s is an isomorphism (in particular flat) for any $s \in S$ by [Ma22, Proposition 7.5]. Now f is an open and closed immersion that gives isomorphisms on fibers. This implies that f is an isomorphism and proves the second equality. Now the first equality is clear because $[\alpha]^* = 0$, $G^0 = T$ and so

$$G_{e_S}^{[\alpha]} = G^{[\alpha]} \times_{G^0} e_S = (\exp_\alpha(\mathbf{W}(\mathfrak{g}_\alpha)) \times_S T) \times_T e_S = \exp_\alpha(\mathbf{W}(\mathfrak{g}_\alpha)).$$

□

REMARK 14.4. Let us recall that by Remark 8.6, $G^{[\alpha]} = (G^{D(M/(\alpha))_S})^{[\alpha] \subset (\alpha)}$. And so $G^{[\alpha]}$ can be obtained as a first stage of fixed points followed by a stage of \mathbb{G}_m -attractor. Similarly $G_{e_S}^{[\alpha]} = (G_{T_\alpha}^{(\alpha)})_{e_S}^{[\alpha]}$ where T_α is defined in [Co14, Lemma 4.1.3] or [SGA3], this explains why [Co14] works with small semisimple groups of rank one as a first stage in order to build root groups using \mathbb{G}_m -attractors. Note that [SGA3] also uses semisimple groups of rank one as a first stage [SGA3, Exp. XX] before defining root groups in the general case [SGA3, Exp. XXII].

15. Attractors and dilatations

Let S be a scheme and let S' be a closed locally principal scheme in S . Let X be a scheme over S with a $D(M)_S$ -action where M is an abelian group. Put $D = X_{S'}$. Let Y be a closed subscheme of $X_{S'}$. Then by [MRR20], we get a scheme $\mathrm{Bl}_Y^D X$ called the dilatation of X with center Y along S' , and a morphism of schemes $\mathrm{Bl}_Y^D X \rightarrow X$. Let N be a submonoid in M . Assume that Y is stable under the action of $D(M)_{S'}$ on $X_{S'}$. Then by Proposition 9.3 $Y^N \rightarrow D^N$ is a closed immersion. Moreover $D^N = (X_{S'})^N = (X^N)_{S'}$ is a locally principal closed subscheme of X^N . So $\mathrm{Bl}_{Y^N}^{D^N} X^N$ is well-defined.

PROPOSITION 15.1. *Assume moreover that $X \rightarrow S$, $(\mathrm{Bl}_Y^D X)^N \rightarrow S$ and $\mathrm{Bl}_{Y^N}^{D^N} X^N \rightarrow S$ are flat. Then $D(M)_S$ acts naturally on $\mathrm{Bl}_Y^D X$, moreover we get a canonical isomorphism*

$$\Theta : (\mathrm{Bl}_Y^D X)^N \cong \mathrm{Bl}_{Y^N}^{D^N} X^N.$$

REMARK 15.2. We refer to [MRR20, Prop. 2.16] for conditions ensuring flatness of dilatations.

Proof. Remark first that since $D(M)_S \rightarrow S$, $X \rightarrow S$, $\mathrm{Bl}_Y^D X \rightarrow S$ are flat, $D(M)_S$, X and $\mathrm{Bl}_Y^D X$ belongs in $\mathrm{Sch}_S^{S'\text{-reg}}$, moreover by [MRR20, §1.2, §3.1] any products of these objects in the category Sch_S or $\mathrm{Sch}_S^{S'\text{-reg}}$ coincide. So to check that we have an action of $D(M)_S$ on $\mathrm{Bl}_Y^D X$ it is enough to show that $D(M)(T)$ acts on $\mathrm{Bl}_Y^D X(T)$ functorially for any $T \in \mathrm{Sch}_S^{S'\text{-reg}}$. So let $T \in \mathrm{Sch}_S^{S'\text{-reg}}$. Let $(g, x) \in D(M)(T) \times \mathrm{Bl}_Y^D X(T)$, note that x corresponds to a morphism $T \xrightarrow{x} X$ such that $T|_{S'} \rightarrow X_{S'}$ factors through Y . We define $g.x$ as the composition

$$T \xrightarrow{(g,x)} D(M)_S \times X \xrightarrow{\text{action}} X.$$

Then $g.x$ restricted to S' factors through Y and so $g.x \in \mathrm{Bl}_Y^D X(T)$. So $D(M)_S$ acts on $\mathrm{Bl}_Y^D X(T)$.

We obtain the following two diagrams

$$\begin{array}{ccc}
(\mathrm{Bl}_Y^D X)^N \rightarrow X^N, \\
\\
(\mathrm{Bl}_Y^D X)^N|_{S'} = (\mathrm{Bl}_Y^D X|_{S'})^N & \longrightarrow & X^N|_{S'} = (X|_{S'})^N \\
& \searrow & \nearrow \\
& Y^N &
\end{array}$$

By the universal property of dilatations, we obtain a morphism

$$\Theta : (\mathrm{Bl}_Y^D X)^N \rightarrow \mathrm{Bl}_{Y^N}^{D^N} X^N.$$

We now prove that it is an isomorphism. Again, let $T \in \mathrm{Sch}_S^{S'-\mathrm{reg}}$ and let T' be $T \times_S S'$. Then T' is a closed and locally principal subscheme of T . Moreover, since $A(N)_{\mathbb{Z}}$ is flat over \mathbb{Z} because $\mathbb{Z}[N]$ is free as a \mathbb{Z} -module, using [StP, Tag 01U9] we obtain that the scheme $A(N)_T$ is flat over T . In particular $A(N)_T$ belongs in $\mathrm{Sch}_T^{T'-\mathrm{reg}}$. Since $\mathrm{Bl}_Y^D X \rightarrow S$ is flat, by [MRR20, Lemma 2.7], we have $(\mathrm{Bl}_Y^D X)_T = \mathrm{Bl}_{Y_{T'}}^{D_{T'}} X_T$. So, on a first hand

$$\begin{aligned}
(\mathrm{Bl}_Y^D X)^N(T) &= \mathrm{Hom}^{D(M)_T}(A(N)_T, (\mathrm{Bl}_Y^D X)_T) \\
&= \mathrm{Hom}^{D(M)_T}(A(N)_T, \mathrm{Bl}_{Y_{T'}}^{D_{T'}} X_T) \\
&= \{A(N)_T \xrightarrow{f} X_T \mid f \text{ is } D(M)_T\text{-equiv. and } A(N)_T|_{T'} \xrightarrow{f_{T'}} X|_{T'} \text{ factors through } Y_{T'}\},
\end{aligned}$$

moreover, on an other hand,

$$\begin{aligned}
(\mathrm{Bl}_{Y^N}^{D^N} X^N)(T) &= \{T \rightarrow X^N \mid T|_{S'} \rightarrow X^N|_{S'} \text{ factors through } Y^N\} \\
&= \{A(N)_T \xrightarrow{f} X \mid f \text{ is } D(M)_T\text{-equiv. and } A(N)_T|_{T'} \xrightarrow{f_{T'}} X|_{T'} \text{ factors through } Y_{T'}\}.
\end{aligned}$$

Now since $\mathrm{Bl}_{Y^N}^{D^N} X^N$ and $(\mathrm{Bl}_Y^D X)^N$ are flat over S , they belongs in $\mathrm{Sch}_S^{S'\text{-reg}}$. So by Yoneda $\mathrm{Bl}_{Y^N}^{D^N} X^N = (\mathrm{Bl}_Y^D X)^N$. This finishes the proof. \square

COROLLARY 15.3. *Let S be a scheme and let S' be an effective Cartier divisor on S . Let X be a smooth scheme over S with a $D(M)_S$ -action where M is an abelian group. Put $D = X_{S'}$. Let Y be a closed subscheme of $X_{S'}$ such that $Y \rightarrow S'$ is smooth. Let $\mathrm{Bl}_Y^D X$ be the dilatation of X with center Y along S' . Let N be a submonoid in M . Assume that Y is stable under the action of $D(M)_{S'}$ on $X_{S'}$. Then $D(M)_{S'}$ acts on $\mathrm{Bl}_Y^D X$ and we get a canonical isomorphism*

$$\Theta : (\mathrm{Bl}_Y^D X)^N \cong \mathrm{Bl}_{Y^N}^{D^N} X^N.$$

Proof. Since $X \rightarrow S, D \rightarrow S'$ and $Y \rightarrow S$ are smooth, by Proposition 8.3 $X^N \rightarrow S, D^N \rightarrow S$ and $Y^N \rightarrow S$ are smooth. So by [MRR20, Proposition 2.16], $\mathrm{Bl}_Y^D X \rightarrow S$ and $\mathrm{Bl}_{Y^N}^{D^N} X^N \rightarrow S$ are smooth. So using Proposition 8.3 again, $(\mathrm{Bl}_Y^D X)^N \rightarrow S$ is smooth. Now since smooth implies flat, the Corollary follows from Proposition 15.1. \square

REMARK 15.4. We note that the fact that dilatations commute with attractors may be used to study valued root datum as in Bruhat-Tits theory. Indeed by Sections 14 root groups of reductive groups are examples of attractors (cf. also Section 11) and dilatations allow to define filtrations.

REFERENCES

- AHR20 J. Alper, J. Hall, D. Rydh: *A Luna étale slice theorem for algebraic stacks*, Ann. of Math. (2) 191(3): 675-738, 2020 [3](#)
- AHR21 J. Alper, J. Hall, D. Rydh: *The étale local structure of algebraic stacks*, <https://arxiv.org/pdf/1912.06162.pdf>, 2021 [3](#), [6](#), [13](#), [14](#)
- Bi73 A. Białynicki-Birula: *Some theorems on actions of algebraic groups*. Ann. of Math. (2) 98 (1973) [3](#)
- Co14 B. Conrad: *Reductive group schemes*, Autour des schémas en groupes. Vol. I, Panor. Synthèses, 2014, <http://math.stanford.edu/~conrad/papers/luminysga3.pdf> [23](#), [24](#)
- CGP10 B. Conrad, O. Gabber, G. Prasad: *Pseudo-reductive groups*, Cambridge University Press, 2010
- Dr13 V. Drinfeld: *On algebraic spaces with an action of \mathbb{G}_m* , <https://arxiv.org/abs/1308.2604>. [3](#), [4](#), [17](#)
- GW U. Görtz, T. Wedhorn: *Algebraic Geometry I Scheme With Examples and Exercises*, book, 2010 [8](#), [16](#), [24](#)
- He80 W. Hesselink: *Concentration under actions of algebraic groups*. Paul Dubreil and Marie-Paule Malliavin Algebra Seminar, 33rd Year (Paris, 1980), pp. 55–89, Lecture Notes in Math., 867, Springer, Berlin, 1981. [4](#)
- HP19 D. Halpern-Leistner, A. Preygel: *Mapping stacks and categorical notions of properness* <https://arxiv.org/abs/1402.3204v2> [3](#)
- HR21 T. Haines, T. Richarz: *The test function conjecture for parahoric local models*, J. Amer. Math. Soc., 2021, [21](#)
- JS18 J. Jelisiejew, L. Sienkiewicz: *Białynicki-Birula decomposition for reductive groups*, J. Math. Pures Appl. (9), 2019, <https://arxiv.org/abs/1805.11558> [3](#), [4](#), [6](#)
- JS20 J. Jelisiejew, L. Sienkiewicz: *Białynicki-Birula decomposition for reductive groups in positive characteristic*, J. Math. Pures Appl. (9), 2021, <https://arxiv.org/abs/2006.02315> [6](#)
- Ju85 J. Jurkiewicz: *Torus embeddings, polyhedra, k^* -actions and homology*. Dissertationes Math. (Rozprawy Mat.) 236 (1985) [4](#)
- Ma22 A. Mayeux: *Remarks on attractors for affine schemes*, arXiv, 2022 [7](#), [8](#), [9](#), [11](#), [24](#)
- MRR20 A. Mayeux, T. Richarz, M. Romagny: *Néron blowups and low-degree cohomological applications* <https://arxiv.org/abs/2001.03597> [24](#), [25](#)
- Og A. Ogus: *Lectures on Logarithmic Algebraic Geometry*, Cambridge UP, 2006 [4](#), [12](#), [18](#)
- Ri16 T. Richarz: *Spaces with \mathbb{G}_m -action, hyperbolic localization and nearby cycles*, J. Algebraic Geom., 2019, <https://arxiv.org/abs/1611.01669> [3](#), [4](#), [15](#), [18](#)
- Ry11 D. Rydh: *Representability of Hilbert schemes and Hilbert stacks of points*, Commun. Algebra 39, No. 7, 2632-2646 (2011). [13](#), [14](#)
- SGA3 M. Demazure, A. Grothendieck: *Séminaire de Géométrie Algébrique du Bois Marie - 1962-64 - Schémas en groupes - (SGA 3)*, Lecture notes in mathematics **151** (1970), Berlin; New York: Springer-Verlag. pp. xv+564. <https://webusers.imj-prg.fr/~patrick.polo/SGA3/> [4](#), [7](#), [8](#), [9](#), [11](#), [17](#), [19](#), [20](#), [24](#)
- StP The Stacks project authors: *The Stacks Project*, <https://stacks.math.columbia.edu>, 2022. [8](#), [11](#), [12](#), [13](#), [14](#), [15](#), [16](#), [17](#), [19](#), [24](#), [25](#)

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