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Generalized Perron roots and solvability of the absolute value equation

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Abstract

Let A be a real $(n \times n)$-matrix. The piecewise linear equation system $z - A|z| = b$ is called an absolute value equation (AVE). It is well-known to be equivalent to the linear complementarity problem (LCP). For AVE and LCP unique solvability is comprehensively characterized in terms of conditions on the spectrum (AVE), resp., the pricipal minors (LCP) of the coefficient matrix. For mere solvability no such characterization exists. We close this gap in the theory on the AVE-side. The aligning spectrum of A consists of real eigenvalues of the matrices SA, where $S \in \mathrm{diag}(\{\pm\}^n)$, which have a corresponding eigenvector in the positive orthant of $\mathbb{R}^n$. For the mapping degree of the piecewise linear function $z \mapsto z - A|z|$ we prove, under some mild genericity assumptions on A: The degree is 1 if all aligning values are smaller than 1, it is 0 if all aligning values are larger than 1, and in general it is congruent to $(k+1) \mod 2$ if k aligning values are larger than 1. The modulus cannot be omitted because the degree can both increase and decrease.

1 Introduction

The linear complementarity problem (LCP) stands at the crossroads of numerous optimization contexts. Not only do many problems in computational mechanics arise naturally in LCP form. Linear and quadratic programs are special cases of the LCP. For a comprehensive introduction to the topic, see [1]. Other relevant optimization problems, such as bimatrix games [1], or arbitrary finite piecewise affine systems [2], can be reduced to solving an LCP.

Linear complementarity problems can be formulated in various equivalent ways, e.g., via complementarity conditions, max-min expressions, or absolute values. In terms of the wealth of publications treating them, and the depth of the associated theory, two formulations stand out. First, the classical form by Cottle and Dantzig, which was first introduced in [3]: Let $M \in \mathrm{M}_n(\mathbb{R})$, where $\mathrm{M}_n(\mathbb{R})$ denotes the space of $n \times n$ real matrices, and $q \in \mathbb{R}^n$. Then the $\mathrm{LCP}(M, q)$ is to find vectors $v, w \in \mathbb{R}^n_{\geq 0}$ with $w^T v = 0$ so that

$$w \;=\; Mv + q\,.$$

Now let $A \in \mathrm{M}_n(\mathbb{R})$ and $b \in \mathbb{R}^n$. Then the piecewise linear equation system

$$z - A|z| \;=\; b\,,$$

where $|\cdot|$ denotes the componentwise absolute value, is the second outstanding formulation, called an absolute value equation $\mathrm{AVE}(A, b)$. The term was first coined by Mangasarian in [4], but the first journal publication to investigate it was [5].

Setting $x \coloneqq v - w$, and consequently $v = \max(0, x)$, and $w = -\min(0, x)$, both formulations can be transformed into another via the identities $\max(0, x) = (|x| + x)/2$ and $\min(0, x) = (x - |x|)/2$ [6]. For for both AVE and LCP unique solvability is comrehensively characterized.

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The LCP(M, q) is uniquely solvable for arbitrary $q \in \mathbb{R}^n$ if and only if M is a P-matrix, that is, if all its principal minors are positive [1]. Let $\mathcal{S}$ be the set of $n \times n$ signature matrices, i.e., diagonal matrices with entries in $\{-1, +1\}$. Then the largest real eigenvalue of any matrix SA, where $S \in \mathcal{S}$, is called the sign-real spectral radius of A. The AVE(A, b) is uniquely solvable if and only if the sign-real spectral radius of A is smaller than one [6, 7].

The sign-real spectral radius can be interpreted as a piecewise linear analogue of contractivity conditions for linear operators, or as a generalized Perron root for matrices without sign-restrictions [7]. Its computation is equivalent to the computation of the weighted componentwise distance to the nearest singular matrix [7]. The latter relation provides a direct connection between the condition of the matrices $I - SA$ and the complexity of the AVE, which is NP-complete in general [8], but lies in $\mathrm{O}(n^3)$ for certain uniquely solvable systems with well-conditioned matrices $I - SA$ [9]. The latter fact makes the AVE particularly interesting in light of recent developments in real algebraic geometry that deal with precisely such connections of complexity and condition, see [10, 11] for founding texts of the research area, and [12, 13] for recent applications.

For mere, possibly non-unique, solvability of AVE and LCP there exists no similarly comprehensive characterization as in the case of unique solvability. We close this gap in the theory on the AVE side. We again consider nonnegative real eigenvalues of the matrices SA, but with the restriction that only those are taken into account that have a corresponding eigenvector in the positive orthant of $\mathbb{R}^n$. The resulting eigenvalues are called aligning values. The set of aligning values, ordered by descending magnitude, is called the aligning spectrum. We denote the largest and the smallest aligning value $\lambda_{\max}$ and $\lambda_{\min}$, respectively. We prove that, under some genericity assumptions on A, the mapping degree of the piecewise linear function

$$F_A : \mathbb{R}^n \to \mathbb{R}^n, \quad z \mapsto z - A|z|$$

is one if $\lambda_{\max} < 1$, it is zero if $\lambda_{\min} > 1$, and otherwise we have

$$\deg F_A \equiv (k+1) \mod 2,$$

where k is the number of aligning values larger than one. The genericity of A can always be achieved by a perturbation of A, which does not change the degree of F_A, as the degree is locally constant in the function space.

Content and structure: In the subsequent section we will introduce some basics to our needs.

2 Preliminaries - degree basics

In this section we present a collection of basic facts about the mapping degree that is tailored to our needs. All statements are taken directly from or are special cases of theorems in [14, p. 111 ff].

Let U be some nonempty connected subset of $\mathbb{R}^n$. We say a function on f is almost smooth if it is continuous and differentiable almost everywhere. Let $f : U \to U$ be a function which is almost smooth, and for $y \in U$ let $f^{-1}(y) = \{x \in U : f(x) = y\}$ be the fiber of y under f. We say y is a regular value of f if f is differentiable with nonsingular Jacobian $d_x f$ at all $x \in f^{-1}(y)$. This includes the case that $f^{-1}(y) = \emptyset$.

If f is proper, that is, if preimages of compact sets under f are compact, then the oriented preimage count

$$\sum_{x \in f^{-1}(y)} \operatorname{sign}(\det(d_x f))$$

of any regular value y of f equals a constant d, which is called the (mapping) degree of f. By the Theorem of Sard Brown regular values lie dense in the range of an almost smooth function. Hence, for closed U and closed f, a nonzero degree implies surjectivity of f.

Note that properness is a sufficient, not a necessary condition for a consistent degree. Any constant function on $\mathbb{R}^n$ has degree 0 without being proper. A positively homogeneous function $F : \mathbb{R}^n \to \mathbb{R}^n$ is proper if and only if it is nondegenerate in the sense that it maps no nonzero point to the origin. If one is only interested in degree-related questions, it makes sense to force a nondegenerate positively homogeneous function onto the Euclidean unit spere $\mathbb{S}^{n-1}$ via

$$\bar{F} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}, \quad x \mapsto \frac{F(x)}{\|F(x)\|_2}. \tag{1}$$

An almost smooth function on $\mathbb{S}^{n-1}$ is proper by default due to the compactness of the sphere and thus has a well defined mapping degree. One can show that the degree of F and its normalization $\bar{F}$ coincide, cf. [1, Chap. 6.2].

In the context of this work a proper homotopy is an almost smooth function $H : U \times [0,1] \to U$ so that for all $t \in [0,1]$ the function $H_t \coloneqq H(\cdot, t)$ is proper. Two proper almost smooth functions F, G on U are properly homotopic if there exists a proper homotopy H so that $H_0 = F$ and $H_1 = G$. The mapping degree is invariant under proper homotopies. We close this subsection with an important fact.

Lemma 1. *Let $F, G : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be continuous mappings so that*

$$\|F(x) - G(x)\|_2 < 2$$

for all $x \in \mathbb{S}^{n-1}$. Then the mapping degrees of F and G conincide.

This is due to the fact that the homotopy

$$H : \mathbb{S}^{n-1} \times [0,1] \to \mathbb{S}^{n-1}, (x,t) \mapsto \frac{tF(x) + (1-t)G(x)}{\|tF(x) + (1-t)G(x)\|_2}$$

is then well defined, as the term in the denominator on the right-hand side cannot become 0.

3 The aligning spectrum

The piecewise linear function

$$F_A : \mathbb{R}^n \to \mathbb{R}^n, \quad z \mapsto z - A|z|$$

is linear and thus differentiable on the orthants of $\mathbb{R}^n$. The locus of non-differentiability are the orthant boundaries. It is thus almost smooth and positively homogeneous. For our investigation we need to answer when F_A is nondegenerate and thus proper, so that we can define the normalization $\bar{F}_A$.

Lemma 2. *Let $A \in \mathrm{M}_n(\mathbb{R})$. Then F_A is degenerate if and only if there exists a $S \in \mathcal{S}$ so that SA has a real eigenvalue 1 with corresponding nonnegative eigenvector.*

Proof. "⇒": Let $z \in \mathbb{R}^n$ be a solution of $z - A|z| = 0$ and let S be a signature of z, so that $Sz = |z|$. Then we have

$$0 = Sz - SA|z| = |z| - SA|z| \iff |z| = SA|z|.$$

"⇐": Let v the eigenvector of SA in question. Then we have $v = |v|$ and thus

$$0 = v - SAv = v - SA|v| \iff 0 = Sv - A|v| = Sv - A|Sv|,$$

which concludes the proof. □

This lemma provides some crucial insights into $\bar{F}_A$, which will be the key motivation for our definition of the aligning spectrum:

Corollary 3. *Let λ be a positive real eigenvalue of some SA which has a corresponding nonnegative eigenvector v with $\|v\|_2 = 1$. Then, if F_A is nondegenerate,*

1. *$-v$ is a fixed point of $\bar{F}_A$,*

2. *v is a fixed point of $\bar{F}_A$, if $\lambda < 1$,*

3. *v is mapped to $-v$ by $\bar{F}_A$, if $\lambda > 1$,*

and $\bar{F}_A$ has no fixed points or points that are mapped to their antipode which are not characterized in this fashion.

So the interesting objects of investigation for the analysis $F_A, \bar{F}_A$, and homotopies of the latter induced by scalings of A are vectors $z \in \mathbb{R}^n$ so that z and $A|z|$ are aligned. Lemma 2 and Corollary 3 thus justify

Definition 4. *Let $A \in \mathrm{M}_n(\mathbb{R})$ and $S \in \mathcal{S}$. We call a nonnegative real eigenvalue λ of SA an aligning value of A if there exists a nonnegative eigenvector corresponding to λ. We call such an eigenvector an aligning vector of A. The set of aligning values, enumerated by descending magnitude, is the aligning spectrum of a, denoted*

$$\mathrm{Spec}^a(A) \;=\; \{\lambda_1, \ldots, \lambda_\ell\}\,.$$

We further set $\lambda_{\max}(A) \coloneqq \lambda_1$ and $\lambda_{\min}(A) \coloneqq \lambda_\ell$.

We note that the aligning spectrum cannot be empty since a continuous mapping on the sphere has at least one fixed point. Set

$$\bar{H}_A :\; \mathbb{S}^{n-1} \times [0,1] \;\to \mathbb{S}^{n-1}\,, \quad z \;\mapsto\; \bar{F}_{tA}\,. \tag{2}$$

Lemma 2 yields

Lemma 5. *Let $A \in \mathrm{M}_n(\mathbb{R})$. Then the homotopy $\bar{H}_A$ from (2) is proper if and only if the interval $[0,1]$ does not contain a reciprocal of an aligning value of A.*

This leads to our first main result.

Theorem 6. *Let $A \in \mathrm{M}_n(\mathbb{R})$. Then the mapping degree of F_A equals 1 one if $\lambda_{\max}(A) < 1$. Further, if A is nonsingular and $\lambda_{\min}(A) > 1$, then the mapping degree of F_A is 0.*

Proof. The first part of the statement is an immediate consequence of Lemma 5 and the correspondence of the degree of F_A and $\bar{F}_A$. Alternatively consider that $\bar{F}_A$ maps no point to its polar opposite and apply Lemma 1.

Concerning the second part: Let A be nonsingular. Then there exist a hyperplane V and an open halfspace V^+ so that all column vecors of A are contaied in V^+. Since $\lambda_{\min} > 1$, F_A and F_{tA} are properly homotopic for all $t \geq 1$. Let $S \in \mathcal{S}$. Then there exists a $t_S > 1$ so that the columns of the matrix $S - tA$ lie in V^+ for all $t \geq t_S$. Hence, for any t larger than the maximum over the t_S, the image of all orthants of $\mathbb{R}^n$ under F_{tA} must be contained in $V^+ \cup \{0\}$. But then F_A is properly homotopic to a function which cannot be surjective and must thus have degree 0. □

Corollary 7. *Let $A \in \mathrm{M}_n(\mathbb{R})$ so that $\lambda_{\max}(A) = 0$. Then for all $t \in \mathbb{R}$ the function F_A has degree 1.*

Proof. Far any $t \in \mathbb{R}$ the function F_A is properly homotopic to F_{tA}. □

4 Summary of the rest of the work

The key observation underlying this section is that there are stable and unstable aligning values. And only the stable ones are relevant to the determination of the mapping degree of F_A. Let

$$D_\varepsilon := \begin{pmatrix} 1 & -0.5-\varepsilon \\ 0.5 & 0 \end{pmatrix}. \tag{3}$$

We have $\lambda_{\max}(D_0) = 0.5$, but for any $\varepsilon > 0$ the corresponding eigenvalue becomes complex and we get $\lambda_{\max}(D_\varepsilon) = 0.5(\sqrt{2}\sqrt{1+\varepsilon} - 1)$ which roughly equals 0.207 for small ε.

The degree of F_{tD_0} equals 1 for $t = 1$ and $t = 2.1$. That is, it does not flip when $\lambda_{\max}(tD_0)$ becomes larger than 1. But for t so that $t(\sqrt{2}\sqrt{1+\varepsilon} - 1) > 1$, the degree of F_{tD_0} becomes 0. In light of the fact that the mapping degree is stable under perturbations, this obsevation makes perfect sense, and it leads us to the following

Definition 8. *Let $A \in \mathrm{M}_n(\mathbb{R})$. We say A is generic if all its aligning values are simple and the corresponding aligning vectors do not lie in an orthant boundary.*

We prove, in essence, that a generic form can be achieved with probability 1 by a random perturbation (which does not affect the degree), as non-generic matrices are confined to some lower-dimensional surface in $\mathrm{M}_n(\mathbb{R})$. This leads to the main result.

Theorem 9. *Let $A \in \mathrm{M}_n(\mathbb{R})$ with $1 \notin \mathrm{Spec}^a(A)$ be generic, and let k be the number of aligning values of A which are larger than 1. The we have*

$$\deg F_A \;\equiv\; (k+1) \mod 2\,.$$

The proof is fairly involved and cannot be presented here. For $k = 0$ the statement is correct due to Theorem 6. The basic idea for an induction is to compare two functions $\bar{F}_{t_1 A}$, $\bar{F}_{t_2 A}$ on the sphere, where t_1 and t_2 are in a small neighborhood of the reciprocal of an aligning value, t_1 being smaller, t_2 larger than the reciprocal. Then Lemma 1 is used to construct a function $G : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ which is homotopic to $\bar{F}_{t_2 A}$, but much simpler to analyze. It is then shown by means which are, in the widest sense Morse-theoretic, that the mapping degree of G differs from that of $\bar{F}_{t_1 A}$ by exactly 1. There are examples for increasing and decreasing degree. Which is why the modulus cannot be omitted from the statement.

References

[1] R.W. Cottle, J.-S. Pang, and R. Stone. *The linear complementarity problem.* Academic Press, 1992.

[2] A. Griewank, J. U. Bernt, M. Radons, and T. Streubel. Solving piecewise linear equations in abs-normal form. *Linear Algebra and Its Applications*, 471:500–530, 2015.

[3] R. W. Cottle and G. B. Dantzig. Complementary pivot theory of mathematical programming. *Linear Algebra and its Applications*, 1:103–125, 1968.

[4] O.L. Mangasarian. Absolute value programming. *Computational Optimization and Applications*, 36(1):43–53, 2007.

[5] J. Rohn. Systems of linear interval equations. *Linear Algebra and Its Applications*, 126:39–78, 1989.

[6] A. Neumaier. *Interval methods for systems of equations.* Cambridge University Press, 1990.

[7] S. M Rump. Theorems of perron-frobenius type for matrices without sign restrictions. *Linear Algebra and Its Applications*, 266:1–42, 1997.

[8] S.J. Chung. Np-completeness of the linear complementarity problem. *J Optim Theory Appl*, 60:393—-399, 1989.

[9] M. Radons. Direct solution of piecewise linear systems. *Theoretical Computer Science*, 626:97–109, 2016.

[10] D. Amelunxen and M. Lotz. Average-case complexity without the black swans. *Journal of Complexity*, 41:82–101, 2017.

[11] P. Bürgisser and F. Cucker. *Condition – The Geometry of Numerical Algorithms*. Springer, 2013.

[12] Peter Bürgisser, Felipe Cucker, and Josué Tonelli-Cueto. Computing the homology of semialgebraic sets. i: Lax formulas. *Foundations of Computational Mathematics*, May 2019.

[13] Algebra meets numerics: Condition and complexity. `https://www.math.tu-berlin.de/fachgebiete_ag_diskalg/fachgebiet_algorithmische_algebra/v_menue/veranstaltungen/2017ws/algebra_meets_numerics_condition_and_complexity`.

[14] E. Outerelo and J.M. Ruiz. *Mapping Degree Theory*. American Mathematical Society, 2009.