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**THE CHEVALLEY–HERBRAND FORMULA
AND THE REAL ABELIAN MAIN CONJECTURE
NEW CRITERION USING CAPITULATION OF THE CLASS GROUP**

GEORGES GRAS

ABSTRACT. The Main Theorem for abelian fields (often called Main Conjecture despite proofs in most cases) has a long history which has found a solution by means of “elementary arithmetic”, as detailed in Washington’s book from Thaine’s method having led to Kolyvagin’s Euler systems. Analytic theory of real abelian fields K says (in the semi-simple case) that the order of the p -class group \mathcal{H}_K is equal to the p -index of cyclotomic units $(\mathcal{E}_K : \mathcal{F}_K)$. We have conjectured (1977) the relations $\#\mathcal{H}_\varphi = (\mathcal{E}_\varphi : \mathcal{F}_\varphi)$ for the isotypic p -adic components using the irreducible p -adic characters φ of K . We develop, in this article, new promising links between: (i) the Chevalley–Herbrand formula giving the number of “ambiguous classes” in p -extensions L/K , $L \subset K(\mu_\ell)$ for the auxiliary prime numbers $\ell \equiv 1 \pmod{2p^N}$ inert in K ; (ii) the phenomenon of capitulation of \mathcal{H}_K in L ; (iii) the real Main Conjecture $\#\mathcal{H}_\varphi = (\mathcal{E}_\varphi : \mathcal{F}_\varphi)$ for all φ . We prove that the real Main Conjecture is trivially fulfilled as soon as \mathcal{H}_K capitulates in L (Theorem 1.1). Computations with PARI programs support this new philosophy of the Main Conjecture. The very frequent phenomenon of capitulation suggests Conjecture 1.2.

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1. INTRODUCTION – STATEMENT OF THE MAIN RESULT

1.1. **Abelian characters.** Let \mathbb{Q}^{ab} be the maximal abelian extension of \mathbb{Q} contained in an algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} ; let \mathbb{Q}_p be the p -adic field and $\overline{\mathbb{Q}_p}$ an algebraic closure of \mathbb{Q}_p containing $\overline{\mathbb{Q}}$.

Let Ψ be the set of irreducible characters of $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$, of degree 1 and finite order, with values in $\overline{\mathbb{Q}_p}$. We define the set Φ of irreducible p -adic characters and the set \mathbf{X} of irreducible rational characters. For a subfield K of \mathbb{Q}^{ab} , we define the subsets Ψ_K , Φ_K , \mathbf{X}_K , whose kernels fixe K . The set \mathbf{X} is in one-to-one correspondence with the set of cyclic subfields of \mathbb{Q}^{ab} .

The notation $\psi \mid \varphi \mid \chi$ (for $\psi \in \Psi$, $\varphi \in \Phi$, $\chi \in \mathbf{X}$) means that φ is a term of χ and ψ a term of φ ; so φ is the sum of the \mathbb{Q}_p -conjugates of ψ and χ the sum of the \mathbb{Q} -conjugates of ψ (cf. [Ser1998]).

Let $\chi \in \mathbf{X}$; we denote by $g_\psi = g_\varphi = g_\chi$ the order of $\psi \mid \varphi \mid \chi$; the field of values of these characters is the cyclotomic field $\mathbb{Q}(\mu_{g_\psi})$.

Let K/\mathbb{Q} be an abelian extension of Galois group g of prime-to- p order. For any $\mathbb{Z}[g]$ -module A of finite type, we denote by \mathcal{A}_χ (resp. \mathcal{A}_φ) the χ -component (resp. the φ -component) of $\mathcal{A} := A \otimes \mathbb{Z}_p$; we get $\mathcal{A}_\chi = \bigoplus_{\varphi \mid \chi} \mathcal{A}_\varphi$, then $\mathcal{A} = \bigoplus_{\chi \in \mathbf{X}_K} \bigoplus_{\varphi \mid \chi} \mathcal{A}_\varphi = \bigoplus_{\varphi \in \Phi_K} \mathcal{A}_\varphi$.

In this semi-simple case all reasonings reduce to the arithmetic of the cyclic subfields of K ; so in what follows, we only consider cyclic K/\mathbb{Q} 's.

1.2. **Main theorem.** We have obtained the following result (Theorem 4.6):

Theorem 1.1. *Let $p \geq 2$ be a prime. Let K/\mathbb{Q} be a real cyclic extension of prime-to- p degree and Galois group g . Denote by $\mathcal{H}_K = \bigoplus_{\varphi \in \Phi_K} \mathcal{H}_\varphi$ the p -class group of K .*

Consider primes $\ell \equiv 1 \pmod{2p^N}$, ℓ totally inert in K , and let K_n be the subfield of $K(\mu_\ell)$ of degree p^n over K , $n \in [1, N]$, where μ_ℓ is the group of ℓ -roots of unity.

Let \mathbf{E}_K (resp. \mathbf{F}_K) be the group of units (resp. of cyclotomic units) of K and put $\mathcal{E}_K = \bigoplus_{\varphi \in \Phi_K} \mathcal{E}_\varphi$ (resp. $\mathcal{F}_K = \bigoplus_{\varphi \in \Phi_K} \mathcal{F}_\varphi$).

Then:

(i) As soon as \mathcal{H}_K capitulates by extension in some K_n , the Main Conjecture holds in K , that is to say, $\#\mathcal{H}_\varphi = (\mathcal{E}_\varphi : \mathcal{F}_\varphi)$ for all $\varphi \in \Phi_K$.

(ii) We have $\#\mathcal{H}_{K_n} = \#\mathcal{H}_K$, for all $n \in [1, N]$, if and only if $\#\mathcal{H}_{K_1} = \#\mathcal{H}_K$. If this stability property holds, we have the following consequences:

- The p -class groups \mathcal{H}_{K_n} are invariant by $\text{Gal}(K_n/K)$ and the norms $\mathbf{N}_{K_n/K} : \mathcal{H}_{K_n} \rightarrow \mathcal{H}_K$ are isomorphisms.
- Let p^e be the exponent of \mathcal{H}_K and assume $N \geq e$; then the p -class group \mathcal{H}_K capitulates in K_e and (from (i)) the Main Conjecture holds.

Denote by \mathbf{N} the arithmetic norm and by \mathbf{J} the transfer (corresponding to extension of classes). Note, once for all, that if \mathcal{H}_K is of exponent p^e and capitulates in K_n , the relation $\mathbf{N}_{K_n/K} \circ \mathbf{J}_{K_n/K}(\mathcal{H}_K) = \mathcal{H}_K^{p^n}$ shows that necessarily $n \geq e$; but incomplete capitulation may occur for any $n \geq 1$.

We have proposed in [Gra2022^e] the following Conjecture of capitulation, without any assumption of splitting on ℓ :

Conjecture 1.2. *Let K be any totally real number field and let \mathcal{H}_K be its p -class group, of exponent p^e . There exist infinitely many primes $\ell \equiv 1 \pmod{2p^N}$, $N \geq e$, such that \mathcal{H}_K capitulates in $K(\mu_\ell)$.*

We shall give Section 5 extensive numerical computations with PARI programs [Pari2013] showing that capitulation in such auxiliary cyclic p -extensions is, surprisingly, very frequent and conjecturally holds for infinitely many ℓ . The criterion (ii) of the theorem allows easy effective verifications; but capitulation may hold without stability in K_1/K .

We do not intend to evoke the case of the abelian capitulations of class groups proved in the literature (Gras [Gra1997], Kurihara [Kur1999], Bosca [Bos2009], Jaulent [Jau2022]); all techniques in these papers need to build abelian p -extensions L_0 of \mathbb{Q} , ramified at various primes and requiring many local arithmetic conditions, whose compositum L with K gives a capitulation field of \mathcal{H}_K ; the method is completely incomparable to ours since it must apply to any real abelian field K , of arbitrary degree, obtained in an iterative process giving that the maximal real subfield of $\mathbb{Q}(\bigcup_{f>0} \mu_f)$ is principal.

However, these results together with Theorem 1.1 and the help of numerical computations, among many other results of the literature, support the fact that the phenomenon of capitulation governs many aspects of abelian arithmetic, independently of the well-known case of capitulation in the Hilbert class field, from Hilbert’s theorem 94 and a lot of improvements (see the surveys [Jau1988, Jau1998, Mai1997, Mai1998] and their references).

In a slightly different, but related, context of the p -adic class field theory, mention for instance that the capitulation of the logarithmic class group of K [Jau1994, Jau1998], in its cyclotomic \mathbb{Z}_p -extension $K_\infty = \bigcup_{n \geq 0} K_n$, is equivalent to Greenberg’s conjecture [Gree1976] saying that the Iwasawa invariants λ, μ for $\varprojlim \mathcal{H}_{K_n}$ are zero [Jau2016, Jau2019^a, Jau2019^b].

Analogous criteria of stability of the $\#\mathcal{H}_{K_n}$ were given by Greenberg then by Fukuda [Fuk1994]; for instance, in the similar context as ours where one considers p non split in K , the condition $\lambda = \mu = 0$ in $L = K_\infty$ is equivalent to the capitulation of \mathcal{H}_K in some K_{n_0} [Gree1976, Theorem 1]; the case of p totally split in K [Gree1976, Theorem 2] relies on equalities $\mathcal{H}_{K_n}^{G_n} = \mathcal{H}_{K_n}^{\text{ram}}$ involving the exact sequence of Theorem 3.1.

In a numerical setting, let's point out works of Kraft–Schoof [KrSc1995^a, KrSc1995^b] (resp. Pagani [Pag2022]), verifying Greenberg's conjecture for some real quadratic fields of conductor $f < 10^4$ and $p = 3$ (resp. $p = 2$), by means of the analytic formulas in some layers of K_∞ with the use of cyclotomic units, giving some algebraic similarities.

But Greenberg's conjecture is still unproved and depends (without any assumption on the decomposition of p) on random algorithmic process, as explain in [Gra2021], governed by the torsion group of the Galois group of the maximal abelian p -ramified pro- p -extension of K (essentially, the second Tate–Chafarevich group of K); this takes place in a deep p -adic context, beyond Leopoldt's conjecture.

Remark 1.3. We have proven the criterion (ii) of stability of Theorem 1.1 in [Gra2022^c, Theorem 3.1, Corollaire 3.2], generalizing similar results [Fuk1994, LOXZ2022, MiYa2021]. More precisely, this criterion can be applied at some layer n_0 and one obtains $\#\mathcal{H}_{K_n} = \#\mathcal{H}_{K_{n_0}}$ for all $n \geq n_0$, if and only if the equality holds for $n = n_0 + 1$; so, this means that $\mathcal{H}_{K_{n_0}}$ capitulates in $K_{n_0+e_{n_0}}$, where $p^{e_{n_0}}$ is the exponent of $\mathcal{H}_{K_{n_0}}$, but a fortiori, \mathcal{H}_K capitulates in $K_{n_0+e_{n_0}}$. We shall give again a proof in our particular simpler framework (Theorem 3.5).

1.3. Methodology. We shall obtain Theorem 1.1 by means of a classical exact sequence describing \mathcal{H}_L^G , in cyclic p -extensions L/K of Galois group G , in terms of the units and image of the extension $\mathbf{J}_{L/K} : \mathcal{H}_K \rightarrow \mathcal{H}_L^G$ of p -classes, which gives, under the phenomenon of capitulation, the needed information about the index of cyclotomic units, taking into account their norm properties in abelian extensions (Corollary 3.2 and Proposition 4.3). For that, we shall need the order of the “ φ -components”, for all $\varphi \in \Phi_K$, of the Chevalley–Herbrand formula giving $\#\mathcal{H}_L^G$, that is to say, the computation of $\#\mathcal{H}_\varphi^G$, where $\mathcal{H}_\varphi^G := (\mathcal{H}_\varphi)^G = (\mathcal{H}^G)_\varphi$. This will give the opportunity to write cohomological exact sequences linking invariant classes and capitulation to the norm properties of the units (local and global); for this, we shall follow [Jau1986, III, p. 167]. This step is crucial since Chevalley–Herbrand formula writes, for p -class groups in L/K cyclic real of degree p^n , as follows [Che1933, pp. 402-406]:

$$\#\mathcal{H}_L^G = \frac{\#\mathcal{H}_K \times \prod_{\mathfrak{q}} e_{\mathfrak{q}}(L/K)}{p^n \times (\mathcal{E}_K : \mathcal{E}_K \cap \mathbf{N}_{L/K}(L^\times))},$$

where $e_{\mathfrak{q}}(L/K)$ is the ramification index in L/K of the prime ideals \mathfrak{q} of K ; the original Chevalley formula depends on the Herbrand theorem [Her1930], introducing the general “Herbrand quotient” (see Lemme 3, p. 375 of Chevalley's Thesis, then [Her1936, Appendice, § 1, p. 57]).¹

One must obtain the decompositions of each factor into φ -components, especially for $\#\mathcal{H}_K$ since \mathcal{H}_K is not a submodule of \mathcal{H}_L^G , the transfer map

¹It may be useful to recall some historical context of such an important formula, quoting the following excerpt from Chevalley's Thesis (footnote 19, p. 402): *Le calcul fait ici est la généralisation au cas “cyclique quelconque” du calcul fait par Takagi dans le cas “cyclique de degré premier”, calcul dont l'idée essentielle se trouve déjà dans le “Zahlbericht” de Hilbert, dans la démonstration du théorème suivant : si un sur-corps relativement cyclique de degré premier de k est non ramifié, il y a au moins un idéal de k qui n'est pas principal dans k mais qui est principal dans le sur-corps. L'extension au cas “cyclique de degré quelconque” a été rendue possible par le théorème des unités de Herbrand.*

$\mathbf{J}_{L/K} : \mathcal{H}_K \rightarrow \mathcal{H}_L^G$ being in general non injective (see, e.g., Example 3.4). For this, we introduce the following elementary principle.

Definition 1.4. *Let K/\mathbb{Q} be an abelian extension of Galois group g and let X be a finite $\mathbb{Z}[g]$ -module. Assume that we know that $\#X = \prod_i \#U_i \times \prod_j (\#V_j)^{-1}$ depending on finite $\mathbb{Z}[g]$ -modules U_i, V_j .*

We say that X (or the formula giving $\#X$) is p -localizable if there exist exact sequences $1 \rightarrow A_k \rightarrow B_k \rightarrow C_k \rightarrow 1$ of finite $\mathbb{Z}[g]$ -modules of the form U_i, V_j and X , such that $\#X$ is obtained by means of the relations $\#B_k = \#A_k \times \#C_k$.

Under this property, the flatness of \mathbb{Z}_p allows to deduce the family of exact sequences of $\mathbb{Z}_p[g]$ -modules $1 \rightarrow \mathcal{A}_k := A_k \otimes \mathbb{Z}_p \rightarrow \mathcal{B}_k := B_k \otimes \mathbb{Z}_p \rightarrow \mathcal{C}_k := C_k \otimes \mathbb{Z}_p \rightarrow 1$, then, taking the isotopic components (for instance by means of the fundamental idempotents e_φ of $\mathbb{Z}_p[g]$, $\varphi \in \Phi_K$), we get the exact sequences $1 \rightarrow \mathcal{A}_{k,\varphi} \rightarrow \mathcal{B}_{k,\varphi} \rightarrow \mathcal{C}_{k,\varphi} \rightarrow 1$ and the formulas $\#\mathcal{B}_{k,\varphi} = \#\mathcal{A}_{k,\varphi} \times \#\mathcal{C}_{k,\varphi}$ yielding $\#X_\varphi$.

For instance, cohomology groups $H^n(G, X)$, $n \in \{1, 2\}$, $G =: \langle \sigma \rangle$, are p -localizable, as soon as X is p -localizable; indeed:

$$H^1(G, X) := \text{Ker}_X(\nu_G)/X^{1-\sigma} \text{ and } H^2(G, X) := X^G/\nu_G(X),$$

where ν_G is the algebraic norm.

1.4. The real abelian Main Conjecture. The Main Conjecture for real abelian fields K (to be called Main Theorem because of its numerous proofs; but this name has become common) essentially says that, when $p \nmid \#g$, one has, for all p -adic irreducible characters $\varphi \in \Phi_K$ [Gra1976, Gra1977]:

$$\#\mathcal{H}_{K,\varphi} = (\mathcal{E}_{K,\varphi} : \mathcal{F}_{K,\varphi}),$$

for the φ -components of \mathcal{H}_K , where $\mathcal{E}_K = \mathbf{E}_K \otimes \mathbb{Z}_p$, $\mathcal{F}_K = \mathbf{F}_K \otimes \mathbb{Z}_p$ denote the groups of global units and of cyclotomic units of K , respectively.

The following obvious property of rational characters is to be considered as the “Main Theorem” for rational components [Leo1954, Chap. I, § 1, 1]:

Theorem 1.5. *Let K/\mathbb{Q} be an abelian extension; let $(A_\chi)_{\chi \in \mathbf{X}_K}$, $(A'_\chi)_{\chi \in \mathbf{X}_K}$, be two families of positive numbers, indexed by the set \mathbf{X}_K of irreducible rational characters of K . If for all subfields k of K , one has $\prod_{\chi \in \mathbf{X}_k} A'_\chi = \prod_{\chi \in \mathbf{X}_k} A_\chi$, then $A'_\chi = A_\chi$ for all $\chi \in \mathbf{X}_K$.*

So this result applies to the well-known complex analytic global formula $\#\mathbf{H}_K = (\mathbf{E}_K : \mathbf{F}_K)$ for K cyclic real, which implies $\#\mathcal{H}_{K,\chi} = (\mathcal{E}_{K,\chi} : \mathcal{F}_{K,\chi})$ for all $\chi \in \mathbf{X}_K$, in the semi-simple case. But when χ is a sum of several p -adic irreducible characters φ , the complex analytic theory does not give precise relation between $\#\mathcal{H}_{K,\varphi}$ and $(\mathcal{E}_{K,\varphi} : \mathcal{F}_{K,\varphi})$. In other words, there is no sufficient information, even with the help of p -adic ζ and L -functions.

Deep geometrical methods (Ribet, Mazur–Wiles) were successful, then more arithmetic ones were used to solve the problem for odd and even characters [CoSu2006, Grei1992, Kol2007, PeRi1990, Rib2008, Rub1990, Thai1988] among others. The beginnings of the story around the proof, by Ribet [Rib1976], of the (non-trivial) converse of the Herbrand theorem [Herb1932] on Bernoulli’s numbers and the minus part of the p -class group of the p th cyclotomic field, is given in the survey [Rib2008].

A more complete and recent story is available in Washington's book [Was1997, Chap. 8] in which Thaine's paper is reproduced before the presentation of the various developments.

The non-semi-simple case of the real abelian Main Conjecture does exist and may be split into two frameworks:

(i) The Iwasawa formulation [Iwa1964], "replacing" K by its cyclotomic \mathbb{Z}_p -extension K_∞ and considering semi-simple Galois actions from base fields of prime-to- p degree, also called Main Conjecture without any precision (Lang–Rubin [Lan1990], Greither [Grei1992], Washington's book [Was1997, §§ 13.6, 15.4]); a typical statement being, for $K = \mathbb{Q}(\mu_p)$ and $X := \varprojlim \mathcal{H}_{K_n}$, the equality $\text{char}(X_\varphi) = f(T, \varphi^*) \cdot u(T)$, with $\varphi \in \Phi_K$ even, $u(T)$ invertible in $\mathbb{Z}_p[[T]]$ and the power series $f(T, \varphi^*)$ such that $L_p(s, \varphi^*) = f((1+p)^s - 1, \varphi^*)$ in terms of p -adic L -functions, where $\varphi \mapsto \varphi^*$ by reflection. Nevertheless, in the real case, Greenberg's conjecture makes it somewhat unnecessary and brings back to the finite cases K_n/K as we have explained.

(ii) The case of cyclic extensions K/\mathbb{Q} when $p \mid [K : \mathbb{Q}]$ and $K = K_\chi$; this case corresponds to our conjecture given in [Gra1976, Gra1977] and still unproved for real fields. We refer to the survey [Gra2022^a] devoted to this non semi-simple case using the specific notion of φ -objects that we had introduced in the 1976's. Indeed, classical works deal with an algebraic definition of the φ -components of p -class groups, denoted $\mathcal{H}_{K,\varphi}^{\text{alg}}$, which presents an inconsistency regarding analytic formulas; that is to say, when $g := \text{Gal}(K/\mathbb{Q})$ is cyclic of order $g_\chi \equiv 0 \pmod{p}$:

$$\mathcal{H}_{K,\varphi}^{\text{alg}} := \mathcal{H}_K \otimes_{\mathbb{Z}_p[g]} \mathbb{Z}_p[\mu_{g_\chi}], \text{ for all } \varphi \mid \chi,$$

with the $\mathbb{Z}_p[\mu_{g_\chi}]$ -action $\tau \in g \mapsto \psi(\tau)$ ($\psi \mid \varphi$ of order g_χ). We then have:

$$\mathcal{H}_{K,\varphi}^{\text{alg}} = \{x \in \mathcal{H}_K, \nu_{K/k}(x) = 1, \forall k \subsetneq K\} \otimes_{\mathbb{Z}_p[g]} \mathbb{Z}_p[\mu_{g_\chi}]$$

(where $\nu_{K/k}$ is the algebraic norm), contrary to our arithmetic definition:

$$\mathcal{H}_{K,\varphi}^{\text{ar}} := \{x \in \mathcal{H}_K, \mathbf{N}_{K/k}(x) = 1, \forall k \subsetneq K\} \otimes_{\mathbb{Z}_p[g]} \mathbb{Z}_p[\mu_{g_\chi}].$$

This notion gives rise to an unexpected semi-simplicity especially in accordance with analytic formulas, which enforce the conjecture in that case:

$$\mathcal{H}_{K,\chi}^{\text{ar}} := \{x \in \mathcal{H}_K, \mathbf{N}_{K/k}(x) = 1, \forall k \subsetneq K\} \otimes \mathbb{Z}_p = \bigoplus_{\varphi \mid \chi} \mathcal{H}_{K,\varphi}^{\text{ar}}.$$

2. THE p -LOCALIZATION OF THE CHEVALLEY–HERBRAND FORMULAS

2.1. The higher rank Chevalley–Herbrand formulas. Let K/\mathbb{Q} be any extension. To avoid technical complications, we assume that the given extension L/K , cyclic of degree p^n , $n \geq 1$, of Galois group $G =: \langle \sigma \rangle$, is such that any prime ideal of K , ramified in L/K , is totally ramified and that infinite places do not ramify (i.e., do not complexify) when $p = 2$. Moreover we assume that the set of ramified primes is non-empty.

The basic principle of Chevalley–Herbrand formula for p -class groups defines a filtration of the form $\{\mathcal{H}_L^i\}_{i \geq 0}$, where $\mathcal{H}_L^0 := 1$,

$$\mathcal{H}_L^1 := \mathcal{H}_L^G, \text{ of order } \#\mathcal{H}_K \times \frac{p^{n(r-1)}}{(\mathcal{E}_K : \mathcal{E}_K \cap \mathbf{N}_{L/K}(L^\times))},$$

$\mathcal{H}_L^{i+1}/\mathcal{H}_L^i := (\mathcal{H}_L/\mathcal{H}_L^i)^G$, up to a bound m for which $\mathcal{H}_L^m = \mathcal{H}_L$ [Gra2017]:

$$(1) \quad \#(\mathcal{H}_L^{i+1}/\mathcal{H}_L^i) = \frac{\#\mathcal{H}_K}{\#\mathbf{N}_{L/K}(\mathcal{H}_L^i)} \times \frac{p^{n(r-1)}}{(\mathbf{\Lambda}_K^i : \mathbf{\Lambda}_K^i \cap \mathbf{N}_{L/K}(L^\times))}, \quad i \geq 0,$$

where $r \geq 1$ is the number of primes of K ramified (totally) in L/K , and $\mathbf{\Lambda}_K^i := \{x \in K^\times, (x) = \mathbf{N}_{L/K}(\mathfrak{A}), \mathfrak{A}P_L \in \mathcal{H}_L^i\} \otimes \mathbb{Z}_p$ is a subgroup of finite type of K^\times containing \mathbf{E}_K (with $\mathbf{\Lambda}_K^0 = \mathbf{E}_K$); the quotient $\mathbf{\Lambda}_K^i/\mathbf{\Lambda}_K^i \cap \mathbf{N}_{L/K}(L^\times)$ is of course a p -group of order a divisor of $p^{n(r-1)}$. The first factor is called the class factor and the second one the norm factor.

In the case $r = 1$ (which will be our context, taking a prime ℓ inert in the cyclic extension K/\mathbb{Q} and $L \subset K(\mu_\ell)$), one obtains only the class factors:

$$(2) \quad \#\mathcal{H}_L^G = \#\mathcal{H}_K \quad \& \quad \#(\mathcal{H}_L^{i+1}/\mathcal{H}_L^i) = \frac{\#\mathcal{H}_K}{\#\mathbf{N}_{L/K}(\mathcal{H}_L^i)}, \quad i \geq 1.$$

There are a lot of generalizations of the original Chevalley–Herbrand formula. See for instance [Jau1986, III, p. 167], [Gra2017] with modulus, [LiYu2020] in the spirit of Chevalley’s theory of idèles, then [Gon2008] showing universality of the Chevalley–Herbrand principle in arithmetic and algebraic geometry.

2.2. The p -localizations for \mathcal{H}_L^G . We choose to privilege the aspect “fixed points formula”, instead of genus theory (that is also p -localizable and may lead to parts of the results), to deduce (in §2.2.2) the p -localization of the filtration $(\mathcal{H}_L^i)_{i \geq 1}$, which is of fixed points type, and to prove the stability Theorem 3.5 using Chevalley–Herbrand formula in the p -tower L/K . We think that this is more suitable in a logical point of view since proofs of the deep statements of class field theory in the classical way rely first on Chevalley–Herbrand formula, whence Herbrand theorem, genus theory dealing only with local norms in non necessarily cyclic p -extensions for which Chevalley–Herbrand formula is not valid referring to *global norms*. So, to justify the integrality of some p -localized expressions, one may use, at the end, the deep Hasse norm theorem saying that, in the cyclic case, $x \in K^\times$ is in $\mathbf{N}_{L/K}(L^\times)$ if and only if x is everywhere local norm (except one place). All these comments about class field theory may be found in all the literature, and the classical way is detailed in our book [Gra2005, §II.6; IV (b); Theorem II.6.2], respectively.

2.2.1. Exact sequence of the ambiguous classes. The p -localization of the Chevalley–Herbrand formula will exist from the definitions of class groups, ideal groups and units. Such p -localizations were given many years ago ([Gra1978, Théorèmes II.1, II.2], [Jau1986, Théorèmes III.1.12, III.1.13]); these papers being written in french, we give again, in a more direct manner, the computations for the convenience of the reader.

To get a formula for the orders of the φ -components of \mathcal{H}_L^G , we follow the process given in Jaulent’s Thesis [Jau1986, Chapitre III, p. 167] (note that large generalizations of such methods, with ramification and decomposition, are also given in [Mai1997, Mai1998] and, in the Galois case L/K , [Gon2006] with many references):

Theorem 2.1. *Let L_0/\mathbb{Q} be a real cyclic extension of degree p^n , $n \geq 1$. Let K/\mathbb{Q} be a real abelian extension of Galois group g and of prime-to- p degree. Put $L = L_0K$ and $G := \text{Gal}(L/K) =: \langle \sigma \rangle$. We identify $\text{Gal}(L/L_0)$ and g .*

Let I_K and I_L (resp. P_K and P_L) be the ideal groups (resp. the subgroups of principal ideals), of K and L , respectively. Then, put $\mathbf{H}_K := I_K/P_K$ and $\mathbf{H}_L := I_L/P_L$.

(i) *We have the exact sequence of $\mathbb{Z}[g]$ -modules:*

$$1 \rightarrow \text{Ker}(\mathbf{J}_{L/K}) \rightarrow \text{H}^1(G, \mathbf{E}_L) \rightarrow \text{Coker}(\mathbf{j}_{L/K}) \rightarrow \\ \text{Coker}(\mathbf{J}_{L/K}) \rightarrow \text{H}^2(G, \mathbf{E}_L) \rightarrow \text{H}^2(G, L^\times),$$

where $\mathbf{J}_{L/K}$ is the transfer map $\mathbf{H}_K \rightarrow \mathbf{H}_L^G$ defined by $\mathfrak{a}P_K \mapsto \mathfrak{a}P_L$ and $\mathbf{j}_{L/K}$ is the extension of ideals $I_K \rightarrow I_L^G$.

(ii) *Let φ be an irreducible p -adic character of K . Then:*

- *If $\varphi = 1$, then $\#\mathcal{H}_{L,\varphi}^G = \#\mathcal{H}_{K,\varphi} = 1$;*
- *If $\varphi \neq 1$, then $\#\mathcal{H}_{L,\varphi}^G = \#\mathcal{H}_{K,\varphi} \times \frac{\#\text{Coker}(\mathbf{j}_{L/K})_\varphi}{(\mathcal{E}_{K,\varphi} : \mathcal{E}_{K,\varphi} \cap \mathbf{N}_{L/K}(L^\times))}$.*

Proof. Note that each prime ℓ , ramified in L/K , is totally ramified.

Consider the exact sequences of $\mathbb{Z}[g]$ -modules:

$$(3) \quad (a) \ 1 \rightarrow \mathbf{E}_L \rightarrow L^\times \rightarrow P_L \rightarrow 1, \quad (b) \ 1 \rightarrow \mathbf{E}_K \rightarrow K^\times \rightarrow P_K \rightarrow 1$$

$$(4) \quad (c) \ 1 \rightarrow P_L \rightarrow I_L \rightarrow \mathbf{H}_L \rightarrow 1, \quad (d) \ 1 \rightarrow P_K \rightarrow I_K \rightarrow \mathbf{H}_K \rightarrow 1.$$

Lemma 2.2. *We have the following properties:*

- (i) $P_L^G/\mathbf{j}_{L/K}(P_K) \simeq \text{H}^1(G, \mathbf{E}_L)$;
- (ii) $\text{H}^1(G, P_L) \simeq \text{Ker}[\text{H}^2(G, \mathbf{E}_L) \rightarrow \text{H}^2(G, L^\times)] = \mathbf{E}_K/\mathbf{E}_K \cap \mathbf{N}_{L/K}(L^\times)$;
- (iii) $\text{H}^1(G, I_L) = 1$.

Proof. We have, from the above exact sequences (3) (a), (b):

$$1 \rightarrow \mathbf{E}_L^G = \mathbf{E}_K \rightarrow L^{\times G} = K^\times \rightarrow P_L^G \rightarrow \text{H}^1(G, \mathbf{E}_L) \rightarrow \text{H}^1(G, L^\times) \\ \rightarrow \text{H}^1(G, P_L) \rightarrow \text{H}^2(G, \mathbf{E}_L) \rightarrow \text{H}^2(G, L^\times).$$

Since $\text{H}^1(G, L^\times) = 1$ (Hilbert's Theorem 90), this yields (i) and (ii).

The claim (iii) is classical since I_L is a $\mathbb{Z}[G]$ -module generated by the prime ideals of L on which the Galois action is canonical. \square

From the above exact sequences (4) (c), (d), we have the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_K & \longrightarrow & I_K & \longrightarrow & \mathbf{H}_K & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \mathbf{j}_{L/K} & & \downarrow \mathbf{J}_{L/K} & & \\ 1 & \longrightarrow & P_L^G & \longrightarrow & I_L^G & \longrightarrow & \mathbf{H}_L^G & \longrightarrow & \text{H}^1(G, P_L). \end{array}$$

The snake lemma gives the exact sequence:

$$1 \rightarrow 1 \rightarrow \text{Ker}(\mathbf{j}_{L/K}) \rightarrow \text{Ker}(\mathbf{J}_{L/K}) \rightarrow \text{H}^1(G, \mathbf{E}_L) \rightarrow \\ \text{Coker}(\mathbf{j}_{L/K}) \rightarrow \text{Coker}(\mathbf{J}_{L/K}) \rightarrow \text{H}^2(G, \mathbf{E}_L) \rightarrow \text{H}^2(G, L^\times);$$

since $\text{Ker}(\mathbf{j}_{L/K}) = 1$ and $\text{H}^2(G, L^\times) \simeq K^\times/\mathbf{N}_{L/K}(L^\times)$, it becomes:

$$1 \rightarrow \text{Ker}(\mathbf{J}_{L/K}) \rightarrow \text{H}^1(G, \mathbf{E}_L) \rightarrow \text{Coker}(\mathbf{j}_{L/K}) \rightarrow$$

$$\text{Coker}(\mathbf{J}_{L/K}) \rightarrow \mathbf{H}^2(G, \mathbf{E}_L) \rightarrow \mathbf{E}_K/\mathbf{E}_K \cap \mathbf{N}_{L/K}(L^\times) \rightarrow 1.$$

Remark 2.3. The representation $\text{Coker}(\mathbf{j}_{L/K})$, as g -module, depends on the splitting of the prime ideals of K ramified in L/K and gives standard φ -components from the relation $\#\text{Coker}(\mathbf{j}_{L/K}) = \prod_{\mathfrak{q}} e_{\mathfrak{q}}(L/K)$; we refer to [Gra1978] or [Jau1986] for the most general localized formulas. From Hasse’s norm theorem, the factors $\frac{\#\text{Coker}(\mathbf{j}_{L/K})_{\varphi}}{(\mathcal{E}_{K,\varphi} : \mathcal{E}_{K,\varphi} \cap \mathbf{N}_{L/K}(L^\times))}$ are always integers in the totally ramified case. In our particular context, where ramified primes ℓ are inert in K/\mathbb{Q} and totally ramified in L/K , $\text{Coker}(\mathbf{j}_{L/K})$ is of character 1, so all the factors $\frac{\#\text{Coker}(\mathbf{j}_{L/K})_{\varphi}}{(\mathcal{E}_{K,\varphi} : \mathcal{E}_{K,\varphi} \cap \mathbf{N}_{L/K}(L^\times))}$ are trivial for $\varphi \neq 1$; in particular, any unit of K is a global norm in L/K . Note that if the p -Hilbert class field H_K^{nr} of K is not disjoint from L/K , the Chevalley–Herbrand formula becomes $\#\mathcal{H}_L^G = [H_K^{\text{nr}} : L \cap H_K^{\text{nr}}] \times \frac{\prod_{\mathfrak{q}} e_{\mathfrak{q}}(L/K)}{[L : L \cap H_K^{\text{nr}}] \times (\mathcal{E}_K : \mathcal{E}_K \cap \mathbf{N}_{L/K}(L^\times))}$ giving another localization (e.g., $L = H_K^{\text{nr}}$ cyclic gives $\mathcal{H}_L^G = 1$).

We obtain the p -localized exact sequences of $\mathbb{Z}_p[g]$ -modules:

$$1 \rightarrow \text{Ker}(\mathbf{J}_{L/K})_{\varphi} \rightarrow \mathbf{H}^1(G, \mathcal{E}_{L,\varphi}) \rightarrow \text{Coker}(\mathbf{j}_{L/K})_{\varphi} \rightarrow$$

$$\text{Coker}(\mathbf{J}_{L/K})_{\varphi} \rightarrow \mathbf{H}^2(G, \mathcal{E}_{L,\varphi}) \rightarrow \mathcal{E}_{K,\varphi}/\mathcal{E}_{K,\varphi} \cap \mathbf{N}_{L/K}(L^\times) \rightarrow 1$$

(by p -localization, $\mathbf{H}^1(G, \mathbf{E}_L)_{\varphi} = \mathbf{H}^1(G, \mathcal{E}_{L,\varphi})$, $\mathbf{H}^2(G, \mathbf{E}_L)_{\varphi} = \mathbf{H}^2(G, \mathcal{E}_{L,\varphi})$).

All these $\mathbb{Z}_p[g]$ -modules are finite, which gives the p -localized formula:

$$\frac{\#(\mathcal{H}_{L,\varphi}^G)}{\#(\mathcal{H}_{K,\varphi})} = \frac{\#\text{Coker}(\mathbf{J}_{L/K})_{\varphi}}{\#\text{Ker}(\mathbf{J}_{L/K})_{\varphi}} = \frac{\#\text{Coker}(\mathbf{j}_{L/K})_{\varphi}}{(\mathcal{E}_{K,\varphi} : \mathcal{E}_{K,\varphi} \cap \mathbf{N}_{L/K}(L^\times))} \times \frac{\#\mathbf{H}^2(G, \mathcal{E}_{L,\varphi})}{\#\mathbf{H}^1(G, \mathcal{E}_{L,\varphi})},$$

where $\frac{\#\mathbf{H}^2(G, \mathcal{E}_{L,\varphi})}{\#\mathbf{H}^1(G, \mathcal{E}_{L,\varphi})}$ is the Herbrand quotient of $\mathcal{E}_{L,\varphi}$ we talked about, whose computation leads, for \mathbf{E}_L , to the global value $\frac{1}{[L : K]}$ in the real case (cf. [Lan1990], [Lan2000, Chap. IX, §§ 1,4]).

Lemma 2.4. *The Herbrand quotient of $\mathcal{E}_{L,\varphi}$ is trivial for all $\varphi \in \Phi_K \setminus \{1\}$.*

Proof. We know that $(\mathbf{E}_L \otimes \mathbb{Q}) \oplus \mathbb{Q}$ is the regular representation $\mathbb{Q}[G \times g]$ (see, e.g., [Gra2005, Theorem I.3.7]); so there exists a “Minkowski unit” ε such that the $\mathbb{Z}[G \times g]$ -module generated by ε is of finite index in \mathbf{E}_L that one may choose prime to p ; so \mathcal{E}_L is such that $\mathcal{E}_L \oplus \mathbb{Z}_p \simeq \mathbb{Z}_p[G][g]$ as g -modules. Thus, $(\mathcal{E}_L \oplus \mathbb{Z}_p)_{\varphi} = \mathcal{E}_{L,\varphi} \simeq \mathbb{Z}_p[\mu_{g,\varphi}][G]$ for $\varphi \neq 1$; whence the result. \square

So, $\#(\mathcal{H}_{L,\varphi}^G) = \#(\mathcal{H}_{K,\varphi}) \times \frac{\#\text{Coker}(\mathbf{j}_{L/K})_{\varphi}}{(\mathcal{E}_{K,\varphi} : \mathcal{E}_{K,\varphi} \cap \mathbf{N}_{L/K}(L^\times))}$, for $\varphi \neq 1$, and $\mathcal{H}_{L,1}^G = \mathcal{H}_{K,1} = 1$, which completes the proof of the Theorem. \square

2.2.2. The main exact sequences associated to the filtration. We give, without proofs, the exact sequences leading to the formulas (1) giving the orders of $\mathcal{H}_L^{i+1}/\mathcal{H}_L^i := (\mathcal{H}_L/\mathcal{H}_L^i)^G$ for $i \geq 1$, and justifying the p -localizations of the formulas (see [Gra2017, Section 3] for the details). Moreover, we will use essentially the case of $(\mathcal{H}_L/\mathcal{H}_L^G)^G$.

Let \mathcal{H} be a sub- G -module of \mathcal{H}_L . Put $\widetilde{\mathcal{H}} = \{h \in \mathcal{H}_L, h^{1-\sigma} \in \mathcal{H}\}$; so $(\mathcal{H}_L/\mathcal{H})^G = \widetilde{\mathcal{H}}/\mathcal{H}$. We have the exact sequences:

$$1 \longrightarrow \mathcal{H}_L^G \longrightarrow \widetilde{\mathcal{H}} \xrightarrow{1-\sigma} (\widetilde{\mathcal{H}})^{1-\sigma} \longrightarrow 1$$

$$1 \longrightarrow {}_{\mathbf{N}}\mathcal{H} \longrightarrow \mathcal{H} \xrightarrow{\mathbf{N}_{L/K}} \mathbf{N}_{L/K}(\mathcal{H}) \longrightarrow 1.$$

Let $\mathcal{I} \subset I_L$ be such that $\mathcal{I}P_L/P_L = \mathcal{H}$; so:

$$\mathbf{N}_{L/K}(\mathcal{H}) = \mathbf{N}_{L/K}(\mathcal{I})P_K/P_K.$$

Let $\mathbf{\Lambda} := \{x \in K^\times, (x) \in \mathbf{N}_{L/K}(\mathcal{I})\}$; the fundamental exact sequence is then:

$$1 \rightarrow (\mathbf{E}_K \mathbf{N}_{L/K}(L^\times)) \cap \mathbf{\Lambda} \rightarrow \mathbf{\Lambda} \xrightarrow{\varphi} {}_{\mathbf{N}}\mathcal{H}/(\widetilde{\mathcal{H}})^{1-\sigma} \rightarrow 1,$$

where, for all $x \in \mathbf{\Lambda}$, $\varphi(x) = \mathfrak{A}P_L \pmod{(\widetilde{\mathcal{H}})^{1-\sigma}}$, for any $\mathfrak{A} \in \mathcal{I}$ such that $\mathbf{N}_{L/K}(\mathfrak{A}) = (x)$. This exact sequence is p -localizable since $\mathbf{\Lambda}$, containing \mathbf{E}_K , is a sub- \mathbb{Z} -module of finite type of K^\times . We deduce from the above:

$$(5) \quad (\widetilde{\mathcal{H}} : \mathcal{H}) = \frac{\#\mathcal{H}_L^G \cdot \#(\widetilde{\mathcal{H}})^{1-\sigma}}{\#\mathbf{N}_{L/K}(\mathcal{H}) \cdot \#{}_{\mathbf{N}}\mathcal{H}} = \frac{\#\mathcal{H}_L^G}{\#\mathbf{N}_{L/K}(\mathcal{H}) \cdot ({}_{\mathbf{N}}\mathcal{H} : (\widetilde{\mathcal{H}})^{1-\sigma})};$$

$$\text{thus } \#(\mathcal{H}_L/\mathcal{H})^G = \frac{\#\mathcal{H}_L^G}{\#\mathbf{N}_{L/K}(\mathcal{H}) \cdot (\mathbf{\Lambda} \mathbf{E}_K \mathbf{N}_{L/K}(L^\times) : \mathbf{E}_K \mathbf{N}_{L/K}(L^\times))}.$$

Then Chevalley–Herbrand formula gives the final p -localized result.

Corollary 2.5. *Let K be a cyclic real field of prime-to- p degree and let $L \subset K(\mu_\ell)$, $\ell \equiv 1 \pmod{2p^N}$, inert in K , with $[L : K] = p^n$, $n \in [1, N]$. Then for all $\varphi \in \Phi_K$, the filtrations defined by $\mathcal{H}_{L,\varphi}^{i+1}/\mathcal{H}_{L,\varphi}^i := (\mathcal{H}_{L,\varphi}/\mathcal{H}_{L,\varphi}^i)^G$, fulfill the higher rank Chevalley–Herbrand formulas:*

$$\#\mathcal{H}_{L,\varphi}^G = \#\mathcal{H}_{K,\varphi} \quad \text{and} \quad \#(\mathcal{H}_{L,\varphi}^{i+1}/\mathcal{H}_{L,\varphi}^i) = \frac{\#\mathcal{H}_{K,\varphi}}{\#\mathbf{N}_{L/K}(\mathcal{H}_{L,\varphi}^i)} \quad \text{for all } i \geq 1.$$

Proof. We then have a unique place (ℓ) of K , totally ramified in L/K . So any $x \in \mathbf{\Lambda}$ being norm of an ideal (that we may choose prime to (ℓ)), it is local norm at any place distinct from (ℓ) ; then the product formula of the Hasse norm symbols [Gra2005, Theorem II.3.4.1] gives that x is everywhere local norm, hence global norm (Hasse’s norm theorem).

This applies to the group of units for which $\mathcal{E}_{K,\varphi} \subset \mathbf{N}_{L/K}(L^\times)$, whence $(\mathcal{E}_{K,\varphi} : \mathcal{E}_{K,\varphi} \cap \mathbf{N}_{L/K}(L^\times)) = 1$ (recall that under the unicity of the ramified prime ideal in L/K , $\text{Coker}(\mathbf{j}_{L/K})_\varphi = 1$ for $\varphi \neq 1$; see Remark 2.3). \square

Of course, this does not mean $\mathcal{H}_{L,\varphi}^G = \mathbf{J}_{L/K}(\mathcal{H}_{K,\varphi})$ since there is most often capitulation of classes; this expresses the subtlety of Chevalley–Herbrand formula for which we shall give another description of $\mathcal{H}_{L,\varphi}^G$ likely to involve the kernel of capitulation.

3. EXACT SEQUENCE OF CAPITULATION

We consider a real cyclic extension K/\mathbb{Q} of prime-to- p degree and $L = KL_0$ with L_0/\mathbb{Q} cyclic of degree p^n , $n \geq 1$. As for the case $L \subset K(\mu_\ell)$, we assume to simplify that any ramified prime is totally ramified in L/K . Put $G := \text{Gal}(L/K) =: \langle \sigma \rangle$.

Let $\mathfrak{A}P_L \in \mathbf{H}_L$ be a class invariant under G ; thus $\mathfrak{A}^{1-\sigma} = \alpha P_L$, $\alpha \in L^\times$, and $\mathbf{N}_{L/K}(\alpha)$ is a unit $\varepsilon \in \mathbf{E}_K \cap \mathbf{N}_{L/K}(L^\times)$; if $\alpha' = \alpha\eta$, $\eta \in \mathbf{E}_L$, is another generator of $\mathfrak{A}^{1-\sigma}$, then $\mathbf{N}_{L/K}(\alpha') = \varepsilon \mathbf{N}_{L/K}(\eta)$. This defines the map:

$$\mathbf{H}_L^G \longrightarrow \mathbf{E}_K \cap \mathbf{N}_{L/K}(L^\times) / \mathbf{N}_{L/K}(\mathbf{E}_L),$$

which associates with $\mathfrak{A}P_L \in \mathbf{H}_L^G$ the class of the unit $\varepsilon = \mathbf{N}_{L/K}(\alpha)$.

Theorem 3.1. *We have, for all $\varphi \in \Phi_K$, the exact sequences:*

$$1 \rightarrow \mathbf{J}_{L/K}(\mathcal{H}_{K,\varphi}) \cdot \mathcal{H}_{L,\varphi}^{\text{ram}} \rightarrow \mathcal{H}_{L,\varphi}^G \rightarrow \mathcal{E}_{K,\varphi} \cap \mathbf{N}_{L/K}(L^\times) / \mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi}) \rightarrow 1,$$

where $\mathcal{H}_L^{\text{ram}} \subseteq \mathcal{H}_L^G$ is generated by the classes of the ramified prime ideals.

Proof. We shall establish the global exact sequence:

$$1 \rightarrow \mathbf{J}_{L/K}(\mathbf{H}_K) \cdot \mathbf{H}_L^{\text{ram}} \rightarrow \mathbf{H}_L^G \rightarrow \mathbf{E}_K \cap \mathbf{N}_{L/K}(L^\times) / \mathbf{N}_{L/K}(\mathbf{E}_L) \rightarrow 1.$$

(i) Image. Let $\varepsilon \in \mathbf{E}_K \cap \mathbf{N}_{L/K}(L^\times)$; put $\varepsilon = \mathbf{N}_{L/K}(\alpha)$, then the ideal αP_L being of norm 1 is of the form $\mathfrak{A}^{1-\sigma}$, $\mathfrak{A} \in I_L$ (Lemma 2.2 (iii)), and its class is invariant giving the pre-image.

(ii) Kernel. Suppose that the image of the invariant class $\mathfrak{A}P_L$ is $\varepsilon = \mathbf{N}_{L/K}(\eta)$, $\eta \in \mathbf{E}_L$; then $\mathbf{N}_{L/K}(\alpha) = \mathbf{N}_{L/K}(\eta)$ and $\mathbf{N}_{L/K}(\alpha\eta^{-1}) = 1$ giving $\alpha\eta^{-1} = \beta^{1-\sigma}$, $\beta \in L^\times$ (Hilbert's Theorem 90) then $\alpha P_L = \mathfrak{A}^{1-\sigma} = (\beta P_L)^{1-\sigma}$. So, the class $\mathfrak{A}P_L$ is the class of the invariant ideal $\mathfrak{A}(\beta)^{-1}$; but the group of invariants ideals is generated by $\mathbf{j}_{L/K}(I_K)$ and the ramified primes of L since ramification is total. Whence the result. \square

Corollary 3.2. *Let K be a cyclic real field of prime-to- p degree and let $L \subset K(\mu_\ell)$, $\ell \equiv 1 \pmod{2p^N}$, inert in K , with $[L : K] = p^n$, $n \in [1, N]$. Then:*

(i) *For all $\varphi \in \Phi_K \setminus \{1\}$, we have $\#\mathcal{H}_{L,\varphi}^G = \#\mathcal{H}_{K,\varphi}$ (Corollary 2.5) and the exact sequences $1 \rightarrow \mathbf{J}_{L/K}(\mathcal{H}_{K,\varphi}) \rightarrow \mathcal{H}_{L,\varphi}^G \rightarrow \mathcal{E}_{K,\varphi} / \mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi}) \rightarrow 1$.*

(ii) *The capitulation of \mathcal{H}_K in L/K is equivalent to:*

$$\mathcal{H}_{L,\varphi}^G \simeq \mathcal{E}_{K,\varphi} / \mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi}), \text{ for all } \varphi \in \Phi_K;$$

if so, $\mathcal{H}_{L,\varphi}^G \simeq \mathbb{Z}_p[\mu_{g_\varphi}] / (p^{a_\varphi}) \mathbb{Z}_p[\mu_{g_\varphi}]$, a_φ being such that $p^{\rho_\varphi a_\varphi} = \#\mathcal{H}_{K,\varphi}$, where $\rho_\varphi = [\mathbb{Q}_p(\mu_{g_\varphi}) : \mathbb{Q}_p]$, and $\#\mathcal{H}_{K,\varphi} = (\mathcal{E}_{K,\varphi} : \mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi}))$.

Proof. Exact sequence in (i) comes from the fact that $\mathcal{H}_L^{\text{ram}}$, generated by the prime ideal $\mathfrak{l} \mid \mathfrak{l}$ for the unique prime ideal $\mathfrak{l} = (\ell)$ of K , is of character $\varphi = 1$ and from the fact that $\text{Coker}(\mathbf{j}_{L/K})_\varphi = (\mathcal{E}_{K,\varphi} : \mathcal{E}_{K,\varphi} \cap \mathbf{N}_{L/K}(L^\times)) = 1$ for $\varphi \neq 1$ (Remark 2.3).

Equivalence (ii) comes from the exact sequence. The last claims come from the monogenicity of $\mathcal{E}_{K,\varphi}$ as $\mathbb{Z}_p[\mu_{g_\varphi}]$ -module and from the equality $\#\mathcal{H}_{L,\varphi}^G = \#\mathcal{H}_{K,\varphi}$. \square

3.1. Test of capitulation – Numerical illustrations. Recall that for $\ell \equiv 1 \pmod{p^N}$, inert in K , we consider $L \subset K(\mu_\ell)$ of degree p^n over K , $n \in [1, N]$. In the computations L is often denoted K_n and G is denoted G_n , etc.

To verify in practice some capitulations in L/K , we use the relation:

$$\mathbf{J}_{L/K}(\mathbf{H}_K) = \mathbf{J}_{L/K}(\mathbf{N}_{L/K}(\mathbf{H}_L)) = \nu_{L/K}(\mathbf{H}_L),$$

since $\mathbf{N}_{L/K} : \mathbf{H}_L \rightarrow \mathbf{H}_K$ is surjective (L/K is totally ramified), and we compute the algebraic norm of the generators h_i of \mathbf{H}_L given by PARI. So, we obtain explicite relations $\nu_{L/K}(h_i) = \prod_j h_j^{a_{i,j}}$ in \mathbf{H}_L , the complete capitulation being given by the identity $\nu_{L/K}(\mathcal{H}_L) = \langle \dots, \prod_j h_j^{a_{i,j}}, \dots \rangle_i \otimes \mathbb{Z}_p = 1$ and incomplete capitulations are deduced from the matrices $(a_{i,j})$.

We will give numerical examples showing what happens, in the isomorphism (ii) of the corollary since $\mathcal{E}_{K,\varphi}/\mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi})$ is monogenic as Galois module, while $\mathcal{H}_{K,\varphi}$ is not in general.

(i) Let's give, first, a few words about the important PARI instruction `bnfispprincipal`, of constant use in the computations, to prove capitulation of a class $\mathfrak{a}P_K$ in L , whence that $\mathfrak{a}P_L$ is principal; it is described as follows by [Pari2013]:

```
bnfispprincipal(bnf,x,{flag=1}):bnf being output by bnfinit
(with flag<=2), gives [v,alpha], where v is the vector of
exponents on the class group generators and alpha is the
generator of the resulting principal ideal. In particular
x is principal if and only if v is the zero vector.
```

Thus, the most important output is the vector of exponents like $[0, 0]$ meaning total capitulation of the selected p -class. Nevertheless, these vectors may be only 0 modulo the order of the p -classes.

(ii) Let's give an example for K cubic and $p = 2$.

Example 3.3. We consider a cyclic cubic field with $p = 2$, for which $\mathcal{H}_K \simeq \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, $\mathcal{H}_{K_1} \simeq \mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathcal{H}_{K_2} \simeq \mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, but there is complete capitulation of \mathcal{H}_K in K_2 for $\ell = 449$ inert in K (CK, CK1, CK2 represent \mathcal{H}_K , \mathcal{H}_{K_1} , \mathcal{H}_{K_2} , respectively):

```
conductor f=2817 PK=x^3 - 939*x + 6886 CK=[12,4]
e11=449 N=2 Nn=2 n=1 CK=[12,4] CK1=[24,8,2,2]
norm in K1/K of the component 1 of CK1: [12,0,0,1]~
norm in K1/K of the component 2 of CK1: [12,0,1,0]~
norm in K1/K of the component 3 of CK1: [0,0,0,0]~
norm in K1/K of the component 4 of CK1: [0,0,0,0]~

e11=449 N=2 Nn=2 n=2 CK=[12,4] CK2=[48,16,2,2]
norm in K2/K of the component 1 of CK2: [0,0,0,0]~
norm in K2/K of the component 2 of CK2: [0,0,0,0]~
norm in K2/K of the component 3 of CK2: [0,0,0,0]~
norm in K2/K of the component 4 of CK2: [0,0,0,0]~
```

(iii) Let's give another example for K quadratic and $p = 3$.

Example 3.4. Let $m = 32009$, $K = \mathbb{Q}(\sqrt{m})$ for which $\mathcal{H}_K \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Take $\ell = 19$ (inert in K). The general Program 6.1 gives an incomplete capitulation in K_1 , then a total capitulation in K_2 (in this last data for $n = 2$, we give the 18 integer coefficients of a generator of the ideal on the integral basis computed by PARI):

```
PK=x^2 - 32009 CK=[3,3]
e11=19 N=2 Nn=2 n=1 CK=[3,3] CK1=[9,3]
norm in K1/K of the component 1 of CK1: [3,0]~
norm in K1/K of the component 2 of CK1: [0,0]~

e11=19 N=2 Nn=2 n=2 CK=[3,3] CK2=[9,3]
norm in K2/K of the component 1 of CK2: [0,0]~
norm in K2/K of the component 2 of CK2: [0,0]~
```

```
[[0,0]~,
[1439216371631838382,473754414131112320,454228965737496519,
-536418025036156085,919689041243214339,207983767848706102,
-1036595574155274193,128338672307382267,840355575133838069,
14736364857686206,6548993298829283,8896045582984518,
3839859983910278,7389856141626720,10096758515087182,
-895338434531174,-1442732225425698,8004210362478777]~]
```

In K_1/K , the exact sequence looks like:

$$1 \rightarrow \mathbf{J}_{K_1/K}(\mathcal{H}_K) \simeq \mathbb{Z}/3\mathbb{Z} \longrightarrow \mathcal{H}_{K_1}^{G_1} \longrightarrow \mathcal{E}_K/\mathbf{N}_{K_1/K}(\mathcal{E}_{K_1}) \simeq \mathbb{Z}/3\mathbb{Z} \rightarrow 1,$$

the structure of $\mathcal{H}_{K_1}^{G_1}$ being a priori unknown. A direct computation shows that $\mathcal{H}_{K_1}^{G_1} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, but it is not $\mathbf{J}_{K_1/K}(\mathcal{H}_K)$ since $\#\text{Ker}(\mathbf{J}_{K_1/K}) = 3$.

In K_2/K , the exact sequence becomes the isomorphism:

$$\mathcal{H}_{K_2}^{G_2} \simeq \mathcal{E}_K/\mathbf{N}_{K_2/K}(\mathcal{E}_{K_2}) \simeq \mathbb{Z}/9\mathbb{Z},$$

since $\mathbf{J}_{K_2/K}(\mathcal{H}_K) = 1$ and $\mathcal{E}_K \simeq \mathbb{Z}_3$. So, we intend to find a generator of $\mathcal{H}_{K_2}^{G_2}$. Taking the class of order 9 (first component of \mathcal{H}_{K_2} given by the instruction `A0 = Kn.clgp[3][1]`), we compute its conjugate by the automorphism `S` of order 9:

```
B0 = nfgaloisapply(Kn, S, A0), then C0 = idealpow(Kn, B0, 8) and
R = idealmul(Kn, A0, C0) for which we apply the test:
U = bnfisprincipal(Kn, R),
```

giving a principal integer with huge integer coefficients:

```
A0=Kn.clgp[3][1];B0=nfgaloisapply(Kn,S,A0);C0=idealpow(Kn,B0,8);
R=idealmul(Kn,A0,C0);U=bnfisprincipal(Kn,R);print(U)
[[0, 0],
[-15352694895259448716005913288179,-4937712840022370286191878614596,
10234788031577460568990927971879,9473644150178364411380147768767,
919093855688240643550377510520,4933150036914472598668945255159,
-8186088867315265238068365774860,-12462519184163404099427848753248,
-10116044365250124206306632744945,-32490191902858719490198341631,
-25043275245516486863055415213,150430762033938424462889500018,
83650512449675221885273689474,-243856275294066992198407658217,
229543001775729020765314244564,-195555194837852410495787525040,
140098686428490793581998673118,26378140206989683079187437611]~]
```

This confirms that $\mathcal{H}_{K_2}^{G_2}$ is cyclic of order 9 (cf. Corollary 3.2 (ii)).

This phenomenon is general (when the capitulation is complete), for a p -class group of the form (say, for K quadratic) $\mathcal{H}_K \simeq \mathbb{Z}/p^{a_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{a_r}\mathbb{Z}$ and gives, at a layer $n \geq a := a_1 + \cdots + a_r$, the isomorphism:

$$\mathcal{H}_{K_n}^{G_n} \simeq \mathcal{E}_K/\mathbf{N}_{K_n/K}(\mathcal{E}_{K_n}) \simeq \mathbb{Z}/p^a\mathbb{Z},$$

but $\#\mathcal{H}_{K_n}^{G_n} = \#\mathcal{H}_K$. This is typical of the Main Conjecture philosophy and will be enforced by the analytic framework, recalled in the next subsection, showing that non-cyclic structures of the base field K (i.e., that of $\mathcal{H}_{K,\varphi}$) leads to canonical ones by extension in L (i.e., as quotients of $\mathcal{E}_{K,\varphi} \simeq \mathbb{Z}_p[\mu_{g,\varphi}]$), whatever the p -rank and the exponent of \mathcal{H}_K .

It seems that this “monogenicity”, by suitable cyclic p -extensions, has not been remarked in the literature. Unfortunately, proof of capitulations are perhaps out of reach despite their obviousness in the practice.

3.2. The stability as sufficient condition of capitulation. Now, we give a sufficient condition of capitulation (see the comments given in Remark 1.3):

Theorem 3.5. *Consider a prime $\ell \equiv 1 \pmod{2p^N}$, inert in the cyclic real field K , and $K_n \subset K(\mu_\ell)$, of degree p^n over K , $n \in [1, N]$.*

Then $\#\mathcal{H}_{K_n} = \#\mathcal{H}_K$ for all n if and only if $\#\mathcal{H}_{K_1} = \#\mathcal{H}_K$. If this criterion applies, then $\mathcal{H}_{K_n}^{G_n} = \mathcal{H}_{K_n}$ for all n , $\text{Ker}(\mathbf{J}_{K_n/K}) = \mathbf{N}_{K_n/K}(\mathcal{H}_{K_n}[p^n])$, and if p^e is the exponent of \mathcal{H}_K , then \mathcal{H}_K capitulates in K_e (assuming $N \geq e$).

Proof. Consider $\text{Gal}(K_n/K_1) = G_n^p$. Then we have the Chevalley–Herbrand formulas $\#\mathcal{H}_{K_n}^{G_n} = \#\mathcal{H}_K$ and $\#\mathcal{H}_{K_n}^{G_n^p} = \#\mathcal{H}_{K_1}$. But $\mathcal{H}_{K_n}^{G_n} \subseteq \mathcal{H}_{K_n}^{G_n^p}$; then under the condition $\#\mathcal{H}_{K_1} = \#\mathcal{H}_K$, we get $\mathcal{H}_{K_n}^{G_n} = \mathcal{H}_{K_n}^{G_n^p}$, equivalent to

$$\mathcal{H}_{K_n}^{1-\sigma_n} = \mathcal{H}_{K_n}^{1-\sigma_n^p} = \mathcal{H}_{K_n}^{(1-\sigma_n) \cdot \theta},$$

where $\theta = 1 + \sigma_n + \dots + \sigma_n^{p-1} \in (p, 1 - \sigma_n)$, a maximal ideal of $\mathbb{Z}_p[G_n]$ since $\mathbb{Z}_p[G_n]/(p, 1 - \sigma_n) \simeq \mathbb{F}_p$; so $\mathcal{H}_{K_n}^{1-\sigma_n} = 1$, thus $\mathcal{H}_{K_n} = \mathcal{H}_{K_n}^{G_n}$ for all $n \in [1, N]$.

Reciprocal is trivial.

From $\mathbf{N}_{K_n/K}(\mathcal{H}_{K_n}) = \mathcal{H}_K$, $\mathcal{H}_{K_n} = \mathcal{H}_{K_n}^{G_n}$ and $\mathbf{J}_{K_n/K} \circ \mathbf{N}_{K_n/K} = \nu_{K_n/K}$, one gets $\mathbf{J}_{K_n/K}(\mathcal{H}_{K_n}) = \mathbf{J}_{K_n/K}(\mathbf{N}_{K_n/K}(\mathcal{H}_{K_n})) = \nu_{K_n/K}(\mathcal{H}_{K_n}) = \mathcal{H}_{K_n}^{p^n}$. Let $c \in \text{Ker}(\mathbf{J}_{K_n/K})$ and put $c = \mathbf{N}_{K_n/K}(C)$, $C \in \mathcal{H}_{K_n}$; so $1 = \mathbf{J}_{K_n/K}(c) = \mathbf{J}_{K_n/K}(\mathbf{N}_{K_n/K}(C)) = C^{p^n}$, and $\text{Ker}(\mathbf{J}_{K_n/K}) \subseteq \mathbf{N}_{K_n/K}(\mathcal{H}_{K_n}[p^n])$.

Reciprocally, if $c = \mathbf{N}_{K_n/K}(C)$, $C^{p^n} = 1$, then:

$$\mathbf{J}_{K_n/K}(c) = \mathbf{J}_n(\mathbf{N}_{K_n/K}(C)) = C^{p^n} = 1;$$

whence:

$$\text{Ker}(\mathbf{J}_{K_n/K}) = \mathbf{N}_{K_n/K}(\mathcal{H}_{K_n}[p^n]) \subseteq \mathcal{H}_K[p^n].$$

For $n = e$, one obtains the capitulation of \mathcal{H}_K in K_e . □

Remarks 3.6. (i) Since all the relations and exact sequences defining the filtration p -localize, the stability relation $\#\mathcal{H}_{K_1, \varphi} = \#\mathcal{H}_{K, \varphi}$ implies the capitulation of $\mathcal{H}_{K, \varphi}$ in K_e .

(ii) From formula of Corollary 2.5, the assumption $\#\mathcal{H}_{K_1} = \#\mathcal{H}_K$ is equivalent to $\frac{\#\mathcal{H}_K}{\#\mathbf{N}_{K_1/K}(\mathcal{H}_{K_1}^{G_1})} = 1$; indeed, if $\#\mathcal{H}_{K_1} = \#\mathcal{H}_K$, then from Theorem

3.5, $\mathcal{H}_{K_1} = \mathcal{H}_{K_1}^{G_1}$, so $\mathbf{N}_{K_1/K}(\mathcal{H}_{K_1}^{G_1}) = \mathbf{N}_{K_1/K}(\mathcal{H}_{K_1}) = \mathcal{H}_K$.

If $\frac{\#\mathcal{H}_K}{\#\mathbf{N}_{K_1/K}(\mathcal{H}_{K_1}^{G_1})} = 1$, then the filtration stops and $\mathcal{H}_{K_1}^{G_1} = \mathcal{H}_{K_1}$, whence $\#\mathcal{H}_{K_1} = \#\mathcal{H}_{K_1}^{G_1} = \#\mathcal{H}_K$.

(iii) The same criterion holds if one replaces K by K_{n_0} for some $n_0 \geq 1$, under the condition $N \geq n_0 + e$; the fact that K_{n_0}/\mathbb{Q} is not of prime-to- p degree does not matter (proof of Theorem 3.5 does not need this assumption and requires that L/K_{n_0} be totally ramified at a unique place; the notation $\mathcal{H}_{K_n, \varphi}$ is still relative to characters of K , which makes sense since $\text{Gal}(K_n/\mathbb{Q}) \simeq G_n \times g$).

4. CRUCIAL LINK BETWEEN CAPITULATION AND MAIN CONJECTURE

4.1. Analytcs – The group of cyclotomic units. This aspect being very classical, we just recall the main definitions and needed results. For the main definitions and properties of cyclotomic units, see [Leo1954, §8 (1)], [Leo1962] or [Was1997, Chap. 8].

Definition 4.1. (i) Let $\chi \in \mathbf{X}$ even of conductor f_χ ; we define the “cyclotomic numbers” $\theta_\chi := \prod_{a \in A_\chi} (\zeta_{2f_\chi}^a - \zeta_{2f_\chi}^{-a})$, with $\zeta_{2f_\chi} := \exp\left(\frac{i\pi}{f_\chi}\right)$, where A_χ is a half-system of representatives of $\text{Gal}(\mathbb{Q}(\mu_{f_\chi})/K_\chi)$ in $(\mathbb{Z}/f_\chi\mathbb{Z})^\times$.

(ii) Let K be a real abelian field and let \mathbf{F}_K be the intersection with \mathbf{E}_K of the multiplicative group generated by the conjugates of θ_χ , for all $\chi \in \mathbf{X}_K$.

Recall that $\theta_\chi^2 \in K_\chi$ and that $\frac{\theta'_\chi}{\theta_\chi} \in \mathbf{E}_{K_\chi}$ for any conjugate θ'_χ of θ_χ . If f_χ is not a prime power, $\theta_\chi \in \mathbf{E}_{K_\chi}$. Since we will consider φ -components of the θ_χ , for $\varphi \neq 1$, one gets always units of K_χ .

These units lead to an analytic computation of $\#\mathbf{H}_\chi$, $\chi \in \mathbf{X}$ even, $\chi \neq 1$ (using Theorem 1.5). One obtains, in the semi-simple case:

Theorem 4.2. *For all $\chi \in \mathbf{X}$, even of prime-to- p order, $\#\mathcal{H}_\chi = (\mathcal{E}_\chi : \mathcal{F}_\chi)$.*

The philosophy of the abelian Main Conjecture is to ask if the analogous relations $\#\mathcal{H}_\varphi = (\mathcal{E}_\varphi : \mathcal{F}_\varphi)$ exist or not, since we only know that:

$$(6) \quad \#\mathcal{H}_\chi = \prod_{\varphi|\chi} \#\mathcal{H}_\varphi = (\mathcal{E}_\chi : \mathcal{F}_\chi) = \prod_{\varphi|\chi} (\mathcal{E}_\varphi : \mathcal{F}_\varphi).$$

4.2. Norm properties of cyclotomic units. We mention, first, the classical norm property of cyclotomic units that are given in many books and articles, but are crucial for our purpose:

Proposition 4.3. *Let $f > 1$ and then let $m \mid f$, with $m > 1$, be any modulus; let $\mathbb{Q}^m := \mathbb{Q}(\zeta_m) \subseteq \mathbb{Q}^f := \mathbb{Q}(\zeta_f)$ be the corresponding cyclotomic fields with $\zeta_t := \exp\left(\frac{2i\pi}{t}\right)$, for all $t \geq 1$. Put $\eta_{\mathbb{Q}^f} := 1 - \zeta_f$, $\eta_{\mathbb{Q}^m} := 1 - \zeta_m$; we have $\mathbf{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\eta_{\mathbb{Q}^f}) = \eta_{\mathbb{Q}^m}^\Omega$, with $\Omega = \prod_{\ell|f, \ell \nmid m} (1 - (\frac{\mathbb{Q}^m}{\ell})^{-1})$, where $(\frac{\mathbb{Q}^m}{\ell}) \in \text{Gal}(\mathbb{Q}^m/\mathbb{Q})$ denotes the Frobenius (or Artin) automorphism of the prime number $\ell \nmid m$, that is to say such that $\zeta_m \mapsto \zeta_m^\ell$.*

Proof. To simplify, denote by τ_a , a prime to f , the Artin automorphism $(\frac{\mathbb{Q}^f}{a})$ defined by $\zeta_f \mapsto \zeta_f^a$, then put $\eta_{\mathbb{Q}^f} =: \eta_f$, $\eta_{\mathbb{Q}^m} =: \eta_m$.

We consider, by induction, the case $f = \ell \cdot m$, with ℓ prime and examine the two cases $\ell \nmid m$ and $\ell \mid m$. We have $\mathbf{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\eta_f) = \prod_a \eta_f^{\tau_a}$ where a runs through the integers $a \in [1, f]$ prime to f and such that $a \equiv 1 \pmod{m}$.

(i) Case $\ell \nmid m$. Put $a = 1 + \lambda \cdot m$, $\lambda \in [0, \ell - 1]$, but we must exclude a unique $\lambda^* \in [0, \ell - 1]$ such that $1 + \lambda^* \cdot m \equiv 0 \pmod{\ell}$; put $1 + \lambda^* m = \mu\ell$. Thus:

$$\begin{aligned} \mathbf{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\eta_f) &= \prod_{\lambda \in [0, \ell - 1], \lambda \neq \lambda^*} (1 - \zeta_f^{1 + \lambda m}) \\ &= \frac{\prod_{\lambda \in [0, \ell - 1]} (1 - \zeta_f \zeta_f^{\lambda m})}{1 - \zeta_f^{\mu\ell}} = \frac{1 - \zeta_f^\ell}{1 - \zeta_m^\mu} = \frac{1 - \zeta_m}{1 - \zeta_m^\mu}. \end{aligned}$$

Since $\mu \equiv \ell^{-1} \pmod{m}$, we get $\mathbf{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\eta_f) = \eta_m^{1 - \tau_\ell^{-1}}$.

(ii) If $\ell \mid m$, any $\lambda \in [0, \ell - 1]$ is suitable, giving $\mathbf{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\eta_f) = \eta_m$. \square

Corollary 4.4. *Let $L/K/\mathbb{Q}$ be real abelian extensions where L is of conductor f and K of conductor m ; set $\eta_L := \mathbf{N}_{\mathbb{Q}^f/L}(\eta_f)$ and $\eta_K := \mathbf{N}_{\mathbb{Q}^m/K}(\eta_m)$. Then:*

$$\mathbf{N}_{L/K}(\eta_L) = \eta_K^\Omega, \quad \text{with } \Omega = \prod_{\ell \mid f, \ell \nmid m} \left(1 - \left(\frac{K}{\ell}\right)^{-1}\right).$$

If moreover K/\mathbb{Q} is a cyclic extension of prime-to- p degree, and if all the primes $\ell \mid f$, $\ell \nmid m$, are inert in K , then $\Omega e_\varphi = \prod_{\ell \mid f, \ell \nmid m} \left(1 - \left(\frac{K}{\ell}\right)^{-1}\right) e_\varphi$ is an invertible element of the algebra $\mathbb{Z}_p[g]e_\varphi \simeq \mathbb{Z}_p[\mu_{g_\varphi}]$, for all $\varphi \in \Phi_K \setminus \{1\}$. In particular, $\mathbf{N}_{L/K}\mathcal{F}_{L,\varphi} = \mathcal{F}_{K,\varphi}$ for all $\varphi \in \Phi_K \setminus \{1\}$.

Proof. Indeed, $\tau_{\ell,K} := \left(\frac{K}{\ell}\right)$ is a generator of $g = \text{Gal}(K/\mathbb{Q})$ since ℓ is inert in K/\mathbb{Q} ; so for $\psi \mid \varphi \neq 1$, $\psi(1 - \tau_{\ell,K}^{-1}) = 1 - \psi(\tau_{\ell,K}^{-1})$ is a unit of $\mathbb{Q}(\mu_{g_\varphi})$ if $\psi(\tau_{\ell,K}^{-1})$ is not of prime power order, otherwise, if g_φ is a power of a prime q , then since $q \neq p$, $(1 - \psi(\tau_{\ell,K}^{-1}))$ is a prime ideal above q in $\mathbb{Q}(\mu_{g_\varphi})$ and $1 - \psi(\tau_{\ell,K}^{-1})$ is a p -adic unit. Whence the norm relation between the p -localized groups of cyclotomic units for $\varphi \neq 1$. \square

Remark 4.5. The link with the Leopoldt definition of cyclotomic units is easy since we get $\zeta_{2f} - \zeta_{2f}^{-1} = -\zeta_{2f}^{-1}(1 - \zeta_{2f}^2) = -\zeta_{2f}^{-1}(1 - \zeta_f) = -\zeta_{2f}^{-1}\eta_f$, which has no consequence for Proposition 4.3 and its Corollary since norms are taken over real fields L , K . Numerically, η_L and η_K must be replaced by suitable square roots due to the use of the half-system A_χ .

4.3. Final statement. So, we can state and prove the main result involving the transfer map $\mathbf{J}_{L/K}$ and its p -localized images $\mathbf{J}_{L/K}(\mathcal{H}_{K,\varphi})$:

Theorem 4.6. *Let K/\mathbb{Q} be a real cyclic extension of prime-to- p degree. Let $\ell \equiv 1 \pmod{2p^N}$, $N \geq 1$, and assume ℓ totally inert in K . Let $L \subset K(\mu_\ell)$ of degree p^n over K , $n \in [1, N]$, and put $G := \text{Gal}(L/K) =: \langle \sigma \rangle$.*

(i) *We have the relations (product of two integers):*

$$(\mathcal{E}_{K,\varphi} : \mathcal{F}_{K,\varphi}) = (\mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi}) : \mathcal{F}_{K,\varphi}) \times \frac{\#\mathcal{H}_{K,\varphi}}{\#\mathbf{J}_{L/K}(\mathcal{H}_{K,\varphi})}, \quad \text{for all } \varphi \in \Phi_K.$$

(ii) *If $\mathcal{H}_{K,\varphi}$ capitulates in L , then $(\mathcal{E}_{K,\varphi} : \mathcal{F}_{K,\varphi}) \geq \#\mathcal{H}_{K,\varphi}$.*

(iii) *The Main Conjecture $\#\mathcal{H}_{K,\varphi} = (\mathcal{E}_{K,\varphi} : \mathcal{F}_{K,\varphi})$ for all $\varphi \in \Phi_K$, holds under the existence, for each $\varphi \in \Phi_K \setminus \{1\}$, of an inert prime $\ell \equiv 1 \pmod{2p^N}$, N large enough, such that $\mathcal{H}_{K,\varphi}$ capitulates in $K(\mu_\ell)$.*

Proof. From Corollary 3.2 (i) to Theorem 3.1, we have, for all $\varphi \neq 1$ the exact sequences $1 \rightarrow \mathbf{J}_{L/K}(\mathcal{H}_{K,\varphi}) \rightarrow \mathcal{H}_{L,\varphi}^G \rightarrow \mathcal{E}_{K,\varphi}/\mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi}) \rightarrow 1$, and $\#\mathcal{H}_{L,\varphi}^G = \#\mathcal{H}_{K,\varphi}$ from Corollary 2.5; whence the relations:

$$\#\mathcal{H}_{L,\varphi}^G = \#\mathcal{H}_{K,\varphi} = (\mathcal{E}_{K,\varphi} : \mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi})) \times \#\mathbf{J}_{L/K}(\mathcal{H}_{K,\varphi}).$$

From Corollary 4.4, $\mathcal{F}_{K,\varphi} = \mathbf{N}_{L/K}(\mathcal{F}_{L,\varphi})$, whence the inclusions:

$$\mathcal{F}_{K,\varphi} \subseteq \mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi}) \subseteq \mathcal{E}_{K,\varphi},$$

where $(\mathcal{E}_{K,\varphi} : \mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi})) = \frac{\#\mathcal{H}_{K,\varphi}}{\#\mathbf{J}_{L/K}(\mathcal{H}_{K,\varphi})}$, proving the claims (i) and (ii).

For (iii), formula (6), $\prod_{\varphi \in \Phi_K} (\mathcal{E}_{K,\varphi} : \mathcal{F}_{K,\varphi}) = \prod_{\varphi \in \Phi_K} \#\mathcal{H}_{K,\varphi}$, implies equalities for all $\varphi \in \Phi_K$. \square

Recall that a sufficient condition of capitulation (whence implying the Main Conjecture) is the stability of the p -class groups in the cyclic p -tower $\bigcup_{n \in [1, N]} K_n$ of $K(\mu_\ell)/K$, that is to say, the existence of n_0 , $0 \leq n_0 \leq N - e(\varphi)$, such that $\#\mathcal{H}_{K_{n_0+1}, \varphi} = \#\mathcal{H}_{K_{n_0}, \varphi}$, where $p^{e(\varphi)}$ is the exponent of $\mathcal{H}_{K, \varphi}$. Let's note that, as soon as there is an incomplete capitulation of $\mathcal{H}_{K, \varphi}$ in some K_n/K , $n \in [1, N]$, the index $(\mathcal{E}_{K, \varphi} : \mathcal{F}_{K, \varphi})$ is non trivial. In practice, one obtains often the whole capitulation of \mathcal{H}_K using a single prime ℓ among, probably, infinitely many.

5. NUMERICAL EXPERIMENTS OVER CYCLIC CUBIC FIELDS

We consider cyclic cubic fields K with $p \neq 3$. Let $L \subset K(\mu_\ell)$, $\ell \equiv 1 \pmod{2p^N}$ inert in K , be a cyclic p -extension of degree p^N of K .

The following program gives, at the beginning, the complete list of cyclic cubic fields of conductor $f \in [\text{bf}, \text{Bf}]$ and selects those having a suitable (non-trivial) p -class group to study the capitulation in L/K . The test is about the order of \mathcal{H}_K , so various structures may occur. The fields $K_n \subseteq L$, of degree p^n , are given by means of the polynomial P of degree $3 \cdot p^n$, $n \in [1, \text{Nn}]$, where $\text{Nn} \leq \text{N}$, not too large, defines the layers in which the computations are done (regrettably the execution time becomes rapidly out of reach).

As the instruction `bnfinit(P, 1)` takes huge time if the degree of P increases, we are limited to $p = 2$ and possibly $p = 5$ and 7 (minimal prime giving two p -adic characters). The purpose being to suggest the randomness of the (very frequent) phenomenon of capitulation, we hope that these cases constitute a good heuristic. The case $p = 2$ is not specific for capitulation aspects and allows more complex structures for \mathcal{H}_K , which is crucial to understand the process in the tower.

5.1. General program for cyclic cubic fields. In the following general program, one must precise the following data:

(i) The numbers N (and $\text{Nn} \leq N$, the number of layers to be tested by the program) to define the primes ℓ , limited by the bound Bell , congruent to 1 modulo $2p^N$; it is possible, to prove capitulations at a larger layer, to take N large, but Nn very small.

(ii) The bounds `bf, Bf` defining an interval for the conductors f .

(iii) The positive numbers `vHK` and `vHKn`, for the instructions:

$$\text{valuation}(\text{HK}, p) < \text{vHK}, \text{valuation}(\text{HK}n, p) < \text{vHK}n,$$

to get only interesting p -class groups for K and K_n ; note that from the Chevalley–Herbrand formula, $\#\mathcal{H}_{K_n}$ (in $\text{HK}n$) is a multiple of $\#\mathcal{H}_K$ (in HK), and one must take `vHK` \leq `vHKn`.

STUDY OF THE CAPITULATION OF HK IN K_n/K FOR CYCLIC CUBIC FIELDS

```
{p=2;N=2;Nn=2;bf=7;Bf=5*10^3;vHK=4;vHKn=6;Bell=500;
\\List of cubic fields of any conductor f:
for(f=bf,Bf,h=valuation(f,3);if(h!=0 & h!=2,next);
F=f/3^h;if(core(F)!=F,next);
F=factor(F);Div=component(F,1);d=matsize(F)[1];for(j=1,d,D=Div[j];
if(Mod(D,3)!=1,break));for(b=1,sqrt(4*f/27),
if(h==2 & Mod(b,3)==0,next);A=4*f-27*b^2;
if(issquare(A,&a)==1,\\a and b such that f=(a^2+27b^2)/4
if(h==0,if(Mod(a,3)==1,a=-a);PK=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(h==2,if(Mod(a,9)==3,a=-a);PK=x^3-f/3*x-f*a/27);
```

```

\\End of computation of PK defining K of conductor f.
K=bnfinit(PK,1);HK=K.no;\\Whole class number of K
\\Test on the order of the p-class group of K:
if(valuation(HK,p)<vHK,next);CK=K.clgp;\\Class group of K
\\Definition of the primes ell inert in K:
forprime(ell=1,Bell,if(Mod(ell-1,2*p^N)!=0 || Mod(f,ell)==0,next));
F=factor(PK+O(ell));if(matsize(F)[1]!=1,next);
\\Definitions of the fields Kn<L, computation of their class group:
for(n=1,Nn,QKn=polsubcyclo(ell,p^n);P=polcompositum(PK,QKn)[1];
Kn=bnfinit(P,1);
HKn=Kn.no;htame=HKn/p^valuation(HKn,p);\\Tame part of HKn
\\Test on the order of the p-class group of Kn:
if(valuation(HKn,p)<vHKn,break);
print();print("conductor f=",f," PK=",PK," CK=",CK[2]);
print("ell=",ell," N=",N," Nn=",Nn," n=",n);
CKn=Kn.clgp;print("CKn=",CKn[2]);
rKn=matsize(CKn[2])[2];\\rank of CKn
\\Calcul de Gal(Kn/K) and search of a generator S of order p^n:
G=nfgaloisconj(Kn);Id=G[1];for(k=2,3*p^n,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
\\Computation of the algebraic norms of the generators of CKn:
for(j=1,rKn,AO=CKn[3][j];A=1;
for(t=1,p^n,As=nfgaloisapply(Kn,S,A);A=idealmul(Kn,AO,As));
A=idealpow(Kn,A,htame);\\Replace A by its p-component
\\Test of capitulation (incomplete or total):
X=bnfisprincipal(Kn,A)[1];print("norm in K",n,"/K
of the component ",j," of CK",n," : ",X))))))}

```

5.2. Case of cyclic cubic fields and $p = 2$. We give an excerpt of the various forms of examples, with the structure $\mathcal{H}_K \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in K and \mathcal{H}_{K_1} of order at least 2^6 (we indicate the nature of the capitulation at the end of the data):

```

conductor f=1777 PK=x^3 + x^2 - 592*x + 724 CK=[4,4]
ell=41 N=3 Nn=2 n=1 CK=[4,4] CK1=[4,4,2,2]
norm in K1/K of the component 1 of CK1: [0,0,0,1]~
norm in K1/K of the component 2 of CK1: [2,2,1,1]~
norm in K1/K of the component 3 of CK1: [0,0,0,0]~
norm in K1/K of the component 4 of CK1: [0,0,0,0]~
Incomplete capitulation in K1

ell=41 N=3 Nn=2 n=2 CK=[4,4] CK2=[8,8,2,2]
norm in K2/K of the component 1 of CK2: [0,0,0,0]~
norm in K2/K of the component 2 of CK2: [0,0,0,0]~
norm in K2/K of the component 3 of CK2: [0,0,0,0]~
norm in K2/K of the component 4 of CK2: [0,0,0,0]~
Complete capitulation in K2 without stabilization in K2/K1

-----
ell=113 N=3 Nn=2 n=1 CK=[4,4] CK1=[4,4,2,2]
norm in K1/K of the component 1 of CK1: [0,0,1,0]~
norm in K1/K of the component 2 of CK1: [0,0,0,1]~
norm in K1/K of the component 3 of CK1: [0,0,0,0]~
norm in K1/K of the component 4 of CK1: [0,0,0,0]~
Incomplete capitulation in K1

ell=113 N=3 Nn=2 n=2 CK=[4,4] CK2=[8,8,2,2]
norm in K2/K of the component 1 of CK2: [0,4,0,0]~
norm in K2/K of the component 2 of CK2: [4,4,0,0]~
norm in K2/K of the component 3 of CK2: [0,0,0,0]~
norm in K2/K of the component 4 of CK2: [0,0,0,0]~

```

Incomplete capitulation in K2

 ell=257 N=3 Nn=2 n=1 CK=[4,4] CK1=[72,24]
 norm in K1/K of the component 1 of CK1: [54,12]~
 norm in K1/K of the component 2 of CK1: [36,18]~
 No capitulation in K1

ell=257 N=3 Nn=2 n=2 CK=[4,4] CK2=[72,24]
 norm in K2/K of the component 1 of CK2: [36,0]~
 norm in K2/K of the component 2 of CK2: [0,12]~
 Incomplete capitulation in K2
 Stability from K1--->capitulation in K3

 ell=337 N=3 Nn=2 n=1 CK=[4,4] CK1=[4,4]
 norm in K1/K of the component 1 of CK1: [2,0]~
 norm in K1/K of the component 2 of CK1: [0,2]~
 Incomplete capitulation in K1
 Stability from K--->capitulation in K2

ell=337 N=3 Nn=3 n=2 CK=[4,4] CK2=[4,4]
 norm in K2/K of the component 1 of CK2: [0, 0]~
 norm in K2/K of the component 2 of CK2: [0, 0]~
 Complete capitulation in K2 as expected

 ell=2129 N=3 Nn=2 n=1 CK=[4,4] CK1=[16,16,2,2]
 norm in K1/K of the component 1 of CK1: [8,4,0,1]~
 norm in K1/K of the component 2 of CK1: [12,12,1,0]~
 norm in K1/K of the component 3 of CK1: [8,8,0,0]~
 norm in K1/K of the component 4 of CK1: [0,8,0,0]~
 No capitulation in K1

ell=2129 N=3 Nn=2 n=2 CK=[4,4] CK2=[16,16,2,2]
 norm in K2/K of the component 1 of CK2: [0,8,0,0]~
 norm in K2/K of the component 2 of CK2: [8,8,0,0]~
 norm in K2/K of the component 3 of CK2: [0,0,0,0]~
 norm in K2/K of the component 4 of CK2: [0,0,0,0]~
 Incomplete capitulation in K2
 Stability from K1--->capitulation in K_3

 conductor f=2817 PK=x^3 - 939*x + 6886 CK=[12,4]
 ell=449 N=2 Nn=2 n=1 CK=[4,4] CK1=[24,8,2,2]
 norm in K1/K of the component 1 of CK1: [12,0,0,1]~
 norm in K1/K of the component 2 of CK1: [12,0,1,0]~
 norm in K1/K of the component 3 of CK1: [0,0,0,0]~
 norm in K1/K of the component 4 of CK1: [0,0,0,0]~
 Incomplete capitulation in K1

ell=449 N=2 Nn=2 n=2 CK=[4,4] CK2=[48,16,2,2]
 norm in K2/K of the component 1 of CK2: [0,0,0,0]~
 norm in K2/K of the component 2 of CK2: [0,0,0,0]~
 norm in K2/K of the component 3 of CK2: [0,0,0,0]~
 norm in K2/K of the component 4 of CK2: [0,0,0,0]~
 Complete capitulation in K2

 conductor f=4297 PK=x^3 + x^2 - 1432*x + 20371 CK=[4,4]
 ell=449 N=2 Nn=2 n=1 CK=[4,4] CK1=[4,4,2,2]
 norm in K1/K of the component 1 of CK1: [0,2,1,1]~
 norm in K1/K of the component 2 of CK1: [2,2,0,1]~
 norm in K1/K of the component 3 of CK1: [0,0,0,0]~

norm in K1/K of the component 4 of CK1: [0,0,0,0]~
 Incomplete capitulation in K1

ell=449 N=2 Nn=2 n=2 CK=[4,4] CK2=[292,4,4,4]
 norm in K2/K of the component 1 of CK2: [146,0,2,0]~
 norm in K2/K of the component 2 of CK2: [0,0,2,2]~
 norm in K2/K of the component 3 of CK2: [146,0,2,0]~
 norm in K2/K of the component 4 of CK2: [146,0,2,0]~
 Incomplete capitulation in K2

 conductor f=5409 PK=x^3 - 1803*x + 29449 CK=[12,4]
 ell=113 N=2 Nn=2 n=1 CK=[4,4] CK1=[12,4,2,2]
 norm in K1/K of the component 1 of CK1: [6,2,0,1]~
 norm in K1/K of the component 2 of CK1: [0,0,1,0]~
 norm in K1/K of the component 3 of CK1: [0,0,0,0]~
 norm in K1/K of the component 4 of CK1: [0,0,0,0]~
 Incomplete capitulation in K1

ell=113 N=2 Nn=2 n=2 CK=[4,4] CK2=[24,8,2,2,2,2]
 norm in K2/K of the component 1 of CK2: [0,4,0,0,0,0]~
 norm in K2/K of the component 2 of CK2: [12,4,0,0,0,0]~
 norm in K2/K of the component 3 of CK2: [0,0,0,0,0,0]~
 norm in K2/K of the component 4 of CK2: [0,0,0,0,0,0]~
 norm in K2/K of the component 5 of CK2: [0,0,0,0,0,0]~
 norm in K2/K of the component 6 of CK2: [0,0,0,0,0,0]~
 Incomplete capitulation in K2

Remark 5.1. For the first example above, the capitulation in K_2 is complete, even if the stability does not occur from the first layer; the step $n = 1$ shows an incomplete capitulation giving, $\mathbf{J}_{K_1/K}(\mathcal{H}_K) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (indeed the exponent of \mathcal{H}_K is 4).

To be more convincing, let's give the coefficients, on the PARI integral basis, of the generators of the representative ideals of the classes $\nu_{K_2/K}(h_i)$ for the four classes h_i of orders 8, 8, 2, 2, respectively (and given as usual by $\text{CKn} = \text{Kn.clgp}$); for this, replace, in the program, $\text{X} = \text{bnfisprincipal}(\text{Kn}, \text{A})[1]$ by $\text{X} = \text{bnfisprincipal}(\text{Kn}, \text{A})$. One gets 12 huge integer coefficients:

conductor f=1777 PK=x^3 + x^2 - 592*x + 724 CK=[4,4]
 ell=41 N=3 Nn=2 n=2 CK=[4,4] CK1=[4,4,2,2] CK2=[8,8,2,2]

norm in K2/K of the component 1 of CK2:
 [[0,0,0,0]~,
 [-31780222254443, -12898232803596, 15554698429537, -11030242667244,
 699234644603, -1433180593820, -97846196830, 480428807611,
 70679128541, 581754438178, 701511836521, 497443446811]~]

norm in K2/K of the component 2 of CK2:
 [[0,0,0,0]~,
 [-409139735188114, -166218206982845, 200303000159397, -142025979393819,
 10482585098180, 32200771380552, -1471046798500, 7172751349077,
 1043934050162, -13082943399099, -15769847218663, -11179282571565]~]

norm in K2/K of the component 3 of CK2:
 [[0,0,0,0]~,
 [4595853941743, 7574362186256, -7431095890343, 3180376719682,
 -878235409486, 520990038484, 351933447679, 127583225152,
 327914236819, -381696290156, -181901226812, -173412643330]~]

norm in K2/K of the component 4 of CK2:

```
[[0,0,0,0]~,
[206178918528161385818507,45009133745540603328818,
109639228343931043367320,18671957635985615261071,
13770653000372358842954,7958894412958725580875,
13213450239959129254028,3845666006771496793309,
2250298427236450403785,-1737437297938711409589,
4232280623726481124024,-720772717060054219812]~]
```

5.3. **Case of cyclic cubic fields and $p = 7$.** For $p = 7$, due to the execution time, let's give some examples of the case $n = 1$ with \mathcal{H}_K of order 7, then one case of order 7^2 , and $B_\ell = 100$.

We obtain complete capitulations in K_1 , except few cases; we give an excerpt of some possibilities:

```
conductor f=313 PK=x^3 + x^2 - 104*x + 371 CK=[7]
ell=29 N=1 Nn=1 n=1 CK=[7] CK1=[7]
norm in K1/K of the component 1 of CK1: [0]~
Complete capitulation in K1
Stability from K
```

```
-----
conductor f=1261 PK=x^3 + x^2 - 420*x - 1728 CK=[21]
ell=43 N=1 Nn=1 n=1 CK=[7] CK1=[21]
norm in K1/K of the component 1 of CK1: [0]~
Complete capitulation in K1
Stability from K
```

```
-----
conductor f=1567 PK=x^3 + x^2 - 522*x - 4759 CK=[7]
ell=29 N=1 Nn=1 n=1 CK=[7] CK1=[49]
norm in K1/K of the component 1 of CK1: [7]~
No capitulation in K1
```

```
-----
conductor f=8563 PK=x^3 + x^2 - 2854*x + 57721 CK=[49]
ell=71 N=1 Nn=1 n=1 CK=[49] CK1=[49]
norm in K1/K of the component 1 of CK1: List([7])
Incomplete capitulation in K1
Stability from K but N too small (e=2)
```

The last case shows an incomplete capitulation giving $\mathbf{J}_{K_1/K}(\mathcal{H}_K) \simeq \mathbb{Z}/7\mathbb{Z}$. Since $N = 1$, there is no possible complete capitulation despite the stability from K . The case of primes ℓ with $N = n = 2$ seems out of reach.

Let's give the generator of the principal ideal obtained after capitulation (first example above):

```
conductor f=313 PK=x^3 + x^2 - 104*x + 371 CK=[7]
ell=29 N=1 Nn=1 n=1 CK=[7] CK1=[7]
norm in K1/K of the component 1 of CK1:
[[0]~,
[4529357,2479589,125622,-2879283,2922668,4270474,-6202812,
-107453,1865872,37436,-613198,1546287,-1637834,1355628,
1276626,886508,944469,-900999,474890,508450,962907]~]
```

5.4. **Application of the sufficient condition of capitulation.** We consider cyclic cubic fields and $p = 2$. We only search examples of primes ℓ giving the stability of the 2-class groups in K_1/K , so that the capitulation automatically applies in $L = K_N$ if N is large enough; so we compute the 2-valuation of $\#\mathcal{H}_K$ (in $N = \text{valuation}(\text{HK}, \mathfrak{p})$), which is the minimal possible bound (N must be even). We illustrate the case $\mathcal{H}_K \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$; a great lot of examples are found giving capitulation in K_1 , even with small ℓ 's (one writes only $f, \text{PK}, \text{ell}, \text{CK}, \text{CK1}$; we put $\text{ell} = \text{prime}(t)$, $t \in [2, \text{nell}]$):

```

{p=2;N=6;bf=7;Bf=10^6;nell=100;for(f=bf,Bf,h=valuation(f,3);
if(h!=0 & h!=2,next);F=f/3^h;if(core(F)!=F,next);F=factor(F);
Div=component(F,1);d=matsize(F)[1];
for(j=1,d,D=Div[j];if(Mod(D,3)!=1,break));
for(b=1,sqrt(4*f/27),if(h==2 & Mod(b,3)==0,next);
A=4*f-27*b^2;if(issquare(A,&a)==1,
if(h==0,if(Mod(a,3)==1,a=-a);PK=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(h==2,if(Mod(a,9)==3,a=-a);PK=x^3-f/3*x-f*a/27);
K=bnfinit(PK,1);HK=K.no;v=valuation(HK,p);if(v!=N,next);
for(t=2,nell,ell=prime(t);if(Mod(ell-1,2*2^N)!=0,next);
if(Mod(f,ell)==0,next);F=factor(PK+O(ell));
if(matsize(F)[1]!=1,next);QK1=polsubcyclo(ell,p);
P=polcompositum(PK,QK1)[1];K1=bnfinit(P,1);
if(valuation(K1.no,p)==N,print("f=",f," PK=",PK," ell=",ell,
" CK=",K.clgp[2]," CK1=",K1.clgp[2]);break)))))}
N=2
f=163 PK=x^3+x^2-54*x-169 ell=29 CK=[2,2] CK1=[2,2]
f=277 PK=x^3+x^2-92*x+236 ell=5 CK=[2,2] CK1=[2,2]
f=349 PK=x^3+x^2-116*x-517 ell=5 CK=[2,2] CK1=[2,2]
f=397 PK=x^3+x^2-132*x-544 ell=5 CK=[2,2] CK1=[2,2]
(...)
f=9709 PK=x^3+x^2-3236*x+21216 ell=5 CK=[6,6] CK1=[6,6]
f=9721 PK=x^3+x^2-3240*x-39244 ell=5 CK=[2,2] CK1=[2,2]
f=9891 PK=x^3-3297*x+70336 ell=29 CK=[6,6] CK1=[6,6]
f=9961 PK=x^3+x^2-3320*x-74523 ell=5 CK=[6,2] CK1=[6,2]
(...)
-----
N=4
f=1777 PK=x^3+x^2-592*x+724 ell=353 CK=[4,4] CK1=[52,4]
f=2817 PK=x^3-939*x+6886 ell=97 CK=[12,4] CK1=[12,12]
f=4297 PK=x^3+x^2-1432*x+20371 ell=97 CK=[4,4] CK1=[4,4]
f=5409 PK=x^3-1803*x+29449 ell=193 CK=[12,4] CK1=[12,4]
(...)
f=98479 PK=x^3+x^2-32826*x-1940401 ell=353 CK=[4,4] CK1=[4,4]
f=98581 PK=x^3+x^2-32860*x+1453157 ell=193 CK=[12,4] CK1=[228,12]
f=99133 PK=x^3+x^2-33044*x+117491 ell=193 CK=[4,4] CK1=[4,4]
f=100807 PK=x^3+x^2-33602*x+321089 ell=97 CK=[12,4] CK1=[12,4]
(...)
-----
N=6
f=10513 PK=x^3+x^2-3504*x-80989 ell=257 CK=[8,8] CK1=[24,8]
f=48769 PK=x^3+x^2-16256*x-7225 ell=257 CK=[24,8] CK1=[24,24]
f=70897 PK=x^3+x^2-23632*x-1389056 ell=257 CK=[24,8] CK1=[72,24,3]
f=80947 PK=x^3+x^2-26982*x+1696889 ell=257 CK=[24,8] CK1=[168,24,3,3]
(...)
f=351063 PK=x^3-117021*x-15407765 ell=257 CK=[24,24,3]CK1=[72,24,3,3]
f=357229 PK=x^3+x^2-119076*x+15228540ell=257 CK=[8,8] CK1=[24,8]
f=492517 PK=x^3+x^2-164172*x-24479919ell=257 CK=[24,8] CK1=[168,168,3]
f=552763 PK=x^3+x^2-184254*x-27842877ell=257 CK=[24,8] CK1=[24,24,3]
(...)

```

Some cases of more complex structures of \mathcal{H}_K do not give stabilization at the first step; this clearly depends on the exponent of the class groups as we have explained; but this sufficient condition is not necessary and capitulation does appear at larger layers; we illustrate (using the general program) the cases $\mathcal{H}_K \simeq (\mathbb{Z}/2\mathbb{Z})^4$ or $\mathcal{H}_K \simeq (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$:

```

conductor f=7687 PK=x^3 + x^2 - 2562*x - 48969 CK=[2,2,2,2]
ell=17 N=3 Nn=3 n=1 CK=[2,2,2,2] CK1=[4,4,2,2]
norm in K1/K of the component 1 of CK1: [2,2,0,0]~

```

norm in K_1/K of the component 2 of CK_1 : $[0,2,0,0]^\sim$
 norm in K_1/K of the component 3 of CK_1 : $[2,0,0,0]^\sim$
 norm in K_1/K of the component 4 of CK_1 : $[0,2,0,0]^\sim$
 Incomplete capitulation in K_1

$\ell=17$ $N=3$ $N_n=3$ $n=2$ $CK=[2,2,2,2]$ $CK_2=[4,4,2,2]$
 norm in K_2/K of the component 1 of CK_2 : $[0,0,0,0]^\sim$
 norm in K_2/K of the component 2 of CK_2 : $[0,0,0,0]^\sim$
 norm in K_2/K of the component 3 of CK_2 : $[0,0,0,0]^\sim$
 norm in K_2/K of the component 4 of CK_2 : $[0,0,0,0]^\sim$
 Complete capitulation in K_2 with stability from K_1

 conductor $f=20887$ $PK=x^3 + x^2 - 6962*x - 225889$ $CK=[4,4,2,2]$
 $\ell=193$ $N=3$ $N_n=3$ $n=1$ $CK=[4,4,2,2]$ $CK_1=[8,8,2,2]$
 norm in K_1/K of the component 1 of CK_1 : $[2,4,0,0]^\sim$
 norm in K_1/K of the component 2 of CK_1 : $[4,6,0,0]^\sim$
 norm in K_1/K of the component 3 of CK_1 : $[4,0,0,0]^\sim$
 norm in K_1/K of the component 4 of CK_1 : $[4,4,0,0]^\sim$
 Incomplete capitulation in K_1

$\ell=193$ $N=3$ $N_n=3$ $n=2$ $CK=[4,4,2,2]$ $CK_2=[8,8,2,2]$
 norm in K_2/K of the component 1 of CK_2 : $[4,0,0,0]^\sim$
 norm in K_2/K of the component 2 of CK_2 : $[0,4,0,0]^\sim$
 norm in K_2/K of the component 3 of CK_2 : $[0,0,0,0]^\sim$
 norm in K_2/K of the component 4 of CK_2 : $[0,0,0,0]^\sim$
 Incomplete capitulation in K_2
 Stability from K_1 --->Complete capitulation in K_3

 conductor $f=31923$ $PK=x^3 - 10641*x + 227008$ $CK=[6,2,2,2]$
 $\ell=97$ $N=3$ $N_n=3$ $n=1$ $CK=[6,2,2,2]$ $CK_1=[12,4,2,2,2,2]$
 norm in K_1/K of the component 1 of CK_1 : $[0,0,1,0,0,0]^\sim$
 norm in K_1/K of the component 2 of CK_1 : $[0,0,0,1,0,0]^\sim$
 norm in K_1/K of the component 3 of CK_1 : $[0,0,0,0,0,0]^\sim$
 norm in K_1/K of the component 4 of CK_1 : $[0,0,0,0,0,0]^\sim$
 norm in K_1/K of the component 5 of CK_1 : $[6,0,0,0,0,0]^\sim$
 norm in K_1/K of the component 6 of CK_1 : $[6,2,0,0,0,0]^\sim$
 No capitulation in K_1

$\ell=97$ $N=3$ $N_n=3$ $n=2$ $CK=[6,2,2,2]$ $CK_2=[312,8,4,4,4,4]$
 norm in K_2/K of the component 1 of CK_2 : $[156,0,2,0,0,2]^\sim$
 norm in K_2/K of the component 2 of CK_2 : $[156,4,0,0,2,2]^\sim$
 norm in K_2/K of the component 3 of CK_2 : $[156,0,0,0,0,0]^\sim$
 norm in K_2/K of the component 4 of CK_2 : $[156,4,0,0,0,0]^\sim$
 norm in K_2/K of the component 5 of CK_2 : $[156,0,0,0,0,0]^\sim$
 norm in K_2/K of the component 6 of CK_2 : $[156,0,0,0,0,0]^\sim$
 No capitulation in K_2

So we must try another ℓ for $f = 31923$:

conductor $f=31923$ $PK=x^3 - 10641*x + 227008$ $CK=[6,2,2,2]$
 $\ell=257$ $N=3$ $N_n=2$ $n=1$ $CK=[6,2,2,2]$ $CK_1=[18,6,2,2,2,2]$
 norm in K_1/K of the component 1 of CK_1 : $[0,0,0,0,0,0]^\sim$
 norm in K_1/K of the component 2 of CK_1 : $[0,0,0,0,0,0]^\sim$
 norm in K_1/K of the component 3 of CK_1 : $[0,3,0,0,0,1]^\sim$
 norm in K_1/K of the component 4 of CK_1 : $[9,0,0,0,1,0]^\sim$
 norm in K_1/K of the component 5 of CK_1 : $[0,0,0,0,0,0]^\sim$
 norm in K_1/K of the component 6 of CK_1 : $[0,0,0,0,0,0]^\sim$
 Incomplete capitulation in K_1

$\ell=257$ $N=3$ $N_n=2$ $n=2$ $CK=[6,2,2,2]$ $CK_2=[36,12,2,2,2,2]$

norm in K_2/K of the component 1 of CK2: $[0,0,0,0,0,0]^\sim$
 norm in K_2/K of the component 2 of CK2: $[0,0,0,0,0,0]^\sim$
 norm in K_2/K of the component 3 of CK2: $[0,0,0,0,0,0]^\sim$
 norm in K_2/K of the component 4 of CK2: $[0,0,0,0,0,0]^\sim$
 norm in K_2/K of the component 5 of CK2: $[0,0,0,0,0,0]^\sim$
 norm in K_2/K of the component 6 of CK2: $[0,0,0,0,0,0]^\sim$
 Complete capitulation in K_2

For this new ℓ , the complete capitulation is obtained in K_2 . We note that the 2-rank of \mathcal{H}_K is 4, which implies, from Corollary 3.2 (ii) and $j = \exp(\frac{2i\pi}{3})$, the isomorphism $\mathcal{H}_{K_2}^{G_2} \simeq \mathbb{Z}_2[j]/(4)$ but never $\mathbb{Z}_2[j]/(2) \times \mathbb{Z}_2[j]/(2)$.

5.5. Statistics. Let's consider an example of cyclic cubic field K such that, for $p = 2$, $\mathcal{H}_K \simeq \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (conductor 20887); we give some heuristics about the capitulation of \mathcal{H}_K in K_1 and K_2 (necessarily incomplete in K_1), for all prime numbers $\ell \equiv 1 \pmod{8}$.

```
{p=2;Nn=2;PK=x^3+x^2-6962*x-225889;Bell=10^3;
K=bnfinit(PK,1);CK=K.clgp[2];print("PK=",PK," CK=",CK);
forprime(e11=5,Bell,N=valuation((e11-1)/2,p);
if(N<Nn+1,next);F=factor(PK+0(e11));if(matsize(F)[1]!=1,next);
for(n=1,Nn,print();print("e11=",e11," N=",N," Nn=",Nn," n=",n);
QKn=polsubcyclo(e11,p^n);P=polcompositum(PK,QKn)[1];
Kn=bnfinit(P,1);CKn=Kn.clgp;print("CK=",CK," CKn",n,"=",CKn[2]);
htame=Kn.no/p^valuation(Kn.no,p);rKn=matsize(CKn[2])[2];
G=nfgaloisconj(Kn);Id=G[1];for(k=2,3*p^n,Z=G[k];ks=1;
while(Z!=Id,Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);
if(ks==p,S=G[k];break));for(j=1,rKn,A0=CKn[3][j];A=1;
for(t=1,p^n,As=nfgaloisapply(Kn,S,A);A=idealmul(Kn,A0,As));
A=idealpov(Kn,A,htame);X=bnfisprincipal(Kn,A)[1];
print("norm in K",n,"/K of the component ",j," of CK",n,": ",X))}
```

The most frequent structure of \mathcal{H}_{K_1} is $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with 272 cases over 396, and there is never complete capitulation in K_1/K (existence of elements of order 4), but for N large enough, capitulation (and even stability) is possible from K_2 , as shown by the following examples with automatic capitulation in K_3 :

```
conductor f=20887 PK=x^3+x^2-6962*x-225889 CK=[4,4,2,2]
e11=193 N=5 Nn=2 n=1 CK=[4,4,2,2] CK1=[8,8,2,2]
norm in K1/K of the component 1 of CK1:  $[2,4,0,0]^\sim$ 
norm in K1/K of the component 2 of CK1:  $[4,6,0,0]^\sim$ 
norm in K1/K of the component 3 of CK1:  $[4,0,0,0]^\sim$ 
norm in K1/K of the component 4 of CK1:  $[4,4,0,0]^\sim$ 

e11=193 N=5 Nn=2 n=2 CK=[4,4,2,2] CK2=[8,8,2,2]
norm in K2/K of the component 1 of CK2:  $[4,0,0,0]^\sim$ 
norm in K2/K of the component 2 of CK2:  $[0,4,0,0]^\sim$ 
norm in K2/K of the component 3 of CK2:  $[0,0,0,0]^\sim$ 
norm in K2/K of the component 4 of CK2:  $[0,0,0,0]^\sim$ 

-----
e11=241 N=3 Nn=2 n=1 CK=[4,4,2,2] CK1=[8,8,2,2]
norm in K1/K of the component 1 of CK1:  $[6,0,0,0]^\sim$ 
norm in K1/K of the component 2 of CK1:  $[0,6,0,0]^\sim$ 
norm in K1/K of the component 3 of CK1:  $[4,4,0,0]^\sim$ 
norm in K1/K of the component 4 of CK1:  $[4,0,0,0]^\sim$ 

e11=241 N=3 Nn=2 n=2 CK=[4,4,2,2] CK2=[8,8,2,2]
norm in K2/K of the component 1 of CK2:  $[4,0,0,0]^\sim$ 
norm in K2/K of the component 2 of CK2:  $[0,4,0,0]^\sim$ 
```

norm in K_2/K of the component 3 of CK2: $[0,0,0,0] \sim$
 norm in K_2/K of the component 4 of CK2: $[0,0,0,0] \sim$

 ell=1777 N=3 Nn=2 n=1 CK=[4,4,2,2] CK1=[8,8,2,2]
 norm in K_1/K of the component 1 of CK1: $[6,4,0,0] \sim$
 norm in K_1/K of the component 2 of CK1: $[4,2,0,0] \sim$
 norm in K_1/K of the component 3 of CK1: $[4,0,0,0] \sim$
 norm in K_1/K of the component 4 of CK1: $[0,4,0,0] \sim$

ell=1777 N=3 Nn=2 n=2 CK=[4,4,2,2] CK2=[56,56,2,2]
 norm in K_2/K of the component 1 of CK2: $[4,0,0,0] \sim$
 norm in K_2/K of the component 2 of CK2: $[0,4,0,0] \sim$
 norm in K_2/K of the component 3 of CK2: $[0,0,0,0] \sim$
 norm in K_2/K of the component 4 of CK2: $[0,0,0,0] \sim$

Then the following structures for \mathcal{H}_{K_1} are often obtained:

$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ (42 cases over 396),
 $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (35 cases over 396),
 $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ (34 cases over 396).

Some cases of:

$\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (8 cases over 396),

may give capitulation. Then we obtain a unique case of each of the following structures for \mathcal{H}_{K_1} :

$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$,
 $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$,
 $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$,
 $\mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$,
 $\mathbb{Z}/32\mathbb{Z} \times \mathbb{Z}/32\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

These examples suggest that for suitable N , capitulation is always obtained in the tower in a wide variety of ways, and that each structure gives, most often, the same kind of results. These phenomena are certainly governed by precise probabilities.

6. NUMERICAL EXPERIMENTS OVER QUADRATIC FIELDS

We consider real quadratic fields $K = \mathbb{Q}(\sqrt{-m})$ with $p \neq 2$. Let $L \subset K(\mu_\ell)$, $\ell \equiv 1 \pmod{2p^N}$ inert in K , be a cyclic p -extension of degree p^N of K . Of course, this case has no interest for verifications of the Main Conjecture since it is true for the trivial reason $\chi = \varphi$; but it remains significant to study the capitulation phenomenon. The program is analogous to the general one with slight modifications due to the quadratic context; we then give some excerpt for $p = 3$ and $p = 5$:

6.1. General program for quadratic fields.

```
{p=3;N=2;Nn=2;bm=2;Bm=10^4;vHK=2;vHKn=2;Bell=500;
for(m=bm,Bm,if(core(m)!=m,next);PK=x^2-m;K=bnfinit(PK,1);
HK=K.no;if(valuation(HK,p)<vHK,next);CK=K.clgp;
forprime(ell=1,Bell,if(Mod(ell-1,2*p^N)!=0,next);
if(Mod(m,ell)==0,next);if(kronecker(m,ell)!=-1,next);
for(n=1,Nn,QKn=polsubcyclo(ell,p^n);P=polcompositum(PK,QKn)[1];
Kn=bnfinit(P,1);HKn=Kn.no;if(valuation(HKn,p)<vHKn,break);
htame=HKn/p^valuation(HKn,p);CKn=Kn.clgp[2];
print("PK=",PK," CK=",CK[2]," ell=",ell,
" N=",N," Nn=",Nn," n=",n," CK",n,"=",CKn);
rKn=matsize(CKn)[2];G=nfgaloisconj(Kn);Id=G[1];
```

```

for(k=2,2*p^n,Z=G[k];ks=1;
while(Z!=Id,Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);
if(ks==p^n,S=G[k];break));for(j=1,rKn,e=CKn[j];
A0=Kn.clgp[3][j];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));A=idealpow(Kn,A,htame);
X=bnfisprincipal(Kn,A)[1];
print("norm in K",n,"/K of the component ",j," of CK",n,":",X))))}

```

6.2. Case of quadratic fields and $p = 3$.

6.2.1. *Examples with $n \in [1, 2]$.* For instance, the configuration:

```

PK=x^2 - 1129 CK=[9]
ell=13 N=1 Nn=1 n=1 CK=[9] CK1=[27]
norm in K1/K of the component 1 of CK1: [3]~
No capitulation in K1

```

since $\mathcal{V}_{K_1/K}(\mathcal{H}_{K_1}) = \mathbf{J}_{K_1/K}(\mathcal{H}_K) = \mathcal{H}_{K_1}^3$ of order 9, there is no capitulation of \mathcal{H}_K in K_1 .

```

PK=x^2 - 1129 CK=[9]
ell=307 N=2 Nn=1 n=1 CK=[9] CK1=[9,3]
norm in K1/K of the component 1 of CK1: [3,0]~
norm in K1/K of the component 2 of CK1: [0,0]~
Incomplete capitulation in K1

```

```

ell=307 N=2 Nn=2 n=2 CK=[9] CK2=[9,9]
norm in K2/K of the component 1 of CK2: [0,0]~
norm in K2/K of the component 2 of CK2: [0,0]~
Complete capitulation in K2

```

```

-----
ell=19 N=2 Nn=1 n=1 CK=[9] CK1=[9]
norm in K1/K of the component 1 of CK1: [3]~
Incomplete capitulation in K1
Stability from K--->capitulation in K2

```

```

ell=19 N=2 Nn=2 n=2 CK=[9] CK2=[9]
norm in K2/K of the component 1 of CK2: [0]~
Complete capitulation in K2

```

```

-----
PK=x^2 - 1129 CK=[9]
ell=73 N=2 Nn=2 n=1 CK=[9] CK1=[189,3]
norm in K1/K of the component 1 of CK1: [84,0]
norm in K1/K of the component 2 of CK1: [0,0]
No capitulation in K1

```

```

ell=73 N=2 Nn=2 n=2 CK=[9] CK2=[567,9]
norm in K2/K of the component 1 of CK2: [252,0]
norm in K2/K of the component 2 of CK2: [0,0]
No capitulation in K2

```

```

-----
PK=x^2 - 3137 CK=[9]
ell=199 N=2 Nn=2 n=1 CK=[9] CK1=[27,3]
norm in K1/K of the component 1 of CK1: [21,0]~
norm in K1/K of the component 2 of CK1: [0,0]~
No capitulation in K1

```

```

ell=199 N=2 Nn=2 n=2 CK=[9] CK2=[27,9]
norm in K2/K of the component 1 of CK2: [9,0]~
norm in K2/K of the component 2 of CK2: [0,0]~
Incomplete capitulation in K2

```

```
-----
PK=x^2 - 8761 CK=[27]
ell=19 N=2 Nn=2 n=1 CK=[27] CK1=[81]
norm in K1/K of the component 1 of CK1: [3]~
No capitulation in K1
```

```
ell=19 N=2 Nn=2 n=2 CK=[27] CK2=[81]
norm in K2/K of the component 1 of CK2: [9]~
Incomplete capitulation in K2
```

We have the case of the following non-cyclic structure of \mathcal{H}_K :

```
PK=x^2 - 32009 CK=[3,3]
ell=19 N=2 Nn=2 n=1 CK=[3,3] CK1=[9,3]
norm in K1/K of the component 1 of CK1: [3,0]
norm in K1/K of the component 2 of CK1: [0,0]
Incomplete capitulation in K1
```

```
ell=19 N=2 Nn=2 n=2 CK=[3,3] CK2=[9,3]
norm in K2/K of the component 1 of CK2: [0,0]~
norm in K2/K of the component 2 of CK2: [0,0]~
Complete capitulation in K2
Stability from K1
```

```
-----
PK=x^2 - 42817 CK=[3,3]
ell=19 N=2 Nn=2 n=1 CK=[3,3] CK1=[9,3]
norm in K1/K of the component 1 of CK1: [3,0]~
norm in K1/K of the component 2 of CK1: [0,0]~
```

```
ell=19 N=2 Nn=2 n=2 CK=[3,3] CK2=[27,3]
norm in K2/K of the component 1 of CK2: [9,0]~
norm in K2/K of the component 2 of CK2: [0,0]~
Incomplete capitulation in K2
```

6.2.2. *Examples of stability from K_1 .* Let's give the program testing the stability in K_1/K ; as for the cubic case with $p = 2$, there are much solutions. We put $\text{ell} = \text{prime}(t)$, $t \in [2, \text{nell}]$:

```
{p=3;bm=2;Bm=10^4;N=3;nell=100;for(m=bm,Bm,if(core(m)!=m,next);
PK=x^2-m;K=bnfinit(PK,1);v=valuation(K.no,p);if(v!=N,next);
CK=K.clgp[2];for(t=1,nell,ell=prime(t);
if(t==nell,print("m=",m," nell insufficient");break);
if(Mod(ell-1,3)!=0,next);if(kronecker(m,ell)!=-1,next);
P=polcompositum(PK,polsubcyclo(ell,p))[1];K1=bnfinit(P,1);
v1=valuation(K1.no,p);if(v1!=N,next);
print("m=",m," PK=",PK," ell=",ell,
" CK=",K.clgp[2]," CK1=",K1.clgp[2]);break)}}
N=2
m=1129 PK=x^2 - 1129 ell=19 CK=[9] CK1=[9]
m=1654 PK=x^2 - 1654 ell=43 CK=[9] CK1=[9]
m=3137 PK=x^2 - 3137 ell=19 CK=[9] CK1=[9]
m=3719 PK=x^2 - 3719 ell=31 CK=[9] CK1=[18,2]
(...)
m=9217 PK=x^2 - 9217 ell=7 CK=[18] CK1=[18]
m=9606 PK=x^2 - 9606 ell=31 CK=[18] CK1=[18,2,2]
m=9799 PK=x^2 - 9799 ell=19 CK=[18] CK1=[18,2,2]
m=9998 PK=x^2 - 9998 ell=37 CK=[9] CK1=[9]
(...)
N=3
m=8761 PK=x^2 - 8761 ell=37 CK=[27] CK1=[27]
m=21433 PK=x^2 - 21433 ell=7 CK=[27] CK1=[27]
```

```

m=30859 PK=x^2 - 30859 e11=7 CK=[27] CK1=[27]
m=31327 PK=x^2 - 31327 e11=19 CK=[27] CK1=[27]
(...)
m=68513 PK=x^2 - 68513 e11=19 CK=[27] CK1=[27]
m=83713 PK=x^2 - 83713 e11=13 CK=[27] CK1=[54,2]
m=90271 PK=x^2 - 90271 e11=31 CK=[27] CK1=[54,2]
m=94865 PK=x^2 - 94865 e11=43 CK=[54] CK1=[54]
(...)

```

The most impressive is that, up to $m \leq 10^{10}$, small primes ℓ are sufficient to get stability for cyclic groups \mathcal{H}_K .

6.3. Case of quadratic fields and $p = 5$. We give some excerpt of numerical results, analogous to the case $p = 3$; most of examples give stability, whence capitulation in some layer. We have taken $N = 2, Nn = 1$. For some rare cases, the capitulation is incomplete in the first layer.

```

PK=x^2 - 24859 CK=[25]
e11=251 N=2 Nn=1 n=1 CK=[25] CK1=[50,2,2,2]
norm in K1/K of the component 1 of CK1: [30,0,0,0]~
norm in K1/K of the component 2 of CK1: [0,0,0,0]~
norm in K1/K of the component 3 of CK1: [0,0,0,0]~
norm in K1/K of the component 4 of CK1: [0,0,0,0]~
Incomplete capitulation in K1
Stability from K--->capitulation in K_2

```

```

e11=401 N=2 Nn=1 n=1 CK=[25] CK1=[1525]
norm in K1/K of the component 1 of CK1: [305]~
Incomplete capitulation in K1

```

```

-----
PK=x^2 - 27689 CK=[25]
e11=101 N=2 Nn=1 n=1 CK=[25] CK1=[25]
norm in K1/K of the component 1 of CK1: [5]~
Incomplete capitulation in K1
Stability from K--->capitulation in K_2

```

```

-----
PK=x^2 - 68119 CK=[50]
e11=251 N=2 Nn=1 n=1 CK=[50] CK1=[250]
norm in K1/K of the component 1 of CK1: [5]~
No capitulation in K1

```

```

-----
PK=x^2 - 68819 CK=[25]
e11=101 N=2 Nn=1 n=1 CK=[25] CK1=[25]
norm in K1/K of the component 1 of CK1: [5]~
Incomplete capitulation in K1
Stability from K--->capitulation in K_2

```

```

e11=151 N=2 Nn=1 n=1 CK=[25] CK1=[125]
norm in K1/K of the component 1 of CK1: [5]~
No capitulation in K1

```

```

-----
PK=x^2 - 69403 CK=[25]
e11=251 N=2 Nn=1 n=1 CK=[25] CK1=[25,5]
norm in K1/K of the component 1 of CK1: [5,0]~
norm in K1/K of the component 2 of CK1: [0,0]~
Incomplete capitulation in K1

```

```

e11=401 N=2 Nn=1 n=1 CK=[25] CK1=[50,2,2,2]
norm in K1/K of the component 1 of CK1: [30,0,0,0]~
norm in K1/K of the component 2 of CK1: [0,0,0,0]~

```

```

norm in K1/K of the component 3 of CK1: [0,0,0,0]~
norm in K1/K of the component 4 of CK1: [0,0,0,0]~
Incomplete capitulation in K1
Stability from K--->capitulation in K_2
-----
PK=x^2 - 88211 CK=[25]
ell=101 N=2 Nn=1 n=1 CK=[25] CK1=[125]
norm in K1/K of the component 1 of CK1: [5]~
No capitulation in K1

ell=151 N=2 Nn=1 n=1 CK=[25] CK1=[25]
norm in K1/K of the component 1 of CK1: [5]~
Incomplete capitulation in K1
Stability from K--->capitulation in K_2
-----
PK=x^2 - 119029 CK=[50]
ell=251 N=2 Nn=1 n=1 CK=[50] CK1=[250]
norm in K1/K of the component 1 of CK1: [5]~
No capitulation in K1

ell=401 N=2 Nn=1 n=1 CK=[50] CK1=[50]
norm in K1/K of the component 1 of CK1: [5]~
Incomplete capitulation in K1
Stability from K--->capitulation in K_2

```

7. ON THE USE OF THE AUXILIARY EXTENSIONS $K(\mu_\ell)/K$

7.1. Remarks on the classical proof of the real Main Conjecture.

Contrary, to our point of view, some proofs of the Main Conjecture, as that of Thaine [Thai1988], have a fundamental difference. The brief overview that we give now must be completed by technical elements that the reader can find especially in [Was1997, §§ 15.2, 15.3].

Let f be the conductor of K . Let $\ell \equiv 1 \pmod{2p^N}$, *totally split in K* and $p^N \geq p^e$, the exponent of \mathcal{H}_K ; put $M_0 = \mathbb{Q}(\mu_\ell)$ and $M := M_0K$. From Proposition 4.3, the cyclotomic unit η_M of M is such that $\mathbf{N}_{M/K}(\eta_M) = 1$ and the link with η_K is only a congruence modulo $(1 - \zeta_\ell)$.

Put $\eta_M = \alpha_M^{1-\sigma}$ where $\alpha_M \in M^\times$ is such that $(\alpha_M) \in I_M^G$; modulo K^\times , we can take α_M integer in M , whence:

$$(7) \quad (\alpha_M) = \mathbf{j}_{M/K}(\mathfrak{a}_K) \cdot \mathfrak{L}_1^{\Omega_\ell},$$

where $\mathfrak{a}_K \in I_K$ and \mathfrak{L}_1 is a fixed prime ideal dividing ℓ in M , thus totally ramified in M/K , with $\Omega_\ell = \sum_{s \in g} r_s \cdot s^{-1}$, $r_s \geq 0$; thus, since $\mathbf{N}_{M/K}(\mathfrak{L}_1) = \mathfrak{l}_1$,

$\mathfrak{L}_1 \mid \mathfrak{l}_1 \mid \ell$ in $M/K/\mathbb{Q}$, this yields $(\alpha_K) := (\mathbf{N}_{M/K}(\alpha_M)) = \mathfrak{a}_K^{\ell-1} \cdot \mathfrak{l}_1^{\Omega_\ell}$. But $\mathfrak{a}_K^{\ell-1}$ is principal since $\ell - 1 \equiv 0 \pmod{p^e}$, whence $\mathfrak{l}_1^{\Omega_\ell}$ principal.

Lemma 7.1. *Except a finite number of primes ℓ , the ideal $\mathfrak{L}_1^{\Omega_\ell}$ in relation (7) gives a non trivial relation, in the meaning that Ω_ℓ is not of the form $\lambda \cdot \nu_{M/M_0}$, $\lambda \geq 0$, giving $\mathfrak{l}_1^{\Omega_\ell} = (\ell)^\lambda$.*

Proof. Assume that $\Omega_\ell = \lambda \cdot \nu_{M/M_0}$; the character of $\mathfrak{L}_1^{\Omega_\ell}$, as $\mathbb{Z}[g]$ -module, is the unit one and any non-trivial φ -component $\alpha_{M,\varphi}$ of α_M is prime to ℓ , thus congruent, modulo any $\mathfrak{L} \mid \ell$ in M , to $\rho_\mathfrak{l} \in \mathbb{Z}$, $\rho_\mathfrak{l} \not\equiv 0 \pmod{\ell}$ (residue degrees 1 in M/\mathbb{Q}). Since $\mathfrak{L}^\sigma = \mathfrak{L}$, we obtain $\eta_{M,\varphi} = \alpha_{M,\varphi}^{\sigma-1} \equiv 1 \pmod{\mathfrak{L}}$.

We have $\eta_{f_\ell} \equiv \eta_f \pmod{(1-\zeta_\ell)}$ where $1-\zeta_\ell$ is a uniformizing parameter at the places above ℓ in M_0 , so that $\eta_M \equiv \eta_K \pmod{(1-\zeta_\ell)}$, which leads to $\eta_{K,\varphi} \equiv \eta_{M,\varphi} \equiv 1 \pmod{\mathfrak{l}}$, for all $\mathfrak{l} \mid \ell$, giving $\eta_{K,\varphi} \equiv 1 \pmod{\ell}$ (absurd for almost all ℓ). \square

Reducing modulo \mathcal{V}_{M/M_0} , one may get $\Omega_\ell \neq 0$, “minimal” in an obvious sense, with $r_s \geq 0$ but not all zero.

Consider $\frac{\alpha_M^s}{(1-\zeta_\ell)^{r_s}}$ modulo \mathfrak{L}_1 and the conjugations $\alpha_M^\sigma = \alpha_M \cdot \eta_M$ and $\frac{1-\zeta_\ell^\sigma}{1-\zeta_\ell} = \frac{1-\zeta_\ell^{\mathfrak{g}_\ell}}{1-\zeta_\ell} \equiv \mathfrak{g}_\ell \pmod{(1-\zeta_\ell)}$ (where \mathfrak{g}_ℓ is a primitive root modulo ℓ such that $\zeta_\ell^\sigma =: \zeta_\ell^{\mathfrak{g}_\ell}$); one gets:

$$\left(\frac{\alpha_M^s}{(1-\zeta_\ell)^{r_s}} \right)^\sigma = \frac{\alpha_M^{\sigma s}}{(1-\zeta_\ell)^{\sigma r_s}} \equiv \frac{\eta_M^s \alpha_M^s}{(\mathfrak{g}_\ell(1-\zeta_\ell))^{r_s}} \equiv \frac{\eta_M^s}{\mathfrak{g}_\ell^{r_s}} \left(\frac{\alpha_M^s}{(1-\zeta_\ell)^{r_s}} \right) \pmod{\mathfrak{L}_1},$$

whence $\mathfrak{g}_\ell^{r_s} \equiv \eta_M^s \equiv \eta_K^s \pmod{\mathfrak{l}_1}$, which identifies the coefficients r_s .

So, this yields a non-trivial relation between the classes of the conjugates of \mathfrak{l}_1 that is p -localizable; this constitute the basis of the reasonings, on condition to add many more technical arguments to get some annihilation of $\mathcal{E}_{K,\varphi}/\mathcal{F}_{K,\varphi}$, then a final equality $\#\mathcal{H}_{K,\varphi} = (\mathcal{E}_{K,\varphi} : \mathcal{F}_{K,\varphi})$.

In this way, Thaine’s method is essentially analytic, working on norm properties and subtle congruences of the cyclotomic units, leading to the principle of Kolyvagin Euler systems, while that using capitulation (if any) is of class field theory framework, especially with Chevalley–Herbrand context, and gives immediately the result *without any supplementary work*.

7.2. Heuristics about the Conjecture 1.2 of capitulation. Suppose, in the totally opposite case, that for all inert primes $\ell \equiv 1 \pmod{2p^N}$, with N arbitrary large, there is never any capitulation in the real cyclic p -tower L/K of $K(\mu_\ell)/K$; we get easily the following result, where we denote by K_n the subfield of L of degree p^n over K , $n \in [0, N]$.

Proposition 7.2. *Assume that, for all $n, m \in [0, N]$, $n \leq m$, the transfer maps $\mathbf{J}_n^m := \mathbf{J}_{K_m/K_n}$ are injective. Put $G_n^m := \text{Gal}(K_m/K_n)$, $\mathcal{H}_n := \mathcal{H}_{K_n}$, $\mathcal{H}_m := \mathcal{H}_{K_m}$. Then $\#\mathcal{H}_m \geq \#\mathcal{H}_n \cdot \#\mathcal{H}_n[p^{m-n}]$ which leads, for all n , to $\#\mathcal{H}_{K_n} \geq \#\mathcal{H}_K \cdot p^{n \cdot r_K}$ where $r_K = \dim_{\mathbb{F}_p}(\mathcal{H}_K/\mathcal{H}_K^p)$ is the p -rank of \mathcal{H}_K .*

Proof. From the exact sequence of $\mathbb{Z}[G_n^m]$ -modules:

$$1 \rightarrow \mathbf{J}_n^m \mathcal{H}_n \rightarrow \mathcal{H}_m \rightarrow \mathcal{H}_m/\mathbf{J}_n^m \mathcal{H}_n \rightarrow 1,$$

we get, from the Chevalley–Herbrand formula $\#\mathcal{H}_m^{G_n^m} = \#\mathcal{H}_n$ and the injectivity of \mathbf{J}_n^m , $\mathcal{H}_m^{G_n^m}/\mathbf{J}_n^m \mathcal{H}_n = 1$ and the exact sequence:

$$1 \rightarrow (\mathcal{H}_m/\mathbf{J}_n^m \mathcal{H}_n)^{G_n^m} \rightarrow H^1(G_n^m, \mathbf{J}_n^m \mathcal{H}_n) \rightarrow H^1(G_n^m, \mathcal{H}_m),$$

where $H^1(G_n^m, \mathbf{J}_n^m \mathcal{H}_n) = (\mathbf{J}_n^m \mathcal{H}_n)[p^{m-n}] \simeq \mathcal{H}_n[p^{m-n}]$ and:

$$\begin{aligned} \#\mathbf{H}^1(G_n^m, \mathcal{H}_m) &= \#\mathbf{H}^2(G_n^m, \mathcal{H}_m) \\ &= \#\mathcal{H}_m^{G_n^m}/\#\mathbf{J}_n^m \circ \mathbf{N}_n^m(\mathcal{H}_m) = \#\mathcal{H}_m^{G_n^m}/\#\mathcal{H}_n = 1 \end{aligned}$$

giving $(\mathcal{H}_m/\mathbf{J}_n^m \mathcal{H}_n)^{G_n^m} \simeq \mathcal{H}_n[p^{m-n}]$. Whence $\#\mathcal{H}_m \geq \#\mathbf{J}_n^m \mathcal{H}_n \cdot \#\mathcal{H}_n[p^{m-n}] = \#\mathcal{H}_n \cdot \#\mathcal{H}_n[p^{m-n}]$. For $m = n + 1$, one obtains $\#\mathcal{H}_{n+1} \geq \#\mathcal{H}_n \cdot p^{\text{rank}_p(\mathcal{H}_n)}$, then the last claim by induction, $\text{rank}_p(\mathcal{H}_n)$ being increasing. \square

This result indicates that, for $L = K_N$, the filtration $(\mathcal{H}_L^i)_{i \geq 0}$ has length unbounded regarding N . Since $\#(\mathcal{H}_L^{i+1}/\mathcal{H}_L^i) = \frac{\#\mathcal{H}_K}{\#\mathbf{N}_{L/K}(\mathcal{H}_L^i)}$, giving a decreasing sequence, in an probabilistic point of view, one sees that the length of the filtration depends essentially on the size of \mathcal{H}_K but not necessarily of N .

7.3. Conclusion. The behavior of p -class groups in real cyclic p -towers L/K (with $L \subset K(\mu_\ell)$, $\ell \equiv 1 \pmod{2p^N}$ inert in K/\mathbb{Q}), suggests that there exist infinitely many such primes for which \mathcal{H}_K capitulates at some layer. The most interesting fact being that, if so, this implies the real Main Conjecture on abelian fields in the semi-simple case with minimal analytic arguments and almost trivial proof from the classical context of Chevalley–Herbrand formulas.

Regarding some works about these questions of “exceptional classes” (i.e., non invariant), it is admitted (and proved in some circumstances) that the filtrations defining the \mathcal{H}_{K_n} , for $n \in [1, N]$, are random and that cases of “unbounded” filtrations are of probabilities tending to 0 with N . For instance, in [KoPa2022, Smi2022] it is proved that p -class groups of cyclic extensions L/\mathbb{Q} of degree p have a standard distribution, the case of filtrations of length 1 (i.e., $\mathcal{H}_L = \mathcal{H}_L^G$) being the most probable; but this is another context since the Chevalley–Herbrand formula $\#\mathcal{H}_L^G = p^{r-1}$ is non-trivial only if the number r of ramified primes is at least 2, in which case $\mathcal{H}_L^G = \mathcal{H}_L^{\text{ram}}$ (see Theorem (3.1)) and $\#((\mathcal{H}_L/\mathcal{H}_L^G)^G) = \frac{p^{r-1}}{(\mathbf{\Lambda} : \mathbf{\Lambda} \cap \mathbf{N}_{L/\mathbb{Q}}(L^\times))}$ (see (1)), where $\mathbf{\Lambda}$ is generated by the r prime numbers ramified in L/\mathbb{Q} ; whence the algorithm giving the $\#(\mathcal{H}_L^{i+1}/\mathcal{H}_L^i) = \frac{p^{r-1}}{(\mathbf{\Lambda}_i : \mathbf{\Lambda}_i \cap \mathbf{N}_{L/\mathbb{Q}}(L^\times))}$. But, in our case ($\mathcal{H}_K \neq 1$ while $r = 1$), on the contrary, the norm factors are trivial and $\#(\mathcal{H}_L^{i+1}/\mathcal{H}_L^i) = \frac{\#\mathcal{H}_K}{\#\mathbf{N}_{L/K}(\mathcal{H}_L^i)}$ depends of the structure of \mathcal{H}_K and allows standard probabilities about order and structure of \mathcal{H}_L .

In other words, the context of “fixed points formulas” in cyclic p -towers, is, in some sense, specific and “easier” or “more reachable” than the general framework on p -class groups in arbitrary number fields, only accessible via complex analytic methods, as the reader can see in the recent papers [EPW2017, Wan2020, PTBW2020, Gra2020, KIWa2022, Gra2022^b, Pier2022], among many others.

Moreover, the ϵ -conjectures are of no help, here, since the discriminants of the fields $K(\mu_\ell)$ are larger than $\ell^{\ell-2} = O(p^{N \cdot p^N})$ giving dreadful bounds for the $\#\mathcal{H}_L$, and the philosophy is on the contrary (as for Greenberg’s conjecture in $L = K_\infty$ for “ $\ell = p$ ”) that $\#\mathcal{H}_L = O(\#\mathcal{H}_K)$ for infinitely many ℓ ’s. To be more audacious, one may imagine an “Iwasawa behavior” leading, for $N \gg 0$, to formulas of the form $\#\mathcal{H}_{K_n} = p^{\lambda(\ell) \cdot n + \nu(\ell)}$ for $n \in [n_0(\ell), N]$, possibly with $\lambda(\ell) = 0$ (stability) for infinitely many ℓ ’s.

In other words, capitulation phenomena in p -towers are of p -adic type in a specific meaning adding randomness and, logically, govern many arithmetic results and conjectures of number theory around class field theory, and appear essentially as new general regularity principle which deserves deepened researches.

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