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Georges Gras. The Chevalley–Herbrand formula and the real abelian Main Conjecture: New criterion using capitulation of the class group. 2022. hal-03739451v1

HAL Id: hal-03739451

<https://hal.science/hal-03739451v1>

Preprint submitted on 27 Jul 2022 (v1), last revised 2 Sep 2022 (v3)

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**THE CHEVALLEY–HERBRAND FORMULA
AND THE REAL ABELIAN MAIN CONJECTURE**
NEW CRITERION USING CAPITULATION OF THE CLASS GROUP

GEORGES GRAS

ABSTRACT. The Main Theorem for abelian fields K (often called Main Conjecture in the literature despite proofs in most cases) has a long history which has found its solution by means of “elementary arithmetic properties”, as detailed in Washington’s book from Thaine’s method having led to Kolyvagin’s Euler systems and many developments. The analytic theory of real abelian fields says (in the semi-simple case $p \nmid [K : \mathbb{Q}]$) that the order of the p -class group \mathcal{H}_K is equal to the index $(\mathcal{E}_K : \mathcal{F}_K)$ of the subgroup of cyclotomic units in the group of units. Believing in the relevance of mathematical theories, we have conjectured (1977) the same relations $\#\mathcal{H}_\varphi = (\mathcal{E}_\varphi : \mathcal{F}_\varphi)$ for the isotypic p -adic components corresponding to the irreducible p -adic characters φ of K . We develop, in this article, new promising links between: (i) the Chevalley–Herbrand formula giving the order of the group of “ambiguous classes” in $L/K \subset K(\mu_\ell)/K$, of degree p^N , for the auxiliary prime number $\ell \equiv 1 \pmod{2p^N}$ inert in K ; (ii) the phenomenon of capitulation of \mathcal{H}_K in L ; (iii) the Main Conjecture $\#\mathcal{H}_\varphi = (\mathcal{E}_\varphi : \mathcal{F}_\varphi)$ for all φ , which is fulfilled as soon as \mathcal{H}_K capitulates in L (Theorem 1.1). Many computations with PARI programs support this new philosophy of the Main Conjecture.

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Date: July 27, 2022.

1991 Mathematics Subject Classification. Primary 11R18, 11R29, 11R27; Secondary 11R37, 12Y05.

Key words and phrases. Chevalley–Herbrand formula, abelian fields, analytic formulas, class field theory, capitulation of p -class groups, main conjecture.

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1. INTRODUCTION – STATEMENT OF THE MAIN RESULT

1.1. **Abelian characters.** Let \mathbb{Q}^{ab} be the maximal abelian extension of \mathbb{Q} contained in an algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} ; let \mathbb{Q}_p be the p -adic field and $\overline{\mathbb{Q}}_p$ an algebraic closure of \mathbb{Q}_p containing $\overline{\mathbb{Q}}$.

Let Ψ be the set of irreducible characters of $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$, of degree 1 and finite order, with values in $\overline{\mathbb{Q}}_p$. We define the set Φ of irreducible p -adic characters and the set \mathbf{X} of irreducible rational characters. For a subfield K of \mathbb{Q}^{ab} , we define the subsets Ψ_K , Φ_K , \mathbf{X}_K , whose kernels fixe K . The set \mathbf{X} is in one-to-one correspondence with the set of cyclic subfields of \mathbb{Q}^{ab} .

The notation $\psi \mid \varphi \mid \chi$ (for $\psi \in \Psi$, $\varphi \in \Phi$, $\chi \in \mathbf{X}$) means that φ is a term of χ and ψ a term of φ ; so φ is the sum of the \mathbb{Q}_p -conjugates of ψ and χ the sum of the \mathbb{Q} -conjugates of ψ (cf. [Ser1998]).

Let $\chi \in \mathbf{X}$; we denote by $g_\psi = g_\varphi = g_\chi$ the order of $\psi \mid \varphi \mid \chi$; the field of values of these characters is by definition the cyclotomic field $\mathbb{Q}(\mu_{g_\psi})$.

Let K/\mathbb{Q} be an abelian extension of Galois group g of prime-to- p order. For any $\mathbb{Z}[g]$ -module A of finite type of K , we denote by \mathcal{A}_χ (resp. \mathcal{A}_φ) the χ -component (resp. the φ -component) of $\mathcal{A} := A \otimes \mathbb{Z}_p$; we get $\mathcal{A}_\chi = \bigoplus_{\varphi \mid \chi} \mathcal{A}_\varphi$, then $\mathcal{A} = \bigoplus_{\chi \in \mathbf{X}_K} \bigoplus_{\varphi \mid \chi} \mathcal{A}_\varphi = \bigoplus_{\varphi \in \Phi_K} \mathcal{A}_\varphi$. These φ -components may be obtained by means of the system of idempotents of the algebra $\mathbb{Z}_p[g]$.

In this semi-simple case all reasonings reduce to the arithmetic of the cyclic subfields of K ; so in what follows, we only consider cyclic extensions K/\mathbb{Q} .

1.2. **Main theorem.** We have obtained the following result (Theorem 5.6 and Corollary 5.7):

Theorem 1.1. *Let $p \geq 2$ be a prime number. Let K/\mathbb{Q} be a real cyclic extension of prime-to- p degree and Galois group g . Denote by $\mathcal{H}_K = \bigoplus_{\varphi \in \Phi_K} \mathcal{H}_\varphi$ the p -class group of K .*

Consider primes $\ell \equiv 1 \pmod{2p^N}$, ℓ totally inert in K , and let K_n be the subfield of $K(\mu_\ell)$ of degree p^n over K , $n \in [1, N]$, where μ_ℓ is the group of ℓ -roots of unity.

Let \mathbf{E}_K (resp. \mathbf{F}_K) be the group of units (resp. of cyclotomic units) of K and put $\mathcal{E}_K = \bigoplus_{\varphi \in \Phi_K} \mathcal{E}_\varphi$ (resp. $\mathcal{F}_K = \bigoplus_{\varphi \in \Phi_K} \mathcal{F}_\varphi$). Then:

(i) As soon as \mathcal{H}_K capitulates by extension in K_n , the Main Conjecture holds in K , that is to say, $\#\mathcal{H}_\varphi = (\mathcal{E}_\varphi : \mathcal{F}_\varphi)$ for all $\varphi \in \Phi_K$.

(ii) We have $\#\mathcal{H}_{K_n} = \#\mathcal{H}_K$ for all $n \in [1, N]$, if and only if $\#\mathcal{H}_{K_1} = \#\mathcal{H}_K$. Moreover, if this stability property holds, we have the following consequences:

- *The p -class groups \mathcal{H}_{K_n} are invariant by $\text{Gal}(K_n/K)$ and the norms $\mathbf{N}_{K_n/K} : \mathcal{H}_{K_n} \rightarrow \mathcal{H}_K$ are isomorphisms.*

- *Let p^e be the exponent of \mathcal{H}_K and assume $N \geq e$; then the p -class group \mathcal{H}_K capitulates in K_e and (from (i)) the Main Conjecture $\#\mathcal{H}_\varphi = (\mathcal{E}_\varphi : \mathcal{F}_\varphi)$, for all $\varphi \in \Phi_K$, holds.*

We have proposed in [Gra2022^b] the following Conjecture of capitulation for any totally real number field K , without any assumption of splitting on ℓ :

Conjecture 1.2. *Let K be any totally real number field and let \mathcal{H}_K be its p -class group. There are infinitely many primes ℓ , $\ell \equiv 1 \pmod{2p^N}$, such that \mathcal{H}_K capitulates in the subfield of $K(\mu_\ell)$ of degree p^N over K .*

We shall give Section 6 extensive numerical computations with PARI programs [Pari2013] showing that capitulation in such auxiliary cyclic p -extensions is, surprisingly, very frequent and conjecturally holds for infinitely many ℓ . The criterion (ii) of the theorem allows easy effective verifications; but capitulation may hold without any stability.

We do not intend to evoke the case of the abelian capitulations of class groups proved in the literature (Gras [Gra1997], Kurihara [Kur1999], Bosca [Bos2009], Jaulent [Jau2022]); all techniques in these papers need to build abelian p -extensions L_0 of \mathbb{Q} , ramified at various primes and requiring many local arithmetic conditions, whose compositum L with K gives a capitulation field of \mathcal{H}_K ; the method is completely incomparable to ours since it must apply to any real abelian field K , of arbitrary degree, obtained in an iterative process giving that the maximal real subfield of $\mathbb{Q}(\bigcup_{n>0} \mu_n)$ is principal.

However, these results together with Theorem 1.1 and the help of extensive computations, among many other results of the literature, support the fact that the phenomenon of capitulation governs many aspects of abelian arithmetic, independently of the well-known case of capitulation in the Hilbert class field, from Hilbert’s theorem 94 and a lot of improvements (see the surveys [Jau1988, Jau1998, Mai1997, Mai1998] and their references giving a good idea of the story).

In a slightly different, but related, context of the p -adic class field theory, mention for instance that the capitulation of the logarithmic class group of K [Jau1994, Jau1998], in its cyclotomic \mathbb{Z}_p -extension $K_\infty = \bigcup_{n \geq 0} K_n$, is equivalent to Greenberg’s conjecture [Gree1976] saying that the Iwasawa invariants λ, μ for $\varprojlim \mathcal{H}_{K_n}$ are zero [Jau2016, Jau2019^a, Jau2019^b]. Analogous criteria of stability of the $\#\mathcal{H}_{K_n}$ were given by Greenberg itself and by Fukuda [Fuk1994]; for instance, in the similar context as ours where one considers p inert in K , the condition $\lambda = \mu = 0$ in $L = K_\infty$ is equivalent to the capitulation of \mathcal{H}_K in some layer K_{n_0} [Gree1976, Theorem 1].

In a numerical setting, let’s point out the work of Kraft–Schoof [KrSc1995^a, KrSc1995^b] (resp. Pagani [Pag2022]), verifying Greenberg’s conjecture for some real quadratic fields of conductor $f < 10^4$ and $p = 3$ (resp. $p = 2$), by means of the analytic formula.

But Greenberg’s conjecture is still unproved and depends on random algorithmic process, as explain in [Gra2021], governed by the torsion group of the Galois group of the maximal abelian p -ramified pro- p -extension of K (essentially, the second Tate–Chafarevich group of K); this takes place in a deep p -adic context, beyond Leopoldt’s conjecture and so on.

Remark 1.3. We have proven the criterion (ii) of stability of Theorem 1.1 in [Gra2022^b, Theorem 3.1, Corollaire 3.2], generalizing similar results [Fuk1994, LOXZ2022, MiYa2021]. More precisely, this criterion can be applied at some layer n_0 and one obtains $\#\mathcal{H}_{K_n} = \#\mathcal{H}_{K_{n_0}}$ for all $n \geq n_0$, if and only if the equality holds for $n = n_0 + 1$; so, this means that $\mathcal{H}_{K_{n_0}}$ capitulates in $K_{n_0+e_{n_0}}$, where $p^{e_{n_0}}$ is the exponent of $\mathcal{H}_{K_{n_0}}$, but a fortiori, \mathcal{H}_K capitulates in $K_{n_0+e_{n_0}}$. We shall give again a proof in our particular simpler framework (Theorem 4.9).

1.3. Methodology. We shall obtain Theorem 1.1 by means of a classical exact sequence describing \mathcal{H}_L^G , in cyclic p -extensions L/K of Galois group G , in terms of the units and image of the extension $\mathcal{H}_K \rightarrow \mathcal{H}_L^G$ of p -classes, which gives, under the phenomenon of capitulation, the needed information about the index of cyclotomic units, taking into account their norm properties in abelian extensions (Corollary 4.6 and Proposition 5.3). For that, we shall need the order of the “ φ -components”, for all $\varphi \in \Phi_K$, of the Chevalley–Herbrand formula

giving $\#\mathcal{H}_L^G$, that is to say, to compute $\#\mathcal{H}_\varphi^G$, where $\mathcal{H}_\varphi^G := (\mathcal{H}_\varphi)^G = (\mathcal{H}^G)_\varphi$ since G is commutative.

This will give the opportunity to write cohomological exact sequences linking invariant classes and capitulation to the norm properties of the units (local and global); for this, we shall follow [Jau1986, III, p. 167]. This step is crucial since Chevalley–Herbrand formula writes, for p -class groups in L/K cyclic of degree p^n , as follows:

$$\#\mathcal{H}_L^G = \frac{\#\mathcal{H}_K \times \prod_{\mathfrak{q}} e_{\mathfrak{q}}(L/K)}{p^n \times (\mathcal{E}_K : \mathcal{E}_K \cap \mathbf{N}_{L/K}(L^\times)},$$

where $e_{\mathfrak{q}}(L/K)$ is the ramification index in L/K of the prime ideals \mathfrak{q} of K ; see the original Chevalley formula [Che1933, pp. 402-406], depending on the Herbrand theorem [Her1930], introducing the general “Herbrand quotient” (see Lemme 3, p. 375 of Chevalley’s Thesis then [Her1936, Appendice, § 1, p. 57]).

One must obtain the decompositions of each factor into φ -components, especially for $\#\mathcal{H}_K$ since \mathcal{H}_K is not a submodule of \mathcal{H}_L^G , the transfer map $\mathbf{J}_{L/K} : \mathcal{H}_K \rightarrow \mathcal{H}_L^G$ being in general non injective (see for instance Example 4.8). For this, we introduce the following elementary principle.

Definition 1.4. *Let K/\mathbb{Q} be an abelian extension of Galois group g and let X be a finite $\mathbb{Z}[g]$ -module. Assume that we know that $\#X = \prod_i \#U_i \times \prod_j (\#V_j)^{-1}$ depending on finite $\mathbb{Z}[g]$ -modules U_i, V_j .*

We say that X (or the formula giving $\#X$) is p -localizable if there exist exact sequences $1 \rightarrow A_k \rightarrow B_k \rightarrow C_k \rightarrow 1$ of finite $\mathbb{Z}[g]$ -modules of the form U_i, V_j and X , such that $\#X$ is obtained by means of the relations $\#B_k = \#A_k \times \#C_k$.

Under this property, the flatness of \mathbb{Z}_p allows to deduce the exact sequences of $\mathbb{Z}_p[g]$ -modules $1 \rightarrow \mathcal{A}_k := A_k \otimes \mathbb{Z}_p \rightarrow \mathcal{B}_k := B_k \otimes \mathbb{Z}_p \rightarrow \mathcal{C}_k := C_k \otimes \mathbb{Z}_p \rightarrow 1$, then, taking the isotopic components (for instance by means of the fundamental idempotents e_φ of $\mathbb{Z}_p[g]$, $\varphi \in \Phi_K$), we get the exact sequences:

$$1 \rightarrow \mathcal{A}_{k,\varphi} \rightarrow \mathcal{B}_{k,\varphi} \rightarrow \mathcal{C}_{k,\varphi} \rightarrow 1.$$

and the formulas $\#\mathcal{B}_{k,\varphi} = \#\mathcal{A}_{k,\varphi} \times \#\mathcal{C}_{k,\varphi}$ yielding $\#X_\varphi$.

For instance, cohomology groups $H^n(G, X)$, $n \in \{1, 2\}$, $G = \langle \sigma \rangle$, are p -localizable, as soon as X is p -localizable; indeed, $H^1(G, X) := \text{Ker}_X(\nu_G)/X^{1-\sigma}$ and $H^2(G, X) := X^G/\nu_G(X)$, where ν_G is the algebraic norm. But, of course, this principle applies in all cohomology.

2. INTRODUCTION TO THE REAL ABELIAN MAIN CONJECTURE

The Main Conjecture for real abelian fields K (to be called Main Theorem because of its numerous proofs; but this name has become common) essentially says that, when $p \nmid \#g$, one has, for all p -adic irreducible characters $\varphi \in \Phi_K$:

$$\#\mathcal{H}_{K,\varphi} = (\mathcal{E}_{K,\varphi} : \mathcal{F}_{K,\varphi}),$$

for the φ -components of \mathcal{H}_K , where $\mathcal{E}_K = \mathbf{E}_K \otimes \mathbb{Z}_p$, $\mathcal{F}_K = \mathbf{F}_K \otimes \mathbb{Z}_p$ denote the groups of global units and of cyclotomic units, respectively.

The following obvious property of rational characters is to be considered as the “Main Theorem” for rational components (e.g., [Leo1954, Chap. I, § 1, 1]):

Theorem 2.1. *Let K/\mathbb{Q} be an abelian extension and let $(A_\chi)_{\chi \in \mathbf{X}_K}$, $(A'_\chi)_{\chi \in \mathbf{X}_K}$, be two families of positive numbers, indexed by the set \mathbf{X}_K of irreducible rational characters of K . If for all subfields k of K , one has $\prod_{\chi \in \mathbf{X}_k} A'_\chi = \prod_{\chi \in \mathbf{X}_k} A_\chi$, then $A'_\chi = A_\chi$ for all $\chi \in \mathbf{X}_K$.*

So this result applies to the well-known complex analytic global formula $\#\mathbf{H}_K = (\mathbf{E}_K : \mathbf{F}_K)$, which implies immediately $\#\mathcal{H}_{K,\chi} = (\mathcal{E}_{K,\chi} : \mathcal{F}_{K,\chi})$ for all $\chi \in \mathbf{X}_K$, in the semi-simple case. But when χ is a sum of several p -adic irreducible characters φ , the complex analytic theory does not give any relation between $\#\mathcal{H}_{K,\varphi}$ and $(\mathcal{E}_{K,\varphi} : \mathcal{F}_{K,\varphi})$.

One may think that p -adic analysis must solve the problem; but in fact the p -adic framework, using the p -adic ζ_p -function and the corresponding L_p -functions, gives essentially the same relations between χ -components. In other words, there is no sufficient information.

Deep geometrical methods (Ribet, Mazur–Wiles) were successful, then more elementary ones were used to solve the problem (see e.g., [CoSu2006, Grei1992, Kol2007, PeRi1990, Rib2008, Rub1990, Thai1988] among others. A more complete story is available in Washington book [Was1997, Chap. 8].

The non-semi-simple case of the real abelian Main Conjecture does exist and may be split into two frameworks.

(i) The Iwasawa formulation [Iwa1964] (“replacing” K by its cyclotomic \mathbb{Z}_p -extension K_∞) also called Main Conjecture without any precision (Lang–Rubin [Lan1990], Greither [Grei1992], exposed in Washington’s book [Was1997, §§ 13.6, 15.4]); nevertheless, in the real case, Greenberg’s conjecture makes it somewhat unnecessary and brings back to the finite cases K_n/K as we have explained.

(ii) The case of cyclic extension K/\mathbb{Q} when $p \mid [K : \mathbb{Q}]$; this case corresponds to our conjecture given in [Gra1976, Gra1977] and still unproved for real fields. We refer to the survey [Gra2022^a] devoted to this non semi-simple case using the specific notion of φ -objects that we had introduced in the 1976’s. Indeed, classical works deal with an algebraic definition of the φ -components of p -class groups, denoted $\mathcal{H}_{K,\varphi}^{\text{alg}}$, which presents an inconsistency with analytic formulas; that is to say, when $G := \text{Gal}(K/\mathbb{Q})$ is cyclic, of order $g_\chi \equiv 0 \pmod{p}$:

$$\mathcal{H}_{K,\varphi}^{\text{alg}} := \mathcal{H}_K \otimes_{\mathbb{Z}_p[G]} \mathbb{Z}_p[\mu_{g_\chi}], \text{ for all } \varphi \mid \chi,$$

with the $\mathbb{Z}_p[\mu_{g_\chi}]$ -action $\sigma \in G \mapsto \psi(\sigma)$ ($\psi \mid \varphi$ of order g_χ). We then have:

$$\mathcal{H}_{K,\varphi}^{\text{alg}} = \{x \in \mathcal{H}_K, \nu_{K/k}(x) = 1, \forall k \subsetneq K\} \otimes_{\mathbb{Z}_p[G]} \mathbb{Z}_p[\mu_{g_\chi}]$$

(where $\nu_{K/k}$ is the algebraic norm), contrary to our arithmetic definition:

$$\mathcal{H}_{K,\varphi}^{\text{ar}} := \{x \in \mathcal{H}_K, \mathbf{N}_{K/k}(x) = 1, \forall k \subsetneq K\} \otimes_{\mathbb{Z}_p[G]} \mathbb{Z}_p[\mu_{g_\chi}].$$

This specific notion gives rise to an unexpected semi-simplicity:

$$\mathcal{H}_{K,\chi}^{\text{ar}} := \{x \in \mathcal{H}_K, \mathbf{N}_{K/k}(x) = 1, \text{ for all } k \subsetneq K\} \otimes_{\mathbb{Z}_p} = \bigoplus_{\varphi \mid \chi} \mathcal{H}_{K,\varphi}^{\text{ar}},$$

especially in accordance with analytic formulas.

3. PREREQUISITES ON CLASS FIELD THEORY

We recall, for the convenience of the reader, well-known aspects about class field theory that we shall consider in this article:

3.1. The higher rank Chevalley–Herbrand formulas. Let K/\mathbb{Q} be a cyclic extension, with $p \nmid [K : \mathbb{Q}]$. To avoid technical complications, we assume that the extension L/K , cyclic of degree p^n , $n \geq 1$, of Galois group $G =: \langle \sigma \rangle$, is such that any prime ideal of K , ramified in L/K , is totally ramified and that infinite places do not ramify when $p = 2$. Moreover we assume that the set of ramified primes is non-empty. Later, we shall take $L \subset \mathbb{Q}^{\text{ab}}$, but this assumption is not needed in this subsection nor that L/\mathbb{Q} is Galois.

Generalizations of the Chevalley–Herbrand formula for p -class groups defines a filtration of the form $\{\mathcal{H}_L^i\}_{i \geq 0}$, where $\mathcal{H}_L^0 := 1$, $\mathcal{H}_L^1 := \mathcal{H}_L^G$, $\mathcal{H}_L^{i+1}/\mathcal{H}_L^i := (\mathcal{H}_L/\mathcal{H}_L^i)^G$, up to a minimal bound m for which $\mathcal{H}_L^m = \mathcal{H}_L$:

$$(3.1) \quad \begin{cases} \#\mathcal{H}_L^G = \#\mathcal{H}_K \times \frac{p^{n(r-1)}}{(\mathcal{E}_K : \mathcal{E}_K \cap \mathbf{N}_{L/K}(L^\times))}, \\ \#(\mathcal{H}_L^{i+1}/\mathcal{H}_L^i) = \frac{\#\mathcal{H}_K}{\#\mathbf{N}_{L/K}(\mathcal{H}_L^i)} \times \frac{p^{n(r-1)}}{(\mathbf{A}_K^i : \mathbf{A}_K^i \cap \mathbf{N}_{L/K}(L^\times))}, \quad i \geq 0, \end{cases}$$

where $r \geq 1$ is the number of prime ideals of K ramified (totally) in L/K , and $\mathbf{A}_K^i := \{x \in K^\times, (x) = \mathbf{N}_{L/K}(\mathfrak{A}), \mathfrak{A}P_L \in \mathcal{H}_L^i\}$ is a subgroup of finite type of

K^\times containing \mathbf{E}_K (with $\Lambda_K^0 = \mathbf{E}_K$); the quotient $\Lambda_K^i / \Lambda_K^i \cap \mathbf{N}_{L/K}(L^\times)$ is of course a p -group of order a divisor of $p^{n(r-1)}$.

In the case where $r = 1$ (which will be our context, taking a prime ℓ inert in the cyclic extension K/\mathbb{Q} and $L \subset \mathbb{Q}(\mu_\ell)$), one obtains:

$$(3.2) \quad \#\mathcal{H}_L^G = \#\mathcal{H}_K \quad \& \quad \#(\mathcal{H}_L^{i+1}/\mathcal{H}_L^i) = \frac{\#\mathcal{H}_K}{\#\mathbf{N}_{L/K}(\mathcal{H}_L^i)}, \quad i \geq 0.$$

There are a lot of generalizations of the original formulas. See for instance [Jau1986, III, p. 167], [Gra2017] with modulus, [LiYu2020] in the spirit of Chevalley's theory of idèles, then [Gon2008] showing universality of the Chevalley–Herbrand principle in arithmetic and geometry, etc.

3.2. Genus exact sequence. Denote by H_K and H_L the p -Hilbert class fields of K and L , respectively. In the canonical isomorphism $\mathcal{H}_L \simeq \text{Gal}(H_L/L)$, the image of \mathcal{H}_L^G fixes a subfield of H_L which has not a clear interpretation; on the contrary, \mathcal{H}_L^G appears as the quotient of $\text{Gal}(H_L/L)$ by the image of $\mathcal{H}_L^{1-\sigma}$ which has a canonical interpretation. This “duality” leads to genus theory as follows.

We still consider a cyclic p -extension L/K , of Galois group $G =: \langle \sigma \rangle$, in which the prime ideals \mathfrak{q} of K , ramified in L/K , are totally ramified.

Let $H_{L/K}$ be the p -genus field of L defined as the maximal sub-extension of H_L , abelian over K , as given by the following diagram, $H_{L/K}$ being the fixed field of the image of $\mathcal{H}_L^{1-\sigma}$, by the Artin map (the commutator subgroup $[\Gamma, \Gamma]$ of $\Gamma = \text{Gal}(H_L/K)$ is $\text{Gal}(H_L/L)^{1-\sigma}$ since $\Gamma/\text{Gal}(H_L/L)$ is cyclic):

$$\begin{array}{ccccc} L & \xrightarrow{\quad} & LH_K & \xrightarrow{\prod_{\mathfrak{q}} s'_{\mathfrak{q}}} & H_{L/K} & \xrightarrow{\simeq \mathcal{H}_L^{1-\sigma}} & H_L \\ G \Big| & & \Big| & & \Big| & & \\ K & \xrightarrow{\simeq \mathcal{H}_K} & H_K & & \langle I_{\mathfrak{q}}(H_{L/K}/K) \rangle_{\mathfrak{q}} & & \end{array}$$

By assumption, the inertia groups $I_{\mathfrak{q}}(L/K)$ in L/K are equal to G .

We consider the map ω which associates with $x \in \mathcal{E}_K$ the family of norm Hasse symbols $(\frac{x, L/K}{\mathfrak{q}}) \in G$. This gives the genus exact sequence interpreting the product formula of Hasse symbols of a unit (see, e.g., [Gra2005, § II.3 (a)], then Proposition IV.4.5.1):

$$1 \rightarrow \mathcal{E}_K / \mathcal{E}_K \cap \mathcal{N}_{L/K} \xrightarrow{\omega} \Omega(L/K) \subseteq \bigoplus_{\mathfrak{q}} I_{\mathfrak{q}}(L/K) \xrightarrow{\pi} \text{Gal}(H_{L/K}/LH_K) \rightarrow 1,$$

where $\Omega(L/K) := \left\{ (s_{\mathfrak{q}})_{\mathfrak{q}} \in \bigoplus_{\mathfrak{q}} I_{\mathfrak{q}}(L/K), \prod_{\mathfrak{q}} s_{\mathfrak{q}} = 1 \right\}$. The image of ω is contained in $\Omega(L/K)$. Then π associates, with $(s_{\mathfrak{q}})_{\mathfrak{q}} \in \bigoplus_{\mathfrak{q}} I_{\mathfrak{q}}(L/K)$, the product $\prod_{\mathfrak{q}} s'_{\mathfrak{q}}$ of the extensions $s'_{\mathfrak{q}}$ of the $s_{\mathfrak{q}}$ in the inertia groups $I_{\mathfrak{q}}(H_{L/K}/K)$ generating $\text{Gal}(H_{L/K}/H_K)$; product formula of class field theory implies that if $(s_{\mathfrak{q}})_{\mathfrak{q}} \in \Omega(L/K)$, $\prod_{\mathfrak{q}} s'_{\mathfrak{q}}$ fixes both H_K and L , hence LH_K . Thus $\pi(\Omega(L/K)) = \text{Gal}(H_{L/K}/LH_K)$ and $\text{Ker}(\pi) = \omega(\mathcal{E}_K)$.

Now, assume $L \subset \mathbb{Q}^{\text{ab}}$ of degree p^n over K and $g = \text{Gal}(K/\mathbb{Q})$ of prime-to- p order. Since $\bigoplus_{\mathfrak{q}} I_{\mathfrak{q}}(L/K) \simeq \bigoplus_{\mathfrak{q}} \mathbb{Z}/p^n\mathbb{Z}$, is a sum of permutation representations, its p -localization is canonical and depends on the decomposition fields, in K/\mathbb{Q} , of the ramified primes q , so that $(\bigoplus_{\mathfrak{q}} I_{\mathfrak{q}}(L/K))_{\varphi} \simeq (\mathbb{Z}_p/p^n\mathbb{Z}_p)^{d_{\varphi}}$, for some explicit $d_{\varphi} \geq 0$.

Whence the p -localized exact sequences for all $\varphi \in \Phi_k \setminus \{1\}$:

$$1 \rightarrow \mathcal{E}_{K,\varphi} / \mathcal{E}_{K,\varphi} \cap \mathcal{N}_{L/K} \longrightarrow (\mathbb{Z}_p/p^n\mathbb{Z}_p)^{d_{\varphi}} \longrightarrow \text{Gal}(H_{L/K}/LH_K)_{\varphi} \rightarrow 1,$$

If moreover K/\mathbb{Q} is cyclic and $L \subset K(\mu_\ell)$, where $\ell \equiv 1 \pmod{2p^N}$ is totally inert in K/\mathbb{Q} , then $d_{\varphi} = 0$ and $\mathcal{E}_{K,\varphi} = \mathcal{E}_K \cap \mathcal{N}_{L/K}$, which means that any $\varepsilon \in \mathcal{E}_{K,\varphi}$ is norm in L/K of an element of L^\times (Hasse's norm theorem); then, $H_{L/K} = LH_K$ and one finds again that, as expected:

$$\#(\mathcal{H}_{L,\varphi} / \mathcal{H}_{L,\varphi}^{1-\sigma}) = \#\mathcal{H}_{L,\varphi}^G = \#\mathcal{H}_{K,\varphi}.$$

4. THE p -LOCALIZATION OF THE CHEVALLEY–HERBRAND FORMULA

4.1. Exact sequence of the ambiguous classes. The p -localization of the Chevalley–Herbrand formula will exist from the definitions of class groups, ideal groups and units. Such p -localizations were given many years ago ([Gra1978, Théorèmes II.1, II.2], [Jau1986, Théorèmes III.1.12, III.1.13]); these papers being written in french, we give again the computations for the convenience of the reader.

To get a formula of the orders of the φ -components of \mathcal{H}_L^G , we follow the process given in Jaulent’s Thesis [Jau1986, Chapitre III, p. 167] (note that large generalizations of such methods, with ramification and decomposition, are also given in [Mai1997, Mai1998] and, in the Galois case L/K , [Gon2006] with many references):

Theorem 4.1. *Let L_0/\mathbb{Q} be a real cyclic extension of degree p^n , $n \geq 1$. Let K/\mathbb{Q} be a real abelian extension of Galois group g of prime-to- p order. Put $L = L_0K$ and $G := \text{Gal}(L/K) =: \langle \sigma \rangle$. We identify $\text{Gal}(L/L_0)$ and g .*

Let I_K, I_L (resp. P_K, P_L) be the ideal groups (resp. the subgroups of principal ideals), of K and L , respectively. Then, put $\mathbf{H}_K := I_K/P_K$ and $\mathbf{H}_L := I_L/P_L$.

(i) *We have the exact sequence of $\mathbb{Z}[g]$ -modules:*

$$(4.1) \quad 1 \rightarrow \text{Ker}(\mathbf{J}_{L/K}) \rightarrow \text{H}^1(G, \mathbf{E}_L) \rightarrow \text{Coker}(\iota_{L/K}) \rightarrow \\ \text{Coker}(\mathbf{J}_{L/K}) \rightarrow \text{H}^2(G, \mathbf{E}_L) \rightarrow \text{H}^2(G, L^\times),$$

where $\mathbf{J}_{L/K}$ is the transfer map $\mathbf{H}_K \rightarrow \mathbf{H}_L^G$ defined by $\mathfrak{a}P_K \mapsto \mathfrak{a}P_L$ and $\iota_{L/K}$ is the extension of ideals $I_K \rightarrow I_L^G$,

(ii) *Let φ be an irreducible p -adic character of K . Then:*

- *If $\varphi = 1$, then $\#\mathcal{H}_{L,\varphi}^G = \#\mathcal{H}_{K,\varphi} = 1$;*
- *If $\varphi \neq 1$, then $\#\mathcal{H}_{L,\varphi}^G = \#\mathcal{H}_{K,\varphi} \times \frac{\#\text{Coker}(\iota_{L/K})_\varphi}{(\mathcal{E}_{K,\varphi} : \mathcal{E}_{K,\varphi} \cap \mathbf{N}_{L/K}(L^\times))}$.*

Proof. We note that each prime number ℓ , ramified in L/K , is totally ramified.

Consider the exact sequences of $\mathbb{Z}[g]$ -modules:

$$(4.2) \quad \begin{array}{ll} (a) & 1 \rightarrow \mathbf{E}_L \rightarrow L^\times \rightarrow P_L \rightarrow 1, \\ (b) & 1 \rightarrow P_L \rightarrow I_L \rightarrow \mathbf{H}_L \rightarrow 1, \end{array} \quad \begin{array}{ll} (c) & 1 \rightarrow \mathbf{E}_K \rightarrow K^\times \rightarrow P_K \rightarrow 1 \\ (d) & 1 \rightarrow P_K \rightarrow I_K \rightarrow \mathbf{H}_K \rightarrow 1. \end{array}$$

Lemma 4.2. *We have the following properties:*

- (i) $P_L^G/\iota_{L/K}(P_K) \simeq \text{H}^1(G, \mathbf{E}_L)$;
- (ii) $\text{H}^1(G, P_L) \simeq \text{Ker}[\text{H}^2(G, \mathbf{E}_L) \rightarrow \text{H}^2(G, L^\times)]$;
- (iii) $\text{H}^1(G, I_L) = 1$.

Proof. We have, from the above exact sequences (4.2) (a), (c):

$$1 \rightarrow \mathbf{E}_L^G = \mathbf{E}_K \rightarrow L^{\times G} = K^\times \rightarrow P_L^G \rightarrow \text{H}^1(G, \mathbf{E}_L) \rightarrow \text{H}^1(G, L^\times) \\ \rightarrow \text{H}^1(G, P_L) \rightarrow \text{H}^2(G, \mathbf{E}_L) \rightarrow \text{H}^2(G, L^\times).$$

Since $\text{H}^1(G, L^\times) = 1$ (Hilbert’s Theorem 90), this yields (i) and (ii).

The claim (iii) is classical since I_L is a $\mathbb{Z}[G]$ -module generated by the prime ideals of L on which the Galois action is canonical. \square

From the above exact sequences (4.2) (b), (d), we have the following commutative diagram:

$$(4.3) \quad \begin{array}{ccccccc} 1 & \longrightarrow & P_K & \longrightarrow & I_K & \longrightarrow & \mathbf{H}_K \longrightarrow 1 \\ & & \downarrow & & \downarrow \iota_{L/K} & & \downarrow \mathbf{J}_{L/K} \\ 1 & \longrightarrow & P_L^G & \longrightarrow & I_L^G & \longrightarrow & \mathbf{H}_L^G \longrightarrow \text{H}^1(G, P_L). \end{array}$$

The snake lemma gives the exact sequence:

$$(4.4) \quad 1 \rightarrow \mathbf{E}_L^G/\mathbf{E}_K \rightarrow \text{Ker}(\iota_{L/K}) \rightarrow \text{Ker}(\mathbf{J}_{L/K}) \rightarrow \text{H}^1(G, \mathbf{E}_L) \rightarrow \\ \text{Coker}(\iota_{L/K}) \rightarrow \text{Coker}(\mathbf{J}_{L/K}) \rightarrow \text{H}^2(G, \mathbf{E}_L) \rightarrow \text{H}^2(G, L^\times)$$

since $\mathbf{E}_L^G/\mathbf{E}_K = \text{Ker}(\iota_{L/K}) = 1$ and $\text{H}^2(G, L^\times) \simeq K^\times/\mathbf{N}_{L/K}(L^\times)$, it becomes:

$$\begin{aligned} 1 \rightarrow \text{Ker}(\mathbf{J}_{L/K}) \rightarrow \text{H}^1(G, \mathbf{E}_L) \rightarrow \text{Coker}(\iota_{L/K}) \\ \rightarrow \text{Coker}(\mathbf{J}_{L/K}) \rightarrow \text{H}^2(G, \mathbf{E}_L) \rightarrow \mathbf{E}_K/\mathbf{E}_K \cap \mathbf{N}_{L/K}(L^\times) \rightarrow 1. \end{aligned}$$

The representation $\text{Coker}(\iota_{L/K})$ depends on the splitting of the prime ideals of K ramified in L/K and gives standard φ -components similar to that obtained for the representation $\bigoplus_{\mathfrak{q}} I_{\mathfrak{q}}(L/K)$ of genus theory; since we do not need the most general formula, we refer to [Gra1978] or [Jau1986]. In our particular context all these φ -components will be trivial for $\varphi \neq 1$.

We know that, from Hasse's norm theorem, $\mathbf{E}_K \cap \mathbf{N}_{L/K}(L^\times) = \mathbf{E}_K \cap \mathcal{N}_{L/K}$, where $\mathcal{N}_{L/K} \subset K^\times$ is the subgroup of local norms in L/K (see § 3.2).

We obtain the p -localized exact sequences of $\mathbb{Z}_p[g]$ -modules:

$$\begin{aligned} 1 \rightarrow \text{Ker}(\mathbf{J}_{L/K})_\varphi \rightarrow \text{H}^1(G, \mathcal{E}_{L,\varphi}) \rightarrow \text{Coker}(\iota_{L/K})_\varphi \rightarrow \\ \text{Coker}(\mathbf{J}_{L/K})_\varphi \rightarrow \text{H}^2(G, \mathcal{E}_{L,\varphi}) \rightarrow \mathcal{E}_{K,\varphi}/\mathcal{E}_{K,\varphi} \cap \mathcal{N}_{L/K} \rightarrow 1; \end{aligned}$$

by p -localization, $\text{H}^1(G, \mathbf{E}_L)_\varphi = \text{H}^1(G, \mathcal{E}_{L,\varphi})$ and $\text{H}^2(G, \mathbf{E}_L)_\varphi = \text{H}^2(G, \mathcal{E}_{L,\varphi})$.

All these $\mathbb{Z}_p[g]$ -modules are finite, which gives the p -localized formula:

$$\frac{\#(\mathcal{H}_{L,\varphi}^G)}{\#(\mathcal{H}_{K,\varphi})} = \frac{\#\text{Coker}(\mathbf{J}_{L/K})_\varphi}{\#\text{Ker}(\mathbf{J}_{L/K})_\varphi} = \frac{\#\text{Coker}(\iota_{L/K})_\varphi}{(\mathcal{E}_{K,\varphi} : \mathcal{E}_{K,\varphi} \cap \mathcal{N}_{L/K})} \times \frac{\#\text{H}^2(G, \mathcal{E}_{L,\varphi})}{\#\text{H}^1(G, \mathcal{E}_{L,\varphi})},$$

where:

$$\frac{\#\text{H}^2(G, \mathcal{E}_{L,\varphi})}{\#\text{H}^1(G, \mathcal{E}_{L,\varphi})}$$

is the Herbrand quotient of $\mathcal{E}_{L,\varphi}$ we talked about before, whose computation requires some algebraic reasonings and leads, for \mathbf{E}_L , to the global value $\frac{1}{[L:K]}$ in the real case (cf. [Lan1990], [Lan2000, Chap. IX, §§ 1,4]).

Lemma 4.3. *The Herbrand quotient of $\mathcal{E}_{L,\varphi}$ is trivial for $\varphi \in \Phi_K \setminus \{1\}$.*

Proof. We know that $(\mathbf{E}_L \otimes \mathbb{Q}) \oplus \mathbb{Q}$ is the regular representation $\mathbb{Q}[G \times g]$ (see, e.g., [Gra2005, Theorem I.3.7]); so there exists a ‘‘Minkowski unit’’ ε such that the $\mathbb{Z}[G \times g]$ -module generated by ε is of finite index in \mathbf{E}_L that one may choose prime to p ; so \mathcal{E}_L is such that $\mathcal{E}_L \oplus \mathbb{Z}_p \simeq \mathbb{Z}_p[G][g]$ as g -modules.

Thus, $(\mathcal{E}_L \oplus \mathbb{Z}_p)_\varphi = \mathcal{E}_{L,\varphi} \simeq \mathbb{Z}_p[\mu_{g_\varphi}][G]$ for $\varphi \neq 1$; whence the result. \square

We obtain:

$$\#(\mathcal{H}_{L,\varphi}^G) = \#(\mathcal{H}_{K,\varphi}) \times \frac{\#\text{Coker}(\iota_{L/K})_\varphi}{(\mathcal{E}_{K,\varphi} : \mathcal{E}_{K,\varphi} \cap \mathcal{N}_{L/K})}, \text{ for } \varphi \neq 1,$$

and $\mathcal{H}_{L,1}^G = \mathcal{H}_{K,1} = 1$, which completes the proof of the Theorem. \square

Corollary 4.4. *If $L \subset K(\mu_\ell)$, with $\ell \equiv 1 \pmod{2p^N}$ inert in the cyclic real field K and $[L:K] = p^n$, $n \in [1, N]$, then $\#\mathcal{H}_{L,\varphi}^G = \#\mathcal{H}_{K,\varphi}$ for all $\varphi \in \Phi_K$.*

More generally, we have $\#(\mathcal{H}_{L,\varphi}^{i+1}/\mathcal{H}_{L,\varphi}^i) = \frac{\#\mathcal{H}_{K,\varphi}}{\#\mathbf{N}_{L/K}(\mathcal{H}_{L,\varphi}^i)}$ for all $i \geq 0$.

Of course, this does not mean $\mathcal{H}_{L,\varphi}^G = \mathbf{J}_{L/K}(\mathcal{H}_{K,\varphi})$ since there is most often capitulation of classes; this expresses the subtlety of Chevalley–Herbrand formula for which we shall give another description of $\mathcal{H}_{L,\varphi}^G$ likely to involve the kernel of capitulation.

4.2. Exact sequence of capitulation. We still consider a real cyclic extension K/\mathbb{Q} of prime-to- p degree, then the compositum $L = KL_0$ with L_0/\mathbb{Q} cyclic of degree p^n , $n \geq 1$. Put $G := \text{Gal}(L/K) =: \langle \sigma \rangle$.

Let $\mathfrak{A}_L \in \mathbf{H}_L$ be a class invariant under G ; thus $\mathfrak{A}^{1-\sigma} = \alpha P_L$, $\alpha \in L^\times$ and $\mathbf{N}_{L/K}(\alpha)$ is a unit $\varepsilon \in \mathbf{E}_K \cap \mathbf{N}_{L/K}(L^\times)$; if $\alpha' = \alpha\eta$, $\eta \in \mathbf{E}_L$, is another generator of $\mathfrak{A}^{1-\sigma}$, then $\mathbf{N}_{L/K}(\alpha') = \varepsilon \mathbf{N}_{L/K}(\eta)$. This defines the map:

$$\mathbf{H}_L^G \longrightarrow \mathbf{E}_K \cap \mathbf{N}_{L/K}(L^\times)/\mathbf{N}_{L/K}(\mathbf{E}_L),$$

which associates with $\mathfrak{A}_L \in H_L^G$ the class of the unit $\varepsilon = \mathbf{N}_{L/K}(\alpha)$.

Theorem 4.5. *We have, for all $\varphi \in \Phi_K$, the exact sequences:*

$$1 \rightarrow \mathbf{J}_{L/K}(\mathcal{H}_{K,\varphi}) \cdot \mathcal{H}_{L,\varphi}^{\text{ram}} \rightarrow \mathcal{H}_{L,\varphi}^G \rightarrow \mathcal{E}_{K,\varphi} \cap \mathbf{N}_{L/K}(L^\times) / \mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi}) \rightarrow 1,$$

where $\mathcal{H}_{L,\varphi}^{\text{ram}} \subseteq \mathcal{H}_L$ is generated by the classes of the ramified prime ideals.

Proof. We have the global exact sequence:

$$1 \rightarrow \mathbf{J}_{L/K}(\mathbf{H}_K) \cdot \mathbf{H}_L^{\text{ram}} \rightarrow \mathbf{H}_L^G \rightarrow \mathbf{E}_K \cap \mathbf{N}_{L/K}(L^\times) / \mathbf{N}_{L/K}(\mathbf{E}_L) \rightarrow 1.$$

(i) Surjectivity. Let $\varepsilon \in \mathbf{E}_K \cap \mathbf{N}_{L/K}(L^\times)$; put $\varepsilon = \mathbf{N}_{L/K}(\alpha)$, then the ideal αP_L being of norm 1 is of the form $\mathfrak{A}^{1-\sigma}$, $\mathfrak{A} \in I_L$ (Lemma 4.2 (iii)), and its class is invariant giving the pre-image.

(ii) Kernel. Suppose that the image of the invariant class $\mathfrak{A}P_L$ is $\varepsilon = \mathbf{N}_{L/K}(\eta)$, $\eta \in \mathbf{E}_L$; then $\mathbf{N}_{L/K}(\alpha) = \mathbf{N}_{L/K}(\eta)$ and $\mathbf{N}_{L/K}(\alpha\eta^{-1}) = 1$ giving $\alpha\eta^{-1} = \beta^{1-\sigma}$, $\beta \in L^\times$ (Hilber’s Theorem 90) then $\alpha P_L = \mathfrak{A}^{1-\sigma} = (\beta P_L)^{1-\sigma}$. So, the class $\mathfrak{A}P_L$ is the class of the invariant ideal $\mathfrak{A}(\beta)^{-1}$; but the group of invariants ideals is generated by $\iota_{L/K}(I_K)$ and the ramified primes of L . Whence the result. Recall that $\mathbf{E}_K \cap \mathbf{N}_{L/K}(L^\times) = \mathbf{E}_K \cap \mathcal{N}_{L/K}$. \square

Corollary 4.6. *We still consider a prime $\ell \equiv 1 \pmod{2p^N}$, inert in the cyclic real field K , and $L \subset K(\mu_\ell)$, of degree p^n over K , $n \in [1, N]$. Then:*

(i) *For all $\varphi \in \Phi_K$, we have $\#\mathcal{H}_{L,\varphi}^G = \#\mathcal{H}_{K,\varphi}$ and the exact sequences*

$$1 \rightarrow \mathbf{J}_{L/K}(\mathcal{H}_{K,\varphi}) \rightarrow \mathcal{H}_{L,\varphi}^G \rightarrow \mathcal{E}_{K,\varphi} / \mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi}) \rightarrow 1.$$

(ii) *The capitulation of \mathcal{H}_K in L/K is equivalent to the isomorphisms:*

$$\mathcal{H}_{L,\varphi}^G \simeq \mathcal{E}_{K,\varphi} / \mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi}) \simeq \mathbb{Z}_p[\mu_{g_\varphi}] / (p)^{a_\varphi}, \quad \text{for all } \varphi \in \Phi_K,$$

a_φ being such that $p^{\rho_\varphi a_\varphi} = \#\mathcal{H}_{K,\varphi}$ where $\rho_\varphi = [\mathbb{Q}_p(\mu_{g_\varphi}) : \mathbb{Q}_p]$.

(iii) *The capitulation of \mathcal{H}_K in L/K is equivalent to:*

$$\#\mathcal{H}_{K,\varphi} = (\mathcal{E}_{K,\varphi} : \mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi})), \quad \text{for all } \varphi \in \Phi_K.$$

Proof. Exact sequence in (i) comes from the fact that $\mathcal{H}_L^{\text{ram}}$, generated by the prime ideal $\mathfrak{l} \mid \mathfrak{l}$ for the unique prime ideal $\mathfrak{l} = (\ell)$ of K , is of character $\varphi = 1$. The first equivalence comes from the exact sequence and the monogenicity of $\mathcal{E}_{K,\varphi}$ as $\mathbb{Z}_p[\mu_{g_\varphi}]$ -module. The second one is obvious since $\#\mathcal{H}_{L,\varphi}^G = \#\mathcal{H}_{K,\varphi}$. \square

Let’s say once again that the Chevalley–Herbrand formula $\#\mathcal{H}_{L,\varphi}^G = \#\mathcal{H}_{K,\varphi}$ cannot suggest a “trivial” isomorphism between $\mathcal{H}_{K,\varphi}$ and $\mathcal{E}_{K,\varphi} / \mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi})$, all the more that $\mathcal{E}_{K,\varphi}$ is a free \mathbb{Z}_p -module of rank r_φ equal to the degree of the field of values of the characters, giving monogenic quotients in the present case while $\mathcal{H}_{K,\varphi}$ may have arbitrary structure.

4.3. Test of capitulation – Numerical illustrations. Recall that for $\ell \equiv 1 \pmod{p^N}$, inert in K , we consider $L \subset K(\mu_\ell)$ of degree p^n over K , $n \in [1, N]$. In the practice of computations L is often denoted K_n and G is denoted G_n , etc.

To verify in practice some capitulations in L/K , we use the relation:

$$\mathbf{J}_{L/K}(\mathbf{H}_K) = \mathbf{J}_{L/K}(\mathbf{N}_{L/K}(\mathbf{H}_L)) = \nu_{L/K}(\mathbf{H}_L),$$

since $\mathbf{N}_{L/K} : \mathbf{H}_L \rightarrow \mathbf{H}_K$ is surjective (L/K is totally ramified), and compute the algebraic norm of the generators h_i of \mathbf{H}_L given by PARI. So, we obtain explicite relations $\nu_{L/K}(h_i) = \prod_j h_j^{a_{i,j}}$ in \mathbf{H}_L , the complete capitulation being given by the identity $\langle \dots, \prod_j h_j^{a_{i,j}}, \dots \rangle_i \otimes \mathbb{Z}_p = 1$ and partial capitulations are deduced from the matrices $(a_{i,j})$.

We will give numerical examples showing what happens, in the isomorphism (ii) of the corollary since $\mathcal{E}_{K,\varphi} / \mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi})$ is monogenic as Galois module, while $\mathcal{H}_{K,\varphi}$ is not in general.

(i) Let’s give, first, a few words about the important PARI instruction `bnfispincipal`, of constant use in the computations, to prove capitulation of a class $\mathfrak{a}P_K$ in L , whence that $\mathfrak{a}P_L$ is principal; it is described as follows by [Pari2013]:

`bnfisprincipal(bnf,x,{flag=1}):bnf` being output by `bnfinit` (with `flag<=2`), gives `[v,alpha]`, where `v` is the vector of exponents on the class group generators and `alpha` is the generator of the resulting principal ideal. In particular `x` is principal if and only if `v` is the zero vector.

Thus, the most important output is the vector of exponents like `[0,0]` meaning total capitulation of the selected p -class. Nevertheless, these vectors may be only 0 modulo the order of the p -classes (see § 4.3).

(ii) Let's give an example for K cubic and $p = 2$.

Example 4.7. We consider a cyclic cubic field with $p = 2$, for which $\mathcal{H}_K \simeq \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and $\mathcal{H}_{K_2} \simeq \mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, but there is complete capitulation of \mathcal{H}_K :

```
conductor f=2817 PK=x^3 - 939*x + 6886 CK=[12,4]
e11=449 N=2 Nn=2 n=2 CK2=[48,16,2,2]
algebraic norm in K2/K of the component 1 of CK2: [16,0,0,0]~
algebraic norm in K2/K of the component 2 of CK2: [0,0,0,0]~
algebraic norm in K2/K of the component 3 of CK2: [0,0,0,0]~
algebraic norm in K2/K of the component 4 of CK2: [0,0,0,0]~
```

so, the exponent 16 is to be read 0 (mod 16) since 16 is the order of the first component of the 2-class group \mathcal{H}_L .

(iii) Let's give an example for K quadratic and $p = 3$.

Example 4.8. Let $m = 32009$, $K = \mathbb{Q}(\sqrt{m})$ for which $\mathcal{H}_K \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Take $\ell = 19$ (inert in K) and let $L = K_2$ be the extension of degree 9 of K in $K(\mu_\ell)$; we denote by K_1 the intermediate field of degree 3. The general Program 7.1 gives a partial capitulation in K_1 , then a total capitulation in K_2 (in this last data for $n = 2$, we give the 18 integer coefficients of a generator of the ideal on the (unknown) integral basis computed by PARI):

```
PK=x^2 - 32009 CK=[3,3]
e11=19 N=2 Nn=2 n=1 CK1=[9,3]
algebraic norm in K1/K of the component 1 of CK1: [3,0]~
algebraic norm in K1/K of the component 2 of CK1: [0,0]~
e11=19 N=2 Nn=2 n=2 CK2=[9,3]
algebraic norm in K2/K of the component 1 of CK2: [0,0]~
algebraic norm in K2/K of the component 2 of CK2: [0,0]~

[[0,0]~, [1439216371631838382,473754414131112320,454228965737496519,
-536418025036156085,919689041243214339,207983767848706102,-1036595574155274193,
128338672307382267,840355575133838069,14736364857686206,6548993298829283,
8896045582984518,3839859983910278,7389856141626720,10096758515087182,
-895338434531174,-1442732225425698,8004210362478777]~]
```

In K_1/K , the exact sequence looks like:

$$1 \rightarrow \mathbf{J}_{K_1/K}(\mathcal{H}_K) \simeq \mathbb{Z}/3\mathbb{Z} \longrightarrow \mathcal{H}_{K_1}^{G_1} \longrightarrow \mathcal{E}_K/\mathbf{N}_{K_1/K}(\mathcal{E}_{K_1}) \simeq \mathbb{Z}/3\mathbb{Z} \rightarrow 1,$$

the structure of $\mathcal{H}_{K_1}^{G_1}$ being a priori unknown. A direct computation shows that $\mathcal{H}_{K_1}^{G_1} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, but it is not $\mathbf{J}_{K_1/K}(\mathcal{H}_K)$ since $\#\text{Ker}(\mathbf{J}_{K_1/K}) = 3$.

In K_2/K , the exact sequence becomes the isomorphism:

$$\mathcal{H}_{K_2}^{G_2} \simeq \mathcal{E}_K/\mathbf{N}_{K_2/K}(\mathcal{E}_{K_2}) \simeq \mathbb{Z}/9\mathbb{Z},$$

since $\mathbf{J}_{K_2/K}(\mathcal{H}_K) = 1$ and $\mathcal{E}_K \simeq \mathbb{Z}_3$. So, we intend to find a generator of $\mathcal{H}_{K_2}^{G_2}$.

Taking the class of order 9 (first component of \mathcal{H}_{K_2} given by the instruction `A0 = Kn.clgp[3][1]`), we compute its conjugate by the automorphism `S` of order 9, `B0 = nfgaloisapply(Kn,S,A0)`, then `C0 = idealpow(Kn,B0,8)` and `R = idealmul(Kn,A0,C0)` for which we apply the test `U = bnfisprincipal(Kn,R)` giving a principal integer with huge integer coefficients:

```
A0=Kn.clgp[3][1];B0=nfgaloisapply(Kn,S,A0);C0=idealpow(Kn,B0,8);
R=idealmul(Kn,A0,C0);U=bnfisprincipal(Kn,R);print(U)
[[0, 0]~, [-15352694895259448716005913288179,-4937712840022370286191878614596,
10234788031577460568990927971879,9473644150178364411380147768767,
919093855688240643550377510520,4933150036914472598668945255159,
-8186088867315265238068365774860,-12462519184163404099427848753248,
```

-10116044365250124206306632744945, -32490191902858719490198341631,
 -25043275245516486863055415213, 150430762033938424462889500018,
 83650512449675221885273689474, -243856275294066992198407658217,
 229543001775729020765314244564, -195555194837852410495787525040,
 140098686428490793581998673118, 26378140206989683079187437611]~]

This confirms that $\mathcal{H}_{K_2}^{G_2}$ is cyclic of order 9.

This phenomenon is general (when the capitulation is complete), for a p -class group of the form (say, for K quadratic) $\mathcal{H}_K \simeq \mathbb{Z}/p^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{n_r}\mathbb{Z}$ and gives, at a layer $n \geq a := n_1 + \cdots + n_r$, the isomorphism:

$$\mathcal{H}_{K_n}^{G_n} \simeq \mathcal{E}_K/\mathbf{N}_{K_n/K}(\mathcal{E}_{K_n}) \simeq \mathbb{Z}/p^a\mathbb{Z},$$

but $\#\mathcal{H}_{K_n}^{G_n} = \#\mathcal{H}_K$.

This is typical of the Main Conjecture philosophy and will be enforced by the analytic framework, recalled in the next subsection, showing that non-cyclic structures of the base field K (i.e., that of $\mathcal{H}_{K,\varphi}$) leads to canonical ones by extension in L (i.e., as quotients of $\mathcal{E}_{K,\varphi} \simeq \mathbb{Z}_p[\mu_{g_\varphi}]$), whatever the p -rank and the exponent of \mathcal{H}_K .

It seems that this “monogenicity”, by suitable cyclic p -extensions, has not been remarked in the literature. Unfortunately, general proofs of capitulations are perhaps out of reach despite its obviousness in the practice.

4.4. The stability as sufficient condition of capitulation. Now, we give a sufficient condition of capitulation (see the comments given in Remark 1.3):

Theorem 4.9. *Consider a prime $\ell \equiv 1 \pmod{2p^N}$, inert in the cyclic real field K , and $K_n \subset K(\mu_\ell)$, of degree p^n over K , $n \in [1, N]$.*

Then $\#\mathcal{H}_{K_n} = \#\mathcal{H}_K$ for all n if and only if $\#\mathcal{H}_{K_1} = \#\mathcal{H}_K$. If this criterion applies, then $\mathcal{H}_{K_n}^{G_n} = \mathcal{H}_{K_n}$ for all n , $\text{Ker}(\mathbf{J}_{K_n/K}) = \mathbf{N}_{K_n/K}(\mathcal{H}_{K_n}[p^n])$, and if p^e is the exponent of \mathcal{H}_K , assuming $N \geq e$, then \mathcal{H}_K capitulates in K_e .

Proof. Consider $\text{Gal}(K_n/K_1) = G_n^p$. Then we have the Chevalley–Herbrand formulas $\#\mathcal{H}_{K_n}^{G_n} = \#\mathcal{H}_K$ and $\#\mathcal{H}_{K_n}^{G_n^p} = \#\mathcal{H}_{K_1}$. But $\mathcal{H}_{K_n}^{G_n} \subseteq \mathcal{H}_{K_n}^{G_n^p}$; then under the condition $\#\mathcal{H}_{K_1} = \#\mathcal{H}_K$, we get $\mathcal{H}_{K_n}^{G_n} = \mathcal{H}_{K_n}^{G_n^p}$, which is equivalent to

$$\mathcal{H}_{K_n}^{1-\sigma_n} = \mathcal{H}_{K_n}^{1-\sigma_n^p} = \mathcal{H}_{K_n}^{(1-\sigma_n) \cdot \theta},$$

where $\theta = 1 + \sigma_n + \cdots + \sigma_n^{p-1} \in (p, 1 - \sigma_n)$, a maximal ideal of $\mathbb{Z}_p[G_n]$ since $\mathbb{Z}_p[G_n]/(p, 1 - \sigma_n) \simeq \mathbb{F}_p$; so $\mathcal{H}_{K_n}^{1-\sigma_n} = 1$, thus $\mathcal{H}_{K_n} = \mathcal{H}_{K_n}^{G_n}$ for all $n \in [1, N]$.

Reciprocal is trivial.

From $\mathbf{N}_{K_n/K}(\mathcal{H}_{K_n}) = \mathcal{H}_K$, $\mathcal{H}_{K_n} = \mathcal{H}_{K_n}^{G_n}$ and $\mathbf{J}_{K_n/K} \circ \mathbf{N}_{K_n/K} = \nu_{K_n/K}$ (the algebraic norm), one obtains:

$$\mathbf{J}_{K_n/K}(\mathcal{H}_K) = \mathbf{J}_{K_n/K}(\mathbf{N}_{K_n/K}(\mathcal{H}_{K_n})) = \nu_{K_n/K}(\mathcal{H}_{K_n}) = \mathcal{H}_{K_n}^{p^n}.$$

Let $c \in \text{Ker}(\mathbf{J}_{K_n/K})$ and put $c = \mathbf{N}_{K_n/K}(C)$, $C \in \mathcal{H}_{K_n}$; then $1 = \mathbf{J}_{K_n/K}(c) = \mathbf{J}_{K_n/K}(\mathbf{N}_{K_n/K}(C)) = C^{p^n}$, so $\text{Ker}(\mathbf{J}_{K_n/K}) \subseteq \mathbf{N}_{K_n/K}(\mathcal{H}_{K_n}[p^n])$; reciprocally, if $c = \mathbf{N}_{K_n/K}(C)$, $C^{p^n} = 1$, then $\mathbf{J}_{K_n/K}(c) = \mathbf{J}_n(\mathbf{N}_{K_n/K}(C)) = C^{p^n} = 1$; whence $\text{Ker}(\mathbf{J}_{K_n/K}) = \mathbf{N}_{K_n/K}(\mathcal{H}_{K_n}[p^n]) \subseteq \mathcal{H}_K[p^n]$.

For $n = e$, one obtains the capitulation of \mathcal{H}_K in K_e . \square

Remarks 4.10. (i) All the relations and exact sequences defining the filtration p -localize. In particular, the stability relation $\#\mathcal{H}_{K_1,\varphi} = \#\mathcal{H}_{K,\varphi}$ implies the capitulation of $\mathcal{H}_{K,\varphi}$ in K_e . But in practice, it suffices to obtain capitulation of \mathcal{H}_K .

(ii) From formula of Corollary 4.4, the assumption $\#\mathcal{H}_{K_1} = \#\mathcal{H}_K$ is equivalent to $\frac{\#\mathcal{H}_K}{\#\mathbf{N}_{K_1/K}(\mathcal{H}_{K_1}^{G_1})} = 1$; indeed, if $\#\mathcal{H}_{K_1} = \#\mathcal{H}_K$, then from Theorem 4.9,

$$\mathcal{H}_{K_1} = \mathcal{H}_{K_1}^{G_1}, \text{ so } \mathbf{N}_{K_1/K}(\mathcal{H}_{K_1}^{G_1}) = \mathbf{N}_{K_1/K}(\mathcal{H}_{K_1}) = \mathcal{H}_K.$$

If $\frac{\#\mathcal{H}_K}{\#\mathbf{N}_{K_1/K}(\mathcal{H}_{K_1}^{G_1})} = 1$, then the filtration stops and $\mathcal{H}_{K_1}^{G_1} = \mathcal{H}_{K_1}$, whence $\#\mathcal{H}_{K_1} = \#\mathcal{H}_{K_1}^{G_1} = \#\mathcal{H}_K$.

(iii) The same criterion holds if one replaces K by K_{n_0} for some $n_0 \geq 1$, under the condition $N \geq n_0 + e$; the fact that K_{n_0}/\mathbb{Q} is not of prime-to- p degree does not matter (proof of Theorem 4.9 does not need this assumption and requires that L/K_n be totally ramified at a single place; the notation $\mathcal{H}_{K_n, \varphi}$ is still relative to characters of K , which makes sense since $\text{Gal}K_n/\mathbb{Q} \simeq G_n \times g$).

5. CRUCIAL LINK BETWEEN CAPITULATION AND THE MAIN CONJECTURE

5.1. Analytcs – The group of cyclotomic units. This aspect being very classical, we just recall the main definitions and needed results. For the main definitions and properties of cyclotomic units, see [Leo1954, § 8 (1)], [Leo1962] or [Was1997, Chap. 8].

Definition 5.1. (i) Let $\chi \in \mathbf{X}$ be even of conductor f_χ ; we define the ‘‘cyclotomic numbers’’ $\theta_\chi := \prod_{a \in A_\chi} (\zeta_{2f_\chi}^a - \zeta_{2f_\chi}^{-a})$, with $\zeta_{2f_\chi} := \exp\left(\frac{i\pi}{f_\chi}\right)$, where A_χ is a half-system of representatives, in $(\mathbb{Z}/f_\chi\mathbb{Z})^\times$, of $\text{Gal}(\mathbb{Q}(\mu_{f_\chi})/K_\chi)$.

(ii) Let K be a real abelian field and let \mathbf{F}_K be the intersection with \mathbf{E}_K of the multiplicative group generated by the conjugates of θ_χ , for all $\chi \in \mathbf{X}_K$.

Recall that $\theta_\chi^2 \in K_\chi$ and that $\frac{\theta'_\chi}{\theta_\chi} \in \mathbf{E}_{K_\chi}$ for any conjugate θ'_χ of θ_χ . If f_χ is not a prime power, $\theta_\chi \in \mathbf{E}_\chi$. Since we will consider φ -components of the θ_χ , for $\varphi \neq 1$, one gets always units.

These units lead to an analytic computation of $\#\mathbf{H}_\chi$, $\chi \in \mathbf{X}$ even, $\chi \neq 1$ (using Theorem 2.1). One obtains in the semi-simple case $p \nmid \#g$:

Proposition 5.2. *For all $\chi \in \mathbf{X}$, χ even, then $\#\mathcal{H}_\chi = (\mathcal{E}_\chi : \mathcal{F}_\chi)$.*

The philosophy of the abelian Main Conjecture is to ask if the analogous relations $\#\mathcal{H}_\varphi = (\mathcal{E}_\varphi : \mathcal{F}_\varphi)$ exist or not, since we only know that:

$$\#\mathcal{H}_\chi = \prod_{\varphi|\chi} \#\mathcal{H}_\varphi = (\mathcal{E}_\chi : \mathcal{F}_\chi) = \prod_{\varphi|\chi} (\mathcal{E}_\varphi : \mathcal{F}_\varphi).$$

5.2. Norm properties of cyclotomic units. We mention, first, the classical norm property of cyclotomic units that are given in many books and articles, but are crucial for our purpose:

Proposition 5.3. *Let $f > 1$ and then let $m \mid f$, with $m > 1$, be any modulus; let $\mathbb{Q}^m := \mathbb{Q}(\zeta_m) \subseteq \mathbb{Q}^f := \mathbb{Q}(\zeta_f)$ be the corresponding cyclotomic fields with $\zeta_t := \exp\left(\frac{2i\pi}{t}\right)$, for all $t \geq 1$. Let $\mathbf{N}_{\mathbb{Q}^f/\mathbb{Q}^m}$ be the arithmetic norm in $\mathbb{Q}^f/\mathbb{Q}^m$. Put $\eta_{\mathbb{Q}^f} := 1 - \zeta_f$, $\eta_{\mathbb{Q}^m} := 1 - \zeta_m$; we have:*

$$\mathbf{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\eta_{\mathbb{Q}^f}) = \eta_{\mathbb{Q}^m}^\Omega, \quad \text{with } \Omega = \prod_{\ell \mid f, \ell \nmid m} (1 - (\frac{\mathbb{Q}^m}{\ell})^{-1}),$$

where $(\frac{\mathbb{Q}^m}{\ell}) \in \text{Gal}(\mathbb{Q}^m/\mathbb{Q})$ denotes the Frobenius (or Artin) automorphism of the prime number $\ell \nmid m$, that is to say such that $\zeta_m \mapsto \zeta_m^\ell$.

Proof. To simplify, denote by τ_a , a prime to f , the Artin automorphism $(\frac{\mathbb{Q}^f}{a})$ defined by $\zeta_f \mapsto \zeta_f^a$, then put $\eta_{\mathbb{Q}^f} := \eta_f$, $\eta_{\mathbb{Q}^m} := \eta_m$.

We consider, by induction, the case $f = \ell \cdot m$, with ℓ prime and examine the two cases $\ell \nmid m$ and $\ell \mid m$. We have $\mathbf{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\eta_f) = \prod_a \eta_f^{\tau_a}$ where a runs through the integers $a \in [1, f]$ prime to f and such that $a \equiv 1 \pmod{m}$.

(i) Case $\ell \nmid m$. Put $a = 1 + \lambda \cdot m$, $\lambda \in [0, \ell - 1]$, but we must exclude a unique $\lambda^* \in [0, \ell - 1]$ such that $1 + \lambda^* \cdot m \equiv 0 \pmod{\ell}$; put $1 + \lambda^* m = \mu\ell$. Thus:

$$\begin{aligned} \mathbf{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\eta_f) &= \prod_{\lambda \in [0, \ell - 1], \lambda \neq \lambda^*} (1 - \zeta_f^{1 + \lambda m}) \\ &= \frac{\prod_{\lambda \in [0, \ell - 1]} (1 - \zeta_f \zeta_f^\lambda)}{1 - \zeta_f^{\mu\ell}} = \frac{1 - \zeta_f^\ell}{1 - \zeta_f^\mu} = \frac{1 - \zeta_m}{1 - \zeta_m^\mu}. \end{aligned}$$

Since $\mu \equiv \ell^{-1} \pmod{m}$, we get $\mathbf{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\eta_f) = \eta_m^{1 - \tau_\ell^{-1}}$.

(ii) If $\ell \mid m$, any $\lambda \in [0, \ell - 1]$ is suitable, giving $\mathbf{N}_{\mathbb{Q}^f/\mathbb{Q}^m}(\eta_f) = \eta_m$. \square

Corollary 5.4. *Let $L/K/\mathbb{Q}$ be an abelian extension where L is of conductor f and K of conductor m ; set $\eta_L = \mathbf{N}_{\mathbb{Q}^f/L}(\eta_f)$ and $\eta_K = \mathbf{N}_{\mathbb{Q}^m/K}(\eta_m)$. Then:*

$$\mathbf{N}_{L/K}(\eta_L) = \eta_K^\Omega, \quad \text{with } \Omega = \prod_{\ell|f, \ell \nmid m} \left(1 - \left(\frac{K}{\ell}\right)^{-1}\right).$$

If moreover K/\mathbb{Q} is a cyclic extension of prime-to- p degree, and if all the primes $\ell \mid f$, $\ell \nmid m$, are inert in K , then $\Omega = \prod_{\ell|f, \ell \nmid m} \left(1 - \left(\frac{K}{\ell}\right)^{-1}\right)$ is an invertible element of $\mathbb{Z}_p[g]$. In particular, $\mathbf{N}_{L/K}\mathcal{F}_{L,\varphi} = \mathcal{F}_{K,\varphi}$ for all $\varphi \in \Phi_K \setminus \{1\}$.

Proof. Indeed, $\tau_{\ell,K} := \left(\frac{K}{\ell}\right)$ is a generator of g since ℓ is inert in K/\mathbb{Q} ; so for $\psi \mid \varphi \neq 1$, $\psi(1 - \tau_{\ell,K}^{-1}) = 1 - \psi(\tau_{\ell,K}^{-1})$ is a unit if $\psi(\tau_{\ell,K}^{-1}) \in \mu_{g_\varphi} \setminus \{1\}$ and is not a prime power, otherwise, if g_φ is a power of a prime q , then since $q \neq p$, $(1 - \psi(\tau_{\ell,K}^{-1}))$ is a prime ideal above q and $1 - \psi(\tau_{\ell,K}^{-1})$ is a p -adic unit. Whence the norm relation between the p -localized groups of cyclotomic units for $\varphi \neq 1$. \square

Remark 5.5. The link with the Leopoldt definition of cyclotomic units is easy since we get $\zeta_{2f} - \zeta_{2f}^{-1} = -\zeta_{2f}^{-1}(1 - \zeta_{2f}^2) = -\zeta_{2f}^{-1}(1 - \zeta_f) = -\zeta_{2f}^{-1}\eta_f$, which has no consequence for the Proposition and its Corollaries since norms are taken over real fields L, K .

5.3. Final statement. So, we can state and prove the main result involving the transfer map $\mathbf{J}_{L/K}$ and its p -localized images $\mathbf{J}_{L/K}(\mathcal{H}_{K,\varphi})$:

Theorem 5.6. *Let K/\mathbb{Q} be a real cyclic extension of prime-to- p degree. Let $\ell \equiv 1 \pmod{2p^N}$, $N \geq 1$, and assume ℓ totally inert in K . Let $L \subset K(\mu_\ell)$ of degree p^n over K , $n \in [1, N]$, and put $G := \text{Gal}(L/K) =: \langle \sigma \rangle$.*

Then, for all $n \in [1, N]$ we have the relations (product of two integers):

$$(\mathcal{E}_{K,\varphi} : \mathcal{F}_{K,\varphi}) = (\mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi}) : \mathcal{F}_{K,\varphi}) \times \frac{\#\mathcal{H}_{K,\varphi}}{\#\mathbf{J}_{L/K}(\mathcal{H}_{K,\varphi})}, \quad \text{for all } \varphi \in \Phi_K.$$

Proof. Recall that, from Corollary 4.6 (i) to Theorem 4.5, we have, for all φ the exact sequences $1 \rightarrow \mathbf{J}_{L/K}(\mathcal{H}_{K,\varphi}) \rightarrow \mathcal{H}_{L,\varphi}^G \rightarrow \mathcal{E}_{K,\varphi}/\mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi}) \rightarrow 1$, whence the relations $\#\mathcal{H}_{L,\varphi}^G = \#\mathcal{H}_{K,\varphi} = (\mathcal{E}_{K,\varphi} : \mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi})) \times \#\mathbf{J}_{L/K}(\mathcal{H}_{K,\varphi})$.

From Corollary 5.4, $\mathcal{F}_{K,\varphi} = \mathbf{N}_{L/K}(\mathcal{F}_{L,\varphi})$, whence the inclusions:

$$\mathcal{F}_{K,\varphi} \subseteq \mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi}) \subseteq \mathcal{E}_{K,\varphi},$$

where $(\mathcal{E}_{K,\varphi} : \mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi})) = \frac{\#\mathcal{H}_{K,\varphi}}{\#\mathbf{J}_{L/K}(\mathcal{H}_{K,\varphi})}$, proving the claim. \square

Corollary 5.7. *The Main Conjecture, for the real abelian number fields K of prime-to- p degree, holds under the existence, for each irreducible p -adic character $\varphi \neq 1$, of an inert prime $\ell \equiv 1 \pmod{2p^N}$, $N \geq 1$, such that $\mathcal{H}_{K,\varphi}$ capitulates in $K(\mu_\ell)$.*

A sufficient condition of capitulation (whence of the Main Conjecture) is the stability of the p -class groups in the cyclic p -tower $\bigcup_{n \in [1, N]} K_n$ of $K(\mu_\ell)/K$, that is to say, the existence of n_0 , $0 \leq n_0 \leq N - e$, such that $\#\mathcal{H}_{K_{n_0+1},\varphi} = \#\mathcal{H}_{K_{n_0},\varphi}$, where p^e is the exponent of \mathcal{H}_K .

Let's note that as soon as there is a partial capitulation in some K_n/K , $n \in [1, N]$, the index $(\mathcal{E}_{K,\varphi} : \mathcal{F}_{K,\varphi})$ is non trivial. In practice, one obtains often the whole capitulation of \mathcal{H}_K using a single prime ℓ among, probably, infinitely many.

6. NUMERICAL EXPERIMENTS OVER CYCLIC CUBIC FIELDS

Let $L = K_n \subset K(\mu_\ell)$, $\ell \equiv 1 \pmod{2p^N}$, be a cyclic p -tower of degree p^n of K , $n \in [1, N]$. We still assume K/\mathbb{Q} cyclic and ℓ totally inert in K .

The following program gives, at the beginning, the complete list of cyclic cubic fields of conductor $\mathfrak{f} \in [\mathfrak{bf}, \mathfrak{Bf}]$ and selects those having a suitable p -class group to study the capitulation in L/K . The test is about the order of \mathcal{H}_K , so various structures may occur. The fields $K_n \subseteq L$ are given by means of

the polynomial P of degree $3 \cdot p^n$, $n \in [1, N_n]$, where $N_n \leq N$, not too large, define the layers $n \in [1, N_n]$ in which the computations are done (regrettably the execution time becomes rapidly out of reach). A test on the order of \mathcal{H}_{K_n} is also proposed.

Since the PARI instruction `bnfinit(P, 1)` may take huge time if the degree of P increases, we are limited to $p = 2$ and possibly $p = 5$ and 7 (the minimal prime giving two p -adic characters). But the purpose being to suggest the randomness of the phenomenon of capitulation, we hope that these cases constitute a good heuristic.

6.1. General program for cyclic cubic fields. In the following general program, one must precise the following data:

(i) The numbers N (and $N_n \leq N$, the number of layers to be tested by the program) to define the primes ℓ limited by the bound `Bell` and congruent to 1 modulo $2p^N$; it is possible, to get capitulations at a sufficient layer, to take N large, but N_n very small.

(ii) The bounds `bf`, `Bf` defining an interval for the conductors f .

(iii) The instructions `valuation(HK, p) < vHK`, `valuation(HKn, p) < vHKn` to get only interesting p -class groups for K and K_n ; note that from the Chevalley–Herbrand formula, $\#\mathcal{H}_{K_n}$ is a multiple of $\#\mathcal{H}_K$, and one must take `vHK` \leq `vHKn`

```
STUDY OF THE CAPITULATION OF HK IN Kn/K FOR CYCLIC CUBIC FIELDS
{p=2;N=2;Nn=2;bf=7;Bf=5*10^3;vHK=4;vHKn=6;Bell=500;
\\List of cubic fields of any conductor f:
for(f=bf,Bf,h=valuation(f,3);if(h!=0 & h!=2,next);F=f/3^h;if(core(F)!=F,next);
F=factor(F);Div=component(F,1);d=matsize(F)[1];for(j=1,d,D=Div[j];
if(Mod(D,3)!=1,break));for(b=1,sqrt(4*f/27),if(h==2 & Mod(b,3)==0,next);
A=4*f-27*b^2;if(issquare(A,&a)==1,\\a and b such that f=(a^2+27b^2)/4
if(h==0,if(Mod(a,3)==1,a=-a);PK=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(h==2,if(Mod(a,9)==3,a=-a);PK=x^3-f/3*x-f*a/27);
\\End of computation of PK defining K of conductor f.
K=bnfinit(PK,1);HK=K.no;
\\Test on the order HK of the p-class group of K:
if(valuation(HK,p)<vHK,next);CK=K.clgp;
\\Definition of the primes ell inert in K:
forprime(ell=1,Bell,if(Mod(ell-1,2*p^N)!=0,next);
F=factor(PK+0(ell));if(matsize(F)[1]!=1,next);
\\Definitions of the fields Kn<L, computation of their class group:
for(n=1,Nn,QKn=polsubcyclo(ell,p^n);P=polcompositum(PK,QKn)[1];
Kn=bnfinit(P,1);HKn=Kn.no;
\\Test on the order HKn of the p-class group of Kn:
if(valuation(HKn,p)<vHKn,break);
print();print("conductor f=",f," PK=",PK," CK=",CK[2]);
print("ell=",ell," N=",N," Nn=",Nn," n=",n);
CKn=Kn.clgp;print("CKn=",CKn[2]);rKn=matsize(CKn[2])[2];
\\Calcul de Gal(Kn/K) and search of a generator S of order p^n:
G=nfgaloisconj(Kn);Id=G[1];for(k=2,3*p^n,Z=G[k];ks=1;
while(Z!=Id,Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
\\Computation of the algebraic norms of the generators of CKn:
for(j=1,rKn,A0=CKn[3][j];A=1;
for(t=1,p^n,As=nfgaloisapply(Kn,S,A);A=idealmul(Kn,A0,As));
\\Test of capitulation (partial or total):
X=bnfisprincipal(Kn,A)[1];
print("algebraic norm in K",n,"/K of the component ",j," of CK",n," : ",X))))))}
```

6.2. Case of cyclic cubic fields and $p = 2$. We give an excerpt of the various forms of examples, with $\mathcal{H}_K \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and \mathcal{H}_{K_1} of order at least 2^6 :

```
conductor f=1777 PK=x^3 + x^2 - 592*x + 724 CK=[4,4]
ell=41 N=2 Nn=2 n=1 CK=[4,4] CK1=[4,4,2,2]
algebraic norm in K1/K of the component 1 of CK1: [0,0,0,1]~
algebraic norm in K1/K of the component 2 of CK1: [2,2,1,1]~
algebraic norm in K1/K of the component 3 of CK1: [0,0,0,0]~
algebraic norm in K1/K of the component 4 of CK1: [0,0,0,0]~
Incomplete capitulation in K1
ell=41 N=2 Nn=2 n=2 CK=[4,4] CK2=[8,8,2,2]
algebraic norm in K2/K of the component 1 of CK2: [0,0,0,0]~
```


algebraic norm in K_2/K of the component 2 of CK_2 : $[0,0,0,0]^\sim$
 algebraic norm in K_2/K of the component 3 of CK_2 : $[0,0,0,0]^\sim$
 algebraic norm in K_2/K of the component 4 of CK_2 : $[0,0,0,0]^\sim$
 Complete capitulation in K_2 without stabilization in K_2/K_1

 conductor $f=1777$ $PK=x^3 + x^2 - 592x + 724$ $CK=[4,4]$
 $e_1=113$ $N=2$ $N_n=2$ $n=1$ $CK=[4,4]$ $CK_1=[4,4,2,2]$
 algebraic norm in K_1/K of the component 1 of CK_1 : $[0,0,1,0]^\sim$
 algebraic norm in K_1/K of the component 2 of CK_1 : $[0,0,0,1]^\sim$
 algebraic norm in K_1/K of the component 3 of CK_1 : $[0,0,0,0]^\sim$
 algebraic norm in K_1/K of the component 4 of CK_1 : $[0,0,0,0]^\sim$
 Incomplete capitulation in K_1
 $e_1=113$ $N=2$ $N_n=2$ $n=2$ $CK=[4,4]$ $CK_2=[8,8,2,2]$
 algebraic norm in K_2/K of the component 1 of CK_2 : $[0,4,0,0]^\sim$
 algebraic norm in K_2/K of the component 2 of CK_2 : $[4,4,0,0]^\sim$
 algebraic norm in K_2/K of the component 3 of CK_2 : $[0,0,0,0]^\sim$
 algebraic norm in K_2/K of the component 4 of CK_2 : $[0,0,0,0]^\sim$
 Incomplete capitulation in K_2

 conductor $f=1777$ $PK=x^3 + x^2 - 592x + 724$ $CK=[4,4]$
 $e_1=257$ $N=2$ $N_n=2$ $n=1$ $CK=[4,4]$ $CK_1=[72,24]$
 algebraic norm in K_1/K of the component 1 of CK_1 : $[18,12]^\sim$
 algebraic norm in K_1/K of the component 2 of CK_1 : $[36,6]^\sim$
 No capitulation in K_1
 $e_1=257$ $N=2$ $N_n=2$ $n=2$ $CK=[4,4]$ $CK_2=[72,24]$
 algebraic norm in K_2/K of the component 1 of CK_2 : $[36,0]^\sim$
 algebraic norm in K_2/K of the component 2 of CK_2 : $[0,12]^\sim$
 Incomplete capitulation in K_2 , stability from $K_1 \rightarrow$ capitulation in K_3

 conductor $f=2817$ $PK=x^3 - 939x + 6886$ $CK=[12,4]$
 $e_1=449$ $N=2$ $N_n=2$ $n=1$ $CK=[4,4]$ $CK_1=[24,8,2,2]$
 algebraic norm in K_1/K of the component 1 of CK_1 : $[20,0,0,1]^\sim$
 algebraic norm in K_1/K of the component 2 of CK_1 : $[12,0,1,0]^\sim$
 algebraic norm in K_1/K of the component 3 of CK_1 : $[0,0,0,0]^\sim$
 algebraic norm in K_1/K of the component 4 of CK_1 : $[0,0,0,0]^\sim$
 Incomplete capitulation in K_1
 $e_1=449$ $N=2$ $N_n=2$ $n=2$ $CK=[4,4]$ $CK_2=[48,16,2,2]$
 algebraic norm in K_2/K of the component 1 of CK_2 : $[16,0,0,0]^\sim$
 algebraic norm in K_2/K of the component 2 of CK_2 : $[0,0,0,0]^\sim$
 algebraic norm in K_2/K of the component 3 of CK_2 : $[0,0,0,0]^\sim$
 algebraic norm in K_2/K of the component 4 of CK_2 : $[0,0,0,0]^\sim$
 Complete capitulation in K_2

 conductor $f=4297$ $PK=x^3 + x^2 - 1432x + 20371$ $CK=[4,4]$
 $e_1=449$ $N=2$ $N_n=2$ $n=1$ $CK=[4,4]$ $CK_1=[4,4,2,2]$
 algebraic norm in K_1/K of the component 1 of CK_1 : $[0,2,1,1]^\sim$
 algebraic norm in K_1/K of the component 2 of CK_1 : $[2,2,0,1]^\sim$
 algebraic norm in K_1/K of the component 3 of CK_1 : $[0,0,0,0]^\sim$
 algebraic norm in K_1/K of the component 4 of CK_1 : $[0,0,0,0]^\sim$
 Incomplete capitulation in K_1
 $e_1=449$ $N=2$ $N_n=2$ $n=2$ $CK=[4,4]$ $CK_2=[292,4,4,4]$
 algebraic norm in K_2/K of the component 1 of CK_2 : $[0,2,0,2]^\sim$
 algebraic norm in K_2/K of the component 2 of CK_2 : $[0,0,2,2]^\sim$
 algebraic norm in K_2/K of the component 3 of CK_2 : $[0,0,2,2]^\sim$
 algebraic norm in K_2/K of the component 4 of CK_2 : $[0,0,2,2]^\sim$
 Incomplete capitulation in K_2

 conductor $f=5409$ $PK=x^3 - 1803x + 29449$ $CK=[12,4]$
 $e_1=113$ $N=2$ $N_n=2$ $n=1$ $CK=[4,4]$ $CK_1=[12,4,2,2]$
 algebraic norm in K_1/K of the component 1 of CK_1 : $[2,2,0,1]^\sim$
 algebraic norm in K_1/K of the component 2 of CK_1 : $[0,0,1,0]^\sim$
 algebraic norm in K_1/K of the component 3 of CK_1 : $[0,0,0,0]^\sim$
 algebraic norm in K_1/K of the component 4 of CK_1 : $[0,0,0,0]^\sim$
 Incomplete capitulation in K_1
 $e_1=113$ $N=2$ $N_n=2$ $n=2$ $CK=[4,4]$ $CK_2=[24,8,2,2,2,2]$
 algebraic norm in K_2/K of the component 1 of CK_2 : $[16,4,0,0,0,0]^\sim$
 algebraic norm in K_2/K of the component 2 of CK_2 : $[12,4,0,0,0,0]^\sim$
 algebraic norm in K_2/K of the component 3 of CK_2 : $[0,0,0,0,0,0]^\sim$
 algebraic norm in K_2/K of the component 4 of CK_2 : $[0,0,0,0,0,0]^\sim$
 algebraic norm in K_2/K of the component 5 of CK_2 : $[0,0,0,0,0,0]^\sim$

algebraic norm in K_2/K of the component 6 of CK2: $[0,0,0,0,0,0]^\sim$
 No capitulation in K_2

Remark 6.1. For the first example above, the capitulation in K_2 is complete, even if the stability does not occur from the first layer; the step $n = 1$ shows a partial capitulation giving, $\mathbf{J}_{K_1/K}(\mathcal{H}_K) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (indeed the exponent of \mathcal{H}_K is 4).

To be more convincing, let's give the coefficients of the extensions of the generating ideals representatives of the two classes of order 4; one gets 12 huge integer coefficients:

```
PK=x^3 + x^2 - 592*x + 724 CK=[4,4]
ell=41 CK=[4,4] CK2=[8,8,2,2]
algebraic norm in K2/K of the component 1 of CK2:
[[0,0,0,0]~, [-31780222254443, -12898232803596, 15554698429537,
-11030242667244, 699234644603, -1433180593820, -97846196830,
480428807611, 70679128541, 581754438178, 701511836521, 497443446811]~]
```

```
algebraic norm in K2/K of the component 2 of CK2:
[[0,0,0,0]~, [-409139735188114, -166218206982845, 200303000159397,
-142025979393819, 10482585098180, 32200771380552, -1471046798500,
7172751349077, 1043934050162, -13082943399099, -15769847218663,
-11179282571565]~]
```

```
algebraic norm in K2/K of the component 3 of CK2:
[[0,0,0,0]~, [4595853941743, 7574362186256, -7431095890343, 3180376719682,
-878235409486, 520990038484, 351933447679, 127583225152, 327914236819,
-381696290156, -181901226812, -173412643330]~]
```

```
algebraic norm in K2/K of the component 4 of CK2:
[[0,0,0,0]~, [206178918528161385818507, 45009133745540603328818,
109639228343931043367320, 18671957635985615261071, 13770653000372358842954,
7958894412958725580875, 13213450239959129254028, 3845666006771496793309,
2250298427236450403785, -1737437297938711409589, 4232280623726481124024,
-720772717060054219812]~]
```

6.3. Case of cyclic cubic fields and $p = 7$. For $p = 7$, due to the execution time, let's give some examples of the case $n = 1$ with \mathcal{H}_K of orders 7, then one case of order 7^2 , and $B_\ell = 100$.

We obtain complete capitulations in K_1 , except few cases; we give an excerpt of some possibilities:

```
conductor f=313 PK=x^3 + x^2 - 104*x + 371 CK=[7]
ell=29 N=1 Nn=1 n=1 CK=[7] CK1=[7]
algebraic norm in K1/K of the component 1 of CK1: [0]~
Complete capitulation in K1
```

```
-----
conductor f=1261 PK=x^3 + x^2 - 420*x - 1728 CK=[21]
ell=43 N=1 Nn=1 n=1 CK=[7] CK1=[21]
algebraic norm in K1/K of the component 1 of CK1: [7]~
Complete capitulation in K1
```

```
-----
conductor f=1567 PK=x^3 + x^2 - 522*x - 4759 CK=[7]
ell=29 N=1 Nn=1 n=1 CK=[7] CK1=[49]
algebraic norm in K1/K of the component 1 of CK1: [7]~
No capitulation in K1
```

```
-----
ell=71 N=1 Nn=1 n=1 CK=[7] CK1=[49]
algebraic norm in K1/K of the component 1 of CK1: [7]~
No capitulation in K1
```

```
-----
conductor f=8563 PK=x^3 + x^2 - 2854*x + 57721 CK=[49]
ell=71 N=1 Nn=1 n=1 CK=[49] CK1=[49]
algebraic norm in K1/K of the component 1 of CK1: List([7])
Partial capitulation
```

The last case shows a partial capitulation giving $\mathbf{J}_{K_1/K}(\mathcal{H}_K) \simeq \mathbb{Z}/7\mathbb{Z}$. Since $N = 1$, there is no possible complete capitulation despite the stability from K_1 . The case of primes ℓ with $N = n = 2$ seems out of reach.

Let's give an example of generator of the principal ideal obtained after capitulation (first example above):

```
conductor f=313 PK=x^3 + x^2 - 104*x + 371 CK=[7]
ell=29 N=1 Nn=1 n=1 CK=[7] CK1=[7]
algebraic norm in K1/K of the component 1 of CK1:
[[0]~, [4529357, 2479589, 125622, -2879283, 2922668, 4270474, -6202812,
-107453, 1865872, 37436, -613198, 1546287, -1637834, 1355628, 1276626,
886508, 944469, -900999, 474890, 508450, 962907]~]
```

6.4. About the sufficient condition of capitulation. We consider cyclic cubic fields and $p = 2$. We only search examples of primes ℓ giving the stability of the 2-class groups in K_1/K , so that the capitulation applies in $L = K_N$ if N is large enough; we compute the 2-valuation of $\#\mathcal{H}_K$ (in $N = \text{valuation}(\text{HK}, p)$), which is the minimal possible bound. We illustrate the case $\mathcal{H}_K \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$; a great lot of examples are found giving capitulation in K_1 , even with $\ell < 100$ (one writes only $f, PK, \text{ell}, CK, CK1$):

```
{p=2;bf=7;Bf=10^4;nell=100;for(f=bf,Bf,h=valuation(f,3);
if(h!=0 & h!=2,next);F=f/3^h;if(core(F)!=F,next);F=factor(F);
Div=component(F,1);d=matsize(F)[1];for(j=1,d,D=Div[j];if(Mod(D,3)!=1,break));
for(b=1,sqrt(4*f/27),if(h==2 & Mod(b,3)==0,next);
A=4*f-27*b^2;if(issquare(A,&a)==1,
if(h==0,if(Mod(a,3)==1,a=-a);PK=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(h==2,if(Mod(a,9)==3,a=-a);PK=x^3-f/3*x-f*a/27);
K=bnfinit(PK,1);HK=K.no;v=valuation(HK,p);if(v!=2,next);
for(t=2,nell,ell=prime(t);if(Mod(ell-1,4)!=0,next);F=factor(PK+O(ell));
if(matsize(F)[1]!=1,next);QK1=polsubcyclo(ell,p);P=polcompositum(PK,QK1)[1];
K1=bnfinit(P,1);if(valuation(K1.no,p)==2,print("f=",f," PK=",PK," ell=",ell,
" CK=",K.clgp[2]," CK1=",K1.clgp[2]);break)))))}
```

```
f=163 PK=x^3 + x^2 - 54*x - 169 ell=29 CK=[2,2] CK1=[2,2]
f=277 PK=x^3 + x^2 - 92*x + 236 ell=5 CK=[2,2] CK1=[2,2]
f=349 PK=x^3 + x^2 - 116*x - 517 ell=5 CK=[2,2] CK1=[2,2]
f=397 PK=x^3 + x^2 - 132*x - 544 ell=5 CK=[2,2] CK1=[2,2]
f=547 PK=x^3 + x^2 - 182*x - 81 ell=5 CK=[2,2] CK1=[2,2]
(...)
f=9703 PK=x^3 + x^2 - 3234*x - 16531 ell=5 CK=[6,2] CK1=[6,2]
f=9709 PK=x^3 + x^2 - 3236*x + 21216 ell=5 CK=[6,6] CK1=[6,6]
f=9721 PK=x^3 + x^2 - 3240*x - 39244 ell=5 CK=[2,2] CK1=[2,2]
f=9891 PK=x^3 - 3297*x + 70336 ell=29 CK=[6,6] CK1=[6,6]
f=9961 PK=x^3 + x^2 - 3320*x - 74523 ell=5 CK=[6,2] CK1=[6,2]
```

Some cases of more complex structures of \mathcal{H}_K do not give stabilization at the first step; this clearly depends on the exponent of the class groups as we have explained; but this sufficient condition is not necessary and capitulation does appear at larger layers; we illustrate the cases $\mathcal{H}_K \simeq (\mathbb{Z}/2\mathbb{Z})^4$ or $\mathcal{H}_K \simeq (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$:

```
conductor f=7687 PK=x^3 + x^2 - 2562*x - 48969 CK=[2,2,2,2]
ell=17 N=3 Nn=3 n=1 CK=[2,2,2,2] CK1=[4,4,2,2]
algebraic norm in K1/K of the component 1 of CK1: [2,2,0,0]~
algebraic norm in K1/K of the component 2 of CK1: [0,2,0,0]~
algebraic norm in K1/K of the component 3 of CK1: [2,0,0,0]~
algebraic norm in K1/K of the component 4 of CK1: [0,2,0,0]~
Partial capitulation in K1
```

```
ell=17 N=3 Nn=3 n=2 CK=[2,2,2,2] CK2=[4,4,2,2]
algebraic norm in K2/K of the component 1 of CK2: [0,0,0,0]~
algebraic norm in K2/K of the component 2 of CK2: [0,0,0,0]~
algebraic norm in K2/K of the component 3 of CK2: [0,0,0,0]~
algebraic norm in K2/K of the component 4 of CK2: [0,0,0,0]~
Stability from K1-->complete capitulation in K2
```

```
-----
conductor f=20887 PK=x^3 + x^2 - 6962*x - 225889 CK=[4,4,2,2]
ell=193 N=3 Nn=3 n=1 CK=[4,4,2,2] CK1=[8,8,2,2]
algebraic norm in K1/K of the component 1 of CK1: [2,4,0,0]~
algebraic norm in K1/K of the component 2 of CK1: [4,6,0,0]~
algebraic norm in K1/K of the component 3 of CK1: [4,0,0,0]~
algebraic norm in K1/K of the component 4 of CK1: [4,4,0,0]~
Partial capitulation in K1
```

```

ell=193 N=3 Nn=3 n=2 CK=[4,4,2,2] CK2=[8,8,2,2]
algebraic norm in K2/K of the component 1 of CK2: [4,0,0,0]~
algebraic norm in K2/K of the component 2 of CK2: [0,4,0,0]~
algebraic norm in K2/K of the component 3 of CK2: [0,0,0,0]~
algebraic norm in K2/K of the component 4 of CK2: [0,0,0,0]~
Stability from K1-->Complete capitulation in K3, partial capitulation in K2

```

```

-----
conductor f=31923 PK=x^3 - 10641*x + 227008 CK=[6,2,2,2]
ell=97 N=3 Nn=3 n=1 CK=[6,2,2,2] CK1=[12,4,2,2,2,2]
algebraic norm in K1/K of the component 1 of CK1: [8,0,1,0,0,0]~
algebraic norm in K1/K of the component 2 of CK1: [0,0,0,1,0,0]~
algebraic norm in K1/K of the component 3 of CK1: [0,0,0,0,0,0]~
algebraic norm in K1/K of the component 4 of CK1: [0,0,0,0,0,0]~
algebraic norm in K1/K of the component 5 of CK1: [6,0,0,0,0,0]~
algebraic norm in K1/K of the component 6 of CK1: [6,2,0,0,0,0]~
No capitulation in K1

```

```

ell=97 N=3 Nn=3 n=2 CK=[6,2,2,2] CK2=[312,8,4,4,4,4]
algebraic norm in K2/K of the component 1 of CK2: [208,0,2,0,0,2]~
algebraic norm in K2/K of the component 2 of CK2: [0,4,0,0,2,2]~
algebraic norm in K2/K of the component 3 of CK2: [156,0,0,0,0,0]~
algebraic norm in K2/K of the component 4 of CK2: [156,4,0,0,0,0]~
algebraic norm in K2/K of the component 5 of CK2: [156,0,0,0,0,0]~
algebraic norm in K2/K of the component 6 of CK2: [156,0,0,0,0,0]~
No capitulation in K2

```

So we must try another ℓ :

```

conductor f=31923 PK=x^3 - 10641*x + 227008 CK=[6,2,2,2]
ell=257 N=3 Nn=2 n=1 CK=[6,2,2,2] CK1=[18, 6, 2, 2, 2, 2]
algebraic norm in K1/K of the component 1 of CK1: [6,4,0,0,0,0]~
algebraic norm in K1/K of the component 2 of CK1: [12,2,0,0,0,0]~
algebraic norm in K1/K of the component 3 of CK1: [9,0,1,1,0,1]~
algebraic norm in K1/K of the component 4 of CK1: [0,0,0,0,1,0]~
algebraic norm in K1/K of the component 5 of CK1: [0,0,0,0,0,0]~
algebraic norm in K1/K of the component 6 of CK1: [9,0,1,1,1,1]~
Partial capitulation in K1

```

```

conductor f=31923 PK=x^3 - 10641*x + 227008 CK=[6,2,2,2]
ell=257 N=3 Nn=2 n=2 CK=[6,2,2,2] CK2=[36,12,2,2,2,2]
algebraic norm in K2/K of the component 1 of CK2: [0,8,0,0,0,0]~
algebraic norm in K2/K of the component 2 of CK2: [0,4,0,0,0,0]~
algebraic norm in K2/K of the component 3 of CK2: [0,0,0,0,0,0]~
algebraic norm in K2/K of the component 4 of CK2: [0,0,0,0,0,0]~
algebraic norm in K2/K of the component 5 of CK2: [0,0,0,0,0,0]~
algebraic norm in K2/K of the component 6 of CK2: [0,0,0,0,0,0]~

```

For this new ℓ , the complete capitulation is obtained in K_2 . We note that the 2-rank of \mathcal{H}_K is 4, which implies, from Corollary 4.6 (ii), the isomorphism $\mathcal{H}_{K_2}^{G_2} \simeq \mathbb{Z}_2[j]/(4)$ but never $\mathbb{Z}_2[j]/(2) \times \mathbb{Z}_2[j]/(2)$, where $j = \exp(\frac{2i\pi}{3})$.

7. NUMERICAL EXPERIMENTS OVER QUADRATIC FIELDS AND $p = 3$

Of course, this case has no interest for verifications of the Main Conjecture since it is true for the trivial reason $\chi = \varphi$; but it remains interesting to study the capitulation phenomenon. The program is analogous to the general one with slight modifications due to the quadratic context; we give some excerpt:

7.1. General program for quadratic fields.

```

{p=3;N=2;Nn=2;bm=2;Bm=10^4;vHK=2;vHKn=2;Bell=500;
for(m=bm,Bm,if(core(m)!=m,next);PK=x^2-m;K=bnfinit(PK,1);HK=K.no;
if(valuation(HK,p)<vHK,next);CK=K.clgp;
forprime(ell=1,Bell,if(Mod(ell-1,2*p^N)!=0,next);if(kronecker(m,ell)!=-1,next);
for(n=1,Nn,QKn=polsubcyclo(ell,p^n);P=polcompositum(PK,QKn)[1];
Kn=bnfinit(P,1);HKn=Kn.no;if(valuation(HKn,p)<vHKn,break);CKn=Kn.clgp[2];
print("PK=",PK," CK=",CK[2]," ell=",ell,
" N=",N," Nn=",Nn," n=",n," CK",n,"=",CKn);
rKn=matsize(CKn)[2];G=nfgaloisconj(Kn);Id=G[1];for(k=2,2*p^n,Z=G[k];ks=1;
while(Z!=Id,Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
for(j=1,rKn,e=CKn[j];A0=Kn.clgp[3][j];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);

```

```
A=idealmul(Kn,A0,As));X=bnfisprincipal(Kn,A)[1];
print("algebraic norm in K",n,"/K of the component ",j," of CK",n,": ",X)))))}
```

7.2. Case of quadratic fields and $p = 3$.

7.2.1. *Examples with $N = 1$.* For instance, the configuration:

```
PK=x^2 - 1129 CK=[9]
ell=13 N=2 Nn=1 n=1 CK=[9] CK1=[27]
algebraic norm in K1/K of the component 1 of CK1: [3]~
means that  $\nu_{K_n/K}(\mathcal{H}_{K_n}) = \mathcal{H}_{K_n}^3$  of order 9; so there is no partial capitulation
of  $\mathcal{H}_{K_n}$ . But in the following case, there is partial capitulation:
```

```
PK=x^2 - 1129 CK=[9]
ell=307 N=2 Nn=1 n=1 CK=[9] CK1=[9,3]
algebraic norm in K1/K of the component 1 of CK1: [3,0]~
algebraic norm in K1/K of the component 2 of CK1: [0,0]~
```

```
PK=x^2 - 1654 CK=[9]
ell=283 N=2 Nn=1 n=1 CK=[9] CK1=[9,3,3]
algebraic norm in K1/K of the component 1 of CK1: [3,1,0]~
algebraic norm in K1/K of the component 2 of CK1: [0,0,0]~
algebraic norm in K1/K of the component 3 of CK1: [0,0,0]~
```

The following case shows a partial capitulation and since $\#\mathcal{H}_{K_n} = \#\mathcal{H}_K$, we have the stability in the tower and the complete capitulation in the extension K_2 of degree 9:

```
PK=x^2 - 1129 CK=[9]
ell=19 N=2 Nn=1 n=1 CK=[9] CK1=[9]
algebraic norm in K1/K of the component 1 of CK1: [3]~
```

Many examples are of the same kind as above; we do not write them.

Finally we have the case of the following structure of \mathcal{H}_K :

```
PK=x^2 - 32009 CK=[3, 3]
ell=19 N=2 Nn=2 n=1 CK=[3, 3] CK1=[9,3]
algebraic norm in K1/K of the component 1 of CK1:[0,0]
algebraic norm in K1/K of the component 2 of CK1:[0,0]
```

We see the complete capitulation despite the lack of stability. The next case shows a partial capitulation giving $\mathbf{J}_{K_n/K}(\mathcal{H}_{K_n}) \simeq \mathbb{Z}/3\mathbb{Z}$.

7.2.2. *Examples with $n = N = 2$.*

```
PK=x^2 - 1129 CK=[9]
ell=73 N=2 Nn=2 n=1 CK=[9] CK1=[189,3]
algebraic norm in K1/K of the component 1 of CK1: [147,0]
algebraic norm in K1/K of the component 2 of CK1: [0,0]
ell=73 N=2 Nn=2 n=2 CK=[9] CK2=[567,9]
algebraic norm in K2/K of the component 1 of CK2: [441,0]
algebraic norm in K2/K of the component 2 of CK2: [0,0]
```

```
-----
PK=x^2 - 3137 CK=[9]
ell=199 N=2 Nn=2 n=1 CK=[9] CK1=[27,3]
algebraic norm in K1/K of the component 1 of CK1: [21,0]~
algebraic norm in K1/K of the component 2 of CK1: [0,0]~
ell=199 N=2 Nn=2 n=2 CK=[9] CK2=[27,9]
algebraic norm in K2/K of the component 1 of CK2: [9,0]~
algebraic norm in K2/K of the component 2 of CK2: [0,0]~
```

```
-----
PK=x^2 - 8761 CK=[27]
ell=19 N=2 Nn=2 n=1 CK=[27] CK1=[81]
algebraic norm in K1/K of the component 1 of CK1: [3]~
ell=19 N=2 Nn=2 n=2 CK=[27] CK2=[81]
algebraic norm in K2/K of the component 1 of CK2: [9]~
```

7.2.3. *Examples of stability from K_1 .* Let's give the program testing the stability in K_1/K ; some cases of non-stability are present. We put $\text{ell} = \text{prime}(t)$, $t \in [2, \text{nell}]$:

```
{p=3;bm=2;Bm=10^4;nell=20;for(m=bm,Bm,if(core(m)!=m,next);PK=x^2-m;
K=bnfinit(PK,1);v=valuation(K.no,p);if(v!=2,next);CK=K.clgp[2];
for(t=1,nell,ell=prime(t);if(t==nell,print("m=",m," nell insufficient");break);
if(Mod(ell-1,3)!=0,next);
```

```

if(kronecker(m,ell)!=-1,next);\\Test of inertia of ell
P=polcompositum(PK,polsubcyclo(ell,p))[1];K1=bnfinit(P,1);
v1=valuation(K1.no,p);if(v1!=2,next);
print("m=",m," PK=",PK," ell=",ell," CK=",K.clgp[2]," CK1=",K1.clgp[2]);break))}

```

```

m=1129 PK=x^2 - 1129 ell=19 CK=[9] CK1=[9]
m=1654 PK=x^2 - 1654 ell=43 CK=[9] CK1=[9]
m=3137 PK=x^2 - 3137 ell=19 CK=[9] CK1=[9]
m=3719 PK=x^2 - 3719 ell=31 CK=[9] CK1=[18,2]
(...)
m=9217 PK=x^2 - 9217 ell=7 CK=[18] CK1=[18]
m=9606 PK=x^2 - 9606 ell=31 CK=[18] CK1=[18,2,2]
m=9799 PK=x^2 - 9799 ell=19 CK=[18] CK1=[18,2,2]
m=9998 PK=x^2 - 9998 ell=37 CK=[9] CK1=[9]

```

The most impressive is that, up to $m \leq 10^{10}$, small primes ℓ are sufficient to get stability for cyclic groups \mathcal{H}_K .

Other examples where there is sometimes no stabilization up to the upper bound for ℓ ; this depends on the structure of \mathcal{H}_K since if \mathcal{H}_K is not cyclic, we know that total capitulation needs to compute in larger layers:

```

m=23659 PK=x^2 - 23659 CK=[6,3]
x^2 - 23659 NO STABILIZATION UP TO nell
ell=73 N=2 Nn=2 n=1 CK=[6,3] CK1=[18,3]
algebraic norm in K1/K of the component 1 of CK1: [3,0]~
algebraic norm in K1/K of the component 2 of CK1: [0,0]~
Incomplete capitulation in K1
-----
m=32009 PK=x^2 - 32009 CK=[3,3]
x^2 - 32009 NO STABILIZATION UP TO nell
ell=19 N=2 Nn=2 n=1 CK=[3,3] CK1=[9,3]
algebraic norm in K1/K of the component 1 of CK1: [3,0]~
algebraic norm in K1/K of the component 2 of CK1: [0,0]~
ell=19 N=2 Nn=2 n=2 CK=[3,3] CK2=[9,3]
algebraic norm in K2/K of the component 1 of CK2: [0,0]~
algebraic norm in K2/K of the component 2 of CK2: [0,0]~
Complete capitulation in K2
-----
m=42817 PK=x^2 - 42817 CK=[3,3]
x^2 - 42817 NO STABILIZATION UP TO nell
ell=19 N=2 Nn=2 n=1 CK=[3,3] CK1=[9,3]
algebraic norm in K1/K of the component 1 of CK1: [3,0]~
algebraic norm in K1/K of the component 2 of CK1: [0,0]~
ell=19 N=2 Nn=2 n=2 CK=[3,3] CK2=[27,3]
algebraic norm in K2/K of the component 1 of CK2: [9,0]~
algebraic norm in K2/K of the component 2 of CK2: [0,0]~
Incomplete capitulation in K2

```

7.3. Case of quadratic fields and $p = 5$. We give some excerpt of numerical results, analogous to the case $p = 3$; most of examples give stability, whence capitulation in some layer. We have taken $N = 2, Nn = 1$. For some rare cases, the capitulation is partial in the first layer.

```

PK=x^2 - 24859 CK=[25]
ell=251 N=2 Nn=1 n=1 CK=[25] CK1=[50,2,2,2]
algebraic norm in K1/K of the component 1 of CK1: [30,0,0,0]~
algebraic norm in K1/K of the component 2 of CK1: [0,0,0,0]~
algebraic norm in K1/K of the component 3 of CK1: [0,0,0,0]~
algebraic norm in K1/K of the component 4 of CK1: [0,0,0,0]~
Partial capitulation
ell=401 N=2 Nn=1 n=1 CK1=[1525]
algebraic norm in K1/K of the component 1 of CK1: [305]~
Partial capitulation
-----
PK=x^2 - 27689 CK=[25]
ell=101 N=2 Nn=1 n=1 CK=[25] CK1=[25]
algebraic norm in K1/K of the component 1 of CK1: [5]~
Partial capitulation
-----
PK=x^2 - 68119 CK=[50]
ell=251 N=2 Nn=1 n=1 CK=[50] CK1=[250]

```

```

algebraic norm in K1/K of the component 1 of CK1: [5]~
Partial capitulation
-----
PK=x^2 - 68819 CK=[25]
ell=101 N=2 Nn=1 n=1 CK=[25] CK1=[25]
algebraic norm in K1/K of the component 1 of CK1: [5]~
Partial capitulation
ell=151 N=2 Nn=1 n=1 CK=[25] CK1=[125]
algebraic norm in K1/K of the component 1 of CK1: [5]~
Partial capitulation
-----
PK=x^2 - 69403 CK=[25]
ell=251 N=2 Nn=1 n=1 CK=[25] CK1=[25,5]
algebraic norm in K1/K of the component 1 of CK1: [5,0]~
algebraic norm in K1/K of the component 2 of CK1: [0,0]~
Partial capitulation
ell=401 N=2 Nn=1 n=1 CK=[25] CK1=[50,2,2,2]
algebraic norm in K1/K of the component 1 of CK1: [30,0,0,0]~
algebraic norm in K1/K of the component 2 of CK1: [0,0,0,0]~
algebraic norm in K1/K of the component 3 of CK1: [0,0,0,0]~
algebraic norm in K1/K of the component 4 of CK1: [0,0,0,0]~
Partial capitulation
-----
PK=x^2 - 88211 CK=[25]
ell=101 N=2 Nn=1 n=1 CK=[25] CK1=[125]
algebraic norm in K1/K of the component 1 of CK1: [5]~
Partial capitulation
ell=151 N=2 Nn=1 n=1 CK=[25] CK1=[25]
algebraic norm in K1/K of the component 1 of CK1: [5]~
Partial capitulation, stability from K2
-----
PK=x^2 - 119029 CK=[50]
ell=251 N=2 Nn=1 n=1 CK=[50] CK1=[250]
algebraic norm in K1/K of the component 1 of CK1: [5]~
Partial capitulation
ell=401 N=2 Nn=1 n=1 CK=[50] CK1=[50]
algebraic norm in K1/K of the component 1 of CK1: [5]~
Partial capitulation
    
```

8. ON THE USE OF THE AUXILIARY EXTENSIONS $K(\mu_\ell)/K$

8.1. Remarks on the classical proof of the Main Conjecture. Contrary, to our point of view, some proofs of the Main Conjecture, as that of Thaine, have a fundamental difference. The brief overview that we give now must be completed by technical elements that the reader can find especially in [Was1997, §§ 15.2, 15.3].

Let f be the conductor of K . Let $\ell \nmid f$, $\ell \equiv 1 \pmod{2p^N}$, *totally split in K* and $p^N \geq p^e$, the exponent of \mathcal{H}_K ; put $M_0 = \mathbb{Q}(\mu_\ell)$ and $M := M_0K$. From Proposition 5.3, the cyclotomic unit η_M of M is such that $\mathbf{N}_{M/K}(\eta_M) = 1$ and the link with η_K is only a congruence modulo $(1 - \zeta_\ell)$.

Put $\eta_M = \alpha_M^{1-\sigma}$ where $\alpha_M \in M^\times$ is such that $(\alpha_M) \in I_M^G$; modulo K^\times , we can take α_M integer in M , whence:

$$(8.1) \quad (\alpha_M) = \mathbf{J}_{M/K}(\mathfrak{a}_K) \cdot \mathfrak{L}_0^{\Omega_\ell},$$

where $\mathfrak{a}_K \in I_K$ and \mathfrak{L}_0 is a fixed prime ideal dividing ℓ in M , thus totally ramified in M/K , with $\Omega_\ell = \sum_{s \in g} r_s \cdot s^{-1}$, $r_s \geq 0$; thus, since $\mathbf{N}_{M/K}(\mathfrak{L}_0) = \mathfrak{l}_0$, $\mathfrak{L}_0 \mid \mathfrak{l}_0 \mid \ell$ in $M/K/\mathbb{Q}$, this yields $(\alpha_K) := (\mathbf{N}_{M/K}(\alpha_M)) = \mathfrak{a}_K^{\ell-1} \cdot \mathfrak{l}_0^{\Omega_\ell}$.

But $\mathfrak{a}_K^{\ell-1}$ is principal since $\ell - 1 \equiv 0 \pmod{p^e}$, whence $\mathfrak{l}_0^{\Omega_\ell}$ principal.

Lemma 8.1. *Except a finite number of primes ℓ , the ideal $\mathfrak{L}_0^{\Omega_\ell}$ of (8.1) gives a non trivial relation, in the meaning that Ω_ℓ is not of the form $\lambda \cdot \nu_{M/M_0}$, $\lambda \geq 0$, giving $\mathfrak{l}_0^{\Omega_\ell} = (\ell)^\lambda$.*

Proof. Assume that $\Omega_\ell = \lambda \cdot \nu_{M/M_0}$; the character of $\mathfrak{L}_0^{\Omega_\ell}$, as $\mathbb{Z}[g]$ -module, is the unit one and any non-trivial φ -component $\alpha_{M,\varphi}$ of α_M is prime to ℓ , thus

congruent, modulo any $\mathfrak{L} \mid \ell$ in M , to $\rho_l \in \mathbb{Z}$, $\rho_l \not\equiv 0 \pmod{\ell}$ (residue degrees 1 in M/\mathbb{Q}). Since $\mathfrak{L}^\sigma = \mathfrak{L}$, we obtain $\eta_{M,\varphi} = \alpha_{M,\varphi}^{\sigma-1} \equiv 1 \pmod{\mathfrak{L}}$.

We have $\eta_{f\ell} \equiv \eta_f \pmod{(1-\zeta_\ell)}$ where $1-\zeta_\ell := \eta_\ell$ is an uniformizing parameter at the places above ℓ in M_0 , so that $\eta_M \equiv \eta_K \pmod{(1-\zeta_\ell)}$, which leads to $\eta_{K,\varphi} \equiv 1 \pmod{\mathfrak{l}}$, for all $\mathfrak{l} \mid \ell$, giving $\eta_{K,\varphi} \equiv 1 \pmod{\ell}$ (absurd for almost all ℓ). \square

Reducing modulo ν_{M/M_0} , one may get $\Omega_\ell \neq 0$, “minimal” in an obvious sense, with $r_s \geq 0$ but not all zero. Consider $\frac{\alpha_M^s}{(1-\zeta_\ell)^{r_s}}$ modulo \mathfrak{L}_0 and the conjugations $\alpha_M^\sigma = \alpha_M \cdot \eta_M$ and $\frac{(1-\zeta_\ell)^\sigma}{(1-\zeta_\ell)} = \frac{1-\zeta_\ell^{\mathfrak{g}_\ell}}{1-\zeta_\ell} \equiv \mathfrak{g}_\ell \pmod{(1-\zeta_\ell)}$ (where \mathfrak{g}_ℓ is a primitive root modulo ℓ such that $\zeta_\ell^\sigma =: \zeta_\ell^{\mathfrak{g}_\ell}$); one gets:

$$\left(\frac{\alpha_M^s}{(1-\zeta_\ell)^{r_s}} \right)^\sigma = \frac{\alpha_M^{\sigma s}}{(1-\zeta_\ell)^{\sigma r_s}} \equiv \frac{\eta_M^s \alpha_M^s}{(\mathfrak{g}_\ell (1-\zeta_\ell))^{r_s}} \equiv \frac{\eta_M^s}{\mathfrak{g}_\ell^{r_s}} \cdot \left(\frac{\alpha_M^s}{(1-\zeta_\ell)^{r_s}} \right) \pmod{\mathfrak{L}_0},$$

whence $\mathfrak{g}_\ell^{r_s} \equiv \eta_M^s \equiv \eta_K^s \pmod{\mathfrak{l}_0}$, which identifies the coefficients r_s .

So we have obtained a non-trivial congruential relation between the classes of the conjugates of \mathfrak{l}_0 that is p -localizable; this constitute the basis of the reasonings, on condition to add many more technical arguments to get some annihilation of $\mathcal{E}_{K,\varphi}/\mathcal{F}_{K,\varphi}$, then a final equality $\#\mathcal{H}_{K,\varphi} = (\mathcal{E}_{K,\varphi} : \mathcal{F}_{K,\varphi})!$

In this way, Thaine’s method is essentially analytic, working on norm properties and subtle congruences of the cyclotomic units, leading to the principle of Kolyvagin Euler systems, while that using capitulation (if any) is of class field theory framework and gives immediately the result without any supplementary work.

8.2. Remarks on the method of proof by capitulation. Suppose that for all inert prime numbers $\ell_N \equiv 1 \pmod{2p^N}$, with N arbitrary large, there is never any capitulation in the real cyclic p -tower L/K of $K(\mu_\ell)/K$; for these p -towers, we get easily the following more general result.

Proposition 8.2. *Let L/K be a real cyclic p -tower of degree p^N in which there is a unique prime ℓ , inert in K , and totally ramified in L/K ; we denote by K_n the subfield of L of degree p^n over K , $n \in [0, N]$. We assume that, for all $n, m \in [0, N]$, with $n \leq m$, the transfer maps $\mathbf{J}_n^m := \mathbf{J}_{K_m/K_n}$ are injective. Put $G_n^m := \text{Gal}(K_m/K_n)$, $\mathcal{H}_n := \mathcal{H}_{K_n}$, $\mathcal{H}_m := \mathcal{H}_{K_m}$.*

Then, one has the inequalities $\#\mathcal{H}_m \geq \#\mathcal{H}_n \cdot \#\mathcal{H}_n[p^{m-n}]$ which lead, for all n , to $\#\mathcal{H}_{K_n} \geq \#\mathcal{H}_K \cdot p^{n \cdot r_K}$, where $r_K = \dim_{\mathbb{F}_p}(\mathcal{H}_K/\mathcal{H}_K^p)$ is the p -rank of \mathcal{H}_K .

Proof. From the exact sequence of $\mathbb{Z}[G_n^m]$ -modules:

$$1 \rightarrow \mathbf{J}_n^m \mathcal{H}_n \rightarrow \mathcal{H}_m \rightarrow \mathcal{H}_m/\mathbf{J}_n^m \mathcal{H}_n \rightarrow 1,$$

we get, from the Chevalley–Herbrand formula $\#\mathcal{H}_m^{G_n^m} = \#\mathcal{H}_n$ and the injectivity of \mathbf{J}_n^m , $\mathcal{H}_m^{G_n^m}/\mathbf{J}_n^m \mathcal{H}_n = 1$ and the exact sequence:

$$1 \rightarrow (\mathcal{H}_m/\mathbf{J}_n^m \mathcal{H}_n)^{G_n^m} \rightarrow \text{H}^1(G_n^m, \mathbf{J}_n^m \mathcal{H}_n) \rightarrow \text{H}^1(G_n^m, \mathcal{H}_m),$$

where $\text{H}^1(G_n^m, \mathbf{J}_n^m \mathcal{H}_n) = (\mathbf{J}_n^m \mathcal{H}_n)[p^{m-n}] \simeq \mathcal{H}_n[p^{m-n}]$ and $\#\text{H}^1(G_n^m, \mathcal{H}_m) = \#\text{H}^2(G_n^m, \mathcal{H}_m) = \#\mathcal{H}_m^{G_n^m}/\#\mathbf{J}_n^m \circ \mathbf{N}_n^m(\mathcal{H}_m) = \#\mathcal{H}_m^{G_n^m}/\#\mathcal{H}_n = 1$ giving:

$$(\mathcal{H}_m/\mathbf{J}_n^m \mathcal{H}_n)^{G_n^m} \simeq \mathcal{H}_n[p^{m-n}].$$

Whence $\#\mathcal{H}_m \geq \#\mathbf{J}_n^m \mathcal{H}_n \cdot \#\mathcal{H}_n[p^{m-n}] = \#\mathcal{H}_n \cdot \#\mathcal{H}_n[p^{m-n}]$. For $m = n + 1$, one obtains $\#\mathcal{H}_{n+1} \geq \#\mathcal{H}_n \cdot p^{\text{rank}_p(\mathcal{H}_n)}$, then the last claim by induction. \square

This result indicates that the filtration \mathcal{H}_L^i defined by:

$$\mathcal{H}_L^0 = 1, \quad \mathcal{H}_L^1 := \mathcal{H}_L^G \quad \text{and} \quad \mathcal{H}_L^{i+1}/\mathcal{H}_L^i := (\mathcal{H}_L/\mathcal{H}_L^i)^G,$$

has length unbounded regarding N since $\#(\mathcal{H}_L^{i+1}/\mathcal{H}_L^i) \leq \#\mathcal{H}_K$ for all i because of the obvious injective maps giving a decreasing sequence:

$$\mathcal{H}_L^{i+1}/\mathcal{H}_L^i \xrightarrow{1-\sigma} \mathcal{H}_L^i/\mathcal{H}_L^{i-1}, \quad \text{for all } i \geq 1.$$

Remark 8.3. For all $n \in [1, N]$ (i fixed), we have the following diagram where the norms $\mathbf{N}_{n+1/n} := \mathbf{N}_{K_{n+1}/K_n}$ defined on \mathcal{H}_{n+1} and $(\mathcal{H}_{n+1})^{(1-\sigma_{n+1})^i}$ are surjective, but not necessarily that defined on \mathcal{H}_{n+1}^i :

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{H}_{n+1}^i & \longrightarrow & \mathcal{H}_{n+1} & \xrightarrow{(1-\sigma_{n+1})^i} & (\mathcal{H}_{n+1})^{(1-\sigma_{n+1})^i} \longrightarrow 1 \\ & & \downarrow & & \mathbf{N}_{n+1/n} \downarrow & & \mathbf{N}_{n+1/n} \downarrow \\ 1 & \longrightarrow & \mathcal{H}_n^i & \longrightarrow & \mathcal{H}_n & \xrightarrow{(1-\sigma_n)^i} & (\mathcal{H}_n)^{(1-\sigma_n)^i} \longrightarrow 1. \end{array}$$

One has $\mathbf{N}_{n+1/n}(\mathcal{H}_{n+1}^i) \subseteq \mathcal{H}_n^i$. For all $i \geq 0$, the integers $\#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i)$ define an increasing n -sequence of divisors of $\#\mathcal{H}_K$, thus stationary at the layer n_i , giving the stability of $\#(\mathcal{H}_n^{i+1}/\mathcal{H}_n^i)$ for all $n \geq n_i$, if $n_i < N$ (i fixed).

Thus $\#(\mathcal{H}_{n_i}^{i+1}/\mathcal{H}_{n_i}^i) =: p^c$ and the i -sequence p^c is decreasing, stationary at a divisor p^c of $\#\mathcal{H}_K$, $c \geq 0$. A stability $\#\mathcal{H}_{n_0+1} = \#\mathcal{H}_{n_0}$, from $n_0 < N$, does exist if and only if the i -sequence of layers n_i stabilizes and if $c = 0$.

One may ask at what condition $\mathbf{J}_{L/K}$ is injective, giving increasing $\#\mathcal{H}_{K_n}$ as we have seen; in fact in most cases there is at least partial capitulation at some layer and, even in the simplest experiments with $\mathbf{J}_{K_1/K}$ the injectivity is very rare. It is clear that it is necessary to the filtration $\mathcal{H}_{K_1}^i$ to be large. We have found, with some difficulties, the following example for cyclic cubic fields and $p = 2$; most of primes ℓ give partial capitulation (case of $\ell = 113$) or stability (case of $\ell = 337$), then the case of $\ell = 2129$ in K_1 :

```
conductor f=1777 PK=x^3 + x^2 - 592*x + 724 CK=[4,4]
ell=113 n=1 CK=[4,4] CK1=[4,4,2,2]
algebraic norm in K1/K of the component 1 of CK1: [0,0,1,0]~
algebraic norm in K1/K of the component 2 of CK1: [0,0,0,1]~
algebraic norm in K1/K of the component 3 of CK1: [0,0,0,0]~
algebraic norm in K1/K of the component 4 of CK1: [0,0,0,0]~
Partial capitulation
ell=337 n=1 CK=[4,4] CK1=[4,4]
algebraic norm in K1/K of the component 1 of CK1: [2,0]~
algebraic norm in K1/K of the component 2 of CK1: [0,2]~
Partial capitulation, stability-->capitulation in K2
ell=2129 n=1 CK=[4,4] CK1=[16,16,2,2]
algebraic norm in K1/K of the component 1 of CK1: [8,4,0,1]~
algebraic norm in K1/K of the component 2 of CK1: [12,12,1,0]~
algebraic norm in K1/K of the component 3 of CK1: [8,8,0,0]~
algebraic norm in K1/K of the component 4 of CK1: [0,8,0,0]~
No capitulation
ell=2129 n=2 CK=[4,4] CK2=[16,16,2,2]
algebraic norm in K2/K of the component 1 of CK2: [0,8,0,0]~
algebraic norm in K2/K of the component 2 of CK2: [8,8,0,0]~
algebraic norm in K2/K of the component 3 of CK2: [0,0,0,0]~
algebraic norm in K2/K of the component 4 of CK2: [0,0,0,0]~
Partial capitulation, stability from K2
```

For $\ell = 2129$, put $\mathcal{H}_{K_1} = \langle h_1, h_2, h_3, h_4 \rangle$ where the classes h_1, h_2, h_3, h_4 are of orders 16, 16, 2, 2, respectively; one computes easily that

$$\nu_{K_1/K}(\mathcal{H}_{K_1}) = \langle h_1^8 h_2^4 h_4, h_1^{12} h_2^{12} h_3, h_1^8 h_2^8, h_2^8 \rangle$$

is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. But we obtain the stability in K_2/K_1 which implies some capitulation of \mathcal{H}_{K_1} in K_4 , but perhaps not complete since $2129 = 1 + 7 \cdot 19 \cdot 2^4$; nevertheless $J_{K_2/K}(\mathcal{H}_K) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, capitulates in K_3 (from the proof of Theorem 4.9). The computation in K_3 seems out of reach.

8.3. Conclusion. The behavior of p -class groups in cyclic p -towers L/K (with $L \subset K(\mu_\ell)$, $\ell \equiv 1 \pmod{2p^N}$ inert in K/\mathbb{Q}), suggests that there exist infinitely many such primes for which \mathcal{H}_K capitulates at some layer. The most interesting fact being that if so, this implies the Main Conjecture on abelian fields in the semi-simple case with minimal analytic arguments and almost trivial proof from the classical context of Chevalley–Herbrand formulas.

Regarding some works about these questions of “exceptional classes” (i.e., non invariant), it is admitted (and proved in some circumstances) that the filtrations defining the \mathcal{H}_{K_n} , for $n \in [1, N]$, are random and that cases of

unbounded filtrations are of probabilities tending to 0 with N . For instance, in [KoPa2022, Smi2022] it is proved that p -class groups of cyclic fields of degree p have a standard distribution, the case of length 1 (i.e., $\mathcal{H}_L = \mathcal{H}_L^G$) being most probable.

Moreover, the context of “genus theory type” in cyclic p -towers, is, in some sense, specific and “easier” than the most deeper general one on p -class groups in arbitrary number fields as the reader can see in the recent papers [Pier2022, PTBW2020], among many others. But the ϵ -conjectures are of no help, here, since the discriminants of the fields $K(\mu_\ell)$ is larger than $\ell^{\ell-2} = O(p^{N \cdot p^N})$ and the philosophy is on the contrary (as for Greenberg’s conjecture) that $\mathcal{H}_L = O(\mathcal{H}_K)$ for infinitely many ℓ ’s.

More generally, the phenomenon of capitulation, logically governs many arithmetic results and conjectures of number theory, and remains essentially a class field theory concept which deserves deepened researches.

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