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# Fundamentals of Compositional Rewriting Theory<sup>★</sup>

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## Abstract

1 A foundational theory of compositional categorical rewriting theory is presented, based on a collection of fibration-  
2 like properties that collectively induce and intrinsically structure the large collection of lemmata used in the proofs  
3 of theorems such as concurrency and associativity. The resulting highly generic proofs of these theorems are given.  
4 It is noteworthy that the proof of the concurrency theorem takes only a few lines and, while that of associativity  
5 remains somewhat longer, it would be unreadably long if written directly in terms of the basic lemmata. In essence,  
6 our framework improves the readability and ease of comprehension of these proofs by exposing latent modularity.  
7 A curated list of known instances of our framework is used to conclude the paper with a detailed discussion of the  
8 conditions under which the Double Pushout and Sesqui-Pushout semantics of graph transformation are compositional.

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<sup>★</sup>This is an invited extended journal version of the ICGT 2021 conference paper entitled “Concurrency Theorems for Non-linear Rewriting Theories” [1, 2].

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9 **1. Introduction**

10 The main contribution of the present paper is a novel framework of [compositional rewriting double categories](#)  
11 [\(crDCs\)](#) which, at its core, is based upon mathematical notions of fibrational structures relevant to categorical rewriting  
12 theories. Our motivation for this development has been the technically highly involved nature of the definitions of  
13 rule compositions and the resulting [concurrency theorems](#) for [generic Double-Pushout \(DPO\) semantics](#) and [Sesqui-  
14 Pushout \(SqPO\) semantics](#) as presented in [1] (with additional details and proofs presented in [2]).

15 In this extended version, we show that one can modularize the statement of the rewriting semantics, its list of  
16 prerequisites and also the statement and proof of the concurrency theorem in a *uniform* fashion by establishing the  
17 notion of [crDCs](#): once, for a given semantics, the notion of [direct derivations](#) is specified (and thus, in a certain sense,  
18 the very definition of the semantics itself is provided), rather than trying to follow the steps prescribed by the categor-  
19 ical rewriting literature in the tradition of the work of Ehrig et al. [3], our novel concept of [crDCs](#) permits to decide  
20 whether or not the given semantics is *compositional* (i.e., admits both a [concurrency theorem](#) and an [associativity  
21 theorem](#)) purely based upon the properties of the [direct derivations](#) themselves.

22 It should be noted that establishing that a given semantics indeed yields a **crDC** structure is still a technically  
 23 involved task (as will become evident when presenting instantiations of our novel framework for some concrete  
 24 examples of rewriting semantics in Section 6); however, the required reasoning has a rather more mechanical character  
 25 than that of the proofs of complex theorems in compositional rewriting theory—which are *automatically guaranteed*  
 26 to hold once a **crDC** structure is verified for a given semantics.

### 27 1.1. Rewriting

28 The theory of graph transformation has been under development for about the last fifty years. Over this time, it  
 29 has gradually evolved from working with specific concrete settings—such as multi or simple graphs, with or without  
 30 attributes—to being expressed in terms of certain classes of categories—such as **adhesive** [4, 5, 6], **quasi-adhesive/rm-**  
 31 **adhesive** [5, 6, 7] and **M-adhesive** [3, 8] categories—that provide sufficient structure to reprove abstractly the key  
 32 theorems that hold in those concrete settings. This theory has found application in a variety of contexts such as  
 33 model-driven software engineering [9], graph databases [10, 11], graph-based knowledge representation [12, 13, 14]  
 34 and executable representation of complex systems [15, 16, 17] as well as more theoretical uses in graph grammars,  
 35 structural graph theory and string diagrams.

36 In this theory, a graph transformation rule  $O \leftarrow r - I$  is interpreted in a given category as a span, i.e., as two  
 37 morphisms  $O \leftarrow o_r - K_r - i_r \rightarrow I$  with a common source object  $K_r$ , called the *context* or *preserved region*, an *input*  
 38 (or LHS) object  $I$  and an *output* (or RHS) object  $O$ . Note that we depart from the classical representation of rules in  
 39 this theory in two ways: the application to rule algebra [18, 19, 20] makes it more natural to reverse the orientation  
 40 of a rule, so as to match with the usual right-to-left ordering of composition of functions, whereupon the traditional  
 41 terminology of ‘LHS’ and ‘RHS’ becomes rather confusing; as such, we prefer the neutral, self-explanatory terms  
 42 ‘input’ and ‘output’. Similarly, we use the terms ‘input-linear’ and ‘output-linear’ instead of the more usual ‘left-  
 43 linear’ and ‘right-linear’ when speaking of rules where  $i_r$  or, respectively,  $o_r$  are monomorphisms of some kind.

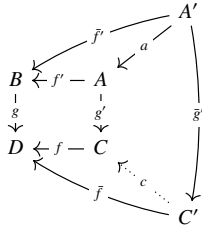
44 In typical concrete settings, the two arrows express the correspondence between the entities (nodes or edges, etc.)  
 45 in  $I$  and those in  $O$ . This naturally suggests a small set of primitive operations—deletion and addition, where  $i_r$  and  $o_r$   
 46 are non-surjective, and cloning and merging, where they are non-injective—that correspond to our intuitive ideas of  
 47 how graphs can be transformed; a rule then interprets a combination of these primitive operations. In order to formalize  
 48 the effect of a rule, several distinct, but closely related, semantics have been proposed, the most prominent of which  
 49 are the Double Pushout (DPO) [21], Single Pushout (SPO) [22] and, more recently, Sesqui-Pushout (SqPO) [23]  
 50 semantics.

51 In all of these approaches, a rule is applied to an object  $X$  through a so-called *matching*  $I \xrightarrow{m} X$ , where  $m$  is  
 52 a monomorphism<sup>1</sup> potentially chosen from a specified restricted class  $\mathcal{M}$ . In the case of DPO or SqPO semantics,  
 53 rewriting proceeds in two steps: first, we use  $i_r$  and  $m$  to construct an intermediate object  $K_{r_m}$ , a monomorphism  
 54  $K_r \xrightarrow{k_{r_m}} K_{r_m}$  and a morphism  $K_{r_m} \xrightarrow{i_{r_m}} X$  (i.e., the square (\*) below); then we use  $o_r$  and  $k_{r_m}$  to construct  
 55 an object  $r_m(X)$  (i.e., the direct derivation of  $X$  along  $r$  with match  $m$ ), a monomorphism  $O \xrightarrow{m^*} r_m(X)$ , and a  
 56 morphism  $K_{r_m} \xrightarrow{o_{r_m}} r_m(X)$  (i.e. the square (†) below):

$$\begin{array}{ccccc}
 O & \xleftarrow{o_r} & K_r & \xrightarrow{i_r} & I \\
 \downarrow m^* & & \downarrow k_{r_m} & & \downarrow m \\
 r_m(X) & \xleftarrow{o_{r_m}} & K_{r_m} & \xrightarrow{i_{r_m}} & X
 \end{array} \quad (1)$$

57 In both DPO and SqPO semantics, the second step is determined by taking the pushout (PO) of  $k_{r_m}$  and  $o_r$  as in square  
 58 (†); the difference between these semantics arises in the first step: the DPO approach specifies that the square (\*) be a  
 59 PO while the SqPO approach specifies it to be the final pullback complement of  $m$  and  $i_r$  [24]: given two composable  
 60 arrows  $A \xrightarrow{f'} \rightarrow B$  and  $B \xrightarrow{g} \rightarrow D$ , a *final pullback complement* (FPC; cf. diagram below) consists of two composable  
 61 arrows  $A \xrightarrow{g'} \rightarrow C$  and  $C \xrightarrow{f} \rightarrow D$  such that (i) the resulting square is a pullback (PB); and (ii) for all PB squares  
 62 such as the outer square in the diagram below, and for all factorizations  $A' \xrightarrow{a} \rightarrow A$  of  $\tilde{f}'$  through  $f'$ , there exists a  
 63 unique arrow  $C' \xrightarrow{c} \rightarrow C$  such that  $f \circ c = \tilde{f}$  and  $c \circ \tilde{g}' = g \circ \tilde{f}'$ .

<sup>1</sup>The DPO approach has sometimes been formulated without requiring  $m$  to be a monomorphism; we do not consider this variant here.



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The notion of FPC is defined by a universal property<sup>2</sup> so that, under SqPO semantics, given  $m$  and  $i_r$ ,  $K_{r_m}$  is *essentially* unique (i.e., unique up to unique isomorphism). However, the construction of  $K_{r_m}$  under the DPO semantics need not be uniquely determined because, in general, there may be several non-isomorphic objects for which the square (\*) is a PO. Nonetheless, in many concrete settings under the assumption that the rule is **input-linear**, there is in fact *at most one way* for square (\*) to be a PO: this follows from the fact that, in the category **Set** of sets and functions, there is exactly one  $K_{r_m}$  for which square (\*) is a PO, provided that  $i_r$  is injective. In these settings, it is easy to show that this square—if it exists, as is characterized by the no-dangling edges condition [3]—satisfies a universal property that is categorically dual to that of an FPC: it is the smallest, i.e., most general,  $K_{r_m}$  together with arrows from and to  $K_r$  and  $X$  respectively for which the resulting square is a PO. This is a mild generalization of the notion of what is called the *initial pushout complement* in [25] or *minimal pushout complement* in [26]; despite the risk of slight confusion, we reuse the terminology of *initial pushout complement* (IPC) as this is standard practice in the naming of categorical duals.

In the case that  $i_r$  is not a monomorphism, there is no longer any guarantee of uniqueness: even in typical concrete settings based on sets and functions, there may be several *minimal* but incomparable candidate PO squares. This leads us to consider the less familiar categorical notion of **multi-IPC** (**mIPC**) which formalizes the notion of a *family* of minimal solutions, i.e., PO squares over  $m$  and  $i_r$ , that are *collectively* universal: any PO square using  $i_r$  and factoring through  $m$  factors uniquely through a unique family member. This multi-universal property is an instance of the general theory of Diers [27]; in the case at hand, it effectively states that the family of  $K_{r_m}$ s contains precisely all possible rewrites of  $X$  by  $i_r$  via  $m$  that are compatible with the DPO semantics. We will investigate this construction in more detail in Sections 4.3 and 5.3.3.

Finally, in *SPO semantics* [22], the rewrite is applied in a single step by taking the PO, in the bi-category of spans, of  $m$  and the rule  $r$ —all subject to the condition that  $i_r$  be a monomorphism, i.e., that the rule is **input-linear**. Due to the well-known fact that SPO semantics in the setting where monic matches are used in fact coincides with a special case of **Sesqui-Pushout** (**SqPO**) semantics [23], we will not consider SPO semantics separately in the present paper. Note in particular that the SqPO approach, unlike SPO, does *not* require input-linearity and works with fully general rules.

An interesting consequence of the definition of DPO semantics is that rule applications are always *reversible* because squares (\*) and (†) are both POs; in effect, the no-dangling condition prevents the rule from being applied in the case that it would otherwise produce an irreversible transformation—or, alternatively, induce a *side-effect*. This is not necessarily the case for SqPO semantics because there is no a fortiori reason that the square (\*), defined by an FPC, be a PO: it is well-known that a rule which deletes a node can always be applied under the SqPO semantics, but induces an irreversible transformation in the case that the deleted node has incident edges in  $X$ . However, such a rule can only be applied under the DPO semantics if the no-dangling condition holds, i.e., the targeted node has no incident edges. Equally, there is no a fortiori reason that the square (†) be an FPC: a rule that merges two nodes generally loses the information about their incident edges that would be required to reverse the transformation—this is another kind of side-effect. The special case of *reversible SqPO*, where the FPC is also always a PO and the PO is also always an FPC, was studied in [28, 29]; in practice, at least for linear rules, this amounts to restricting to the DPO semantics since this constraint is then equivalent to the no-dangling condition.

<sup>2</sup>It can be seen as a categorical generalization of the notion of set difference: the FPC is the largest, i.e., least general,  $C$  together with arrows  $A - g' \rightarrow C$  and  $C - f \rightarrow D$  for which the resulting square is a PB.

104 The use of input-non-linear rules under the SqPO semantics allows for the expression of the natural operation  
 105 of the *cloning* of a node or an edge (when this is meaningful), as explained in [23, 26, 30]. More recently, such  
 106 rules have also been used to express operations such as concept refinement in schemata for graph databases [11] and,  
 107 more generally, in graph-based knowledge representation [14]. In combination with output-non-linear rules, as for  
 108 (non-linear) DPO rewriting, the SqPO semantics thus allows the expression of all natural *primitive operations on*  
 109 *graphs*: *addition* and *deletion* of nodes and edges; and *cloning* and *merging* of nodes and edges. Moreover, the SqPO  
 110 semantics also allows for all natural side-effects, or failures of reversibility, that arise intuitively in the case of deletion  
 111 and merging, as mentioned above, and also in the case of cloning; this has notably been exploited in the definition  
 112 of the semantics of the Kappa language where all rules are linear—so no cloning or merging—but deletion may have  
 113 side-effects [31], and in the KAMI bio-curation framework [13], where fully general rules are used with the SqPO  
 114 semantics.

## 115 1.2. Compositional rewriting

116 In the discussion above, we have seen that the definition of a setting for graph transformation requires us to specify  
 117 a number of things, including: the category in which we work, the classes of rules and matchings under consideration,  
 118 the semantics we use to apply rules, etc. In this paper, we adopt the stance that a choice of parameters in this design  
 119 space should be compatible with *compositionality* in the sense that a notion of *rule composition* exists, which amounts  
 120 to requiring that the [concurrency theorem](#) holds, and that this satisfies an appropriate form of [associativity theorem](#).  
 121 These properties enable the static analysis of collections of rules, such as rule algebras [18, 19, 20] or the causal  
 122 analysis found in the Kappa language [31, 32], that ultimately depend on the notion of *tracets* [33] that follows from  
 123 having such an associative rule composition.

### 124 1.2.1. Rule composition

125 The composition of two rules,  $O_1 \leftarrow r_1 - I_1$  and  $O_2 \leftarrow r_2 - I_2$ , is a third rule  $r_2 \circ r_1$  whose effect on an object  $X$   
 126 should be the same as that of applying first  $r_1$  to  $X$  then  $r_2$  to the resulting  $r_{m_1}(X)$ . However, the effect of applying first  
 127  $r_1$  then  $r_2$  depends critically on the way in which the images of the matchings  $m_1^*$  and  $m_2$ , of  $O_1$  and  $I_2$  respectively,  
 128 overlap in  $r_{m_1}(X)$ : given an initial matching  $m_1$  of  $I_1$  in  $X$ , the resulting  $m_1^*$  is uniquely determined; however, there may  
 129 be many possible choices for the matching  $m_2$  of  $I_2$  in  $r_{m_1}(X)$ . As such, in general, there is not one single composite  
 130 rule  $r_2 \circ r_1$  but rather one such composite for each possible overlap of  $m_1^*$  with some such  $m_2$ .

131 In order to express these ideas independently of the particular choice of  $X$ , we use Diers' notion of multi-sum [27]  
 132 as the means to express the *family* of all possible overlaps of matchings from  $O_1$  and  $I_2$ . This is a generalization of the  
 133 familiar categorical notion of co-product which replaces the single co-product  $O_1 \rightarrow O_1 + I_2 \leftarrow I_2$  with a family of  
 134 co-spans satisfying a multi-universal property: essentially, any given co-span from  $O_1$  and  $I_2$  factors through exactly  
 135 one family member and does so uniquely<sup>3</sup>.

136 This kind of multi-universal construction often arises in concrete settings, based on the category of sets, where  
 137 we wish to restrict our attention to injective functions. In this case, although the inclusions  $O_1 \rightarrow O_1 + I_2 \leftarrow I_2$  are  
 138 injective, the universal arrow is generally non-injective (unless the images of  $O_1$  and  $I_2$  are disjoint). The multi-sum  
 139 construction side-steps this problem by providing all possible overlaps of the images of  $O_1$  and  $I_2$ , including the case  
 140 where they are disjoint, so that all identifications—that would lead to violations of injectivity—necessary for the usual  
 141 universal arrow can instead be accounted for by choosing the appropriate family member.

142 The general notion of rule composition can thus be stated purely at the level of  $r_1$  and  $r_2$ , provided that we are  
 143 working in a setting where the multi-sum of  $O_1$  and  $I_2$  is guaranteed to exist, and it provides one composite rule  
 144 per family member of that multi-sum. The *synthesis* part of the concurrency theorem then states that a sequential  
 145 application, of  $r_1$  then  $r_2$ , can be simulated in a single step by identifying the relevant multi-sum element and using  
 146 the appropriate induced composite rule; and the *analysis* part of the theorem states, conversely, that the direct appli-  
 147 cation of such a composite rule can be decomposed back into a sequential application of its constituents with overlap  
 148 determined by the corresponding multi-sum element.

<sup>3</sup>We give a formal definition in Section 2.1 which, although its precise statement differs from this informal account by allowing for essential uniqueness, remains essentially equivalent.

### 149 1.2.2. Compositionality

150 In this paper, we seek to provide foundations for *compositional* rewriting that apply equally to DPO and SqPO  
151 semantics. In particular, we provide a single proof of the [concurrency theorem](#) in Section 3.3) and a single proof  
152 of an appropriate associativity property of the induced notion of rule composition, i.e., an [associativity theorem](#), in  
153 Section 3.4) that work for linear and non-linear rules under both semantics. This associativity property is important  
154 as it guarantees that, for any sequence of rule applications, their overall composite transformation can be computed,  
155 by iterating the concurrency theorem, in any order without changing the result (up to isomorphism): this is precisely  
156 what we mean by ‘compositionality’.

157 In this paper, we consider a rule application as a single unit (analogous to SPO semantics) rather than decomposing,  
158 as usual, into two stages that proceed via an intermediate object. This choice eases the path to a characterization of the  
159 necessary categorical structure for compositional rewriting in terms of (i) the existence of certain kinds of fibrations;  
160 and (ii) a small number of additional axioms specific to rewriting.

161 From this point of view, the large collection of lemmata used in these proofs—as, for example, collected in the  
162 appendix of [2]—fall into two groups: a first group of fundamental results that do not specifically relate to rewriting;  
163 and a second group that specializes this theory precisely to the case of rewriting. The advantage of this new approach  
164 is that it enables the use of macros that express the key steps in proofs at a higher level of abstraction than usual and,  
165 indeed, the resulting proof of the [concurrency theorem](#) is very compact. The proof of the [associativity theorem](#)  
166 is significantly longer and technically more involved—although this seems to be intrinsic to its nature—but would have  
167 been essentially impossible to express at a lower level of abstraction. Our new approach makes a clear and clean  
168 separation of the basic building blocks from their means of combination; we return to this point in the conclusion.

### 169 Outline of the paper

170 The paper is structured as follows. In Section 2, we present all the preliminary material necessary for the definition  
171 of compositional rewriting double categories with a particular emphasis on the required fibrational structures. In  
172 Section 3, we define this novel concept and apply it to state and prove the concurrency theorem and the associativity  
173 theorem in a universal fashion. In Section 4, we investigate the fibrational structure of various categories of squares  
174 (pullbacks, pushouts and final pullback complements). In Section 5, we study some classes of categories that admit  
175 constructions that are necessary in order to formulate compositional rewriting theories, and, in Section 6, we focus  
176 on the DPO- and SqPO-semantics in order to clarify under what conditions, and for what classes of rules, these  
177 semantics are compositional. Finally, we conclude with a detailed comparison of the approach in this paper with  
178 that of its conference version [1, 2] as well as a discussion of related and future work. The reader interested in our  
179 fibration-based proofs of concurrency and associativity can therefore read Sections 2 and 3 only; one more interested  
180 in how our new framework can be put to use might prefer to skim those sections and focus principally on Sections 4,  
181 5 and 6.

## 182 2. On multi-sums and fibrational structures

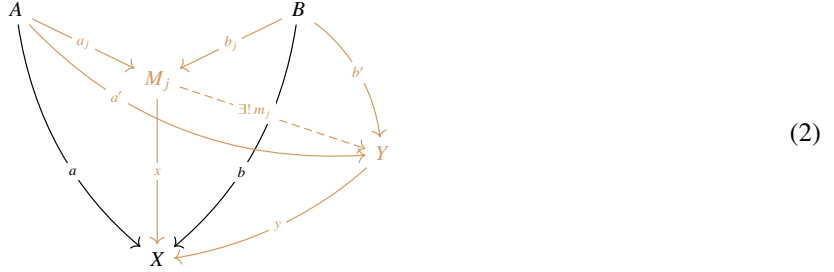
183 In this section, we provide some prerequisite material for our compositional rewriting theory framework. We begin  
184 with the notion of *multi-sum* which is an instance of the general theory of multi-co-limits developed by Diers [27].  
185 We then introduce the mathematical theory for a number of fibrational structures, namely the well-known notions  
186 of [Grothendieck fibration](#) and [Grothendieck opfibration](#), but also [multi-opfibrations](#) and [residual multi-opfibrations](#)  
187 which, to the best of our knowledge, are original results of our work.

### 188 2.1. Multi-sums

189 An atypical feature of fibrational structures relevant for compositional rewriting theories is the following type of  
190 mathematical property, which may eventually play an important role in the static analysis of rewriting systems.

191 **Definition 2.1.** Let  $\mathbf{C}$  be a category. A [multi-sum](#)  $\sum_{\mathcal{M}}(A, B)$  of two objects  $A$  and  $B$  of  $\mathbf{C}$  is a family of cospans  
192  $\{A - a_j \rightarrow M_j \leftarrow b_j - B\}_{j \in J}$  such that for every cospan  $A - a \rightarrow X \leftarrow b - B$ , there exists a  $j \in J$  and morphism  
193  $M_j - x \rightarrow X$  such that  $a = x \circ a_j$  and  $b = x \circ b_j$ , and with the following [\(multi-\) universal property](#): for every cospan

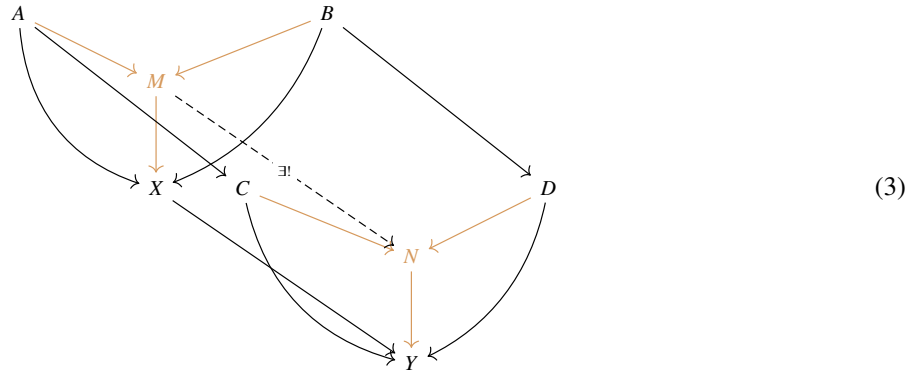
194  $A \leftarrow a' \rightarrow Y \leftarrow b' \leftarrow B$  and morphism  $Y \leftarrow y \rightarrow X$  such that  $a = y \circ a'$  and  $b = y \circ b'$ , there exists a unique morphism  
 195  $M_j \leftarrow m_j \rightarrow Y$  such that  $a' = m_j \circ a_j$  and  $b' = m_j \circ b_j$ :



196 We say that  $\mathbf{C}$  *has multi-sums* if every pair of objects has a multi-sum.

197 While we postpone the presentation of some concrete examples of multi-sum structures to Section 5.3.1, suffice it  
 198 here to introduce a technical result that will be necessary in our ensuing constructions:

199 **Lemma 2.2 (Multi-sum extension).** *Let  $\mathbf{C}$  be a category that *has multi-sums* and that *has pullbacks*. Then for every  
 200 commutative diagram such as in (3) below, where  $A \rightarrow M \leftarrow B$  and  $C \rightarrow N \leftarrow D$  are multi-sum elements, there exists  
 201 a universal arrow  $M \rightarrow N$  that makes the diagram commute.*

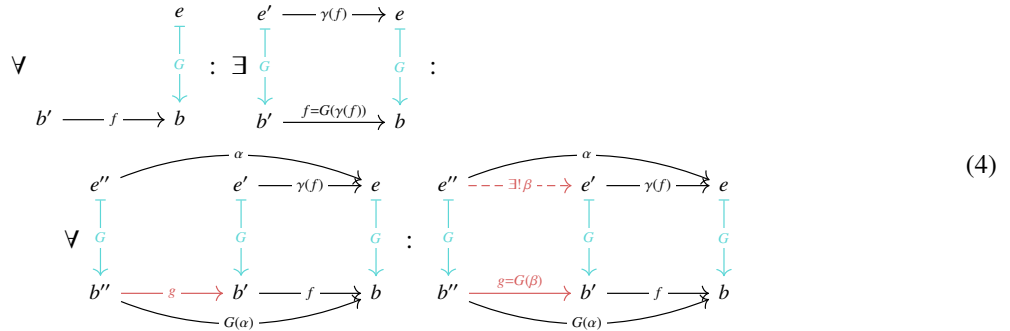


202 The proof of this lemma, and indeed of all the results in Section 2, can be found in Appendix B.1.

### 203 2.2. Grothendieck fibrations and opfibrations

204 In the technical constructions developed in this paper, we will require certain generalizations of the notion of  
 205 **Grothendieck opfibration**. We will therefore employ a notation for fibrations that slightly differs from the standard  
 206 conventions in category theory (cf. e.g. [34, 35, 36, 37]). Let us therefore briefly recall the definitions of **Grothendieck**  
 207 **fibrations** and **Grothendieck opfibrations** for the readers convenience, expressed in our notational conventions:

208 **Definition 2.3.** A functor  $G : \mathbf{E} \rightarrow \mathbf{B}$  is a **Grothendieck fibration** if the following property holds:





209 The second line encodes that  $\gamma(f)$  is a *Cartesian morphism*, hence we will refer to it as a *Cartesian lifting* of  $f$ .

210 **Definition 2.4.** A functor  $G : \mathbf{E} \rightarrow \mathbf{B}$  is a *Grothendieck opfibration* if the following property holds:

$$\begin{array}{c}
 \forall \\
 \begin{array}{ccc}
 \begin{array}{c} e \\ \downarrow G \\ b \end{array} & \xrightarrow{f} & \begin{array}{c} b' \\ \downarrow G \end{array} \\
 \end{array}
 \end{array}
 : \exists
 \begin{array}{ccc}
 \begin{array}{c} e \\ \downarrow G \\ b \end{array}
 & \xrightarrow{\varepsilon(f)} & \begin{array}{c} e' \\ \downarrow G \\ b' \end{array}
 \end{array}
 :
 \end{array}
 \quad (5)$$
  

$$\begin{array}{c}
 \forall \\
 \begin{array}{ccc}
 \begin{array}{c} e \\ \downarrow G \\ b \end{array}
 & \xrightarrow{f} & \begin{array}{c} b' \\ \downarrow G \end{array} \\
 \end{array}
 \end{array}
 : \exists
 \begin{array}{ccc}
 \begin{array}{c} e \\ \downarrow G \\ b \end{array}
 & \xrightarrow{\varepsilon(f)} & \begin{array}{c} e' \\ \downarrow G \\ b' \end{array} \\
 \begin{array}{c} e' \\ \downarrow G \\ b' \end{array}
 & \xrightarrow{f=G(\varepsilon(f))} & \begin{array}{c} e'' \\ \downarrow G \\ b'' \end{array} \\
 \begin{array}{c} e \\ \downarrow G \\ b \end{array}
 & \xrightarrow{G(\alpha)} & \begin{array}{c} e'' \\ \downarrow G \\ b'' \end{array}
 \end{array}
 \end{array}
 \quad (6)$$

211 The second line encodes that  $\varepsilon(f)$  is an *op-Cartesian morphism*, so we will refer to it as an *op-Cartesian lifting* of  $f$ .

212 It is well known that the above definitions imply that the (op-)Cartesian lifting of  $f$  is essentially unique, i.e.,  
 213 unique up to unique isomorphism; see, for example, Proposition 1.1.4 of [35].

### 214 2.3. Multi-opfibrations

215 The definition of a *Grothendieck opfibration* may be generalized in the following form, whereby instead of requiring  
 216 the existence of *op-Cartesian lifts* neither existence nor essential uniqueness are required. This particular variant of  
 217 a fibrational structure postulates instead the existence of a (possibly empty) family of *multi-op-Cartesian lifts*, subject  
 218 to a somewhat more intricate universal property. As will be demonstrated in Section 4.3, this generalized notion is  
 219 the appropriate fibrational concept capable of formalizing so-called *multi-initial pushout complements*, which in turn  
 220 play a key role in categorical rewriting semantics.

221 **Definition 2.5.** A functor  $M : \mathbf{E} \rightarrow \mathbf{B}$  is a *multi-opfibration* if the following property holds:

$$\begin{array}{c}
 \forall \\
 \begin{array}{ccc}
 \begin{array}{c} e \\ \downarrow M \\ b \end{array}
 & \xrightarrow{f} & \begin{array}{c} b' \\ \downarrow M \end{array} \\
 \end{array}
 \end{array}
 : \exists \left\{ \begin{array}{ccc}
 \begin{array}{c} e \\ \downarrow M \\ b \end{array}
 & \xrightarrow{\mu_j(f)} & \begin{array}{c} e'_j \\ \downarrow M \\ b' \end{array} \\
 \end{array} \right\}_{j \in J_{f,e}}
 \end{array}
 :
 \end{array}
 \quad (6)$$
  

$$\begin{array}{c}
 \forall \\
 \begin{array}{ccc}
 \begin{array}{c} e \\ \downarrow M \\ b \end{array}
 & \xrightarrow{f} & \begin{array}{c} b' \\ \downarrow M \end{array} \\
 \end{array}
 \end{array}
 : \exists
 \begin{array}{ccc}
 \begin{array}{c} e \\ \downarrow M \\ b \end{array}
 & \xrightarrow{\mu_j(f)} & \begin{array}{c} e'_j \\ \downarrow M \\ b' \end{array} \\
 \begin{array}{c} e'_j \\ \downarrow M \\ b' \end{array}
 & \xrightarrow{f=M(\mu_j(f))} & \begin{array}{c} e'' \\ \downarrow M \\ b'' \end{array} \\
 \begin{array}{c} e \\ \downarrow M \\ b \end{array}
 & \xrightarrow{M(\alpha)} & \begin{array}{c} e'' \\ \downarrow M \\ b'' \end{array}
 \end{array}
 \end{array}
 \quad (6)$$
  

$$\begin{array}{c}
 \forall \\
 \begin{array}{ccc}
 \begin{array}{c} e \\ \downarrow M \\ b \end{array}
 & \xrightarrow{f} & \begin{array}{c} b' \\ \downarrow M \end{array} \\
 \end{array}
 \end{array}
 : \exists
 \begin{array}{ccc}
 \begin{array}{c} e \\ \downarrow M \\ b \end{array}
 & \xrightarrow{\mu_j(f)} & \begin{array}{c} e'_j \\ \downarrow M \\ b' \end{array} \\
 \begin{array}{c} e'_j \\ \downarrow M \\ b' \end{array}
 & \xrightarrow{\beta_j} & \begin{array}{c} e'' \\ \downarrow M \\ b'' \end{array} \\
 \begin{array}{c} e \\ \downarrow M \\ b \end{array}
 & \xrightarrow{M(\alpha)} & \begin{array}{c} e'' \\ \downarrow M \\ b'' \end{array}
 \end{array}
 \end{array}
 \quad (6)$$

222 In words:

- 223 (i) For every  $b - f \rightarrow b'$  in  $\mathbf{B}$  and  $e \in \mathbf{E}$  with  $M(e) = b$ , there exists a (possibly empty) family  $J_{f,e}$  of *multi-op-*  
 224 *Cartesian liftings*  $e - \mu_j(f) \rightarrow e'_j$  (with  $M(\mu_j(f)) = f$ ).
- 225 (ii) *Universal property of multi-opfibrations*: Multi-op-Cartesianity of the liftings entails that for all  $e - \alpha \rightarrow e''$  in  
 226  $\mathbf{E}$  and  $b' - g \rightarrow b''$  in  $\mathbf{B}$  with  $M(\alpha) = g \circ f$ , there exists a  $j \in J_{f,e}$  such that there exists a unique  $e'_j - \beta_j \rightarrow e''$   
 227 with  $\alpha = \beta_j \circ \mu_j(f)$  and  $M(\beta_j) = g$ .
- 228 (iii) *Essential uniqueness*: for a given *multi-op-Cartesian lifting* as in (ii), if there exists some  $e - \chi \rightarrow \tilde{e}$  and  
 229  $\tilde{e} - \tau \rightarrow e''$  such that  $\tau \circ \chi = \alpha$ ,  $M(\chi) = f$  and  $M(\tau) = g$ , then there exists a unique morphism  $e'_j - \phi \rightarrow \tilde{e}$  such  
 230 that  $\chi = \phi \circ \mu_j(f)$ ,  $\beta_j = \tau \circ \phi$ , and  $M(\phi) = id_{b'}$ .

231 We say that a *multi-opfibration* is *strong* if the morphisms  $\phi$  in (iii) above are *isomorphisms*.

232 It is useful to note that a *Grothendieck opfibration* is a special case of a *multi-opfibration*, namely when for every  
 233  $f \in \mathbf{B}$ , the family of *multi-op-Cartesian lifts* is non-empty, and such that all members of a given family are in the same  
 234 equivalence class under the universal property (i.e., for all  $j, k \in J_{e,f}$ , there exists an isomorphism  $e'_j - \phi_{jk} \rightarrow e'_k$  such  
 235 that  $M(\phi_{jk}) = id_{b'}$  and  $\mu_k(f) = \phi_{jk} \circ \mu_j(f)$ ). However, a *Grothendieck opfibration* is in general *not* a special case of  
 236 a *strong multi-opfibration*. This is relevant since *strong multi-opfibrations* enjoy two important technical properties  
 237 (*isomorphism lifting* and *pullback lifting*, see below) that are crucial for obtaining compositional rewriting theories:

238 **Lemma 2.6.** *Let  $M : \mathbf{E} \rightarrow \mathbf{B}$  be a strong multi-opfibration. Then the following lifting property of isomorphisms is*  
 239 *satisfied:*

$$\begin{array}{ccc}
 \begin{array}{ccc} e & \xrightarrow{\alpha} & e'' \\ \downarrow M & & \downarrow M \\ b & \xrightarrow{f} & b' \xrightarrow{g} b'' \\ \downarrow M & & \downarrow M \\ b & \xrightarrow{f} & b' \xrightarrow{g} b'' \\ \downarrow M & & \downarrow M \\ b & \xrightarrow{f} & b' \xrightarrow{g} b'' \end{array} & : \forall & \begin{array}{ccc} e & \xrightarrow{\alpha} & e'' \\ \downarrow M & & \downarrow M \\ b & \xrightarrow{f} & b' \xrightarrow{g} b'' \\ \downarrow M & & \downarrow M \\ b & \xrightarrow{f} & b' \xrightarrow{g} b'' \\ \downarrow M & & \downarrow M \\ b & \xrightarrow{f} & b' \xrightarrow{g} b'' \end{array}
 \end{array}
 \quad (7)$$

$(g \in \text{iso}(\mathbf{B}) \Rightarrow \beta_j \in \text{iso}(\mathbf{E})) \wedge (f \in \text{iso}(\mathbf{B}) \Rightarrow \varepsilon_j(f) \in \text{iso}(\mathbf{E}))$

240 We conclude the general discussion of multi-opfibrations with the following technical result which will be used in  
 241 the proof of the *associativity theorem* for compositional rewriting theories in Section 3.4:

242 **Lemma 2.7 (Pullback-lifting lemma for strong multi-opfibrations).** *Let  $\mathbf{E}$  be a category that has pullbacks, and*  
 243 *let  $M : \mathbf{E} \rightarrow \mathbf{B}$  be a strong multi-opfibration. Then the following property holds:*

$$\begin{array}{ccc}
 \begin{array}{ccc} e & \xrightarrow{\alpha} & e''' \\ \downarrow M & & \downarrow M \\ b & \xrightarrow{f} & b' \\ \downarrow M & & \downarrow M \\ b & \xrightarrow{f} & b' \end{array} & : \forall & \begin{array}{ccc} e & \xrightarrow{\alpha} & e''' \\ \downarrow M & & \downarrow M \\ b & \xrightarrow{f} & b' \\ \downarrow M & & \downarrow M \\ b & \xrightarrow{f} & b' \end{array}
 \end{array}
 \quad (8)$$

244 *More explicitly, for every diagram such as on the left of (8), whose bottom part contains a pullback square in  $\mathbf{B}$ , the*  
 245 *following properties hold:*

- 246 (i) *There exists an  $\mathbf{E}$ -morphism  $e - \varepsilon_j(f) \rightarrow e'_j$  such that there exists a unique  $\mathbf{E}$ -morphism  $e'_j - \beta_j \rightarrow e'''$  with*  
 247  *$M(\varepsilon_j(f)) = f$  and  $M(\beta_j) = h_1 \circ g_1 = h_2 \circ g_2$ , and such that the diagram commutes.*

248 (ii) There then exist  $\mathbf{E}$ -morphisms  $e'_j - \varepsilon_{j,k}(g_1) \rightarrow e''_{j,k}$  and  $e'_j - \varepsilon_{j,\ell}(g_2) \rightarrow e''_{j,\ell}$  such that there exist unique  $\mathbf{E}$ -  
 249 morphisms  $e''_{j,k} - \gamma_{j,k} \rightarrow e'''$  and  $e''_{j,\ell} - \delta_{j,\ell} \rightarrow e'''$  such that  $M(\varepsilon_{j,k}(g_1)) = g_1$ ,  $M(\varepsilon_{j,\ell}(g_2)) = g_2$ ,  $M(\gamma_{j,k}) = h_1$  and  
 250  $M(\delta_{j,\ell}) = h_2$ , and such that the diagram commutes.

251 (iii) Moreover, the square in  $\mathbf{E}$  into  $e'''$  is a pullback.

## 252 2.4. Residual multi-opfibrations

253 The following concept constitutes yet a further generalization of fibrational concepts – while **multi-opfibrations**  
 254 generalize **Grothendieck opfibrations** via replacing (essentially unique) **op-Cartesian lifts** with **multi-op-Cartesian lifts**,  
 255 one encounters in compositional rewriting theory situations where moreover morphisms may in general not possess  
 256 such liftings, but only certain *extensions* of morphisms, referred to as **residues** in the definition below. The reason for  
 257 introducing this concept will become evident only when considering the salient examples of fibrational properties of  
 258 **final pullback complement squares**, and of **sesqui-pushout direct derivations** in the later parts of this paper.

259 **Definition 2.8.** A functor  $R : \mathbf{E} \rightarrow \mathbf{B}$  is a **residual multi-opfibration** if the following property holds:

$$\begin{array}{c}
 \forall \begin{array}{c} e \\ \downarrow R \\ b \end{array} \xrightarrow{f} b' \\
 \exists \left\{ \begin{array}{c} e \xrightarrow{\rho_j(f)} e'_j \\ \downarrow R \quad \downarrow R \\ b \xrightarrow{f} b' \xrightarrow{f_{\star j}} b'_j \\ \quad \quad \quad \downarrow R(\rho_j(f)) \end{array} \right\}_{j \in J_{f,e}} :
 \end{array}
 \quad (9)$$
  

$$\begin{array}{c}
 \forall \begin{array}{c} e \xrightarrow{\alpha} e'' \\ \downarrow R \quad \downarrow R \\ b \xrightarrow{f} b' \xrightarrow{g} b'' \\ \quad \quad \quad \downarrow R(\alpha) \end{array} :
 \end{array}
 \quad \begin{array}{c}
 \begin{array}{c} e \xrightarrow{\alpha} e'' \\ \downarrow R \quad \downarrow R \\ b \xrightarrow{f} b' \xrightarrow{g} b'' \\ \quad \quad \quad \downarrow R(\alpha) \end{array} :
 \end{array}$$
  

$$\begin{array}{c}
 \forall \begin{array}{c} e \xrightarrow{\alpha} e'' \\ \downarrow R \quad \downarrow R \\ b \xrightarrow{f} b' \xrightarrow{g} b'' \\ \quad \quad \quad \downarrow R(\alpha) \end{array} :
 \end{array}
 \quad \begin{array}{c}
 \begin{array}{c} e \xrightarrow{\alpha} e'' \\ \downarrow R \quad \downarrow R \\ b \xrightarrow{f} b' \xrightarrow{g} b'' \\ \quad \quad \quad \downarrow R(\alpha) \end{array} :
 \end{array}$$

260 In words:

- 261 (i) For every  $b - f \rightarrow b'$  in  $\mathbf{B}$  and  $e \in \mathbf{E}$  with  $R(e) = b$ , there exists a (possibly empty) family  $J_{f,e}$  of **residual**  
 262 **multi-op-Cartesian liftings**  $e - \rho_j(f) \rightarrow e'_j$  (with  $R(\rho_j(f)) = f_{\star j} \circ f$ , and with  $f_{\star j}$  referred to as a **residue** with  
 263 respect to  $(e; f)$ ).
- 264 (ii) **Universal property of residual multi-opfibrations:** Residual multi-op-Cartesianity of the liftings entails that for  
 265 all  $e - \alpha \rightarrow e''$  in  $\mathbf{E}$  and  $b' - g \rightarrow b''$  in  $\mathbf{B}$  with  $R(\alpha) = g \circ f$ , there exists a  $j \in J_{f,e}$  such that there exists a unique  
 266  $e'_j - \beta_j \rightarrow e''$  with  $\alpha = \beta_j \circ \rho_j(f)$  and  $g = R(\beta_j) \circ f_{\star j}$ .
- 267 (iii) **Essential uniqueness:** For all  $b - h \rightarrow \tilde{b}$ ,  $e - \chi \rightarrow \tilde{e}$  and  $\tilde{e} - \tau \rightarrow e''$  such that  $R(\chi) = h \circ f$  and  $g = R(\tau) \circ h$ , there  
 268 exists a unique  $e'_j - \psi \rightarrow \tilde{e}$  such that  $\chi = \psi \circ \rho_j(f)$ ,  $\beta_j = \tau \circ \psi$  (which then implies moreover that  $h = R(\psi) \circ f_{\star j}$   
 269 and  $R(\beta_j) = R(\tau) \circ R(\psi)$ ).

270 We record the following technical result for **residual multi-opfibrations** which will be crucial later in the paper  
 271 when it plays a central role in the proof of the **associativity theorem** of Section 3.4:

272 **Lemma 2.9.** *Let  $R : \mathbf{E} \rightarrow \mathbf{B}$  be a residual multi-opfibration. Then residues have the following universal property:*

$$\begin{array}{c}
 \forall \\
 \begin{array}{ccc}
 e & \xrightarrow{\rho_j} & e'_j \\
 \downarrow R & & \downarrow R \\
 b & \xrightarrow{f} b' \xrightarrow{f_{\star j}} & b'_j \\
 \downarrow R(\rho_j(f)) & & \downarrow R(\rho_j(f))
 \end{array}
 \end{array}
 : \exists
 \begin{array}{ccc}
 e & \xrightarrow{\chi} \bar{e} & \xrightarrow{\tau} e'_j \\
 \downarrow R & \downarrow R & \downarrow R \\
 b & \xrightarrow{f} b' \xrightarrow{f_{\star j}} & b'_j \\
 \downarrow R(\rho_j(f)) & & \downarrow R(\rho_j(f))
 \end{array}
 \Rightarrow \tau \in \text{iso}(\mathbf{E}) \wedge R(\tau) \in \text{iso}(\mathbf{B}) \quad (10)$$

273 In particular, this property entails that if a residue  $f_{\star k}$  factorizes a residue  $f_{\star j}$  as  $f_{\star k} = R(\beta_k) \circ f_{\star j}$  for some  $\beta_k \in \mathbf{E}$ ,  
 274 then the residues  $f_{\star j}$  and  $f_{\star k}$  (both of the same morphism  $f \in \mathbf{B}$ ) are related by an isomorphism  $R(\beta_k) \in \text{iso}(\mathbf{B})$ , as  
 275 are their liftings  $\rho_j(f) = \beta_k \circ \rho_k(f)$  via  $\beta_k \in \text{iso}(\mathbf{E})$ .

276 Finally, we note the following specialization of **residual multi-opfibration**:

277 **Definition 2.10.** A functor  $R : \mathbf{E} \rightarrow \mathbf{B}$  is a **residual opfibration** if it is a **residual multi-opfibration** such that for all  
 278 objects  $e$  of  $\mathbf{E}$  and morphisms  $R(e) - f \rightarrow b$  of  $\mathbf{B}$  the family of **residual multi-op-Cartesian lifts** is non-empty, and  
 279 such that all lifts in the family are equivalent up to universal isomorphisms (i.e., for all  $(\rho_j(f), f_{\star j})$  and  $(\rho_k(f), f_{\star k})$ ,  
 280 there exists a unique isomorphism  $\varphi$  in  $\mathbf{E}$  such that  $\rho_k(f) = \varphi \circ \rho_j(f)$  and  $f_{\star k} = R(\varphi) \circ f_{\star j}$ ). We will sometimes refer  
 281 to such lifts as **residual op-Cartesian** for brevity.

### 282 3. Fundamentals of compositional rewriting theories

283 Taking the notion of **double categories** as a convenient “book-keeping” device, we will demonstrate in this key  
 284 section of the present paper that a very general class of compositional rewriting theories—including in particular the  
 285 “non-linear” variants of DPO- and SqPO-semantics [1]—may be elegantly expressed and studied from a fibrational  
 286 viewpoint. More precisely, based upon and motivated by the fibrational structures presented in Section 2, we introduce  
 287 the novel notion of **compositional rewriting double category (crDC)**. We then demonstrate that **crDCs** provide a  
 288 very high-level representation of categorical rewriting theories with compositionality properties in the sense that every  
 289 **crDC** admits a **concurrency theorem** (Section 3.3) and an **associativity theorem** (Section 3.4). The crucial point of our  
 290 novel approach to proving compositional properties via **crDCs** is that the aforementioned **concurrency theorems** and  
 291 **associativity theorems** may be established in an entirely *universal* form, i.e., entirely independently of the concrete  
 292 rewriting semantics underlying a given **crDC**.

#### 293 3.1. Double categories

294 Throughout this paper, we work exclusively with the “algebraic” order in compositions of morphisms and com-  
 295 mutative squares (i.e.,  $g \circ f$  rather than the “diagram” order notation  $f; g$  common in category theory). For reasons of  
 296 convenience, we will swap the roles of the classes of morphisms that have a weakly associative composition, usually  
 297 the *vertical* morphisms [38], to be the class of *horizontal* morphisms. (We opted for this particular convention so that it  
 298 is essentially a 90 degrees clockwise rotation of the standard mathematical one.) Finally, since we will be exclusively  
 299 interested in *finitary* categories, we will often not mention finitariness explicitly in what follows.

300 **Definition 3.1 (Cf. e.g. [38, 39, 40]).** A **double category (DC)**  $\mathbb{D}$  is a weakly internal category in the 2-category  $\mathcal{CAT}$   
 301 of all categories [41]<sup>4</sup>.

<sup>4</sup>Some authors prefer the term “pseudo double category”, cf. also nLab article on double categories.

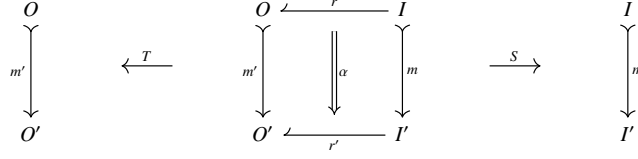


Figure 1: Convention for source and target functors for double categories.

302 In particular, this entails that a double category consists of a category  $\mathbb{D}_0$  of *objects* and *vertical morphisms*, and  
 303 a category  $\mathbb{D}_1$  of *horizontal morphisms* and *squares* of  $\mathbb{D}$ , equipped with functors  $S, T : \mathbb{D}_1 \rightarrow \mathbb{D}_0$ , referred to as  
 304 *source* and *target* functors, respectively (cf. Figure 1), and with a functor  $U : \mathbb{D}_0 \rightarrow \mathbb{D}_1$  which maps every object  
 305  $A$  of  $\mathbb{D}_0$  to a *horizontal unit*  $U_A$  (depicted in Figure 2(d) as identity horizontal morphisms), and every morphism  $f$   
 306 of  $\mathbb{D}_0$  to a *horizontal unit square*  $U_f$  (depicted in Figure 2(d) as *squares* annotated with the symbol  $id_{\dots}$  for better  
 307 readability). We denote vertical morphisms by  $\rightarrow$  and horizontal morphisms by  $\leftarrow$ , respectively. We denote by  $\diamond_v$   
 308 the *vertical composition* of squares as in Figure 2(a) (i.e., the associative composition operation of  $\mathbb{D}_1$ ).  $\mathbb{D}$  moreover  
 309 carries a weakly associative *horizontal composition* of squares (cf. Figure 2(b))  $\diamond_h : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \rightarrow \mathbb{D}_1$ . Finally, for  
 310 technical convenience, we assume without loss of generality<sup>5</sup> that both types of compositions are strictly unitary (cf.  
 311 Figures 2(c) and 2(d)).

### 3.2. Compositional rewriting double categories

312 **Definition 3.2.** A double category (DC)  $\mathbb{D}$  is a *compositional rewriting DC (crDC)* if it has the following properties:

- 313 (i)  $\mathbb{D}_0$  has multi-sums.
- 314 (ii)  $\mathbb{D}_0$  and  $\mathbb{D}_1$  have pullbacks.
- 315 (iii) The horizontal composition functor  $\diamond_h : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \rightarrow \mathbb{D}_1$  is an *isoglobular residual opfibration*, namely a  
 316 *residual opfibration* such that<sup>6</sup> all residues are *globular isomorphisms* (i.e., isomorphisms  $\varphi$  of  $\mathbb{D}_1$  such that  
 317  $T(\varphi)$  and  $S(\varphi)$  are identity morphisms).
- 318 (iv) The source functor  $S : \mathbb{D}_1 \rightarrow \mathbb{D}_0$  is a *strong multi-opfibration*.
- 319 (v) The target functor  $T : \mathbb{D}_1 \rightarrow \mathbb{D}_0$  is a *residual multi-opfibration*.

320 **Remark 3.3.** It is worthwhile unpacking the fibrational property of the horizontal composition functor in a crDC into  
 321 a more explicit form in view of later applications in the proof of the *associativity theorem* (cf. Section (3.4)):

- 322 • Squares in a crDC  $\mathbb{D}$  have the following *horizontal decomposition property*:

323

$$\begin{array}{c} \Delta \\ \downarrow n \\ \blacktriangle \end{array} \begin{array}{c} \xleftarrow{r_2} \\ \xleftarrow{\alpha_{21}} \\ \xleftarrow{r_1} \end{array} \begin{array}{c} \circ \\ \downarrow \alpha_{21} \\ \circ \end{array} \begin{array}{c} \xrightarrow{r_1} \\ \xrightarrow{\alpha_{21}} \\ \xrightarrow{r_2} \end{array} \begin{array}{c} \square \\ \downarrow m \\ \blacksquare \end{array} : \exists \begin{array}{c} \Delta \\ \downarrow n \\ \blacktriangle \end{array} \begin{array}{c} \xleftarrow{r_2} \\ \xleftarrow{\alpha_2} \\ \xleftarrow{r'_2} \end{array} \begin{array}{c} \circ \\ \downarrow s \cdot m' \\ \circ \end{array} \begin{array}{c} \xrightarrow{r_1} \\ \xrightarrow{\alpha_1} \\ \xrightarrow{r'_1} \end{array} \begin{array}{c} \square \\ \downarrow m \\ \blacksquare \end{array} : \alpha_2 \diamond_h \alpha_1 = \sigma \diamond_v \alpha_{21} \quad (11)$$

324 In particular, utilizing the notation  $\cong_{\bullet}$  for equality up to *globular isomorphism*, one has  $s \cong_{\bullet} r'_2 \diamond_h r'_1$ .

<sup>5</sup>We follow here the viewpoint of [42], whereby utilizing the strictification theorem of pseudo double categories [38, Thm. 7.5], this amounts to implicitly utilizing a pseudo-functor into an equivalent double category where unitarity is strict, thus not reducing generality of our constructions.

<sup>6</sup>The definition in fact amounts to a special case of a so-called Street opfibration; this aspect and further variations of fibrational structures in rewriting theory are studied in [43].

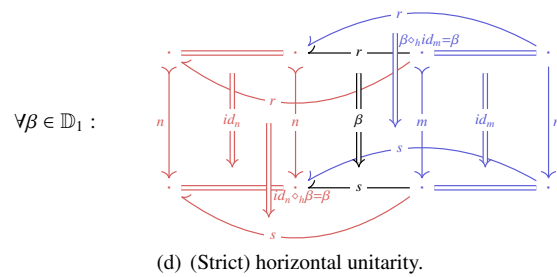
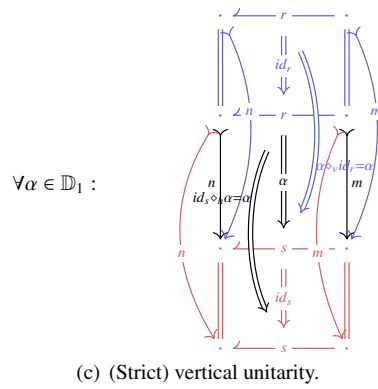
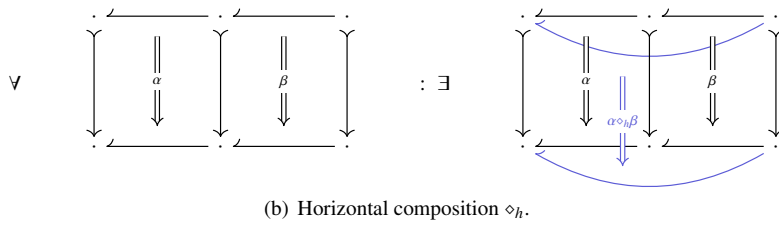
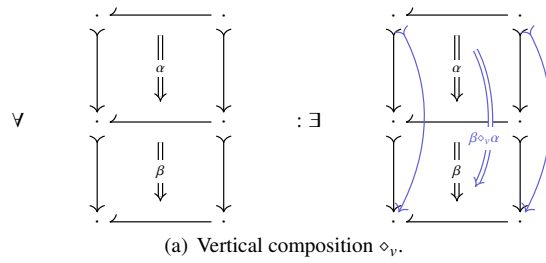


Figure 2: On the definition of **double categories**.

- 325 • Unpacking the definition of the **universal property of residual multi-opfibrations** for the case at hand, we find  
 326 the following **complex decomposition property** (where in the diagram below  $\gamma_2 \diamond_h \gamma_1 = \beta_{21} \diamond_v \alpha_{21}$ ):

$$\alpha_2 \diamond_h \alpha_1 = \sigma \diamond_v \alpha_{21} \quad \wedge \quad \beta_2 \diamond_h \beta_1 = \beta_{21} \diamond_v \sigma^{-1} \quad (12)$$

### 327 3.3. Concurrency theorem

328 Let us finally put the fibrational structures, introduced in Section 2, and the above concept of **compositional**  
 329 **rewriting double category** to use by proving a first theorem—the **concurrency theorem**—that plays a key role in the  
 330 static analysis of rewriting systems.

331 **Theorem 3.4.** *Let  $\mathbb{D}$  be a **compositional rewriting double category**. Then the following statements hold:*

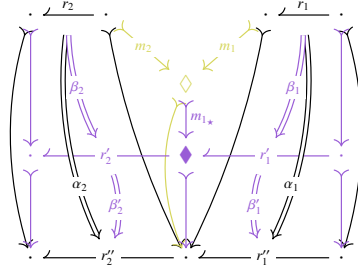
$$(13)$$

- 332 • **Synthesis:** For every pair of **squares**  $(\alpha_1, \alpha_2)$  of  $\mathbb{D}$  “adjacent at the foot” (left diagram above), there exist  
 333 a **multi-sum** element  $(m_2, m_1) \in \sum_{\mathcal{M}}(S(r_2), T(r_1))$ , a **residue**  $m_{1*}$  for  $(r_1; m_1)$ , and **squares**  $(\beta_1, \beta_2, \beta'_{21})$  (with  
 334  $T(\beta_1) = m_{1*} \circ m_1$  and  $S(\beta_2) = m_{1*} \circ m_2$ ), uniquely determined up to universal isomorphisms, such that  
 335  $r''_{21} \cong_{\bullet} r'_2 \diamond_h h'_1$ .
- 336 • **Analysis:** For every **multi-sum** element  $(m_2, m_1) \in \sum_{\mathcal{M}}(S(r_2), T(r_1))$ , **residue**  $m_{1*}$  for  $(r_1; m_1)$ , and **squares**  
 337  $(\beta_1, \beta_2, \beta'_{21})$  (with  $T(\beta_1) = m_{1*} \circ m_1$  and  $S(\beta_2) = m_{1*} \circ m_2$ ), there exist **squares**  $(\alpha_1, \alpha_2)$  of  $\mathbb{D}$ , determined  
 338 uniquely up to universal isomorphisms, such that  $r''_2 \diamond_h r'_1 \cong_{\bullet} r'_{21}$ , where  $r'_2 = \text{codom}(\alpha_2)$ ,  $r'_1 = \text{codom}(\alpha_1)$ , and  
 339  $r'_{21} = \text{codom}(\beta'_{21})$ .

340 Consequently, modulo a suitable notion of isomorphisms (induced by essential uniqueness of the respective construc-  
 341 tions), the resulting sets of equivalence classes are isomorphic.

342 **PROOF. Synthesis part:** Construct the diagram in (14) from the premise as follows:

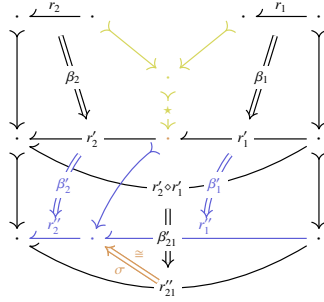
- 343 • Via the **universal property of multi-sums**, there exists a cospan of  $\mathbb{D}_0$ -morphisms  $(m_2, m_1)$  into an object  $\diamond$  and  
 344 a mediating  $\mathbb{D}_0$ -morphism  $\diamond \rightarrow \cdot$ .
- 345 • Since the target functor  $T : \mathbb{D}_1 \rightarrow \mathbb{D}_0$  is a **residual multi-opfibration**, there exists a residue  $m_{1*} : \diamond \rightarrow \blacklozenge$  with  
 346 respect to  $(r_1; m_1)$  and a  $\mathbb{D}_0$ -morphism  $\blacklozenge \rightarrow \cdot$  such that  $T(\beta_1) = m_{1*} \circ m_1$  and  $\alpha_1 = \beta'_1 \diamond_v \beta_1$ .
- 347 • Since the source functor  $S : \mathbb{D}_1 \rightarrow \mathbb{D}_0$  is a **multi-opfibration**, there exist squares  $\beta_2$  and  $\beta'_2$  such that  $S(\beta_1) =$   
 348  $m_{1*} \circ m_2$  and  $\alpha_2 = \beta'_2 \diamond_v \beta_2$ . Thus the claim follows by letting  $\beta'_{21} := \beta'_2 \diamond_h \beta'_1$ .



(14)

349 **Analysis part:** Construct the diagram in (15) as follows:

- 350 • By the [horizontal decomposition property](#) of squares in  $\mathbb{D}$ , there exist squares  $\beta'_2$  and  $\beta'_1$  such that  $\sigma \circ \beta'_{21} =$   
 351  $\beta'_2 \diamond_h \beta'_1$  (for some [globular isomorphism](#)  $\sigma$ ).
- 352 • The claim follows by letting  $\alpha_i := \beta'_i \diamond_v \beta_i$  for  $i = 1, 2$ , since  $r''_2 \diamond_h r''_1 \cong \bullet r''_{21}$ .



(15)

353 □

354 While an interesting mathematical structure on double categories in its own right, the deeper meaning and im-  
 355 portance of the [concurrency theorem](#) in formulating compositional rewriting semantics becomes apparent when in-  
 356 terpreting squares in a [compositional rewriting double category](#) as rewriting steps (traditionally referred to as *direct*  
 357 *derivations* in rewriting theory). To this end, consider a two-step rewriting sequence, where the result of the first step  
 358 is the starting object of the second step, as depicted in the form of the squares  $\alpha_2$  and  $\alpha_1$  in (13). The [concurrency](#)  
 359 [theorem](#) then implies that there exists a one-step rewrite, depicted as the square  $\beta_{21}$  in (13), from the start object to the  
 360 end object of the two-step sequence, and uniquely determined up to universal isomorphisms. Moreover, the particular  
 361 property of the one-step rewrite operation is that it is taken along some [composite rule](#) (here:  $r'_2 \diamond_h r'_1$ ), which—again  
 362 up to universal isomorphisms—is uniquely determined from the data of the two-step rewrite sequence. One may thus  
 363 interpret the top half of the right diagram in (13) (i.e., the squares  $\beta_2$  and  $\beta_1$ , the  $\mathbb{D}_1$ -object  $r'_2 \diamond_h r'_1$  as well as the [multi-](#)  
 364 [sum](#) and [residue](#)) as defining a notion of [rule composition](#). Indeed, as we shall illustrate in Section 6.4, the abstract  
 365 [crDC](#)-based formulation of the [concurrency theorem](#) instantiates precisely to the traditional concepts of concurrency  
 366 and rule compositions when considering [linear Double-Pushout semantics](#) [3], but also provides an abstraction of  
 367 compositional rewriting for more general [semi-linear](#) and [generic Double-Pushout](#) as well as [Sesqui-Pushout seman-](#)  
 368 [tics](#), as first introduced in [1].

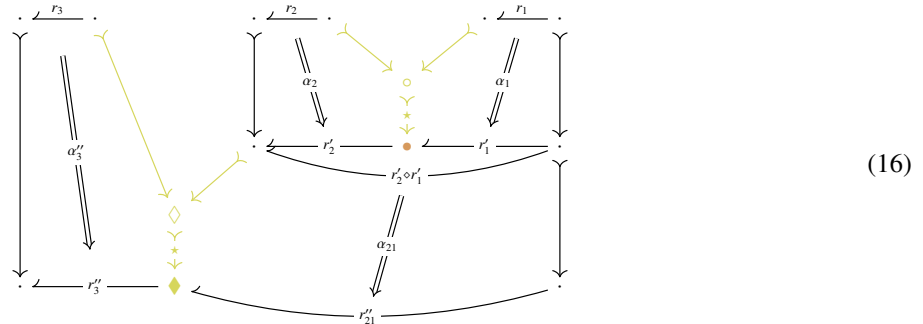
369 Let us finally note that in comparison to the instantiations of compositional rewriting theories to concrete choices  
 370 of semantics, our abstract [crDC](#)-based approach as presented here allows an efficient modularization of the proof of the  
 371 [concurrency theorem](#) by clearly separating the concrete definitions of compositional rewriting theories (i.e., proving  
 372 that a certain semantics and choice of base category gives rise to a [crDC](#)) from the universal structures provided by  
 373 [crDCs](#).  
 374



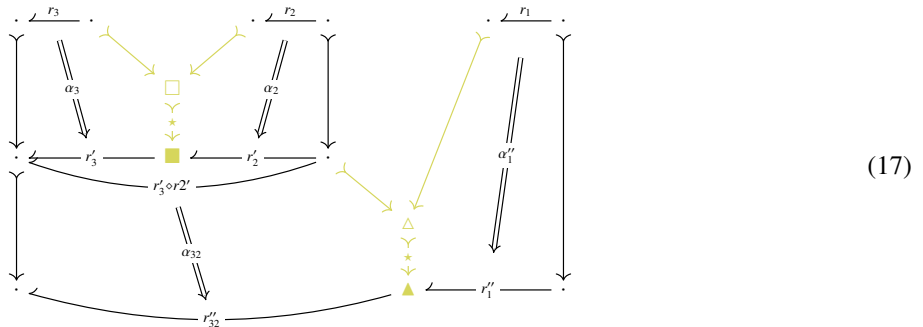
375 *3.4. Associativity theorem*

376 Unlike for the case of the [concurrency theorem](#), the statement and proof of which were a straightforward and  
 377 efficient application of the multi-sum and fibrational concepts, our second main theorem has indeed such a complex  
 378 statement that its proof relies much more non-trivially upon fibrational structures—which serve in a certain sense as a  
 379 form of “proof macros”—without which the proof would be extremely long and presumably difficult to follow.

380 **Theorem 3.5.** *Let  $\mathbb{D}$  be a [compositional rewriting double category](#). Then every diagram as in (16) below (interpreted  
 381 as encoding a [composition](#) of rules  $r_2$  and  $r_1$ , and of the [composite](#) with  $r_3$ ),*



382 *determines uniquely up to universal isomorphisms a diagram as in (17) below (interpreted as encoding a [composition](#)*  
 383 *of rules  $r_3$  with  $r_2$ , and of the [composite](#) with  $r_1$ ), and vice versa:*



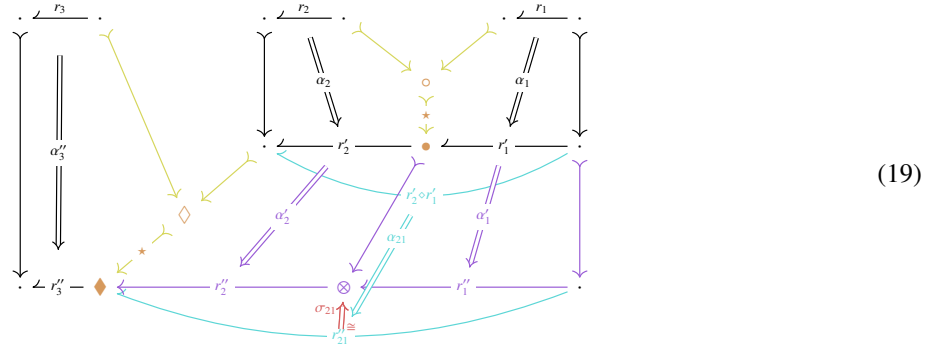
384 *Moreover, the equivalence is such that in addition*

$$r''_3 \diamond_h r''_{21} \cong r''_{32} \diamond_h r''_1. \quad (18)$$

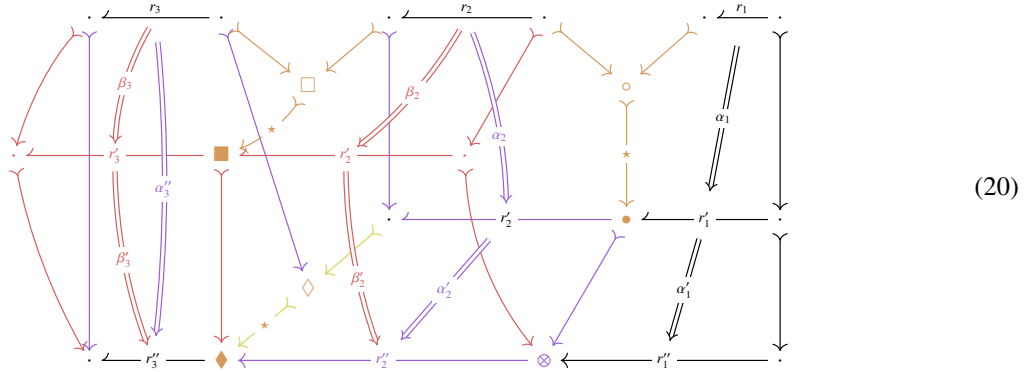
385 *Thus for a suitable notion of equivalence up to isomorphisms (induced by the essential uniqueness of the respective*  
 386 *constructions), there exists an isomorphism between the sets of equivalence classes of nested [composites](#) of the three*  
 387 *rules in the two different nesting orders. This amounts to a notion of associativity for the [rule composition](#) operation.*

388 **PROOF.** For the  $\Rightarrow$  direction of the equivalence, construct the following diagram from the premise by applying the  
 389 [horizontal decomposition property](#) to the square  $\alpha_{21}$ , obtaining a globular isomorphism  $\sigma_{21}$  in  $\mathbb{D}_1$  and squares  $\alpha'_2$  and

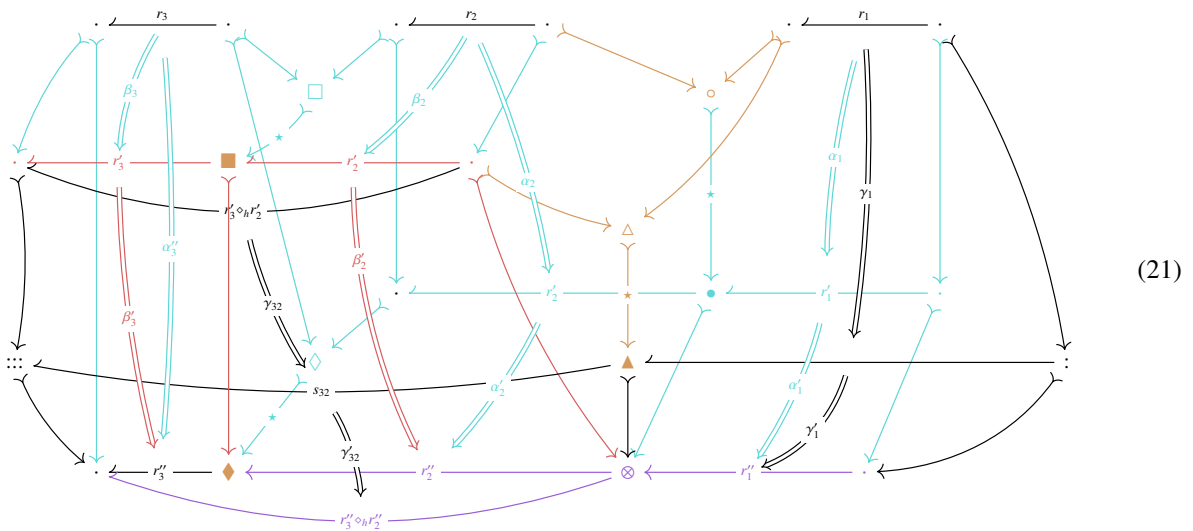
390  $\alpha'_1$  such that  $\sigma_{21} \diamond_v \alpha_{21} = \alpha_2 \diamond_h \alpha_1$  (and with  $r''_2 \diamond_h r''_1 \cong r''_{21}$ ):



391 Next, we apply a part of the **synthesis** construction of the **concurrency theorem**, in that we synthesize from the  
 392 squares  $\alpha'_3$  and  $\alpha'_2 \diamond_v \alpha_2$  a **composite** of rules  $r_3$  and  $r_2$  (as encoded via the squares  $\beta_3$  and  $\beta_2$ , with the **composite rule**  
 393 itself omitted for clarity) and squares  $\beta'_3$  and  $\beta'_2$  such that  $\alpha'_3 = \beta'_3 \diamond_v \beta_3$  and  $\alpha'_2 \diamond_v \alpha_2 = \beta'_2 \diamond_v \beta_2$ :

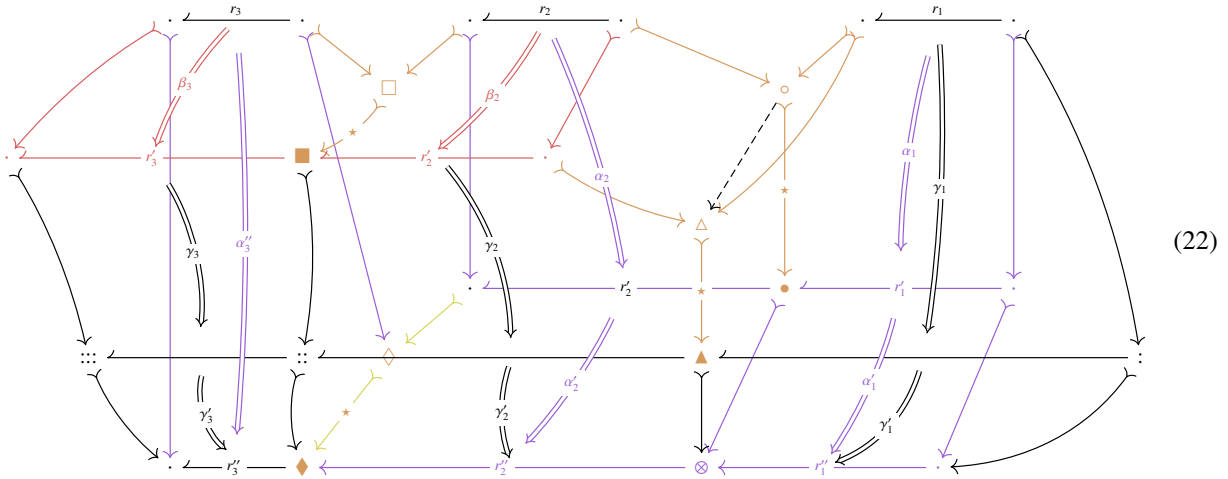


394 By applying the **synthesis** construction of the **concurrency theorem** to the pair of squares  $\beta'_3 \diamond_h \beta'_2$  and  $\alpha'_1 \diamond_v \alpha_1$ , we  
 395 may obtain the diagram in (21) below (where  $\gamma'_{32} \diamond_v \gamma_{32} = \beta'_3 \diamond_h \beta'_2$ , and  $\gamma'_1 \diamond_v \gamma_1 = \alpha'_1 \diamond_v \alpha_1$ ):



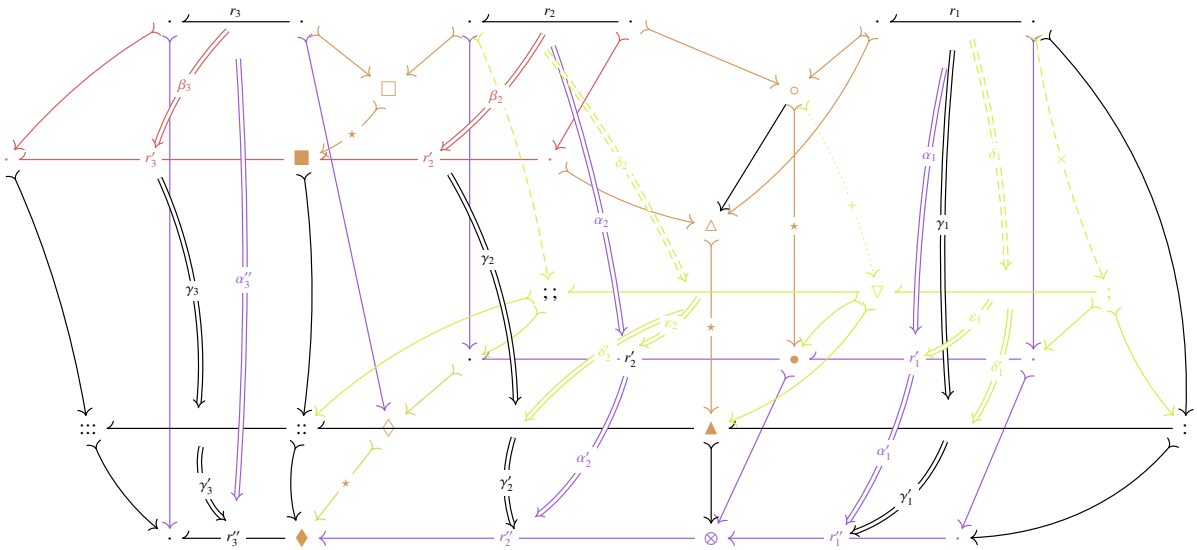
396 We now apply the following sequence of manipulations to obtain the diagram in (22):

- 397 • Via the **multi-sum extension Lemma**, there exists a  $\mathbb{D}_0$ -morphism  $\circ \rightarrow \Delta$  between the multi-sum objects  $\circ$  and  $\Delta$   
398 (and analogously between the multi-sum objects  $\square$  and  $\diamond$ , albeit this is irrelevant for the proof and thus omitted  
399 from the diagrams).
- 400 • Via the **complex decomposition property**, there exist squares  $\gamma_3, \gamma'_3, \gamma'_2$  and  $\gamma_2$  and a **globular isomorphism**  $\tau_{32}$   
401 such that  $\gamma_3 \diamond_h \gamma_2 = \tau_{32} \diamond_v \gamma_{32}$  and  $\gamma'_3 \diamond_h \gamma'_2 = \gamma'_{32} \diamond_v \tau_{32}^{-1}$ .



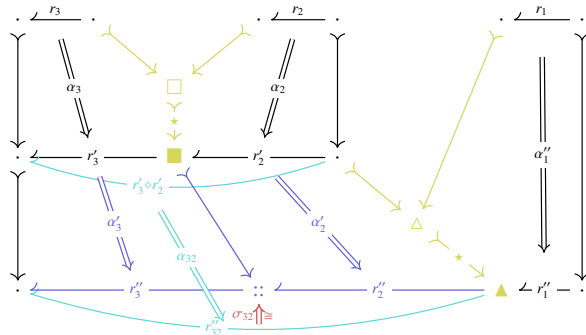
402 For the final step of the  $\Rightarrow$  part of the proof, we construct the diagram in (23) below via the following steps:

- 403 • Take a pullback (admissible since  $\mathbb{D}_0$  has pullbacks) in order to obtain the object marked  $\nabla$  on the back rightmost  
404 part of the diagram in (23), yielding a number of morphisms as indicated (all of which are in  $\mathbb{D}_0$ , again since  $\mathbb{D}_0$   
405 has pullbacks).
- 406 • Since the source functor is a strong multi-opfibration, by applying the **pullback lifting lemma for strong multi-**  
407 **opfibrations** we obtain squares  $\delta_1, \delta'_1$  and  $\varepsilon_1$  such that  $\alpha_1 = \varepsilon_1 \diamond_v \delta_1$  and  $\gamma_1 = \delta'_1 \diamond_v \delta_1$ . The lemma also implies that  
408 since the square from the object marked  $\nabla$  was by construction a pullback, so is the square from  $\nabla$ , which by the  
409 **universal property of pullbacks** yields the existence of a morphism into  $\nabla$  (marked  $+$ ), which is a  $\mathbb{D}_0$ -morphism  
410 since  $\mathbb{D}_0$  has pullbacks.
- 411 • Applying the **pullback lifting lemma for strong multi-opfibrations** once again, we may obtain the configuration  
412 in the middle of the diagram in (23), i.e., squares  $\delta_2, \delta'_2$  and  $\varepsilon_2$  such that  $\alpha_2 = \varepsilon_2 \diamond_v \delta_2$  and  $\gamma_2 \diamond_v \beta_2 = \delta'_2 \diamond_v \delta_2$ . The  
413 lemma also entails that since the commutative square from  $\nabla$  is a pullback, the square from the object marked  
414  $\nabla$  is a pullback, too, and there exists the  $\mathbb{D}_0$ -morphism  $\text{codom}(r_2) \rightarrow \nabla$ .
- 415 • By the **universal property of residues**, since  $\alpha_1 = \varepsilon_1 \diamond_v \delta_1$ , and since the residue  $\circ \rightarrow \bullet$  marked  $\star$  (which  
416 forms the second factor of  $T(\alpha_1)$ ) factors through  $\nabla \rightarrow \bullet$  (i.e., through  $T(\varepsilon_1)$ ), we find that the square  $\varepsilon_1$  is an  
417 isomorphism in  $\mathbb{D}_1$ . By the **lifting property of isomorphisms for strong multi-opfibrations**, the square  $\varepsilon_2$  is then  
418 an isomorphism in  $\mathbb{D}_1$ , too.
- 419 • The latter point entails that we may form the cospan  $\cdot \rightarrow S(\gamma_3 \diamond_v \beta_3) \rightarrow \leftarrow T(\delta'_2 \diamond_v \varepsilon_2^{-1}) \rightarrow \cdot$  of morphisms in  $\mathbb{D}_0$ ;  
420 hence by the **universal property of multi-sums**, there exists a  $\mathbb{D}_0$ -morphism  $\diamond \rightarrow \cdot$ .
- 421 • The existence of the morphism  $\diamond \rightarrow \cdot$  together with  $\alpha'_2 \diamond_h \alpha'_1 = (\gamma'_2 \diamond_h \gamma'_1) \diamond_v (\delta'_2 \diamond_h \delta'_1) \diamond_v (\varepsilon_2^{-1} \diamond_h \varepsilon_1^{-1})$  implies  
422 via the **universal property of residues** that  $\gamma'_2 \diamond_h \gamma'_1$  is an isomorphism in  $\mathbb{D}_1$ .
- 423 • Since  $\gamma'_2 \diamond_h \gamma'_1$  is an isomorphism in  $\mathbb{D}_1$ ,  $S(\gamma'_2 \diamond_h \gamma'_1) = S(\gamma'_1)$  is an isomorphism in  $\mathbb{D}_0$ ; thus applying the **lifting**  
424 **property of isomorphisms for strong multi-opfibrations** repeatedly, we find that the squares  $\gamma'_1, \gamma'_2$  and  $\gamma'_3$  are all  
425 isomorphisms in  $\mathbb{D}_1$ , which concludes the proof of the  $\Rightarrow$  part of the theorem.



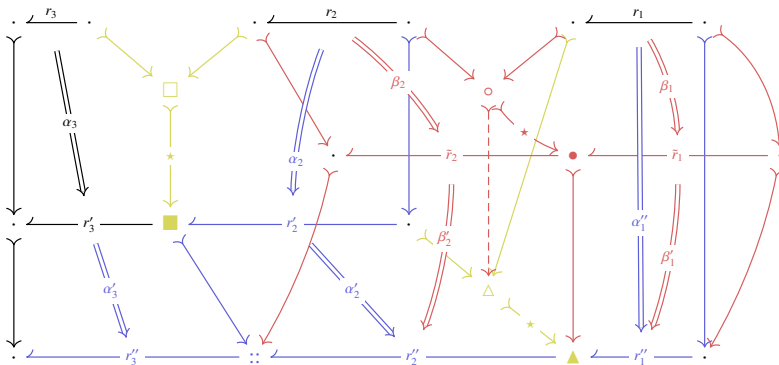
(23)

426 For the  $\Leftarrow$  part of the claim, via the **horizontal decomposition property** (here for the square  $\alpha_{32}$ ), we obtain a  
 427 globular isomorphism  $\sigma_{32}$  and squares  $\alpha'_3$  and  $\alpha'_2$  such that  $\sigma_{32} \diamond_v \alpha_{32} = \alpha'_3 \diamond_h \alpha'_2$  (and with  $r'_{32} \cong r'_3 \diamond_h r'_2$ ):



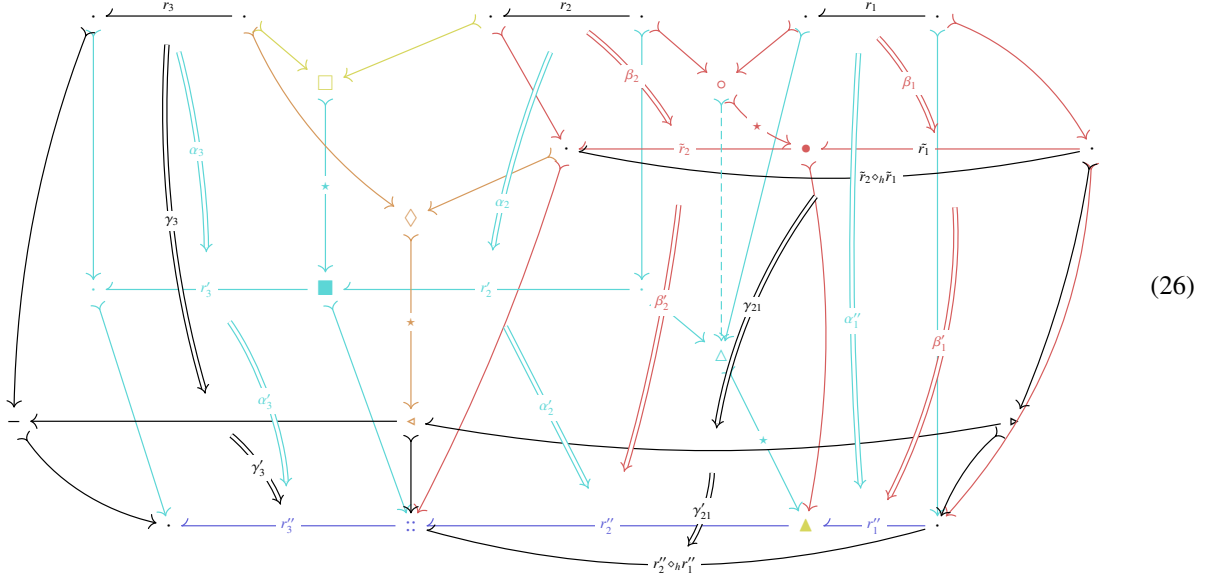
(24)

428 To proceed, we may now apply the **synthesis** part of the **concurrency theorem** to the sequence formed by the  
 429 composite square  $\alpha'_2 \diamond_v \alpha_2$  and the square  $\alpha'_1$ , again not explicitly carrying out the horizontal composition of squares  
 430 in the last step of the construction. We thus arrive at a diagram as in (25) below, with the squares  $\beta_1$ ,  $\beta'_1$ ,  $\beta_2$  and  
 431  $\beta'_2$  arising from the aforementioned construction (where the existence of the  $\mathbb{D}_0$ -morphism  $\circ \rightarrow \Delta$  follows from the  
 432 **multi-sum extension Lemma**):



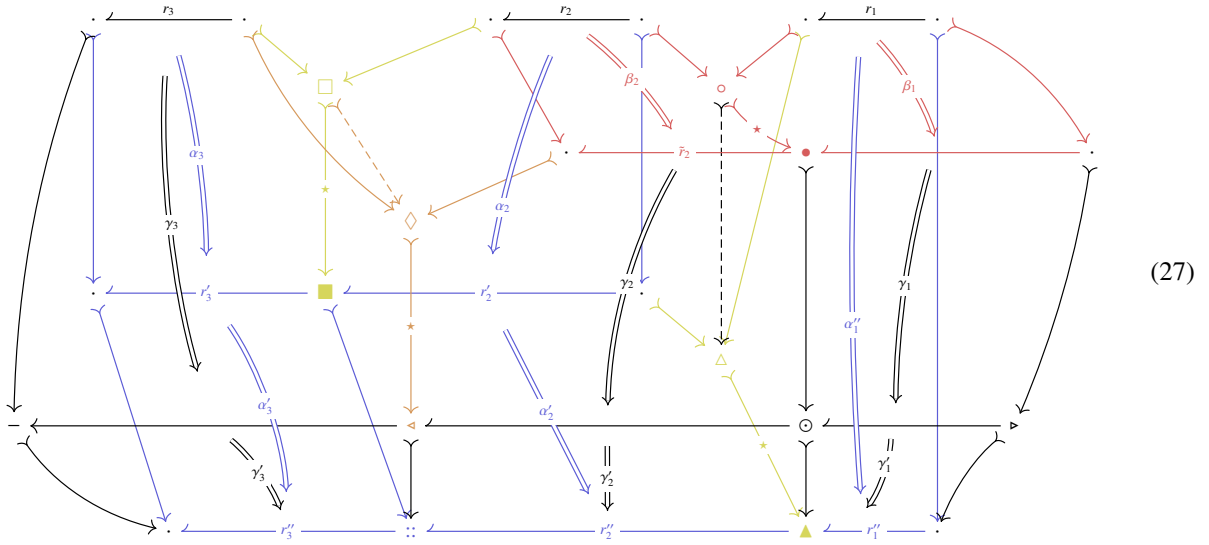
(25)

433 By applying the **synthesis** construction of the **concurrency theorem** to the pair of squares  $\alpha'_3 \diamond_v \alpha_3$  and  $\beta'_2 \diamond_h \beta'_1$ , we  
 434 may obtain the diagram in (26) below (where  $\gamma'_3 \diamond_v \gamma_3 = \alpha'_3 \diamond_v \alpha_3$ , and  $\gamma'_{21} \diamond_v \gamma_{21} = \beta'_2 \diamond_h \beta'_1$ ):



435 Next, we apply the following sequence of manipulations to obtain the diagram in (27):

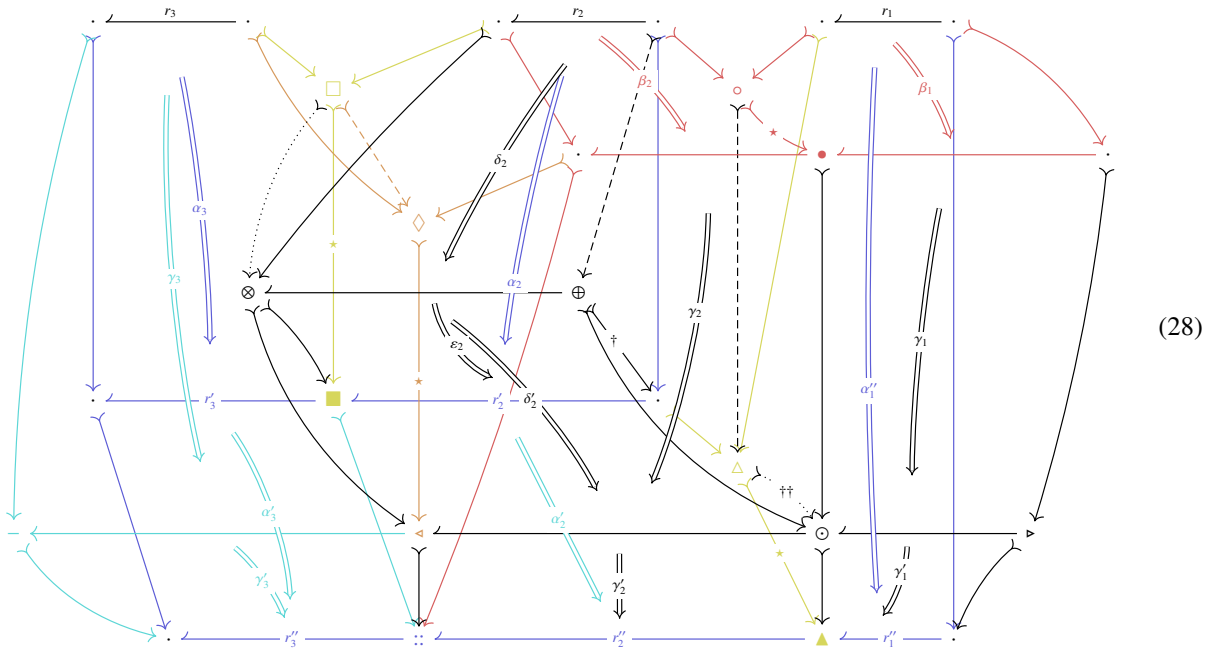
- 436 • Via the **multi-sum extension Lemma**, there exists a  $\mathbb{D}_0$ -morphism  $\square \mapsto \diamond$  between the multi-sum objects  $\square$  and  
 437  $\diamond$ .
- 438 • Via the **complex decomposition property**, there exist squares  $\gamma_2, \gamma'_2, \gamma'_1$  and  $\gamma_1$  and a **globular isomorphism**  $\tau_{21}$   
 439 such that  $\gamma_2 \diamond_h \gamma_1 = \tau_{21} \diamond_v \gamma_{21}$  and  $\gamma'_2 \diamond_h \gamma'_1 = \gamma'_{21} \diamond_v \tau_{21}^{-1}$ .



440 In order to complete the proof, we proceed as follows (yielding the diagram in (28)):

- 441 • Take a pullback to obtain the span  $\cdot \leftarrow \oplus \rightarrow \odot$ . Since  $\mathbb{D}_0$  has pullbacks, the span consists of two  $\mathbb{D}_0$ -morphisms.  
 442 By the **universal property of pullbacks**, there exists a morphism  $\cdot \rightarrow \oplus$  (marked as a dashed arrow in (28)), which  
 443 again since  $\mathbb{D}_0$  has pullbacks is also a  $\mathbb{D}_0$ -morphism.

- 444 • Applying the [pullback lifting lemma for strong multi-opfibrations](#) to the pullback square over the object  $\oplus$ , we  
445 obtain squares  $\delta_2, \delta'_2$  and  $\varepsilon_2$  such that  $\alpha_2 = \varepsilon_2 \diamond_v \delta_2$  and  $\gamma_2 \diamond_v \beta_2 = \delta'_2 \diamond_v \delta_2$ . The lemma also entails that the  
446 square under the object marked  $\otimes$  is a pullback. By the [universal property of pullbacks](#), this entails the existence  
447 of a morphism  $\square \rightarrow \otimes$ , which is a  $\mathbb{D}_0$ -morphism since  $\mathbb{D}_0$  has pullbacks.
- 448 • By the [universal property of residues](#), since  $\alpha_2 = \varepsilon_2 \diamond_v \delta_2$ , and since the residue  $\square \rightarrow \blacksquare$  marked  $\star$  (which  
449 forms the second factor of of  $T(\alpha_2)$ ) factors through  $\otimes \rightarrow \blacksquare$  (i.e., through  $T(\varepsilon_2)$ ), we find that the square  $\varepsilon_2$  is  
450 an isomorphism in  $\mathbb{D}_1$ . Thus in particular the morphism  $\oplus \rightarrow \cdot$  marked  $\dagger$  in (28), i.e.,  $S(\varepsilon_2)$ , is an isomorphism.
- 451 • The latter fact entails that by applying the [universal property of multi-sums](#) to the cospan  $dom(r'_2) \rightarrow \oplus \rightarrow$   
452  $\circ \leftarrow codom(r_1)$ , there exists a  $\mathbb{D}_0$ -morphism  $\Delta \rightarrow \circ$  (marked  $\dagger\dagger$  in (28)).
- 453 • By the [universal property of residues](#), since  $\alpha'_1 = \gamma'_1 \diamond_v (\gamma_1 \diamond_v \beta_1)$ , and since the residue  $\Delta \rightarrow \blacktriangle$  marked  $\star$  (which  
454 forms the second factor of of  $T(\alpha_1)$ ) factors through  $\circ \rightarrow \blacktriangle$  (i.e., through  $T(\gamma'_1)$ ), we find that the square  $\gamma'_1$   
455 is an isomorphism in  $\mathbb{D}_1$ , and thus in particular  $T(\gamma'_1)$  is an isomorphism in  $\mathbb{D}_0$ ; thus by repeated application  
456 of the [lifting property of isomorphisms for strong multi-opfibrations](#), the squares  $\gamma'_2$  and  $\gamma'_3$  are found to be  
457 isomorphisms in  $\mathbb{D}_1$ . This concludes the proof of the  $\Leftarrow$  part of the theorem.



458 □

459 This proof of the [associativity theorem](#) in a crDC provides a strong indication that modularizing the categorical  
460 structures in this form renders vastly complex mathematical developments feasible and, at the same time, provides  
461 some deep structural insights. Most importantly, our characterization of a given categorical rewriting semantics to  
462 qualify as being *compositional* is based exclusively on verifying properties of just the [squares](#) of a [double category](#)  
463 (which model *direct derivations*) and on the existence of [multi-sums](#), i.e., only on the very definition of the rewriting  
464 semantics being formalized as a [compositional rewriting double category](#).

#### 465 4. Examples of fibrational structures relevant for rewriting theory

466 In this section, we demonstrate that a number of constructions of commutative squares that form the building  
467 blocks of standard categorical rewriting semantics in fact carry fibrational structures; this will eventually allow us to

468 instantiate our general compositional rewriting theory to these standard semantics. After a quick review of the notion  
 469 of stable system of monics, we define various categories of pullback, pushout or final pullback complement squares  
 470 where composition is defined by either horizontal or vertical pasting. In the remainder of the section, we then analyze  
 471 the fibrational structures on the four natural boundary functors (domain, codomain, target and source) from these  
 472 categories of squares.

#### 473 4.1. Categories of squares

474 A system of monics  $\mathcal{M}$  in a category  $\mathbf{C}$  is a collection of monomorphisms that *includes all isomorphisms* and is  
 475 *stable under composition*. Throughout the remainder of this paper, we will reserve the notation  $\succrightarrow$  for monos in  $\mathcal{M}$ ,  
 476 and  $\hookrightarrow$  for generic monomorphisms. We say that  $\mathbf{C}$  *has pullbacks along  $\mathcal{M}$ -morphisms* if pullbacks of cospans of the  
 477 form  $A \rightarrow B \leftarrow B'$  always exist in  $\mathbf{C}$ .

478 For a category  $\mathbf{C}$ ,  $\mathcal{M}$  is a *stable system of monics* [44] if  $\mathbf{C}$  has pullbacks along  $\mathcal{M}$ -morphisms and  $\mathcal{M}$  is *stable*  
 479 *under pullback*: if  $A \leftarrow m' - A' \rightarrow B'$  is a pullback of  $A \rightarrow B \leftarrow m - B'$  then  $m' \in \mathcal{M}$ . The morphisms in  $m \in \mathcal{M}$   
 480 satisfy the following *decomposition property of  $\mathcal{M}$ -morphisms* [44]: if  $m = m' \circ f$  where  $m'$  is a mono then  $f \in \mathcal{M}$ .  
 481

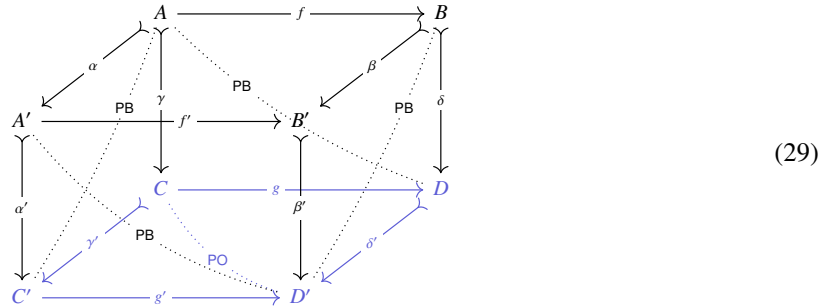
482 For later convenience, we introduce the following auxiliary definitions, which permit us to succinctly express  
 483 whether or not a given category admits pullbacks, pushouts or final pullback complements for generic input data, or  
 484 only when the morphisms on the input are of a certain nature relative to a stable system of monics:

485 **Definition 4.1.** Let  $\mathbf{C}$  be a category.

- 486 (i)  $\mathbf{C}$  *has pullbacks* if  $\mathbf{C}$  admits pullbacks of all cospans.
- 487 (ii)  $\mathbf{C}$  *has pushouts* if  $\mathbf{C}$  admits pushouts of all spans.
- 488 (iii)  $\mathbf{C}$  *has final pullback complements (FPCs)* if  $\mathbf{C}$  admits FPCs along arbitrary sequences of composable morphisms  
 489  $A \rightarrow B \rightarrow B'$ .

490 If  $\mathbf{C}$  has a stable system of monics  $\mathcal{M}$ , we define also the following variants and additional concepts:

- 491 (i')  $\mathbf{C}$  *has pushouts along  $\mathcal{M}$ -morphisms* if pushouts of spans of the form  $A \leftarrow B \rightarrow B'$  exist in  $\mathbf{C}$ .
- 492 (ii')  $\mathbf{C}$  *has final pullback complements (FPCs) along  $\mathcal{M}$ -morphisms* iff FPCs of sequences of composable morphisms  
 493 of the form  $A \rightarrow B \succrightarrow B'$  exist in  $\mathbf{C}$ .
- 494 (iii')  *$\mathcal{M}$ -morphisms are stable under pushout* in  $\mathbf{C}$  if whenever  $A' \rightarrow B' \leftarrow \beta - B$  is a pushout of a span of the form  
 495  $A' \leftarrow \alpha - A \rightarrow B$ , then  $\beta \in \mathcal{M}$ .
- 496 (iv') *pushouts along  $\mathcal{M}$ -morphisms are stable under  $\mathcal{M}$ -pullbacks*<sup>7</sup> in  $\mathbf{C}$  if for all diagrams of the form below,



497 where the bottom square is a *pushout along an  $\mathcal{M}$ -morphism*, and the vertical squares are *pullbacks along*  
 498  *$\mathcal{M}$ -morphisms*, then the top square is a pushout.

<sup>7</sup>Throughout this paper, in order to avoid confusion, we follow the convention that “stable under *pullback*” exclusively refers to the stability of morphisms when considering individual pullback squares (as in the definition of  *$\mathcal{M}$ -morphisms stable under pullback*), while “stable under *pullbacks*” always refers to stability properties that involve commutative cubes with vertical squares being pullbacks (as in the definition of *stability of  $\mathcal{M}$ -pushouts under  $\mathcal{M}$ -pullbacks*).

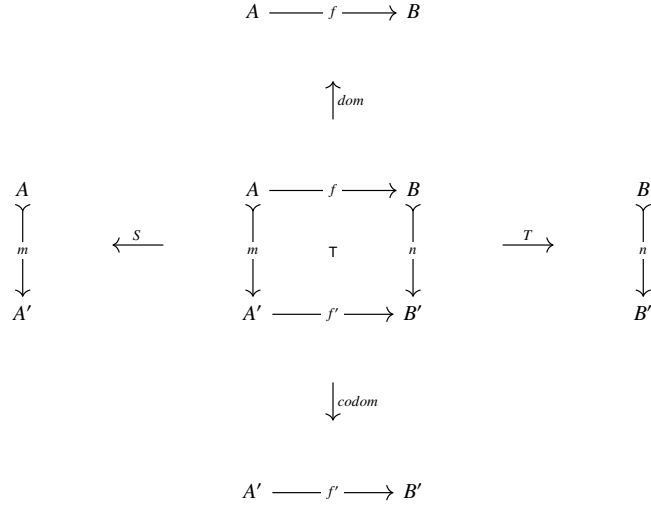


Figure 3: Boundary functors.

499 **Definition 4.2.** Let  $\mathbf{C}$  be a category with a [stable system of monics](#)  $\mathcal{M}$ . Let  $T$  be a type of commutative squares,  
500 for which we consider PB ([pullbacks](#)), PO ([pushouts](#)), or FPC ([final pullback complements](#)). Then we define the  
501 following categories:

- 502 (i)  $T_h(\mathbf{C}, \mathcal{M})$  has as objects the morphisms of  $\mathcal{M}$ , and as morphisms commutative squares of type  $T$  along arbitrary  
503 morphisms of  $\mathbf{C}$ , and a morphism composition induced by *horizontal pasting* of squares of type  $T$ .
- 504 (ii)  $T_v(\mathbf{C}, \mathcal{M})$  has as objects the morphisms of  $\mathbf{C}$ , and as morphisms commutative squares of type  $T$  along  $\mathcal{M}$ -  
505 morphisms, and a morphism composition induced by *vertical pasting* of squares of type  $T$ .

506 In Figure 3, we depict a square of type  $T$  (with  $\mathcal{M}$ -morphism drawn vertically) and the action of four “boundary  
507 functors” that naturally arise from the above definitions:

- 508 (a) The [domain](#) functor  $dom : T_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  and the [codomain](#) functor  $codom : T_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$ .
- 509 (b) The [source](#) functor  $S : T_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  and the [target](#) functor  $codom : T_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$ , where  $\mathbf{C}|_{\mathcal{M}}$  has the  
510 same objects as  $\mathbf{C}$ , and as morphisms those of  $\mathcal{M}$ .

511 The following result, whose proof can be found in Appendix B.2, establishes that these categories are indeed  
512 well-defined.

513 **Lemma 4.3.** *The categories  $T_h(\mathbf{C}, \mathcal{M})$  and  $T_v(\mathbf{C}, \mathcal{M})$  for  $T \in \{\text{PB}, \text{PO}, \text{FPC}\}$  as introduced in Definition 4.2 are  
514 well-defined, i.e., their composition operations are well-typed, associative and unital.*

#### 515 4.2. Fibrational properties of the domain and codomain functors

516 We now begin to investigate a number of interesting fibrational structures carried by the boundary functors of the  
517 various categories of squares, considering first the case of the domain functor.

518 **Theorem 4.4.** *Let  $\mathbf{C}$  be a category with a [stable system of monics](#)  $\mathcal{M}$ , and with the following additional properties:*

- 519 1.  $\mathbf{C}$  has [pullbacks](#).
- 520 2.  $\mathbf{C}$  has [pushouts](#) and [final pullback complements \(FPCs\)](#) along  $\mathcal{M}$ -morphisms.
- 521 3. [Pushouts](#) along  $\mathcal{M}$ -morphisms are [stable under pullbacks](#).
- 522 4. [Pushouts](#) along  $\mathcal{M}$ -morphisms are [pullbacks](#).



523 Then the domain functor  $dom : PB_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  from the category of pullback squares along  $\mathcal{M}$ -morphisms and  
 524 under horizontal composition to the underlying category  $\mathbf{C}$  satisfies the following properties:

- 525 (i)  $dom : PB_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  is a *Grothendieck fibration*, with the *Cartesian liftings* given by *FPCs*.  
 526 (ii)  $dom : PB_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  is a *Grothendieck opfibration*, with the *op-Cartesian liftings* given by *pushouts*.  
 527 (iii)  $dom : PB_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  satisfies a *Beck-Chevalley condition (BCC)*: adopting the notation  $m - (f, f') \rightarrow n$  for  
 528 morphisms in  $PB_h(\mathbf{C}, \mathcal{M})$  (cf. Figure 3), consider a commutative square in  $PB_h(\mathbf{C}, \mathcal{M})$  that is mapped by  $dom$   
 529 into a pullback square in  $\mathbf{C}$ :

$$\begin{array}{ccc}
 m & \xrightarrow{(f, f')} & n \\
 \downarrow (h, h') & & \downarrow (g, g') \\
 o & \xrightarrow{(i, i')} & p
 \end{array}
 \xrightarrow{dom}
 \begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow h & \text{PB} & \downarrow g \\
 C & \xrightarrow{i} & D
 \end{array}
 \quad (30)$$

530 Then the following two equivalent conditions hold:

- 531 • (BCC-1):  $(f, f')$  is *op-Cartesian* if  $(i, i')$  is *op-Cartesian* and  $(g, g')$  and  $(h, h')$  are *Cartesian*.
- 532 • (BCC-2):  $(g, g')$  is *Cartesian* if  $(h, h')$  is *Cartesian* and  $(f, f')$  and  $(i, i')$  are *op-Cartesian*.

533 **PROOF.** As the first two parts of the proof will demonstrate, the  $dom$  functor is in a certain sense a prototypical example  
 534 of a Grothendieck bifibration, in that the fibration and opfibration structures arise directly from universal properties of  
 535 FPCs and pushouts, respectively.

536 *Ad (i) —  $dom$  is a Grothendieck fibration:* this statement follows by specializing the defining equation (4) to the  
 537 case of the  $dom$  functor. The existence of *Cartesian liftings* is guaranteed since the category  $\mathbf{C}$  by assumption *has*  
 538 *FPCs along  $\mathcal{M}$ -morphisms*, while the requisite universal property that qualifies the liftings as being Cartesian (i.e., the  
 539 second line of (4)) is satisfied via the *universal property of FPCs*.

540 *Ad (ii) —  $dom$  is a Grothendieck opfibration:* specializing the defining equation (5) to the case of the  $dom$  functor,  
 541 we find that the *op-Cartesian liftings* exist in the form of pushouts (which are guaranteed to exist since  $\mathbf{C}$  by assumption  
 542 *has pushouts along  $\mathcal{M}$ -morphisms*), while the universal property which qualifies these liftings as op-Cartesian (i.e.,  
 543 the second line of (5)) is satisfied via *pullback-pushout decomposition*.

544 *Ad (iii) — Beck-Chevalley condition (BCC):* The proof can be found in Appendix B.2. □

545 As the above results indicate, the domain functor  $dom : PB_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  is (for suitable categories  $\mathbf{C}$ ) a  
 546 Grothendieck bifibration, i.e., simultaneously a Grothendieck fibration and opfibration. An interesting variant of  
 547 this type of result—which moreover has important computational meaning in its own right—arises when considering  
 548 the domain functors from the categories  $PO_h$  and  $FPC_h$  instead, which permits to state fibrational properties under  
 549 considerably weaker assumptions on the underlying categories  $\mathbf{C}$ :

550 **Theorem 4.5.** *Let  $\mathbf{C}$  be a category with a stable system of monics  $\mathcal{M}$ .*

- 551 (i) *If  $\mathbf{C}$  has pushouts along  $\mathcal{M}$ -morphisms, the functor  $dom : PO_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  is a Grothendieck opfibration.*
- 552 (ii) *If  $\mathbf{C}$  has FPCs along  $\mathcal{M}$ -morphisms, the functor  $dom : FPC_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  is a Grothendieck fibration.*

553 **PROOF.** It is straightforward to demonstrate that, for case (i), *pushouts along  $\mathcal{M}$ -morphisms* provide the *op-Cartesian*  
 554 *liftings* (as was also the case for  $dom : PB_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$ ), while the *op-Cartesianity* properties of the liftings are  
 555 realized in the form of *pushout-pushout decomposition*. For case (ii), *FPCs along  $\mathcal{M}$ -morphisms* provide the *Cartesian*  
 556 *liftings*, while the *Cartesianity* properties of liftings are realized in the form of *horizontal FPC decomposition*. □

557 Let us briefly compare the results of Theorem 4.4 and Theorem 4.5. The *op-Cartesianity* of *op-Cartesian liftings*  
 558 for the functor  $dom : PB_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  relies on *pullback-pushout decomposition* while, for  $dom : PO_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$ ,  
 559 *pushout-pushout decomposition*, valid in any category, is all that is required. On the other hand, the *Cartesianity*

560 of [Cartesian liftings](#) for the functor  $dom : \text{PB}_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  relies on the [universal property of FPCs](#) while it is  
561 a consequence of [horizontal FPC decomposition](#) for  $dom : \text{FPC}_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$ . Since the requisite properties of  
562 FPCs hold in any category that admits FPCs, it appears interesting to note that the strong requirements necessary for  
563  $dom : \text{PB}_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  to carry bifibrational structures appear to be caused mainly by the [Grothendieck opfibration](#)  
564 part of the structure.

565 **Remark 4.6.** In contrast to the domain functors discussed in the previous section, only the codomain functor  $codom : \text{PB}_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$   
566 appears to admit some fibrational structure (see below), while  $codom : \text{PO}_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  and  
567  $codom : \text{FPC}_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  fail to do so. Since none of these three codomain functors play a role in our constructions,  
568 this causes no technical problems, but we found it interesting to mention the following result here for symmetry  
569 nonetheless.

570 **Theorem 4.7.** *Let  $\mathbf{C}$  be a category with a [stable system of monics](#). Then  $codom : \text{PB}_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  is a [Grothendieck](#)  
571 [fibration](#).*

572 **PROOF.** [Cartesian liftings](#) are provided by taking pullbacks, while the [Cartesianty](#) of the liftings amounts to [pullback-](#)  
573 [pullback decomposition](#). □

#### 574 4.3. Fibrational properties of the target functors

575 As we show in this section, the target functors will have rather different fibrational structures:

- 576 •  $T : \text{PB}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  carries no fibrational structures.
- 577 •  $T : \text{FPC}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  carries a [Grothendieck opfibration](#) structure.
- 578 •  $T : \text{PO}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  carries a [strong multi-opfibration](#) structure.

579 We begin with the following theorem, that deals with the case of  $T : \text{FPC}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$ , whose full proof can  
580 be found in Appendix B.2.

581 **Theorem 4.8.** *Let  $\mathbf{C}$  be a category with a [stable system of monics](#)  $\mathcal{M}$  and that [has FPCs along  \$\mathcal{M}\$ -morphisms](#). Then  
582 the target functor  $T : \text{FPC}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  is a [Grothendieck opfibration](#).*

583 It is interesting to note that the proof strategy for [op-Cartesianty](#) (cf. Appendix B.2) would fail if we were to work  
584 in the category  $\text{PB}_v(\mathbf{C}, \mathcal{M})$  rather than in  $\text{FPC}_v(\mathbf{C}, \mathcal{M})$ , since the existence of the isomorphism  $A' - \eta \rightarrow P$  and the  
585 uniqueness of  $A' - \alpha' \rightarrow A''$  relied upon the [universal property of FPCs](#) (i.e., both of the FPC in the front and in the  
586 back of the diagram). Indeed, if we were to consider the analogue of the diagrams in (B.18) in  $\text{PB}_v(\mathbf{C}, \mathcal{M})$ , i.e., where  
587 the front vertical square would be merely a pullback, taking a pullback as indicated would only yield that the squares  
588 under and over  $P - p' \rightarrow B'$  are pullbacks. By the [universal property of FPCs](#) (of the back vertical FPC square, i.e.,  
589 the one of the lifting), we could only conclude that there exists a unique mediating arrow  $P - \eta' \rightarrow A'$ , but this arrow  
590 will in general not be an isomorphism, hence we can indeed not prove [op-Cartesianty](#) of the liftings in  $\text{PB}_v(\mathbf{C}, \mathcal{M})$ .

591 Let us now turn our attention to the remaining variant of the target functor, i.e.,  $T : \text{PO}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$ . This  
592 yields a first example of a [multi-opfibration](#). In order to formulate this result, we require the following multi-universal  
593 notion:

594 **Definition 4.9.** Let  $\mathbf{C}$  be a category with a [stable system of monics](#)  $\mathcal{M}$ . For all composable sequences of morphisms  
595 of the form  $A - f \rightarrow B - \beta \rightarrow B'$  (i.e., with  $\beta \in \mathcal{M}$ ), we define the following class:

$$\mathcal{P}(f, \beta) := \{(A - \alpha \rightarrow A', A' - f' \rightarrow B') \in \text{mor}(\mathbf{C}) \times \text{mor}(\mathbf{C}) \mid \alpha \in \mathcal{M} \wedge (f', \beta) = \text{PO}(\alpha, f)\}. \quad (31)$$

596 More explicitly,  $\mathcal{P}(f, \beta)$  consists of all composable sequences of morphisms  $A - \alpha \rightarrow A' - f' \rightarrow B'$  such that there  
597 exists a pushout square in  $\mathbf{C}$  whose boundary is given by  $(\alpha, f')$  and  $(f, \beta)$ . We refer to  $\mathcal{P}(f, \beta)$  as the  $(\mathcal{M}$ -) [multi-initial](#)

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*pushout complement (mIPC)* of  $(f, \beta)$  if the class satisfies the following *universal property*:

$$\begin{array}{c}
 \begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow \alpha'' & \searrow \text{PO} & \downarrow \beta \\
 A'' & \xrightarrow{f''} & B'' \\
 \downarrow \alpha' & \searrow \text{PO} & \downarrow \beta' \\
 A' & \xrightarrow{f'} & B'
 \end{array}
 & \exists (\alpha, f') \in \mathcal{P}(f, \beta) : \exists ! \alpha' \in \mathcal{M} : &
 \begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow \alpha & \searrow \text{PO} & \downarrow \beta \\
 A' & \xrightarrow{f'} & B' \\
 \downarrow \alpha' & \searrow \text{PO} & \downarrow \beta' \\
 A'' & \xrightarrow{f''} & B''
 \end{array}
 \\
 \\
 \begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow \gamma & \searrow \text{PO} & \downarrow \beta \\
 C' & \xrightarrow{g'} & B' \\
 \downarrow \gamma' & \searrow \text{PO} & \downarrow \beta' \\
 A'' & \xrightarrow{f''} & B''
 \end{array}
 & \exists ! A' - \varphi \rightarrow C' \in \text{iso}(\mathbf{C}) : \gamma = \varphi \circ \alpha \wedge \alpha' = \gamma' \circ \varphi
 \end{array}
 \tag{32}$$

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600

We say that  $\mathbf{C}$  has *has multi-initial pushout complements (mIPCs) along  $\mathcal{M}$ -morphisms* if  $\mathbf{C}$  has an mIPC for every composable sequence of morphisms of the form  $A - f \rightarrow B - \beta \rightarrow B'$ .

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**Remark 4.10.** It is worthwhile pointing out that just as for ordinary (“non-multi-”) pushout complements, a **multi-IPC** for a given composable sequence of morphisms may be an *empty set*. For example, in **Graph**, the category of **directed multigraphs**, the **multi-IPC** of the sequence  $\emptyset \rightarrow \bullet \rightarrow \bullet \rightarrow \circ$  is empty, a well-known phenomenon in the graph rewriting literature, interpreted as the impossibility to apply a vertex-deletion operation in DPO-semantics to a vertex with incident edges (i.e., since deletion of a vertex with incident edges would leave “*dangling*” edges).

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The following lemma establishes sufficient conditions to guarantee that  $\mathbf{C}$  has mIPCs along  $\mathcal{M}$ -morphisms; the proof is given in Appendix B.2.

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**Lemma 4.11.** *Let  $\mathbf{C}$  be a category with a stable system of monics  $\mathcal{M}$ . If pushouts along  $\mathcal{M}$ -morphisms are stable under  $\mathcal{M}$ -pullbacks, and if pushouts along  $\mathcal{M}$ -morphisms are pullbacks, then  $\mathbf{C}$  has multi-initial pushout complements (mIPCs) along  $\mathcal{M}$ -morphisms.*

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After this somewhat lengthy excursion, a direct comparison of the notion of **strong multi-opfibration** (Definition 6) and of **multi-initial pushout complement** yields the following important result:

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615

**Theorem 4.12.** *Let  $\mathbf{C}$  be a category with a stable system of monics  $\mathcal{M}$ . If pushouts along  $\mathcal{M}$ -morphisms are stable under  $\mathcal{M}$ -pullbacks in  $\mathbf{C}$ , and if pushouts along  $\mathcal{M}$ -morphisms are pullbacks, then the target functor  $T : \text{PO}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  is a strong multi-opfibration.*

616  
617

**PROOF.** The **multi-op-Cartesian liftings** are provided by **multi-initial pushout complements (mIPCs)**, whose existence and uniqueness up to isomorphism are guaranteed under the stated assumptions according to Lemma 4.11.  $\square$

618

#### 4.4. Fibrational properties of the source functors

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Finally, let us investigate the fibrational structures of the source functors. This gives rise to the following results:

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621  
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- $S : \text{PB}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  carries no fibrational structures.
- $S : \text{PO}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  carries a **Grothendieck opfibration** structure.
- $S : \text{FPC}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  carries a **residual multi-opfibration** structure.

623 **Theorem 4.13.** Let  $\mathbf{C}$  be a category with a *stable system of monics*  $\mathcal{M}$ , that has *pushouts along  $\mathcal{M}$ -morphisms*, and  
 624 such that  *$\mathcal{M}$ -morphisms are stable under pushout*. Then the source functor  $S : \text{PO}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  is a *Grothendieck*  
 625 *opfibration*, with the *op-Cartesian liftings* provided by pushouts.

626 **PROOF.** It suffices to instantiate the definition of *Grothendieck opfibration* to the case at hand:

$$\begin{array}{c} A \longrightarrow B \\ \downarrow \alpha \\ A' \end{array} \xrightarrow{f} \begin{array}{c} A \longrightarrow B \\ \downarrow \alpha \\ A' \end{array} \quad : \quad \exists \begin{array}{c} A \longrightarrow B \\ \downarrow \alpha \\ A' \end{array} \begin{array}{c} \text{PO} \\ \downarrow \alpha \\ B' \end{array} \begin{array}{c} \text{PO} \\ \downarrow \alpha \\ B'' \end{array} \quad : \quad \forall \begin{array}{c} A \longrightarrow B \\ \downarrow \alpha \\ A' \end{array} \begin{array}{c} \text{PO} \\ \downarrow \alpha \\ B' \end{array} \begin{array}{c} \text{PO} \\ \downarrow \alpha \\ B'' \end{array} \quad : \quad \forall \begin{array}{c} A \longrightarrow B \\ \downarrow \alpha \\ A' \end{array} \begin{array}{c} \text{PO} \\ \downarrow \alpha \\ B' \end{array} \begin{array}{c} \text{PO} \\ \downarrow \alpha \\ B'' \end{array} \begin{array}{c} \text{PO} \\ \downarrow \alpha \\ B''' \end{array} \quad (33)$$

627 Here, the existence of *op-Cartesian liftings* is provided by the assumption that  $\mathbf{C}$  has *pushouts along  $\mathcal{M}$ -morphisms*,  
 628 while the *op-Cartesianity* of the liftings follows from the *universal property of pushouts* (yielding the existence of a  
 629 unique morphism  $B' - \beta' \rightarrow B''$ ), *pushout-pushout decomposition* (which ensures the bottom square in the rightmost  
 630 diagram in (33) is a pushout), and finally from the assumption that  *$\mathcal{M}$ -morphisms are stable under pushout* (ensuring  
 631 that  $\beta' \in \mathcal{M}$ , so that the pushout square over it indeed qualifies as a morphism in  $\text{PO}_v(\mathbf{C}, \mathcal{M})$ ).  $\square$

632 It is worthwhile considering whether the above proof strategy for the *Grothendieck opfibration* structure of  $S : \text{PO}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$   
 633 could be adapted to the case of the source functor  $S : \text{PB}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$ . However, even under the additional assumption that pushouts along  $\mathcal{M}$ -morphisms are pullbacks, we could *not* prove that  $\beta' \in \mathcal{M}$  for the  
 634 analogue of the last diagram in (33) where the outer square is merely a pullback (this was true for (33), because here  
 635 we could rely upon the assumed *stability of  $\mathcal{M}$ -morphisms under pushout*). Nevertheless, it is interesting to observe  
 636 that  $S : \text{PB}_v(\mathbf{C}|_{\mathcal{M}}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  (i.e., restricting to pullback squares where all morphisms are in  $\mathcal{M}$ ) *does* have the  
 637 structure of a *Grothendieck opfibration*, with the *op-Cartesian liftings* given by pushouts, and *op-Cartesianity* ensured  
 638 if the *pullback-pushout decomposition* lemma holds (which requires certain additional assumptions on  $\mathbf{C}$ ).  
 639

640 Finally, let us consider the case of the source functor  $S : \text{FPC}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  from the category of FPCs along  
 641  $\mathcal{M}$ -morphisms with vertical pasting. This case requires the introduction of a novel universal construction, that of  
 642  *$\mathcal{M}$ -final pullback complement pushout augmentation*. Before giving the definition, we first quote some prerequisite  
 643 standard concepts from category theory pertaining to factorization structures on morphisms, which will be used in  
 644 this paper for instance in the form of *epi- $\mathcal{M}$ -factorizations*, but also to demonstrate a certain factorization structure on  
 645 FPC squares (seen as morphisms in  $\text{FPC}_v(\mathbf{C}, \mathcal{M})$ ).

646 **Definition 4.14** ([45], **Def. 14.1**). For a category  $\mathbf{C}$ , let  $E$  and  $M$  be classes of morphisms. By convention, in com-  
 647 mutative diagrams, let morphisms in  $E$  be depicted as  $\twoheadrightarrow$ , and morphisms in  $M$  by  $\triangleright$ . Then  $(E, M)$  is called a  
 648 *factorization structure for morphisms* in  $\mathbf{C}$ , and  $\mathbf{C}$  is called  *$(E, M)$ -structured* iff

- 649 (i) both  $E$  and  $M$  are *closed under composition with isomorphisms*,
- 650 (ii)  $\mathbf{C}$  has  *$(E, M)$ -factorizations of morphisms* (i.e., for every morphism  $f$  in  $\mathbf{C}$ , there exist  $m \in M$  and  $e \in E$  such  
 651 that  $f = m \circ e$ ),
- 652 (iii)  $\mathbf{C}$  has the *unique  $(E, M)$ -diagonalization property*:

$$\forall \begin{array}{ccc} A & \xrightarrow{e} & B \\ \downarrow f & & \downarrow g \\ C & \triangleright m & \rightarrow D \end{array} : \exists! d \begin{array}{ccc} A & \xrightarrow{e} & B \\ \downarrow f & \swarrow d & \downarrow g \\ C & \triangleright m & \rightarrow D \end{array} \quad (34)$$

653 In words: for all commutative squares as in (34) above, where  $e \in E$  and  $m \in M$ , there exists a unique morphism  
 654  $d$  (referred to as the *diagonal*) such that  $f = d \circ e$  and  $g = m \circ d$ .

655 **Proposition 4.15** ([45], Prop. 14.4 and 14.6). Let  $\mathbf{C}$  be a category that is  $(E, M)$ -structured. Then the following  
 656 properties hold:

657 (i)  $E \cap M = \text{iso}(\mathbf{C})$ .

658 (ii) the classes  $E$  and  $M$  are both closed under composition.

659 (iii)  $(E, M)$ -factorizations are essentially unique:

660 (a) If  $e_1 \circ m_1 = e_2 \circ m_2$  (for  $m_1, m_2 \in M$  and  $e_1, e_2 \in E$ ), then there exists an isomorphism  $h$  such that  $e_2 = h \circ e_1$   
 661 and  $m_1 = m_2 \circ h$ .

662 (b) If  $f = e \circ m$  (for  $e \in E$  and  $m \in M$ ), and if  $e' = h \circ e$  and  $m' = m \circ h$  for an isomorphism  $h$ , then  $(e', m')$  is  
 663 also an  $(E, M)$ -factorization of  $f$ .

664 **Definition 4.16.** Let  $\mathbf{C}$  be a category with a stable system of monics  $\mathcal{M}$ , and such that<sup>8</sup>  $\mathbf{C}$  is  $(\mathcal{E}, \mathcal{M})$ -structured. Given  
 665 a pushout square along an  $\mathcal{M}$ -morphism as in back of the diagram below,



666 we define an  $\mathcal{M}$ -final pullback complement pushout augmentation (FPA) as a triple of morphisms  $(\gamma, g, e)$  such that

667 (i)  $\gamma \in \mathcal{M}$ ,  $e \in \mathcal{E}$ ,  $e \circ \beta \in \mathcal{M}$ , and  $g \circ \gamma = e \circ f'$ ,

668 (ii)  $(\gamma \circ \alpha, g)$  is an FPC of  $(f, e \circ \beta)$ , and

669 (iii)  $(id_B, \beta)$  is a pullback of  $(e \circ \beta, e)$ .

670 We denote the class of all FPAs of a given pushout square  $(f, \beta, \alpha, f')$  by  $\text{FPA}(f, \beta, \alpha, f')$ .

671 **Remark 4.17.** It appears worthwhile to note that for a diagram as in (35), since the left vertical square is a pullback  
 672 (given that  $\gamma$  is in  $\mathcal{M}$  and thus a monomorphism), and since the front vertical square is an FPC and thus a pullback,  
 673 by pullback-pullback composition the composite of the right and back vertical squares is a pullback; therefore, we  
 674 find that the morphism  $\gamma$  must coincide with the unique morphism from  $A'$  to  $C$  that exists by the universal property  
 675 of FPCs. The subtlety in the definition of FPAs then lies in the nature of  $\gamma$  as being in  $\mathcal{M}$ : supposing for a moment  
 676 that  $\gamma$  is a generic morphism, if the pushout square in the back is known to be a pullback (which will be the case in  
 677 all categories of interest), then the composite of the right and back vertical is again a pullback, thus  $\gamma$  again has to  
 678 coincide with the unique morphism that exists by the universal property of FPCs; however, we would not be able to  
 679 conclude from this set of assumptions that  $\gamma$  must be in  $\mathcal{M}$ , hence this is indeed found to be a non-trivial part of our  
 680 set of assumptions.

681 Throughout this paper, we will exclusively be interested in situations where the definition above may be slightly  
 682 simplified:

683 **Lemma 4.18.** Let  $\mathbf{C}$  be a category with a stable system of monics  $\mathcal{M}$ , and such that  $\mathbf{C}$  is  $(\mathcal{E}, \mathcal{M})$ -structured. If pushouts  
 684 along  $\mathcal{M}$ -morphisms are stable under  $\mathcal{M}$ -pullbacks in  $\mathbf{C}$ , condition (iii) in Definition 4.16 is automatically satisfied,  
 685 i.e.,  $(id_B, \beta)$  is a pullback of  $(e \circ \beta, e)$  in the notations of (35).

<sup>8</sup>Here and in the following, it is important to emphasize that we do not require any particular properties of the class of morphisms  $\mathcal{E}$  other than that it is a class such that  $\mathbf{C}$  is  $(\mathcal{E}, \mathcal{M})$ -structured. While in many applications of interest  $\mathcal{E}$  will coincide with the class of epimorphisms of  $\mathbf{C}$  or a subclass thereof, in some cases  $\mathcal{E}$  will not even be a class of epimorphisms [8].

686 PROOF. The proof follows by invoking [pullback-pushout decomposition](#) (which holds due to the assumption that  
 687 [pushouts along  \$\mathcal{M}\$ -morphisms are stable under  \$\mathcal{M}\$ -pullbacks](#) in  $\mathbf{C}$ ) to the commutative diagram in (35): since the  
 688 front and left vertical squares compose into a pullback, the vertical morphisms are in  $\mathcal{M}$ , and since the back vertical  
 689 square is a [pushout along an  \$\mathcal{M}\$ -morphism](#), the right vertical square is a pullback.  $\square$

690 The concept of [FPC-pushout-augmentations \(FPAs\)](#) introduced above gives rise to an interesting factorization  
 691 system on FPCs as the following theorem explains; its proof is in Appendix B.2.

692 **Theorem 4.19.** *Let  $\mathbf{C}$  be a category with a [stable system of monics  \$\mathcal{M}\$](#) , that is  $(\mathcal{E}, \mathcal{M})$ -structured, that has [pushouts](#)  
 693 [and FPCs along  \$\mathcal{M}\$ -morphisms](#), such that  [\$\mathcal{M}\$ -morphisms are stable under pushout](#), and such that [pushouts along](#)  
 694  [\$\mathcal{M}\$ -morphisms are stable under  \$\mathcal{M}\$ -pullbacks](#). Then the category  $\text{FPC}_v(\mathbf{C}, \mathcal{M})$  is [\(auto-augmented, inert\)-structured](#).  
 695 Here, the class of [auto-augmented FPCs](#) is defined as*

$$\begin{array}{ccc}
 \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \alpha & & \downarrow \beta \\ A' & \xrightarrow{f'} & B' \end{array} & \text{FPC} & \in \text{mor}(\text{FPC}_v(\mathbf{C}, \mathcal{M}))|_{\text{auto-augmented}} \\
 & & \Leftrightarrow \exists \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \alpha & & \downarrow \beta \\ A' & \xrightarrow{f'} & B' \end{array}
 \end{array}
 \quad (36)$$

696 In words: an FPC square along an  $\mathcal{M}$ -morphism (seen as a morphism in  $\text{FPC}_v(\mathbf{C}, \mathcal{M})$ ) is auto-augmented iff when  
 697 taking a pushout of the span within the FPC, the mediating morphism into the cospan object of the FPC is a morphism  
 698 in  $\mathcal{E}$ .<sup>9</sup> Moreover, the class of [inert FPCs](#) is defined as

$$\text{mor}(\text{FPC}_v(\mathbf{C}, \mathcal{M}))|_{\text{inert}} := \left\{ \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \alpha & & \downarrow \beta \\ A' & \xrightarrow{f'} & B' \end{array} \middle| \alpha \in \mathcal{E} \cap \mathcal{M} = \text{iso}(\mathbf{C}) \right\} \quad (37)$$

699 We refer the interested readers to Example 5.29 of Section 5.3.4 for an illustration of the practical meaning of the  
 700 [\(auto-augmented, inert\) factorization](#) of FPCs, for the case of directed simple graphs, where it will be demonstrated  
 701 that, in a certain sense, the factorization provides a static analysis of the classes of cloning with implicit deletion that  
 702 can be modeled by FPCs.

703 Besides its quintessential role in factorizations of FPCs, the concept of [FPC-pushout-augmentations \(FPAs\)](#) is also  
 704 crucial for the fibrational structure of the source functor  $S : \text{FPC}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  as the following theorem shows. Its  
 705 proof can be found in Appendix B.2.

706 **Theorem 4.20.** *Let  $\mathbf{C}$  be a category with a [stable system of monics  \$\mathcal{M}\$](#) , that is  $(\mathcal{E}, \mathcal{M})$ -structured, that has [pullbacks](#),  
 707 [pushouts and FPCs along  \$\mathcal{M}\$ -morphisms](#), such that  [\$\mathcal{M}\$ -morphisms are stable under pushout](#), and such that [pushouts](#)  
 708 [along  \$\mathcal{M}\$ -morphisms are stable under  \$\mathcal{M}\$ -pullbacks](#). Then  $S : \text{FPC}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  is a [residual multi-opfibration](#).*

709 Let us finally note here that, in all of our applications, we will only consider base categories which are [finitary](#), so  
 710 that one may indeed provide algorithms for the various universal constructions that yield finite sets of solutions up to  
 711 isomorphisms.

## 712 5. Examples of categories suitable for defining categorical constructions with fibrational properties

713 This section is structured into two parts: in Sections 5.1 and 5.2, we present two classes of categories that may  
 714 ultimately serve as a basis for defining compositional rewriting theories (cf. Section 6), i.e., categories with [adhesivity](#)  
 715 [properties](#) and [quasi-topoi](#), respectively; and in Section 5.3, we demonstrate that these classes of categories admit  
 716 certain key categorical constructions of with fibrational properties.

<sup>9</sup>Note that since we admit arbitrary morphisms of  $\mathbf{C}$  for the horizontal morphisms, the mediating morphism would in general be a morphism with a non-trivial  $\mathcal{E}$ - $\mathcal{M}$ -factorization, hence for this morphism to be an  $c\mathcal{E}$ -morphism is indeed a non-trivial requirement.

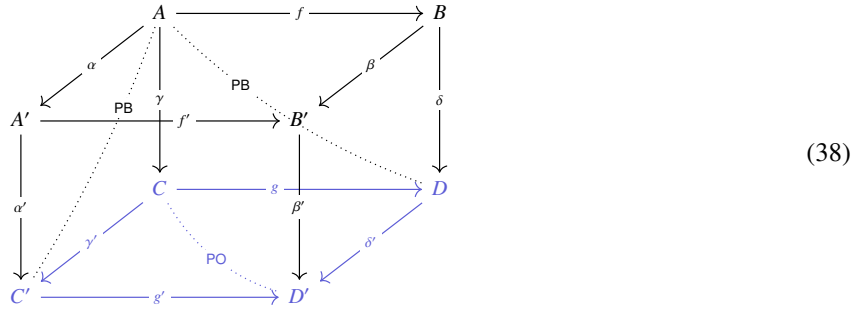
717 *5.1. Categories with adhesivity properties*

718 Starting in the early 2000s, the seminal work of Lack and Sobocinski [4, 5, 7] introducing **adhesive** and **quasi-**  
 719 **adhesive** categories, which was later generalized by Ehrig et al.[3, 46, 47] to **adhesive HLR** and **weak adhesive HLR**  
 720 categories and their variants, constituted a significant breakthrough in formalizing and standardizing the theory of  
 721 Double-Pushout (DPO) rewriting. In this section, we will quote the salient definitions as well as key results from this  
 722 research, with the purpose of providing a curated list of categories of practical interest that carry one of the variants of  
 723 adhesivity properties mentioned above. We refer the interested readers to [3, 47] (cf. also [18]) for further background  
 724 materials.

725 In order to formulate the various notions of adhesivity, we require the following definitions:

726 **Definition 5.1 (Notions of van Kampen (VK) squares).** Let  $\mathbf{C}$  be a category. Then a pushout square is a *van Kam-*  
 727 *pen (VK) square* iff for any commutative diagram as in (38) below, where the bottom square highlighted in blue is the  
 728 aforementioned pushout square, and where the back squares are pullbacks, the following conditions hold:

- 729 • (VK-a) If the front and the right squares are pullbacks, then the top square is a pushout.
- 730 • (VK-b) If the top square is a pushout, then the front and the right squares are pullbacks.



731 If  $\mathbf{C}$  has a **stable system of monics**  $\mathcal{M}$ , we define the following weakenings of the notion of VK squares:

- 732 • **horizontal weak VK squares** are defined as pushouts whose morphism are all in  $\mathcal{M}$ , and that are **VK squares**.
- 733 • **vertical weak VK squares** are defined as pushouts which satisfy the defining properties of **VK squares** modulo  
 734 the constraint that all vertical morphisms in (38) are in  $\mathcal{M}$  (i.e., when  $\alpha', \beta', \gamma, \delta \in \mathcal{M}$ ).

735 **Definition 5.2.** We define the following variants of *adhesivity properties* for categories:

- 736 1. Let  $\mathbf{A}$  be a category. Then  $\mathbf{A}$  is said to be an **adhesive category** [4] iff it has the following properties:
  - 737 (A-i)  $\mathbf{A}$  has pullbacks.
  - 738 (A-ii)  $\mathbf{A}$  has pushouts along monomorphisms.
  - 739 (A-iii) Pushouts along monomorphisms in  $\mathbf{A}$  are van Kampen squares.
- 740 2. Let  $\mathbf{Q}$  be a category, and let  $\text{rm}(\mathbf{Q})$  denote the class of regular monomorphisms of  $\mathbf{Q}$ . Then  $\mathbf{Q}$  is said to be  
 741 **quasi-adhesive** [5] (sometimes also referred to as **rm-adhesive** [6]) iff  $\mathbf{Q}$  satisfies the following properties:
  - 742 (Q-i)  $\mathbf{A}$  has pullbacks.
  - 743 (Q-ii)  $\mathbf{A}$  has pushouts along regular monomorphisms.
  - 744 (Q-iii) Pushouts along regular monomorphisms in  $\mathbf{A}$  are van Kampen squares.
- 745 3. Let  $\mathbf{L}$  be a category that admits a **stable system of monics**  $\mathcal{M}$ . Then  $\mathbf{L}$  is said to be an **adhesive high-level**  
 746 **replacement (HLR) category** [3] iff  $\mathbf{L}$  satisfies the following properties:
  - 747 (L-i)  $\mathbf{L}$  has pullbacks along  $\mathcal{M}$ -morphisms.

- 748 (L-ii)  $\mathbf{L}$  has pushouts along  $\mathcal{M}$ -morphisms, and  $\mathcal{M}$ -morphisms are stable under pushout.  
749 (L-iii) Pushouts along  $\mathcal{M}$ -morphisms in  $\mathbf{L}$  are van Kampen squares.
- 750 4. Let  $\mathbf{H}$  be a category that admits a stable system of monics  $\mathcal{M}$ . Then  $\mathbf{H}$  is said to be a *horizontal weak adhesive HLR category* [47] iff  $\mathbf{H}$  satisfies the following properties:  
751 *HLR category* [47] iff  $\mathbf{H}$  satisfies the following properties:
- 752 (H-i)  $\mathbf{H}$  has pullbacks along  $\mathcal{M}$ -morphisms.  
753 (H-ii)  $\mathbf{H}$  has pushouts along  $\mathcal{M}$ -morphisms, and  $\mathcal{M}$ -morphisms are stable under pushout.  
754 (H-iii)  $\mathbf{H}$  has horizontal weak VK squares.
- 755 5. Let  $\mathbf{V}$  be a category that admits a stable system of monics  $\mathcal{M}$ . Then  $\mathbf{V}$  is said to be a *vertical weak adhesive HLR category* [47] (often alternatively referred to as an  $\mathcal{M}$ -adhesive category) iff  $\mathbf{V}$  satisfies the following properties:  
756 *category* [47] (often alternatively referred to as an  $\mathcal{M}$ -adhesive category) iff  $\mathbf{V}$  satisfies the following properties:
- 757 (V-i)  $\mathbf{V}$  has pullbacks along  $\mathcal{M}$ -morphisms.  
758 (V-ii)  $\mathbf{V}$  has pushouts along  $\mathcal{M}$ -morphisms, and  $\mathcal{M}$ -morphisms are stable under pushout.  
759 (V-iii) Pushouts along  $\mathcal{M}$ -morphisms in  $\mathbf{V}$  are vertical weak van Kampen squares.
- 760 6. Let  $\mathbf{W}$  be a category that admits a stable system of monics  $\mathcal{M}$ . Then  $\mathbf{W}$  is said to be a *weak adhesive HLR category* [47] iff  $\mathbf{W}$  has the properties of both a horizontal and a vertical weak adhesive HLR category.  
761 *category* [47] iff  $\mathbf{W}$  has the properties of both a horizontal and a vertical weak adhesive HLR category.

762 Finally, since in many of the proofs that rely upon the above *adhesivity properties* one in fact needs different sub-  
763 statements of the axioms (X-iii) (i.e., of the VK-type axioms), we will use the notation (*X-iii-a*) for the part of axiom  
764 (X-iii) referring to stability under pullbacks, and to (*X-iii-b*) for the variant of the statement of axiom (VK-b) in the  
765 definition of *van Kampen squares*.

766 **Remark 5.3.** The above list of definitions of categories with *adhesivity properties* might appear to have a certain  
767 “asymmetry” in that for *adhesive* and for *quasi-adhesive categories*, stability of the relevant class of monics under  
768 pushout is not explicitly stated. However, one may prove [5, Prop. 6.4] that this stability in fact follows from the other  
769 axioms for these kinds of adhesivity.

770 The motivation of the seemingly peculiarly long list of *adhesivity properties* for categories is indeed given by the  
771 intricate nature of requirements on categories to admit various notions of rewriting semantics (cf. Section 6). It should  
772 also be noted that an oddity in this line of research is [47] that to date only a single example of a category is known  
773 that is a *vertical*, but not a *horizontal weak adhesive HLR category* (namely the category **ISet** of list sets as introduced  
774 by Heindel in [48]), while all other known examples of categories with weak forms of *adhesivity properties* are indeed  
775 *weak adhesive HLR categories*. This is illustrated in Table 2, which is an adaptation of a similar table presented  
776 in [18], and which lists both examples of categories with *adhesivity properties* as well as examples of *quasi-topoi* (cf.  
777 Section 5.2).

778 Before presenting some examples of categories with *adhesivity properties* in further detail, it is worthwhile stat-  
779 ing the following sufficient condition for when vertical implies horizontal weak HLR adhesivity, which requires the  
780 following well-known result:

781 **Theorem 5.4 ([3], Thm. 4.26(1)).** *Let  $\mathbf{C}$  be a category with one of the variants of *adhesivity properties* for some*  
782 *stable system of monics  $\mathcal{M}$  (which for the case of  $\mathbf{C}$  being an *adhesive category* is  $\mathcal{M} = \text{mono}(\mathbf{C})$ ). Then *pushouts**  
783 *along  $\mathcal{M}$ -morphisms are pullbacks in all cases but one, i.e., when  $\mathbf{C}$  is a *horizontal weak adhesive HLR category*, in*  
784 *which case pushouts of spans of  $\mathcal{M}$ -morphisms are pullbacks.*

785 **Lemma 5.5.** *Let  $\mathbf{V}$  be a *vertical weak adhesive HLR category* with respect to a *stable system of monics  $\mathcal{M}$* . Then a*  
786 *sufficient condition for  $\mathbf{V}$  to also carry the structure of a *horizontal weak adhesive HLR category* (and thus overall of*  
787 *a *weak adhesive HLR category*) is that *pushouts along  $\mathcal{M}$ -morphisms are stable under pullbacks.**



788 PROOF. Since by assumption [pushouts along  \$\mathcal{M}\$ -morphisms](#) are [stable under pullbacks](#), it remains to prove that in the  
 789 diagram below, where the top and bottom squares are pushouts of spans of  $\mathcal{M}$ -morphisms, and where the left and back  
 790 squares are pullbacks, the front and right squares are pullbacks:

(39)

791 Since yet again by assumption [pushouts along  \$\mathcal{M}\$ -morphisms](#) are [stable under pullbacks](#), and since in a [vertical weak](#)  
 792 [adhesive HLR category](#), [pushouts along  \$\mathcal{M}\$ -morphisms](#) are [pullbacks](#), the claim follows by invoking [pullback-pushout](#)  
 793 [decomposition](#).  $\square$

794 The paradigmatic example of an [adhesive category](#) is the following one:

795 **Definition 5.6.** The category **Graph** of [directed multigraphs](#) is defined as the presheaf category  $\mathbf{Graph} := (\mathbb{G}^{op} \rightarrow$   
 796  $\mathbf{Set})$ , where  $\mathbb{G} := (\cdot \rightrightarrows \star)$  is a category with two objects and two morphisms [4]. Objects  $G = (V_G, E_G, s_G, t_G)$  of  
 797 **Graph** are given by a set of vertices  $V_G$ , a set of directed edges  $E_G$  and the source and target functions  $s_G, t_G : E_G \rightarrow$   
 798  $V_G$ . Morphisms of **Graph** between  $G, H \in \mathbf{obj}(\mathbf{Graph})$  are of the form  $\varphi = (\varphi_V, \varphi_E)$ , with  $\varphi_V : V_G \rightarrow V_H$  and  
 799  $\varphi_E : E_G \rightarrow E_H$  such that  $\varphi_V \circ s_G = s_H \circ \varphi_E$  and  $\varphi_V \circ t_G = t_H \circ \varphi_E$ .

800 **Theorem 5.7.** The category **Graph** is an [adhesive category](#) and (by definition) a presheaf topos [4] (and thus in  
 801 particular a [quasi-topos](#)), with strict-initial object  $\emptyset = (\emptyset, \emptyset, \emptyset \rightarrow \emptyset, \emptyset \rightarrow \emptyset)$  the empty graph, and with the following  
 802 additional properties:

- 803 • Morphisms are in the classes  $\mathbf{mono}(\mathbf{Graph})/\mathbf{epi}(\mathbf{Graph})/\mathbf{iso}(\mathbf{Graph})$  if they are component-wise injective/surjective/bijective  
 804 functions, respectively. All monos in **Graph** are regular, and **Graph** therefore possesses an epi-mono-factorization.
- 805 • For each  $G \in \mathbf{obj}(\mathbf{Graph})$  [30, Sec. 2.1],  $\eta_G : G \rightarrow T(G)$  is defined as the embedding of  $G$  into  $T(G)$ , where  
 806  $T(G)$  is defined as the graph with vertex set  $V'_G := V_G \uplus \{\star\}$  and edge set  $E_G \uplus E'_G$ . Here,  $E'_G$  contains one  
 807 directed edge  $e_{n,p} : v_n \rightarrow v_p$  for each pair of vertices  $(v_n, v_p) \in V'_G \times V'_G$ .

808 Many of the examples listed in Table 2 are obtained via the following construction:

809 **Definition 5.8** ([3], Def. A.41). Let  $F : \mathbf{A} \rightarrow \mathbf{C}$  and  $G : \mathbf{B} \rightarrow \mathbf{C}$  be two functors, and let  $I$  be an index set. Then the  
 810 [comma category](#)  $\mathbf{ComCat}(F, G; I)$  is defined as a category whose objects are of the form

$$\mathbf{obj}(\mathbf{ComCat}(F, G; I)) := \{(A, B, op = \{op_i\}_{i \in I}) \mid \forall i \in I : op_i \in \mathbf{mor}(F(A), G(B))\}, \quad (40)$$

811 and whose morphisms  $f : (A, B; op) \rightarrow (A', B'; op')$  consist of pairs of morphisms  $f_A : A \rightarrow A'$  and  $f_B : B \rightarrow B'$  such  
 812 that  $G(f_B) \circ op_i = op'_i \circ F(f_A)$  for all  $i \in I$ .

813 The main interest in this definition of [comma categories](#) is that they enjoy a number of important properties  
 814 that render them extremely useful for determining whether various datatypes of practical importance have [adhesivity](#)  
 815 [properties](#):

816 **Theorem 5.9.** Let  $\mathbf{ComCat}(F, G; I)$  be a [comma category](#), for  $F : \mathbf{A} \rightarrow \mathbf{C}$  and  $G : \mathbf{B} \rightarrow \mathbf{C}$  two functors, and where  
 817  $I$  is an index set.

- 818 (i) Morphisms  $f = (f_A, f_B)$  in  $\mathbf{ComCat}(F, G; I)$  are mono-/epi-isomorphisms iff they are component-wise mono-  
 819 /epi-isomorphisms, respectively [3, Fact A.43].

820 (ii) If  $\mathbf{A}$  and  $\mathbf{B}$  have pushouts and  $F$  preserves pushouts, then  $\mathbf{ComCat}(F, G; \mathcal{I})$  has pushouts, and these are con-  
 821 structed component-wise [3, Fact A.43].

822 (iii) If  $\mathbf{A}$  and  $\mathbf{B}$  have pullbacks and  $G$  preserves pullbacks, then  $\mathbf{ComCat}(F, G; \mathcal{I})$  has pullbacks, and these are  
 823 constructed component-wise [3, Fact A.43].

824 (iv) If  $(\mathbf{A}, \mathcal{M}_1)$  and  $(\mathbf{B}, \mathcal{M}_2)$  are adhesive HLR categories,  $F : \mathbf{A} \rightarrow \mathbf{C}$  preserves pushouts along  $\mathcal{M}_1$ -morphisms  
 825 and  $G : \mathbf{B} \rightarrow \mathbf{C}$  preserves pullbacks, then  $\mathbf{ComCat}(F, G; \mathcal{I})$  is an adhesive HLR category with respect to the  
 826 stable system of monics  $\mathcal{M} = (\mathcal{M}_1 \times \mathcal{M}_2) \cap \text{mor}(\mathbf{ComCat}(F, G; \mathcal{I}))$  [3, Thm. 4.15.4].

827 (v) If  $(\mathbf{A}, \mathcal{M}_1)$  and  $(\mathbf{B}, \mathcal{M}_2)$  are adhesive HLR categories,  $F : \mathbf{A} \rightarrow \mathbf{C}$  preserves pushouts along  $\mathcal{M}_1$ -morphisms and  
 828  $G : \mathbf{B} \rightarrow \mathbf{C}$  preserves pullbacks along  $\mathcal{M}_2$ -morphisms, then  $\mathbf{ComCat}(F, G; \mathcal{I})$  is a weak adhesive HLR category  
 829 with respect to the stable system of monics  $\mathcal{M} = (\mathcal{M}_1 \times \mathcal{M}_2) \cap \text{mor}(\mathbf{ComCat}(F, G; \mathcal{I}))$  [3, Thm. 4.15.4].

830 Generalizing from directed graphs to hypergraphs, it is interesting to note that the various notions of hypergraphs  
 831 yield different notions of adhesivity properties. We present here one of the standard constructions in the literature:

832 **Definition 5.10.** Let **HyperGraph** denote the category of directed ordered hypergraphs [3, Fact 4.17], defined as the  
 833 comma category  $(Id_{\mathbf{Set}}, \square^*; \{1, 2\})$ . Here,  $Id_{\mathbf{Set}}$  denotes the identity functor on the category **Set**, while  $\square^* : \mathbf{Set} \rightarrow \mathbf{Set}$   
 834 denotes the free monoid functor (which assigns to each set  $A$  the free monoid  $A^*$  on  $A$ , and to each set morphism  
 835  $f : A \rightarrow B$  the free monoid morphism  $f^* : A^* \rightarrow B^*$ ). More explicitly, an object of **HyperGraph** is a tuple  
 836  $H = (V_H, E_H, s_H, t_H)$ , where  $V_H$  is the set of vertices,  $E_H$  the set of hyperedges, and where  $s_H, t_H : E \rightarrow V^*$  are  
 837 the source and target functions (assigning to each edge an ordered list of source and target vertices). A morphism  
 838  $\varphi : H \rightarrow H'$  in **HyperGraph** is given by a pair of morphisms  $\varphi_V : V_H \rightarrow V_{H'}$  and  $\varphi_E : E_H \rightarrow E_{H'}$  such that the  
 839 diagram in (41) below commutes.

$$\begin{array}{ccccc}
 & & & & E_H \\
 & & & & \swarrow s_H \\
 V_H & \xrightarrow{\square^*} & V_H^* & & \\
 \downarrow \varphi_V & & \downarrow \varphi_V^* & \swarrow t_H & \downarrow \varphi_E \\
 V_{H'} & \xrightarrow{\square^*} & V_{H'}^* & & E_{H'} \\
 & & & \swarrow s_{H'} & \downarrow \varphi_E \\
 & & & & E_H \\
 & & & & \downarrow \varphi_E \\
 & & & & E_{H'}
 \end{array}
 \tag{41}$$

840 **Proposition 5.11** ([3, Fact 4.17]). The category **HyperGraph** is an adhesive HLR category with respect to the stable  
 841 system of monics  $\mathcal{M}_{\mathbf{HyperGraph}}$  given by morphisms  $\varphi = (\varphi_V, \varphi_E)$  where  $\varphi_V$  and  $\varphi_E$  are both monomorphisms.

842 We conclude our brief presentation of examples by mentioning a number of slightly more sophisticated cases.  
 843 Many interesting examples of categories with adhesivity properties may be obtained by using the construction of  
 844 presheaves (cf. e.g. [49, Sec. 5] for a review within the context of categorical rewriting theory). Remarkable examples  
 845 include the category of asynchronous graphs as introduced in [50], which permit to model certain structures in game  
 846 semantics, and various notions of attributed and symbolic graphs as discussed in [49]. Many other examples concern  
 847 comma category constructions, with a number of illustrative examples provided in Table 1. More intricate examples  
 848 still have been developed in the context of so-called hierarchical graphs, which are obtained via comma-categorical  
 849 constructions along various notions of super-power functors, and whose adhesivity properties have been studied in [51,  
 850 52] (see also [53]).

## 851 5.2. Quasi-topoi

852 Quasi-topoi have been considered in the context of rewriting theories as a natural generalization of adhesive  
 853 categories in [5]. While several adhesive categories of interest to rewriting are topoi, including in particular the

$F$	Description
$Id_{\text{Set}}$	<b>directed multigraphs</b>
$\square^*$	directed “ordered” hypergraphs with multiple incidences ( <b>HyperGraph</b> [3, Fact 4.17] aka <b>PNet</b> [7, Ex. 7])
$\mathcal{M}$	directed “unordered” hypergraphs with multiple incidences (= <b>PTNets</b> of [3, Fact 4.21])
$\mathcal{P}$	directed “unordered” hypergraphs with simple incidences (= <b>ElemNets</b> of [3, Fact 4.20])
$F$	Description
$\mathcal{P}^{(1,2)}$	<b>undirected multigraphs</b> [18]
$\square^*$	undirected “ordered” hypergraphs with multiple incidences (i.e. lists)
$\mathcal{M}$	undirected “unordered” hypergraphs with multiple incidences
$\mathcal{P}$	undirected “unordered” hypergraphs with simple incidences

Table 1: Collection of examples for categories with **adhesivity properties** based upon two “schemas” of **comma category** constructions. Here, we employ the notations  $\square^*$  for the free monoid functor,  $\mathcal{M}$  (also denoted  $\oplus^*$  in [3]) for the free commutative monoid functor,  $\mathcal{P}$  for the covariant powerset functor, and  $\mathcal{P}^{(1,2)}$  for the restricted version thereof (cf. e.g. [52]).

category **Graph** of directed multigraphs (cf. Definition 5.6), it is not difficult to find examples of categories equally relevant to rewriting theory that fail to be topoi. A notable such example is the category **SGraph** of directed simple graphs (cf. Definition 5.17).

Let us first recall a number of results from the work of Cockett and Lack [44, 57] on restriction categories. We will only need a very small fragment of their theory, namely the definition and existence guarantees for  $\mathcal{M}$ -partial map classifiers, so we will follow mostly [30]. We will in particular not be concerned with the notion of  $\mathcal{M}$ -partial maps itself.

**Definition 5.12** ([30], **Sec. 2.1**; compare [57], **Sec. 2.1**). For a **stable system of monics**  $\mathcal{M}$  in a category  $\mathbf{C}$ , an  **$\mathcal{M}$ -partial map classifier**  $(T, \eta)$  is a functor  $T : \mathbf{C} \rightarrow \mathbf{C}$  and a natural transformation  $\eta : ID_{\mathbf{C}} \rightarrow T$  such that

1. for all  $X \in \text{obj}(\mathbf{C})$ ,  $\eta_X : X \rightarrow T(X)$  is in  $\mathcal{M}$ .
2. for each span  $(A \xleftarrow{m} X \xrightarrow{f} B)$  with  $m \in \mathcal{M}$ , there exists a unique morphism  $A \xrightarrow{\varphi(m,f)} T(B)$  such that  $(m, f)$  is a pullback of  $(\varphi(m, f), \eta_B)$ .

**Proposition 5.13** ([30], **Prop. 6**). For every  $\mathcal{M}$ -partial map classifier  $(T, \eta)$ ,  $T$  preserves pullbacks, and  $\eta$  is **Cartesian**, i.e., for each  $X \xrightarrow{f} Y$ ,  $(\eta_X, f)$  is a pullback of  $(T(f), \eta_Y)$ .

**Definition 5.14** ([7], **Def. 9**). A category  $\mathbf{C}$  is a **quasi-topos** iff

1. it has finite limits and colimits.
2. it is locally Cartesian closed.
3. it has a regular-subobject-classifier.

Category (underlying data type)	quasi-topos adhesive	quasi-adhesive adhesive HLR	hor. weak adh. HLR	vert. weak adh. HLR	references
<b>Set</b> (sets)	✓	✓	✓	✓	[5]
<b>Graph</b> (directed multigraphs)	✓	✓	✓	✓	[5]
<b>HyperGraph</b> (directed ordered hypergraphs)	✓	✓	✓	✓	[7, Ex. 7]
<b>Sig</b> (algebraic signatures)	✓	✓	✓	✓	[7, Ex. 6]
$\hat{\mathbf{S}}$ (presheaves on category $\mathbf{S}$ )	✓	✓	✓	✓	[49, 54]
$\hat{\mathbf{T}}_\Sigma$ (term graphs over a signature $\Sigma$ )	?	✓	✓	✓	[55]
<b>TripleGraph</b> (functor category $[\mathbf{S}_3, \mathbf{Graph}]$ )	?		✓	✓	[3, Fact 4.18]
<b>AGraph<math>_\Sigma</math></b> (attributed graphs over signature $\Sigma$ )	?		✓	✓	[3, Thm. 11.11], [8, 49]
<b>SymbGraph<math>_D</math></b> (symbolic graphs over $\Sigma$ -algebra $D$ )	?		✓	✓	[49], [56, Thm. 2]
<b>uGraph</b> (undirected multigraphs)	?		✓	✓	[18]
<b>ElemNets</b> (elementary Petri nets)	?	(!)	✓	✓	[8]
<b>PTnets</b> (place/transition nets)	?		✓	✓	[3, Fact 4.21], [8]
<b>Spec</b> (algebraic specifications)	✓		✓	✓	[3, Fact 4.24], [7, Ex. 6]
<b>SGraph</b> (directed simple graphs)	✓		✓	✓	[7, Prop. 17], <a href="#">Corollary 5.15(q-v)</a>
<b>Set<math>_F</math></b> (coalgebras for $F : \mathbf{Set} \rightarrow \mathbf{Set}$ )	(*)			(†)	[7], [52, Thm. 1]
<b>ISets</b> (list sets)	?			✓	[48]

Table 2: Examples of categories exhibiting various forms of [adhesivity properties](#). The symbol ? indicates when a certain property is (to the best of our knowledge) not known to hold. Note that for the HLR variants of adhesivity properties, the information not contained in the table is the precise nature (cf. references provided) of the [stable system of monics](#)  $\mathcal{M}$  for which the adhesivity properties hold. Moreover, the precise conditions (\*) and (†) under which the category  $\mathbf{Set}_F$  of  $F$ -coalgebras has quasi-topos or adhesivity properties are provided in [7] and [52, Thm. 1], respectively.

872 Based upon a variety of different results from the rich literature on quasi-topoi, we will now exhibit that quasi-topoi  
 873 indeed possess all technical properties required in order for [non-linear SqPO-rewriting to be well-posed](#):

874 **Corollary 5.15.** *Every quasi-topos  $\mathbf{C}$  enjoys the following properties:*

875 (q-i) *It has (by definition) a [stable system of monics](#)  $\mathcal{M} = \text{rm}(\mathbf{C})$  (the class of regular monos), which coincides with  
 876 the class of [extremal monomorphisms](#) [45, Cor. 28.6], i.e., if  $m = f \circ e$  for  $m \in \text{rm}(\mathbf{C})$  and  $e \in \text{epi}(\mathbf{C})$ , then  
 877  $e \in \text{iso}(\mathbf{C})$ .*

878 (q-ii) *It has (by definition) an  $\mathcal{M}$ -partial map classifier  $(T, \eta)$ .*

879 (q-iii) *It is [rm-quasi-adhesive](#), i.e., it has [pushouts along regular monomorphisms](#), these are [stable under pullbacks](#),  
 880 and [pushouts along regular monomorphisms are pullbacks](#) [6].*

881 (q-iv) *It is a [vertical weak adhesive HLR category](#) (sometimes referred to as  $\mathcal{M}$ -adhesive category) [48, Lem. 13].*

882 (q-v) *The latter entails according to Lemma 5.5 that every quasi-topos is in fact a [weak adhesive HLR category](#).*

883 (q-vi) *For all pairs of composable morphisms  $A \xrightarrow{f} B$  and  $B \xrightarrow{m} C$  with  $m \in \mathcal{M}$ , there [exists a final pullback-](#)  
 884 [complement \(FPC\)](#)  $A \xrightarrow{n} F \xrightarrow{g} C$ , and with  $n \in \mathcal{M}$  ([30, Thm. 1]; cf. Theorem 5.23).*

885 (q-vii) *It possesses an [epi- \$\mathcal{M}\$ -factorization](#) [45, Prob. 28.10]: each morphism  $A \xrightarrow{f} B$  factors as  $f = m \circ e$ , with  
 886 morphisms  $A \xrightarrow{e} \bar{B}$  in  $\text{epi}(\mathbf{C})$  and  $\bar{B} \xrightarrow{m} B$  in  $\mathcal{M}$  (uniquely up to isomorphism in  $\bar{B}$ ).*

887 (q-viii) *It possesses a [strict initial object](#)  $\emptyset \in \text{obj}(\mathbf{C})$  [58, A1.4], i.e., for every object  $X \in \text{obj}(\mathbf{C})$ , there exists a  
 888 morphism  $i_X : \emptyset \rightarrow X$ , and if there exists a morphism  $X \rightarrow \emptyset$ , then  $X \cong \emptyset$ .*

889 *If in addition the strict initial object  $\emptyset$  is  [\$\mathcal{M}\$ -initial](#), i.e., if for all objects  $X \in \text{obj}(\mathbf{C})$  the unique morphism  $i_X : \emptyset \rightarrow X$   
 890 is in  $\mathcal{M}$ , then  $\mathbf{C}$  has [disjoint coproducts](#), i.e., for all  $X, Y \in \text{obj}(\mathbf{C})$ , the pushout of the  $\mathcal{M}$ -span  $X \leftarrow \emptyset \rightarrow Y$  is  
 891  $X \rightarrow X + Y \leftarrow Y$  (cf. [59, Thm. 3.2], which also states that this condition is equivalent to requiring  $\mathbf{C}$  to be a [solid](#)  
 892 [quasi-topos](#)), and the coproduct injections are  $\mathcal{M}$ -morphisms as well. Finally, if pushouts along regular monos of  $\mathbf{C}$   
 893 are van Kampen,  $\mathbf{C}$  is a [rm-adhesive category](#) [6, Def. 1.1].*

894 **Remark 5.16.** An interesting (and, as it turns out, highly relevant) curiosity of the above list of properties enjoyed by  
 895 every [quasi-topos](#) is that, while [quasi-topoi](#) in general *fail* to be [adhesive HLR categories](#) (cf. e.g. [7, 8] for the famous  
 896 and paradigmatic example of **SGraph**, the category of [directed simple graphs](#)), they *do* satisfy axiom (L-iii-a), i.e.,  
 897 [pushouts along regular monomorphisms are stable under pullbacks](#). Therefore, as we will demonstrate in Section 6,  
 898 ([finitary](#)) [quasi-topoi](#) are a suitable type of category for all variants of [Sesqui-Pushout \(SqPO\) semantics](#), while they  
 899 in general do *not* have sufficient properties to support [generic Double-Pushout \(DPO\) semantics](#) (cf. Table 3).

900 The prototypical example of quasi-topoi in rewriting is the following notion of directed simple graphs:

901 **Definition 5.17.** Let **SGraph**, the [category of directed simple graphs](#)<sup>10</sup>, be defined as the category of binary relations  
 902  $\mathbf{BRel} \cong \mathbf{Set} // \Delta$  [7]. Here,  $\Delta : \mathbf{Set} \rightarrow \mathbf{Set}$  is the pullback-preserving diagonal functor defined via  $\Delta X := X \times X$ , and  
 903  $\mathbf{Set} // \Delta$  denotes the full subcategory of the slice category  $\mathbf{Set}/\Delta$  defined via restriction to objects  $m : X \rightarrow \Delta X$  that are  
 904 monomorphisms. More explicitly, an object of  $\mathbf{Set} // \Delta$  is given by  $S = (V, E, \iota)$ , where  $V$  is a set of vertices,  $E$  is a set  
 905 of directed edges, and where  $\iota : E \rightarrow V \times V$  is an injective function. A morphism  $f = (f_V, f_E)$  between objects  $S$  and  
 906  $S'$  is a pair of functions  $f_V : V \rightarrow V'$  and  $f_E : E \rightarrow E'$  such that  $\iota' \circ f_E = (f_V \times f_V) \circ \iota$  (see (43)).

907 The category **SGraph** satisfies the following well-known properties:

<sup>10</sup>Some authors prefer to not consider directly the category **BRel**, but rather define **SGraph** as some category equivalent to **BRel**, where simple graphs are of the form  $\langle V, E \rangle$  with  $E \subseteq V \times V$ . This is evidently equivalent to directly considering **BRel**, whence we chose to not make this distinction in this paper.

908 **Theorem 5.18.** *The category **SGraph** is not adhesive, but it is a quasi-topos [7], and with the following additional*  
 909 *properties:*

910 (S-i) *In **SGraph** [7] (compare [26, Prop. 9]), morphisms  $f = (f_V, f_E)$  are monic (epic) if  $f_V$  is monic (epic), while*  
 911 *isomorphisms satisfy that both  $f_V$  and  $f_E$  are bijective. Regular monomorphisms in **SGraph** are those for which*  
 912  *$(\iota, f_E)$  is a pullback of  $(\Delta(f_V), \iota')$  [7, Lem. 14(ii)], i.e., a monomorphism is regular iff it is edge-reflecting. As is*  
 913 *the case for any quasi-topos, **SGraph** possesses an epi-regular mono-factorization.*

914 (S-ii) *The regular mono-partial map classifier  $(T, \eta)$  of **SGraph** is defined as follows [45, Ex. 28.2(3)]: for every*  
 915 *object  $S = (V, E, \iota) \in \text{obj}(\mathbf{SGraph})$ ,*

$$T(S) := (V_\star = V \uplus \{\star\}, E_\star = E \uplus (V \times \{\star\}) \uplus (\{\star\} \times V) \uplus \{\{\star, \star\}, \iota_\star\}), \quad (42)$$

916 *where  $\iota_\star$  is the evident inclusion map, and moreover  $\eta_S : S \rightarrow T(S)$  is the (by definition edge-reflecting)*  
 917 *inclusion of  $S$  into  $T(S)$ .*

918 (S-iii) **SGraph** *possesses a regular mono-initial object  $\emptyset = (\emptyset, \emptyset, \emptyset \rightarrow \emptyset)$ .*

919 **PROOF.** While most of these results are standard, we briefly demonstrate that the epi-regular mono-factorization of  
 920 **SGraph** [7] is “inherited” from the epi-mono-factorization of the adhesive category **Set**. To this end, given an arbitrary  
 921 morphism  $f = (f_V, f_E)$  in **SGraph** as on the left of (43), the epi-mono-factorization  $f_V = m_V \circ e_V$  lifts via application  
 922 of the diagonal functor  $\Delta$  to a decomposition of the morphism  $f_V \times f_V$ . Pulling back  $(\Delta(m_V), \iota')$  results in a span  $(\tilde{\iota}, f'_E)$   
 923 and (by the [universal property of pullbacks](#)) an induced morphism  $f'_E$  that makes the diagram commute. By stability of  
 924 monomorphisms under pullbacks,  $\tilde{\iota}$  is a monomorphism, thus the square marked (\*) precisely constitutes the data of a  
 925 regular monomorphism in **SGraph**, while the square marked (†) is an epimorphism in **SGraph** (since  $e_V \in \text{epi}(\mathbf{Set})$ ).  
 926 □

(43)

927 A number of additional examples of quasi-topoi relevant for rewriting applications is referenced in Table 2. Most  
 928 of these examples arise via so-called *Artin gluing* as demonstrated in [7].

### 929 5.3. Explicit constructions of universal categorical operations (multi-sums, FPCs, multi-IPCs and FPAs)

930 The focus of this section is to provide some results on constructions (as opposed to merely existence) of some of  
 931 the key concepts necessary for practically working with compositional rewriting theories.

#### 932 5.3.1. $\mathcal{M}$ -multi-sums

933 An important technical ingredient for our constructions is the notion of  $\mathcal{M}$ -multi-sum (see Definition 2.1), a special  
 934 case of the general theory of multi-(co-)limits due to Diers [27]. In order to provide a more constructive version of this  
 935 definition, it is in practice often useful to consider a certain notion of finiteness of objects in the underlying categories  
 936 in the sense of [8]:

937 **Definition 5.19** ([8], Def. 2.8 & Def. 4.1). Let  $\mathbf{C}$  be a category with a [stable system of monics](#)  $\mathcal{M}$ .

938 (F-i) An object  $A$  of  $\mathbf{C}$  is said to be *finitary* (or finitely  $\mathcal{M}$ -well-powered) if it has only finitely many  $\mathcal{M}$ -subobjects  
 939 up to isomorphism. Here, an  *$\mathcal{M}$ -subobject* of  $A$  is an  $\mathcal{M}$ -morphism  $X \succ x \rightarrow A$ , and an  $\mathcal{M}$ -subobject  $Y \succ y \rightarrow A$   
 940 is defined to be isomorphic to  $x$  if there exists an isomorphism  $X \xrightarrow{\phi} Y$  such that  $y = \phi \circ x$ .  $\mathbf{C}$  is a *finitary*  
 941 *category* (w.r.t.  $\mathcal{M}$ ) if every object of  $\mathbf{C}$  is *finitary*.

942 (F-ii) The *finitary restriction* of  $\mathbf{C}$ , denoted  $\mathbf{C}_{\text{fin}}$ , is defined by the restriction to *finitary objects* and morphisms thereof.

943 The importance of the work presented in [8] for constructing compositional rewriting theories is in particular that  
 944 it provides an elegant method to demonstrate that finitary restrictions of suitable base categories preserve the requisite  
 945 *adhesivity properties*, and in addition yield categories that are guaranteed to possess a certain form of factorization  
 946 system:

947 **Theorem 5.20 ([8], Thm. 4.6 & Fact 3.4).** *Let  $\mathbf{C}$  be a vertical weak adhesive HLR category with respect to a stable*  
 948 *system of monics  $\mathcal{M}$ . Denote by  $\mathbf{C}_{\text{fin}}$  the finitary restriction of  $\mathbf{C}$ .*

- 949 (i)  $\mathbf{C}_{\text{fin}}$  has a *stable system of monics*  $\mathcal{M}_{\text{fin}} = \mathcal{M} \cap \mathbf{C}_{\text{fin}}$ .
- 950 (ii)  $\mathbf{C}_{\text{fin}}$  is a *vertical weak adhesive HLR category* with respect to  $\mathcal{M}_{\text{fin}}$ .
- 951 (iii)  $\mathbf{C}_{\text{fin}}$  is  $(\mathcal{E}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ -structured, where  $\mathcal{E}_{\text{fin}}$  denotes the class of *extremal morphisms* (w.r.t.  $\mathcal{M}_{\text{fin}}$ ) defined as<sup>11</sup>

$$\mathcal{E}_{\text{fin}} := \{e \in \mathbf{C}_{\text{fin}} \mid \forall m, f \in \mathbf{C}_{\text{fin}} : e = m \circ f : m \in \mathcal{M}_{\text{fin}} \Rightarrow m \in \text{iso}(\mathbf{C}_{\text{fin}})\}. \quad (44)$$

952 With these preparations, we may verify that as presented below, under suitable conditions on the underlying  
 953 category,  *$\mathcal{M}$ -multi-sums* (as e.g. also considered in [60]) coincide with the concept referred to as  $(\mathcal{E}, \mathcal{M})$ -*pair factor-*  
 954 *izations* in the graph rewriting literature [3]:

955 **Lemma 5.21 ([3]; [8], Fact A.3.7).** *Let  $\mathbf{C}$  be a finitary vertical weak adhesive HLR category with respect to a stable*  
 956 *system of monics  $\mathcal{M}$ , and denote by  $\mathcal{E}$  the class of extremal morphisms with respect to  $\mathcal{M}$ .*

- 957 (i) **Existence:** *If  $\mathbf{C}$  has binary coproducts, then every cospan of  $\mathcal{M}$ -morphisms  $A \dashv a \rightarrow Z \leftarrow b \dashv B$  factors es-*  
 958 *entially uniquely through a cospan of  $\mathcal{M}$ -morphisms  $A \dashv y_A \rightarrow Y \leftarrow y_B \dashv B$  and an  $\mathcal{M}$ -morphism  $Y \dashv m \rightarrow Z$ ,*  
 959 *where  $m$  is obtained via the  $\mathcal{E}$ - $\mathcal{M}$ -factorization  $A+B \dashv e \twoheadrightarrow Y \dashv m \rightarrow Z$  of the induced morphism  $A+B \dashv [a, b] \rightarrow$*   
 960  *$Z$ , and where  $y_A = e \circ \text{in}_A$  and  $y_B = e \circ \text{in}_B$ .*
- 961 (ii) **Construction:** *if  $\mathbf{C}$  in addition has an  $\mathcal{M}$ -initial object  $\emptyset$ , then  $\Sigma_{\mathcal{M}}(A, B)$  consists of cospans of  $\mathcal{M}$ -morphisms*  
 962 *obtained as pushouts  $A \dashv p_A \rightarrow P \leftarrow p_B \dashv B$  of  $\mathcal{M}$ -spans  $A \leftarrow x_A \dashv X \dashv x_B \rightarrow B$  (i.e., “ $\mathcal{M}$ -partial overlaps”)*  
 963 *extended by  $\mathcal{E}$ -morphisms  $P \dashv q \twoheadrightarrow Q$  such that  $q_A = q \circ p_A$  and  $q_B = q \circ p_B$  are in  $\mathcal{M}$ .*
- 964 (iii) **Refinements:** *if  $\mathbf{C}$  in addition has pullbacks, and if pushouts along  $\mathcal{M}$ -morphisms in  $\mathbf{C}$  are stable under pull-*  
 965 *backs, then the extension morphisms  $P \dashv q \twoheadrightarrow Q$  are morphisms in  $\mathcal{E} \cap \text{mono}(\mathbf{C})$  (so-called “refinements”).*

966 **Corollary 5.22.** *Every quasi-topos with  $\mathcal{M}$ -initial object  $\emptyset$  has  $\mathcal{M}$ -multi-sums and refinements according to Lemma 5.21.*

967 Since in an adhesive category all monomorphisms are regular [4], in this case the multi-sum construction simplifies  
 968 to the statement that every monic cospan can be uniquely factorized as a cospan obtained as the pushout of a monic  
 969 span composed with a monomorphism. It is however worthwhile emphasizing that for generic quasi-topoi  $\mathbf{C}$  one may  
 970 have  $\mathcal{M} \neq \text{mono}(\mathbf{C})$ , as is the case in particular for the quasi-topos **SGraph** of simple graphs. We illustrate this  
 971 phenomenon in Figure 4, via presenting the multi-sum construction for  $A = B = \bullet$ .

<sup>11</sup>It is instructive to compare this definition to the case of a category with *epi-mono-factorizations*: here, since  $m \circ f$  being an epimorphism implies that  $m$  is an epimorphism, then if  $m$  is also a monomorphism, this indeed implies that it is an isomorphism. However, it is important to note that as highlighted in [8], there exist finitary categories where  $\mathcal{E}_{\text{fin}}$  is *not* a class of epimorphisms.

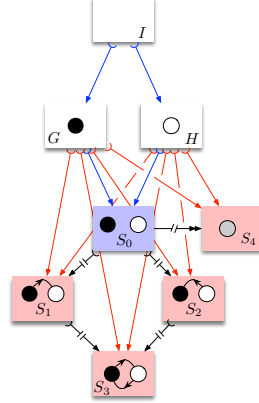


Figure 4: Example for an  $\mathcal{M}$ -multi-sum  $\Sigma_{\mathcal{M}}(\bullet, \bullet)$  of one-vertex graphs in  $\mathbf{SGraph}$  (i.e., for  $\mathcal{M} = \text{rm}(\mathbf{SGraph})$ ). Note in particular the monic-epis that extend the two-vertex graph  $S_0$  into the graphs  $S_1, S_2$  and  $S_3$ , all of which have the same vertices as  $S_0$  (recalling that a morphism in  $\mathbf{SGraph}$  is monic/epic if it is so on vertices), yet additional edges, so that in particular none of the morphisms  $S_0 \rightarrow S_j$  for  $j = 1, 2, 3$  is edge-reflecting.

972 5.3.2. FPCs along  $\mathcal{M}$ -morphisms

973 Let us now turn to the question of the existence of FPCs. To this end, it will prove useful to recall from [30] the  
 974 following constructive result:

975 **Theorem 5.23 ([30], Thm. 1).** For a category  $\mathbf{C}$  with  $\mathcal{M}$ -partial map classifier  $(T, \eta)$ , the final pullback complement  
 976 (FPC) of a composable sequence of arrows  $A \xrightarrow{f} B$  and  $B \xrightarrow{m} C$  with  $m \in \mathcal{M}$  is guaranteed to exist, and is constructed  
 977 via the following algorithm:

- 978 1. Let  $\bar{m} := \varphi(m, id_B)$  (i.e., the morphism that exists by the universal property of  $(T, \eta)$ , cf. square (1) below).
- 979 2. Construct  $T(A) \xleftarrow{\bar{n}} F \xrightarrow{g} C$  as the pullback of  $T(A) \xrightarrow{T(f)} T(B) \xleftarrow{\bar{m}} C$  (cf. square (2) below); by the universal  
 980 property of pullbacks, this in addition entails the existence of a morphism  $A \xrightarrow{n} F$ .

981 Then  $(n, g)$  is the FPC of  $(f, m)$ , and  $n$  is in  $\mathcal{M}$ .

$$\begin{array}{ccc}
 A \xrightarrow{f} B & & \\
 \downarrow m & & \\
 C & & \\
 \rightsquigarrow & & \\
 \begin{array}{ccccc}
 A & \xrightarrow{f} & B & & \\
 \downarrow \eta_A & & \downarrow \eta_B & & \\
 T(A) & \xrightarrow{T(f)} & T(B) & \xleftarrow{\bar{m}} & C \\
 \uparrow \bar{n} & & \uparrow \bar{m} & & \uparrow \eta_C \\
 F & \xrightarrow{g} & C & \xleftarrow{m} & B \\
 \downarrow \eta_F & & \downarrow \eta_C & & \\
 T(A) & \xrightarrow{T(f)} & T(B) & \xleftarrow{\bar{m}} & C
 \end{array}
 \end{array} \tag{45}$$

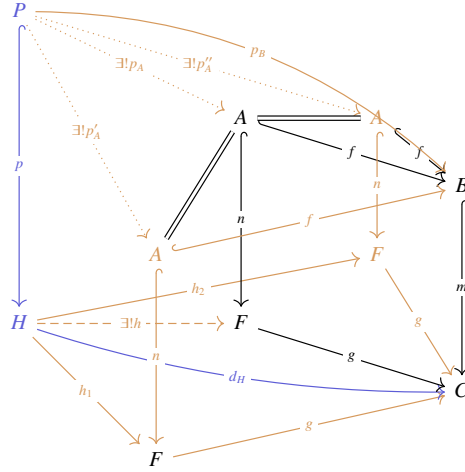
982 An interesting (and to the best of our knowledge open) question that arises in considering (input-) linear Sesqui-  
 983 Pushout (SqPO) rewriting semantics concerns the nature of the morphism  $g$  in the above definition of FPCs under the  
 984 constraint that  $f$  is an  $\mathcal{M}$ -morphism, which we refer to as *stability of  $\mathcal{M}$ -morphisms under FPCs*. We may provide a  
 985 first partial answer for the case where  $\mathcal{M} = \text{mono}(\mathbf{C})$ :

986 **Lemma 5.24.** Let  $\mathbf{C}$  be a category with a stable system of monics  $\mathcal{M} = \text{mono}(\mathbf{C})$  that comprises all monomorphisms  
 987 of  $\mathbf{C}$ , that has pullbacks along monomorphisms, and that has a  $\text{mono}(\mathbf{C})$ -partial map classifier. Then an FPC  $A \xrightarrow{n} F \xrightarrow{g} C$   
 988 of a composable sequence of monomorphisms  $A \xrightarrow{f} B \xrightarrow{m} C$  satisfies the property that both  $n$  and  
 989  $g$  are monomorphisms.

990 **PROOF.** The property that  $n \in \text{mono}(\mathbf{C})$  follows from the stability of monomorphisms under pullback, hence it remains  
 991 to show that  $g$  is a monomorphism. This is by definition equivalent to showing that for arbitrary morphisms  $H \xrightarrow{h_1} F$



992 and  $H - h_2 \rightarrow F$  such that  $g \circ h_1 = g \circ h_2$ , it follows that  $h_1 = h_2$ . To this end, let  $d_H = g \circ h_1 = g \circ h_2$ , and obtain the  
 993 span  $\langle p, p_B \rangle$  via taking a pullback of the cospan  $\rangle d_H, m \langle$ :



(46)

- 994 • By the **universal property of pullbacks**, there exist unique morphisms  $p'_A$  and  $p''_A$  that make the front and back  
 995 parts of the diagram commute, and since  $f$  is a monomorphism and thus the span  $\langle id_A, id_A \rangle$  is a pullback of the  
 996 cospan  $\rangle f, f \langle$ , there also exists a unique morphism  $p_A$  that makes the inner part of the diagram commute. The  
 997 latter entails in particular that  $p_A = p'_A = p''_A$ .
- 998 • Since by assumption  $(n, g)$  is an FPC of  $(f, m)$ , by the **universal property of FPCs** there exists a unique morphism  
 999  $H - h \rightarrow F$  that makes the inner diagram commute, and such that in particular  $h \circ g = h_1 \circ g = h_2 \circ g$ . Thus  
 1000 uniqueness entails that  $h = h_1 = h_2$ , which proves the claim.

1001 The above result ensures in particular that **adhesive categories** with mono-partial map classifiers yield suitable  
 1002 “linear” SqPO rewriting semantics (as already noted in [23]). For more general scenarios, we quote here the following  
 1003 result from the literature on restriction categories:

1004 **Proposition 5.25 ([57], Prop. 4.16).** *Let  $\mathbf{C}$  be a category with a **stable system of monics**  $\mathcal{M}$  and an  $\mathcal{M}$ -partial map  
 1005 classifier  $(T, \eta)$ . Then  $T(m) \in \mathcal{M}$  for all  $m \in \mathcal{M}$  iff  $T(\eta_C) \in \mathcal{M}$  for all objects  $C$  in  $\mathbf{C}$ .*

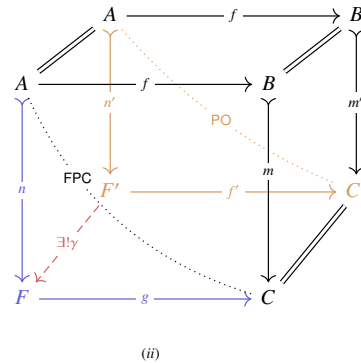
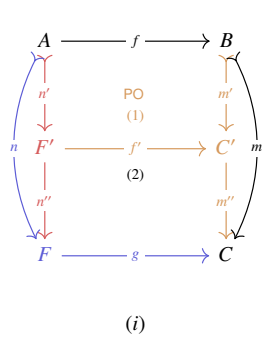
1006 We suspect that the above result should provide a method for determining whether or not, e.g., comma categories  
 1007 constructed from categories with partial map classifiers that preserve the respective stable systems of monics will  
 1008 have the sought-after stability properties of monics under FPCs, as is the case for the example of **directed simple  
 1009 graphs**, yet we leave further investigations to future work.

### 1010 5.3.3. Multi-initial pushout complements

1011 While the standard literature on graph rewriting (cf. [3]) provides some examples of explicit constructions of  
 1012 pushout complements, we require in general a construction of **multi-initial pushout complements**, as in Definition 4.9,  
 1013 for the “non-linear” variants of rewriting semantics. In this subsection, we demonstrate sufficient conditions under  
 1014 which a construction for such mIPCs can be made explicit.

1015 **Proposition 5.26.** *Let  $\mathbf{C}$  be a category with a **stable system of monics**  $\mathcal{M}$  that has pushouts along  $\mathcal{M}$ -morphisms and  
 1016 that possesses an  $\mathcal{M}$ -partial map classifier  $(T, \eta)$ . Assume further that pushouts along  $\mathcal{M}$ -morphisms are pullbacks in  
 1017  $\mathbf{C}$ . Then for any composable sequence of morphisms  $A - f \rightarrow B - m \rightarrow C$  with  $m$  in  $\mathcal{M}$ , the following construction  
 1018 provides  $\mathcal{P}(f, m)$ :*

- 1019 1. Take the FPC  $A - n \rightarrow F - g \rightarrow C$  of  $(f, m)$  (cf. diagram (i) in (47) below).
- 1020 2. For every factorization  $A - n' \rightarrow F' - n'' \rightarrow F$  of  $A - n \rightarrow F$ , where  $n'$  is an  $\mathcal{M}$ -morphism, take a pushout  
 1021  $F' - f' \rightarrow C' \leftarrow m' \rightarrow B$ , which by the **universal property of pushouts** entails that there exists a universal  
 1022 morphism  $C' - m'' \rightarrow C$ . If  $m''$  is an isomorphism, then  $A - n' \rightarrow F' - m'' \circ f' \rightarrow C$  is an element of  $\mathcal{P}(f, m)$ .



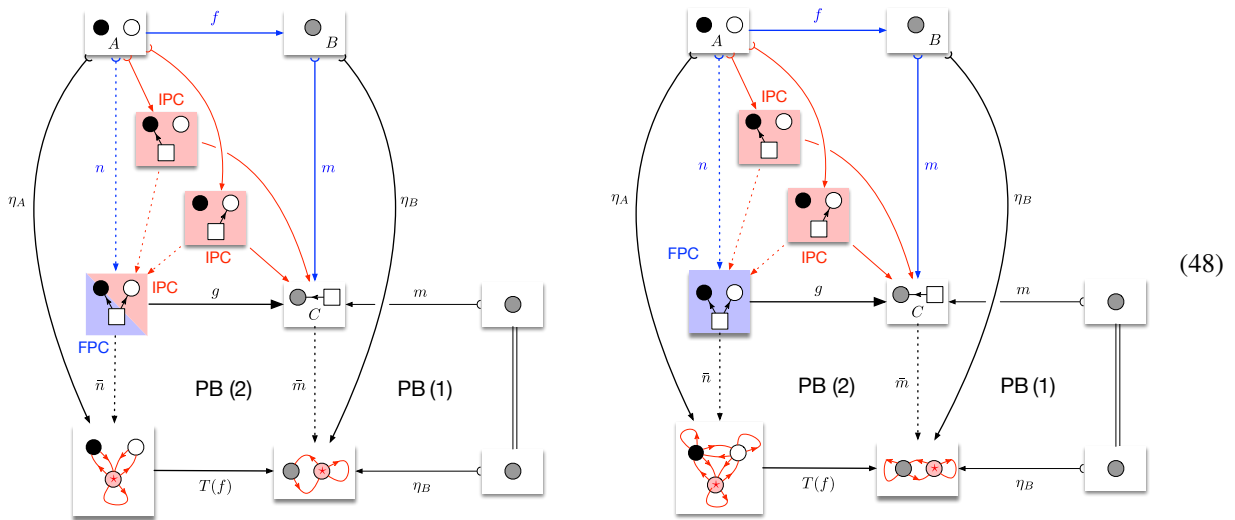
(47)

1023

1024 **PROOF.** Since  $\mathbf{C}$  is assumed to possess an  $\mathcal{M}$ -partial map classifier, by Theorem 5.23 this entails that  $\mathbf{C}$  has FPCs  
 1025 along  $\mathcal{M}$ -morphisms. For any element of  $\mathcal{P}(f, m)$ , i.e., for a pushout square as the vertical back square in diagram (ii)  
 1026 of (47), if  $(n, g)$  is an FPC of  $(f, m)$ , then by the universal property of FPCs there exists a unique morphism  $F' - \gamma \rightarrow F$   
 1027 that makes the diagram commute. Consequently, up to essential uniqueness, every pushout square that is an element  
 1028 of  $\mathcal{P}(f, \beta)$  fits a diagram of shape (i) in (47).  $\square$

1029 The above construction is typically assumed to be applied in situations where the underlying category is finitary  
 1030 with respect to a stable system of monics  $\mathcal{M}$ . According to [45, Prop. 14.9(2)], a sufficient condition for a composition  
 1031 of morphisms  $f \circ g$  that is an  $\mathcal{M}$ -morphism to also satisfy that  $g \in \mathcal{M}$  is that  $f$  is a monomorphism (not necessarily in  
 1032  $\mathcal{M}$ ). In concrete examples (see below), it seems that all factorizations used in constructions of mIPCs in the finitary  
 1033 setting are of the aforementioned form, yet it remains unclear to us whether this is in fact the only possible situation,  
 1034 hence we defer a full investigation of this point to future work.

1035 An example of an  $\mathcal{M}$ -multi-IPC construction both in **SGraph** and in **Graph** is given in the diagram below. Note  
 1036 that in **Graph**, the  $\mathcal{M}$ -multi-IPC does not contain the FPC contribution (since in **Graph** the pushout of the relevant  
 1037 span would yield to a graph with a multi-edge).



1038 We conclude this subsection by mentioning the following useful result, which clarifies that under suitable as-  
 1039 sumptions on the underlying category, mIPCs along sequences of composable  $\mathcal{M}$ -morphisms are essentially unique  
 1040 when they exist (in which case it is customary to speak of (“non-multi-”) pushout complements, compare e.g. [3,  
 1041 Fact 4.27.4])

1042 **Proposition 5.27.** Let  $\mathbf{C}$  be a category with a stable system of monics  $\mathcal{M}$  and with an  $\mathcal{M}$ -partial map classifier  $(T, \eta)$ ,  
 1043 such that pushouts along  $\mathcal{M}$ -morphisms are stable under  $\mathcal{M}$ -pullbacks, such that pushouts along  $\mathcal{M}$ -morphisms are

1044 pullbacks, and such that  $\mathcal{M}$ -morphisms are stable under FPCs. Then if the  $\mathcal{M}$ -multi-initial pushout complement  
 1045  $\mathcal{P}(f, \beta)$  for a composable sequence of  $\mathcal{M}$ -morphisms  $A \xrightarrow{f} B \xrightarrow{\beta} B'$  (i.e., where both  $f$  and  $\beta$  are in  $\mathcal{M}$ ) is  
 1046 non-empty, then  $\mathcal{P}(f, \beta)$  is essentially unique, and moreover every element of  $\mathcal{P}(f, \beta)$  yields a square that is both a  
 1047 pushout and an FPC.

1048 **PROOF.** It suffices to note that assumptions on  $\mathbf{C}$  ensure that **pullback-pushout decomposition** is applicable, and that  
 1049 according to Proposition 5.26, every element of an mIPC fits into a diagram of the shape (i) in (47), where  $f, f'$  and  
 1050  $c$  are in  $\mathcal{M}$ , and where  $p$  is an isomorphism: since the outer square in (47)(i) is an FPC and thus a pullback, the top  
 1051 square is a pushout, and the horizontal morphisms are in  $\mathcal{M}$ , the bottom square is a pullback. Since isomorphisms are  
 1052 stable under pullback,  $\gamma$  is an isomorphism, hence every pushout that is an element of  $\mathcal{P}(f, \beta)$  is isomorphic (via the  
 1053 respective morphism  $\gamma$ ) to the FPC square, demonstrating that if  $\mathcal{P}(f, \beta)$  is non-empty, then every element yields both  
 1054 a pushout and an FPC.  $\square$

### 1055 5.3.4. Final pullback complement pushout augmentations (FPAs)

1056 Let us now consider the final construction of interest, that of **FPAs**, as given in Definition 4.16. This naturally  
 1057 suggests the following explicit construction:

1058 **Lemma 5.28.** *Let  $\mathbf{C}$  be a category with a stable system of monics  $\mathcal{M}$  that is a vertical weak adhesive HLR category  
 1059 and finitary with respect to  $\mathcal{M}$  (which entails by Theorem 5.20(iii) that  $\mathbf{C}$  is  $(\mathcal{E}, \mathcal{M})$ -structured, for  $\mathcal{E}$  the class of  
 1060 extremal morphisms w.r.t.  $\mathcal{M}$ ), such that pushouts along  $\mathcal{M}$ -morphisms are stable under  $\mathcal{M}$ -pullbacks in  $\mathbf{C}$ , and such  
 1061 that pushouts along  $\mathcal{M}$ -morphisms are pullbacks. Then given a pushout along an  $\mathcal{M}$ -morphism such as the square in  
 1062 the diagram below, one may construct the FPAs of this pushout as follows:*

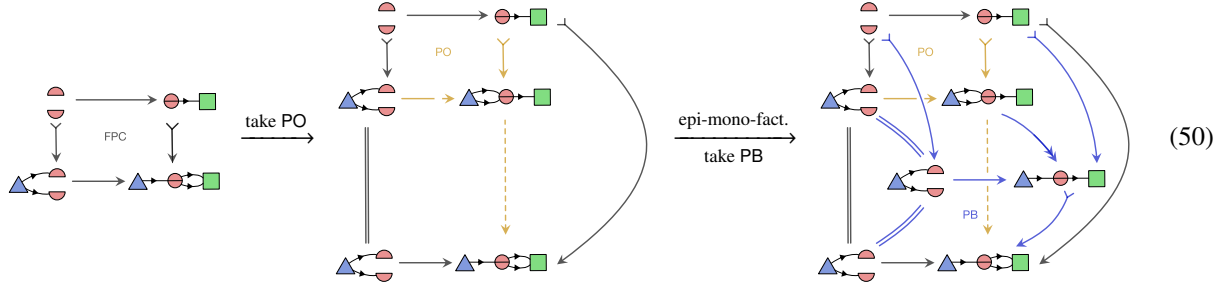
- 1063 1. For every  $\mathcal{E}$ -morphism  $P \xrightarrow{e} E$  such that  $e \circ \beta$  is in  $\mathcal{M}$ , and such that  $(\beta, id_B)$  is a pullback of  $(e, e \circ \beta)$ , take  
 1064 an FPC  $(\varphi, g)$  of  $(f, e \circ \beta)$ .
- 1065 2. Since by assumption *pushouts along  $\mathcal{M}$ -morphisms are pullbacks*, by *pullback-pullback composition*  $(\alpha, f)$  is a  
 1066 pullback of  $(e \circ f', e \circ \beta)$ , hence by the *universal property of FPCs*, there exists a unique morphism  $A' \xrightarrow{\gamma} F$   
 1067 into the FPC object  $F$ . If  $\gamma$  is in  $\mathcal{M}$ , then  $(\gamma, g, e)$  is in the *FPA* of the pushout square.



1068 In the above definition, while we assumed that  $\mathbf{C}$  is **finitary** with respect to the **stable system of monics**  $\mathcal{M}$ , i.e.,  
 1069  $\mathcal{M}$ -well-powered, in order for construction to be practicable, we strictly speaking also have to assume that  $\mathbf{C}$  is  $\mathcal{E}$ -  
 1070 *well-copowered* (“co-finitary”). The latter entails that every object  $X$  of  $\mathbf{C}$  has only finitely many **quotients**, where a  
 1071 *quotient* of  $X$  is an isomorphism class of  $\mathcal{E}$ -morphisms  $X \xrightarrow{e} E$ , with  $X \xrightarrow{e'} \rightarrow E$  isomorphic to  $e$  if there exists  
 1072 an isomorphism  $E \xrightarrow{\varepsilon} E'$  such that  $e' = \varepsilon \circ e$ . One might wonder whether under the assumptions of the above  
 1073 definition, the assumptions on  $\mathbf{C}$  (i.e.,  $\mathbf{C}$  being **finitary**, **having FPCs along  $\mathcal{M}$ -morphisms**, being a **vertical weak  
 1074 adhesive HLR category** and thus  $(\mathcal{E}, \mathcal{M})$ -structured) might be sufficient in order for  $\mathbf{C}$  to be  $\mathcal{E}$ -well-copowered, albeit  
 1075 this does not seem to be a standard result in category theory to the best of our knowledge (perhaps apart from the  
 1076 special case of Grothendieck topoi [58], which are **adhesive categories** [54] and thus in particular also **vertical weak  
 1077 adhesive HLR categories**).

1078 **Example 5.29.** In order to illustrate the notion of **FPAs** in their application to **(auto-augmented, inert)-factorizations**  
 1079 of **final pullback complements**, consider the example of an FPC in the category **Graph** of **directed multigraphs** as

1080 shown in the diagram below left:



1081 In order to exhibit the FPC factorization, first take a pushout of the span of the FPC, which by the [universal property of](#)  
 1082 [pushouts](#) yields the dashed arrow. Since **Graph** is [\(epi, mono\)-structured](#), we may take the [epi-mono-factorization](#)  
 1083 of the dashed arrow, followed by taking a pullback along the monomorphism produced by the factorization. By [vertical](#)  
 1084 [FPC-pullback decomposition](#), the bottom square as well as the vertical composite of the middle and the top square are  
 1085 FPCs. One may verify that the aforementioned composite square is indeed an [auto-augmented FPC](#), while the bottom  
 1086 square is an [inert FPC](#), in the sense of Theorem 4.19. If we interpret the original FPC as a direct derivation under  
 1087 [sesqui-pushout \(SqPO\) semantics](#) (i.e., along a rule of the form  $r = (K \leftarrow id_K - K - i \rightarrow I)$ ), the [auto-augmented FPC](#)  
 1088 encodes the minimal context for a SqPO-type direct derivation along the given rule to produce the graph at the bottom  
 1089 left of the original FPC square. On the other hand, the [inert FPC](#) encodes a context extension that does not modify the  
 1090 outcome of the SqPO-type direct derivation, since it only adds edges to the context that are in effect implicitly deleted  
 1091 during the direct derivation.

## 1092 6. Examples of compositional rewriting semantics

1093 In this section, we investigate a number of variants of [DPO-semantics](#) and [SqPO-semantics](#), and illustrations  
 1094 thereof, for categorical rewriting, parametrized by the choice of the class of rules considered. The results of this  
 1095 section are summarized in Table 3, which makes explicit the conditions under which DPO and SqPO rewriting are  
 1096 compositional.

### 1097 6.1. DPO and SqPO semantics

1098 In much of the traditional work on graph- and categorical rewriting theories [3], while it was appreciated early in  
 1099 its development that SqPO-rewriting permits the [cloning of subgraphs](#) [23], and that both SqPO- and DPO-semantics  
 1100 permit the [fusion of subgraphs](#) (i.e., via [input-linear](#), but output-non-linear rules), the non-uniqueness of pushout  
 1101 complements along non-monic morphisms for the DPO- and the lack of a [concurrency theorem](#) in the SqPO-case  
 1102 in general has prohibited a detailed development of non-linear rewriting theories to date. Interestingly, the SqPO-  
 1103 type concurrency theorem for [linear](#) rules as developed in [19] exhibits the same obstacle for the generalization to  
 1104 non-linear rewriting as the DPO-type concurrency theorem, i.e., the non-uniqueness of certain pushout complements.

1105 Our proof for non-linear rules identifies in addition a new and highly non-trivial “back-propagation effect”, which  
 1106 will be highlighted in Section 6.4 (cf. also Example 6.12 for an in-detail heuristic discussion of this effect). It may be  
 1107 worthwhile emphasizing that there exists previous work that aimed at circumventing some of the technical obstacles of  
 1108 non-linear rewriting either via specializing the semantics e.g. from double pushout to a version based upon so-called  
 1109 [minimal](#) pushout complements [26], or from sesqui-pushout to [reversible](#) SqPO-semantics [28, 29] or other variants  
 1110 such as [AGREE-rewriting](#) [30]. In contrast, we will in the following introduce the “true” extensions of both SqPO-  
 1111 and DPO-rewriting to the non-linear setting, with our constructions based upon [multi-sums](#), [multi-IPCs](#) and [FPAs](#).

1112 We focus here on the following eight variants of categorical rewriting semantics:

1113 **Definition 6.1.** Let  $\mathbf{C}$  be a category with a [stable system of monics](#)  $\mathcal{M}$ .

1114 (i) A [rule](#), denoted  $O \leftarrow r - I$ , is a span  $r = (O \leftarrow o_r - K_r - i_r \rightarrow I)$  in  $\mathbf{C}$ . We refer to a rule as

- 1115 • [output-linear](#) if  $o_r$  is in  $\mathcal{M}$ ,

Property	Double-Pushout semantics			Sesqui-Pushout semantics			
	linear	semi-linear	generic	linear	output-linear	input-linear	generic
$\mathbb{D}_0$ has multi-sums	✓ (Lemma 5.21)						
$\exists$ horizontal/vertical units	✓ (Corollary 6.2)						
vertical composition	✓ (by pushout-pushout composition)			✓ (by PO-PO- and vertical FPC composition)			
horizontal composition (Proposition 6.3)	$\mathbf{C}$ has pullbacks along $\mathcal{M}$ -morphisms $\wedge$ (V-iii-a)	$\wedge$ (W-iii-a)	$\mathbf{C}$ has pullbacks $\wedge$ (L-iii-a)	$\mathbf{C}$ has pullbacks $\wedge$ (V-iii-a)   $\wedge$ (H-iii-a)   $\wedge$ (V-iii-a)   $\wedge$ (L-iii-a)			
horizontal decomposition (Proposition 6.5)	$\mathbf{C}$ has pushouts along $\mathcal{M}$ -morphisms $\wedge$ has pullbacks along $\mathcal{M}$ -morphisms $\wedge$ (V-iii-b)		$\wedge$ has pullbacks $\wedge$ (L-iii-b)	$\mathbf{C}$ has pullbacks $\wedge$ has pushouts and FPCs along $\mathcal{M}$ -morphisms $\wedge$ (V-iii-a)   $\wedge$ (H-iii-a)   $\wedge$ (V-iii-a)   $\wedge$ (L-iii-a)			
$\mathbb{D}_1$ has pullbacks (Proposition 6.6)	(V-iii-a)						
$S$ is a multi-opfibration (Theorem 6.7)	$\mathbf{C}$ is a vertical weak adhesive HLR category			$\mathbf{C}$ is vertical weak adhesive HLR and has FPCs along $\mathcal{M}$ -morphisms			
$T$ is a residual multi-opfibration (Theorem 6.7)	$\mathbf{C}$ is a vertical weak adhesive HLR category			$\mathbf{C}$ is vertical weak adhesive HLR and has FPCs along $\mathcal{M}$ -morphisms			

Table 3: Requirements on the underlying category for giving rise to compositional rewriting semantics of the various kinds. For all cases, we minimally assume that  $\mathbf{C}$  has a *stable system of monics*, with respect to which  $\mathbf{C}$  is *finitary*, with respect to which the variants of *adhesivity properties* are required to hold, and such that  $\mathbb{D}_0 := \mathbf{C}|_{\mathcal{M}}$  has pullbacks. The latter is equivalent to requiring that  $\mathbf{C}$  has *pullbacks of cospans of  $\mathcal{M}$ -morphisms*, which is true for all of the listed adhesivity properties. We moreover use the abbreviation *(W-iii)* to denote (V-iii)  $\wedge$  (H-iii).

- 1116 • *input-linear* if  $i_r$  is in  $\mathcal{M}$ , and
- 1117 • *linear* if both  $o_r$  and  $i_r$  are in  $\mathcal{M}$ .

1118 We will also refer to arbitrary spans as *generic* rules.

1119 (ii) In *Double-Pushout (DPO) semantics*, a *direct derivation* is defined as a commutative diagram as in (51) below,  
1120 where the vertical morphisms are in  $\mathcal{M}$ , and where the square marked  $(\dagger_\alpha)$  is a pushout, while the square marked  
1121  $(*_\alpha)$  is an element of an  *$\mathcal{M}$ -multi-IPC* (and thus in particular also a pushout). A category  $\mathbf{C}$  is thus *suitable for*  
1122 *DPO-semantics* if it has *multi-initial pushout complements (miPCs)* along  $\mathcal{M}$ -morphisms, if it has pushouts  
1123 along  $\mathcal{M}$ -morphisms, and if  $\mathcal{M}$ -morphisms are stable under pushout.

1124 (iii) In *Sesqui-Pushout (SqPO) semantics*, a *direct derivation* is defined as a commutative diagram as in (51) below,  
1125 where the vertical morphisms are in  $\mathcal{M}$ , and where the square marked  $(\dagger_\alpha)$  is a pushout, while the square marked  
1126  $(*_\alpha)$  is a *final pullback complement (FPC)*. A category  $\mathbf{C}$  is thus *suitable for SqPO-semantics* if it has FPCs along  
1127  $\mathcal{M}$ -morphisms, if it has pushouts along  $\mathcal{M}$ -morphisms, and if  $\mathcal{M}$ -morphisms are stable under pushout.

$$\begin{array}{ccc}
\begin{array}{ccc} O & \xleftarrow{r} & I \\ \downarrow m^* & \Downarrow \alpha & \downarrow m \\ O' & \xleftarrow{r'} & I' \end{array} & := & \begin{array}{ccccc} O & \xleftarrow{o_r} & K_r & \xrightarrow{i_r} & I \\ \downarrow m^* & \text{(\dagger}_\alpha) & \downarrow k_\alpha & \text{(\ast}_\alpha) & \downarrow m \\ O' & \xleftarrow{o_{r'}} & K_{r'} & \xrightarrow{i_{r'}} & I' \end{array}
\end{array} \quad (51)$$

1128 It is conventional to refer to the  $\mathcal{M}$ -morphisms  $m$  and  $m^*$  as *match* and *co-match*, respectively. Finally, we will refer to  
1129 either of the two semantics as *generic* if no special restrictions are imposed upon the underlying *rules*, and as *linear*,  
1130 *output-linear* or *input-linear* if rules are restricted to being *linear*, *output-linear* or *input-linear*, respectively. We will  
1131 sometimes also use the term *semi-linear* as an abbreviation for “*output-linear* or *input-linear*”.

1132 As discussed in further detail in Section 1.1, each of these eight types of semantics permits a different set of features,  
1133 e.g., for the rewriting of *directed multigraphs*, where the type of linearity of the rules entails whether or not fusing  
1134 or cloning of subgraphs are possible, and where the choice of SqPO- versus DPO-semantics yields a difference also  
1135 in whether or not edges may be implicitly deleted (in addition to modifying the precise type of cloning semantics for  
1136 the non-input-linear variants of the semantics). It should also be noted that evidently there are many more kinds of  
1137 categorical rewriting semantics available in the literature, including cases where the *matches* and *co-matches* are not  
1138 required to be  $\mathcal{M}$ -morphisms, yet we focus here on the aforementioned eight variants for concreteness as a sufficiently  
1139 diverse set of test cases for our new theoretical framework for categorical rewriting theories.

1140 We will now follow the proof-strategy set forth via the formalism introduced in Section 3 in order to determine  
1141 efficiently sets of sufficient conditions under which the eight different semantics of Definition 6.1 give rise to *compo-*  
1142 *sitional rewriting double categories* (crDCs), and thus to compositional rewriting theories. In this way, we are able to  
1143 demonstrate the high level of modularization afforded by our novel approach, and at the same time highlight some of  
1144 the similarities and crucial mathematical differences between the various rewriting semantics.

## 1145 6.2. Double-categorical structures

1146 For all eight semantics of Definition 6.1, we will let  $\mathbb{D}_0$  be defined as  $\mathbf{C}|_{\mathcal{M}}$ , i.e., the restriction of  $\mathbf{C}$  along  $\mathcal{M}$   
1147 (with objects the objects of  $\mathbf{C}$ , and morphisms the morphisms of  $\mathcal{M}$ ). We let  $\mathbb{D}_1$  be defined as *rules* for the *horizontal*  
1148 *morphisms* (i.e., the objects of  $\mathbb{D}_1$ ), and via *direct derivations* (i.e., diagrams of the form in (51)) for the *squares* (i.e.,  
1149 the morphisms of  $\mathbb{D}_1$ ). This identifies the crDCs we will construct as double categories obtained via restriction of the  
1150 double category  $\mathit{Span}(\mathbf{C})$  of spans (cf. e.g. [61, Ex.12.3.16]), with  $\mathbb{D}_0 = \mathbf{C}$ , with spans of  $\mathbf{C}$  as *horizontal morphisms*  
1151 (i.e., as objects of  $\mathbb{D}_1$ ), and with commutative diagrams of the form below (without any restrictions on the squares  
1152 other than commutativity) for the *squares* of  $\mathit{Span}(\mathbf{C})$ .

$$\begin{array}{ccc}
\begin{array}{ccc} O & \xleftarrow{r} & I \\ \downarrow m^* & \Downarrow \alpha & \downarrow m \\ O' & \xleftarrow{r'} & I' \end{array} & := & \begin{array}{ccccc} O & \xleftarrow{o_r} & K_r & \xrightarrow{i_r} & I \\ \downarrow m^* & \text{(\dagger}_\alpha) & \downarrow k_\alpha & \text{(\ast}_\alpha) & \downarrow m \\ O' & \xleftarrow{o_{r'}} & K_{r'} & \xrightarrow{i_{r'}} & I' \end{array}
\end{array} \quad (52)$$

1153 Rather than having to work through a lengthy set of coherence conditions for our crDCs to indeed qualify as double  
1154 categories, the fact that they are all obtained as restrictions of  $\mathit{Span}(\mathbf{C})$  simplifies this task down to verifying the  
1155 following properties, which ensure that the restrictions are compatible with the existence of horizontal and vertical  
1156 units, and with horizontal and vertical compositions:

1157 **Corollary 6.2.** *For all eight semantics of Definition 6.1, the resulting definitions of  $\mathbb{D}_0$  and  $\mathbb{D}_1$  have *horizontal and*  
1158 *vertical units* in the following form:*

$$\begin{array}{ccc}
\begin{array}{ccc} I & \xleftarrow{U_I} & I \\ \downarrow m & \Downarrow U_m & \downarrow m \\ I' & \xleftarrow{U_{I'}} & I' \end{array} & := & \begin{array}{ccc} I & \xleftarrow{=} & I \\ \downarrow m & \Downarrow m & \downarrow m \\ I' & \xleftarrow{=} & I' \end{array} , & \begin{array}{ccc} O & \xleftarrow{r} & I \\ \parallel 1_O & \Downarrow id_r & \parallel 1_I \\ O & \xleftarrow{r} & I \end{array} & := & \begin{array}{ccccc} O & \xleftarrow{o_r} & K_r & \xrightarrow{i_r} & I \\ \parallel & \parallel & \parallel & \parallel & \parallel \\ O' & \xleftarrow{o_{r'}} & K_{r'} & \xrightarrow{i_{r'}} & I' \end{array}
\end{array} \quad (53)$$

1159 PROOF. The only non-trivial statement to prove is that the diagrams in (53) qualify as direct derivations of the respective  
 1160 types in a given semantics according to Definition 6.1. But this follows immediately from the results of Lemma 4.3,  
 1161 whereby all commutative squares of the types occurring in the direct derivations depicted in (53) are simultaneously  
 1162 pushouts and final pullback complements. Moreover, since by assumption  $\mathcal{M}$  is a **stable system of monics**, it contains  
 1163 in particular all isomorphisms, which completes the proof that the direct derivations in (53) are well-formed.

1164 As mentioned in Table 3, **vertical composition** is guaranteed to be well-posed in all cases because of **pushout**  
 1165 **composition** and **vertical FPC composition**. In contrast, it is considerably more intricate to prove that **horizontal**  
 1166 **composition** is well-posed, which is the first instance where **adhesivity properties** are required in different forms  
 1167 depending on the precise nature of the chosen rewriting semantics:

1168 **Proposition 6.3.** *Under the additional assumptions on  $\mathbf{C}$  presented in Table 3, each of the rewriting semantics of*  
 1169 *Definition 6.1 yields a well-posed horizontal composition for direct derivations.*

PROOF. Since the **crDCs** for the different semantics are obtained via suitable restrictions of the double category  $\mathbb{S}pan(\mathbf{C})$ , the operation  $\diamond_h$  is for all situations induced via span composition of the horizontal morphisms:

$$\begin{array}{ccc}
 \begin{array}{c} \xrightarrow{r_2} \\ \downarrow o \\ \downarrow \alpha_2 \\ \downarrow n \\ \downarrow r'_2 \end{array} & \diamond_h & \begin{array}{c} \xrightarrow{r_1} \\ \downarrow n \\ \downarrow \alpha_1 \\ \downarrow m \\ \downarrow r'_1 \end{array} \\
 & & := \begin{array}{c} \xleftarrow{r_{21}} \\ \downarrow o \\ \downarrow \alpha_{21} \\ \downarrow m \\ \downarrow r'_{21} \end{array}
 \end{array} \quad (54a)$$

1170 In the commutative diagram in (54b), the notation  $PB^*$  indicates that the definition of  $\diamond_h$  involves *choices* of pullbacks  
 1171 for each cospan, so that  $\diamond_h$  will in particular only be a pseudo-functor.

1172  
 1173 With regards to prerequisites on the underlying category, the definition of  $\diamond_h$  via taking pullbacks requires that  
 1174 the underlying category **has pullbacks** for the case of **generic** rules, while for all variants of linearity it suffices that  $\mathbf{C}$   
 1175 **has pullbacks along  $\mathcal{M}$ -morphisms**. Moreover, since by assumption  $\mathcal{M}$  is a **stable system of monics**,  **$\mathcal{M}$ -morphisms**  
 1176 **are stable under pullback**, hence the types of the composite spans are indeed compatible with the types specified in  
 1177 **generic, output-linear, input-linear** or **linear** rewriting semantics, respectively.

1178  
 1179 Next, the **universal property of pullbacks** entails the existence of a unique morphism  $\diamond - p \rightarrow \blacklozenge$  that makes the  
 1180 diagram in (54b) commute. By **pullback-pullback decomposition**, the front left and right vertical squares marked  $(\dagger_p)$   
 1181 and  $(*_p)$  in (54b) are pullbacks.

1182  
 1183 It remains to demonstrate that the squares marked  $(\dagger_p)$  and  $(*_p)$  in (54b) are not only pullbacks, but indeed of the  
 1184 correct type (i.e., pushouts or FPCs, respectively) as required for the chosen rewriting semantics.

1185 (i) For the square marked  $(\dagger_p)$ , since in all eight types of semantics according to Definition 6.1 the square marked  
 1186  $(\dagger_{\alpha_1})$  is a pushout, we require the appropriate notion of stability of this type of pushout under pullbacks (compare  
 1187 Table 3). More precisely, the distinction depends on the character of the horizontal morphisms in the pushout  
 1188 square  $(\dagger_{\alpha_1})$ , and in the nature of the morphisms in the pullback squares over  $(\dagger_{\alpha_1})$  (i.e.,  $i_{r_2}$ ,  $i'_{r_2}$ ,  $p_1$  and  $p'_1$ ),  
 1189 which depending on the rewriting semantics are either generic morphisms or  $\mathcal{M}$ -morphisms:

- 1190 • For **generic** semantics,  $(\dagger_p)$  is a **pushout along an  $\mathcal{M}$ -morphism**, and the morphisms in the pullback squares  
1191 over  $(\dagger_{\alpha_1})$  are generic morphisms, hence we require  $\mathbf{C}$  to satisfy that **pushouts along  $\mathcal{M}$ -morphisms** are  
1192 **stable under pullbacks** (i.e., axiom (L-iii-a) of the definition of **adhesive HLR categories**).
- 1193 • For **output-linear** semantics,  $(\dagger_p)$  is a pushout of a span of  $\mathcal{M}$ -morphisms, and the morphisms in the  
1194 pullback squares over  $(\dagger_{\alpha_1})$  are generic morphisms, hence we require  $\mathbf{C}$  to satisfy axiom (H-iii-a) of the  
1195 definition of **horizontal weak adhesive HLR categories**.
- 1196 • For **input-linear** semantics,  $(\dagger_p)$  is a **pushout along an  $\mathcal{M}$ -morphism**, and the morphisms in the pullback  
1197 squares over  $(\dagger_{\alpha_1})$  are  $\mathcal{M}$ -morphisms, hence we require  $\mathbf{C}$  to satisfy axiom (V-iii-a) of the definition of  
1198 **vertical weak adhesive HLR categories**.
- 1199 • For **linear** semantics,  $(\dagger_p)$  is a pushout of a span of  $\mathcal{M}$ -morphisms, and the morphisms in the pullback  
1200 squares over  $(\dagger_{\alpha_1})$  are  $\mathcal{M}$ -morphisms, hence we require  $\mathbf{C}$  to satisfy axiom (V-iii-a) of the definition of  
1201 **vertical weak adhesive HLR categories**.<sup>12</sup>
- 1202 (ii) For the square marked  $(*_{\alpha_2})$  in (54b), since this square is a **final pullback complement (FPC)** for **sesqui-pushout**  
1203 **semantics**, and under the condition that  $\mathbf{C}$  **has pullbacks** (which is also one of the necessary assumptions for  
1204 the construction of FPCs via an  **$\mathcal{M}$ -partial map classifier**, cf. Theorem 5.23), we obtain that the square marked  
1205  $(*_p)$  is an FPC by **stability of FPCs under pullbacks**. For **double-pushout semantics**, we may repeat the analysis  
1206 of the previous step (i) to demonstrate that since the square marked  $(*_{\alpha_2})$  is a pushout for this semantics, under  
1207 suitable conditions on  $\mathbf{C}$  the square marked  $(*_p)$  is a pushout as well. In particular, we find for the **output-**  
1208 **linear** and **input-linear** variants of **DPO-semantics** that  $\mathbf{C}$  has to satisfy axiom (W-iii-a) of the definition of **weak**  
1209 **adhesive HLR categories**, i.e., both of axioms (V-iii-a) and (H-iii-a).

1210 Finally, by **pushout composition** and **horizontal FPC composition**, respectively, one may demonstrate that the hori-  
1211 zontal composite of  $(\dagger_{\alpha_2})$  and  $(\dagger_p)$  is a pushout, while  $(*_p)$  and  $(*_{\alpha_1})$  compose into a pushout for **DPO-semantics**, and  
1212 into an FPC for **SqPO-semantics**, which concludes the proof.  $\square$

1213 **Remark 6.4.** In earlier work on the **linear** variant of **sesqui-pushout semantics** [19, 62], instead of requiring that  $\mathbf{C}$  **has**  
1214 **pullbacks**, an alternative argument was utilized in order to prove that square  $(*_p)$  in (54b) is an FPC: after completing  
1215 step (i) in order to prove that  $(\dagger_p)$  is a pushout as above, and utilizing that pushouts of  $\mathcal{M}$ -spans are also FPCs (compare  
1216 Proposition 5.27), the pushout squares  $(\dagger_{\alpha_1})$  and  $(\dagger_p)$  are FPCs. Thus by **horizontal FPC composition** the composite of  
1217 squares  $(*_{\alpha_2})$  and  $(\dagger_p)$  is an FPC, hence applying **horizontal FPC decomposition**, one may demonstrate that  $(*_p)$  is an  
1218 FPC. However, there exists to the best of our knowledge no example of a category that **has FPCs along  $\mathcal{M}$ -morphisms**  
1219 where FPCs are not constructed via an  **$\mathcal{M}$ -partial map classifier** as in Theorem 5.23, and since the latter theorem  
1220 requires that the category **has pullbacks**, it appears to be more efficient to apply **stability of FPCs under pullbacks** in  
1221 order to prove that  $(*_p)$  is an FPC.

### 1222 6.3. Properties specific to compositional rewriting double categories

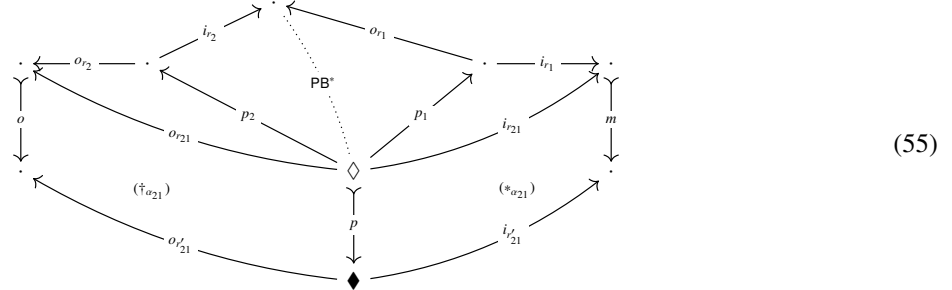
1223 Having established the conditions on the underlying category under which direct derivations of one of the eight  
1224 semantics of Definition 6.1 give rise to a double category, it remains to determine whether additional conditions are  
1225 required such that these double categories indeed qualify as **compositional rewriting double categories**. The results of  
1226 this part of the derivation are summarized in Table 3.

1227 **Proposition 6.5.** *Let  $\mathbf{C}$  be a category suitable for one of the rewriting semantics of Definition 6.1, and such that  $\mathbf{C}$*   
1228 *also satisfies the relevant additional assumptions stated in Table 3. Then the **horizontal composition functor**  $\diamond_h$  of the*  
1229 ***crDC** for the given choice of  $\mathbf{C}$  and rewriting semantics is a **isoglobular residual opfibration**.*

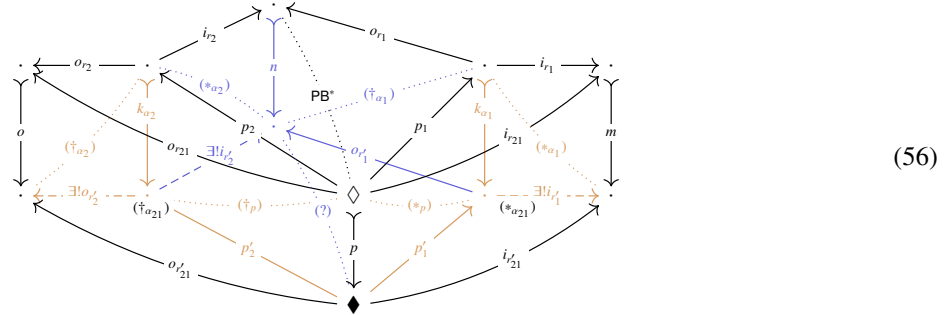
<sup>12</sup>Coincidentally, it would also be sufficient for  $\mathbf{C}$  to satisfy axiom (H-iii-a) of the definition of **horizontal weak adhesive HLR categories**;  
however, it will become evident in the following that axiom (V-iii-a) is in fact required for other properties of **crDCs** to be satisfied (cf. Table 3).



1230 PROOF. Let us first verify that, under the given assumptions,  $\diamond_h$  possesses **residual op-Cartesian lifts** whose **residues**  
 1231 are **globular isomorphisms**. Consider thus a diagram of the form below, where  $\text{PB}^*$  denotes a pullback as chosen in  
 1232 the definition of  $\diamond_h$ ,  $(\dagger_{\alpha_{21}})$  is a pushout, and  $(*\alpha_{21})$  is a pushout for **DPO-semantics** and an **FPC** for **SqPO-semantics**:

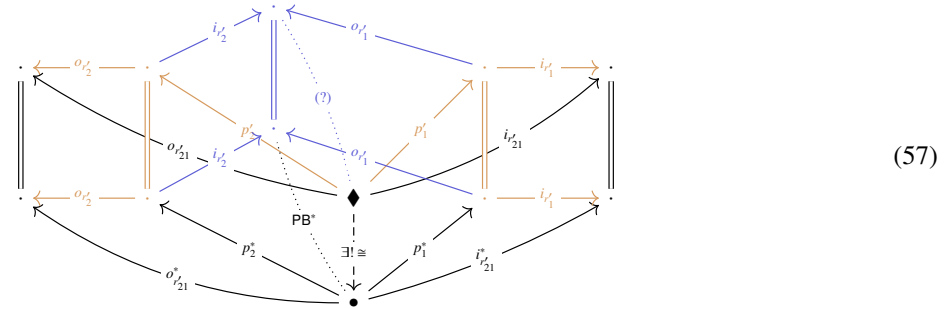


1233 We have to prove that for each of the semantics of Definition 6.1, one may obtain essentially uniquely a horizontal  
 1234 composition of direct derivations. To this end, consider first the case of **DPO-semantics**, for which we transform the  
 1235 diagram of (55) into the diagram below:



- 1236 • Take pushouts to obtain the squares marked  $(\dagger_p)$  and  $(*_p)$ , which by the **universal property of pushouts** entails  
 1237 that there exist unique morphisms  $o_{r'_2}$  and  $i_{r'_1}$ . Moreover, by **pushout-pushout decomposition**, the squares marked  
 1238  $(\dagger_{\alpha_2})$  and  $(*\alpha_1)$  are pushouts.
- 1239 • Take another pushout to obtain the square<sup>13</sup> marked  $(\dagger_{\alpha_1})$ , which by the **universal property of pushouts** entails  
 1240 that there exists a unique morphism  $i_{r'_2}$ . Moreover, by **pushout-pushout decomposition**, the square marked  $(*\alpha_2)$   
 1241 is a pushout.
- 1242 • It then remains to invoke the version of the van Kampen square property applicable to the given variant of  
 1243 **DPO-semantics** (i.e., axiom (L-iii-b) for the **generic** and axiom (V-iii-b) for the other variants, cf. Table 3) in  
 1244 order to demonstrate that the bottom square marked (?) is indeed a pullback.

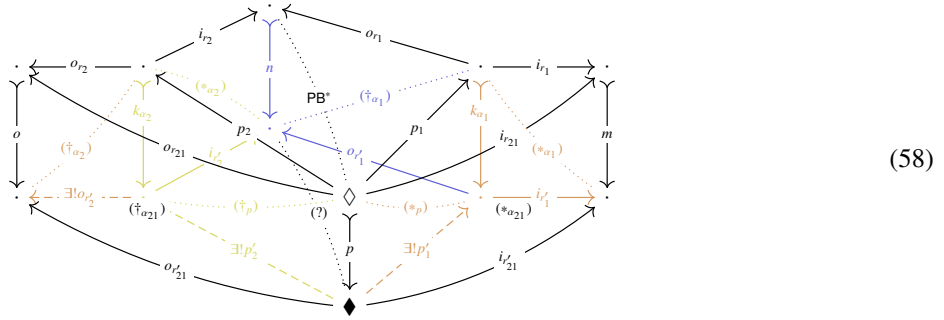
1245 Finally, the pullback square marked (?) will in general *not* coincide with the pullback of the cospan chosen as part  
 1246 of the definition of  $\diamond_h$ ; therefore, it remains to form the diagram below (where  $\text{PB}^*$  marks the chosen pullback):



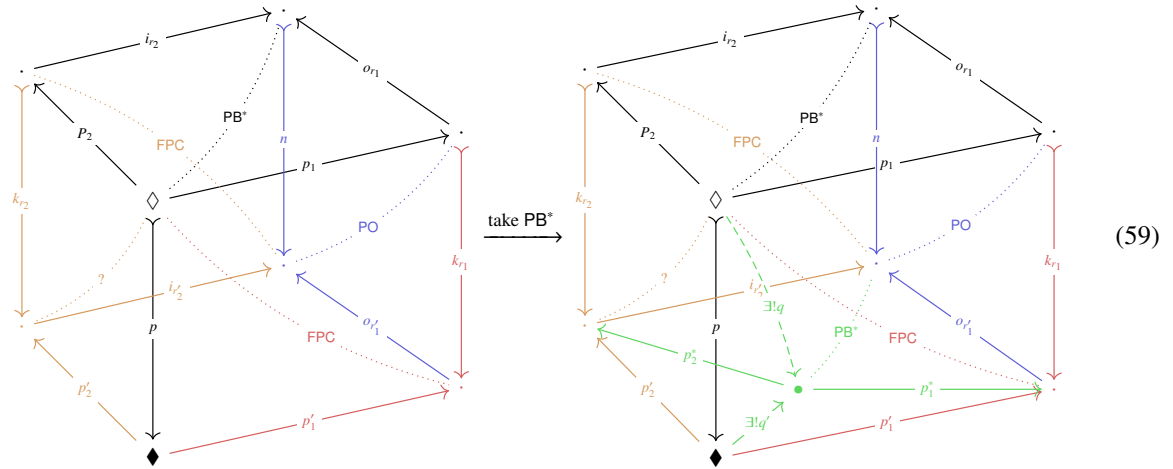
<sup>13</sup>Evidently, we could have equivalently obtained the square marked  $(*\alpha_2)$  first by taking a pushout.

- 1247 • By the **universal property of pullbacks**, there exists a unique mediating isomorphism between the two pullback  
 1248 squares denoted (?) and  $PB^*$ .
- 1249 • Since each of the vertical squares with isomorphisms for vertical morphisms in (57) is a pushout square (and  
 1250 also an FPC square, cf. proof of Lemma 4.3), we conclude that the morphism from the span  $(o_{r'_{21}}, i'_{r'_{21}})$  to  
 1251  $(o_{r'_{21}}^*, i_{r'_{21}}^*)$  (i.e., the frontmost curved vertical squares) is indeed a **globular isomorphism** in the **crDC** for the  
 1252 chosen DPO-type semantics.

1253 For the case of **SqPO-semantics**, we transform the diagram in (55) as follows (where once again  $PB^*$  marks a  
 1254 pullback as chosen in the definition of  $\diamond_h$ ):

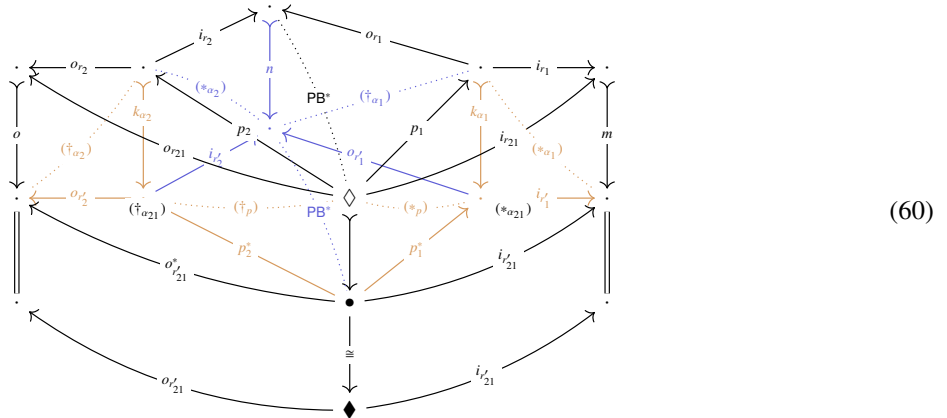


- 1255 • Take an **FPC** to obtain the square marked  $(*_{\alpha_1})$ , which by the **universal property of FPCs** entails the existence  
 1256 of a unique morphism  $p'_1$ , and thus by **horizontal FPC decomposition** that the square marked  $(*_p)$  is an FPC.
- 1257 • Take a pushout to obtain the square marked  $(\dagger_{\alpha_1})$ , and an FPC to obtain the square marked  $(*_{\alpha_2})$ , which by the  
 1258 **universal property of FPCs** yields also a unique morphism  $p'_2$ , and thus by **pullback-pullback decomposition**,  
 1259 the square marked  $(\dagger_p)$  is a pullback.
- 1260 • For the case of **generic SqPO-semantics**, by invoking the Beck-Chevalley-Condition (**BCC-1**) of Theorem 4.4,  
 1261 which allows us to conclude that the square marked  $(\dagger_p)$  is an FPC, and the bottom square marked (?) is a pull-  
 1262 back. It is then related to the chosen pullback  $PB^*$  according to the definition of  $\diamond_h$  by a universal isomorphism  
 1263 (i.e., by a span isomorphism).
- 1264 • For the other types of **SqPO-semantics**, we may develop more general variants of the Beck-Chevalley-Condition  
 1265 (**BCC-1**) by suitably adapting the proof strategy of Theorem 4.4. To this end, consider the diagrammatic state-  
 1266 ment presented in (59) below (which is a 3D-rotated and relabeled version of the statement in (B.14) in order to  
 1267 facilitate the comparison to the diagram in (58)), where  $PB^*$  marks a chosen pullback according to the defini-  
 1268 tion of  $\diamond_h$ . In all three cases, the proof strategy consists in (i) taking a chosen pullback (marked  $PB^*$ ) to obtain  
 1269 the second diagram in (59) (where by the **universal property of pullbacks** entails that there exist unique arrows  
 1270  $\diamond - q \rightarrow \bullet$  and  $\blacklozenge - q' \rightarrow \bullet$ ); (ii) using **pullback-pullback decomposition** to prove that all squares of the interi-  
 1271 or commutative cube are pullbacks; (iii) invoking a suitable variant of stability of pushouts under pullbacks  
 1272 to show that the front left inner vertical square is a pushout; and finally (iv) to apply **stability of FPCs under**  
 1273 **pullbacks** in order to demonstrate that the front right inner vertical square is an FPC, such that by the **universal**  
 1274 **property of FPCs** the morphism  $\diamond - q \rightarrow \bullet$  is an isomorphism. It thus remains to clarify the variant of stability  
 1275 property of pushouts necessary for each kind of semantics:
- 1276 – For **output-linear SqPO-semantics**, all morphisms of the back right vertical square are guaranteed to be  
 1277  $\mathcal{M}$ -morphisms, hence the claim follows if  $\mathbf{C}$  satisfies axiom (H-iii-a).
  - 1278 – For **input-linear SqPO-semantics**, the morphisms  $i_{r_2}$ ,  $i'_{r'_2}$ ,  $p_1$ , and (by **stability of  $\mathcal{M}$ -morphisms under**  
 1279 **pullback**)  $p'_1$  are guaranteed to be  $\mathcal{M}$ -morphisms, hence the claim follows if  $\mathbf{C}$  satisfies axiom (V-iii-a).
  - 1280 – Since **linear SqPO-semantics** is a special case both of **output-linear** and **input-linear SqPO-semantics**, the  
 1281 claim follows if  $\mathbf{C}$  satisfies either (H-iii-a) or (V-iii-a).



1282 Finally, an argument analogous to the one demonstrated in (57) then reveals that the isomorphism  $q'$  in (59) gives rise  
 1283 to a **globular isomorphism** in the chosen **crDC** of SqPO-type.

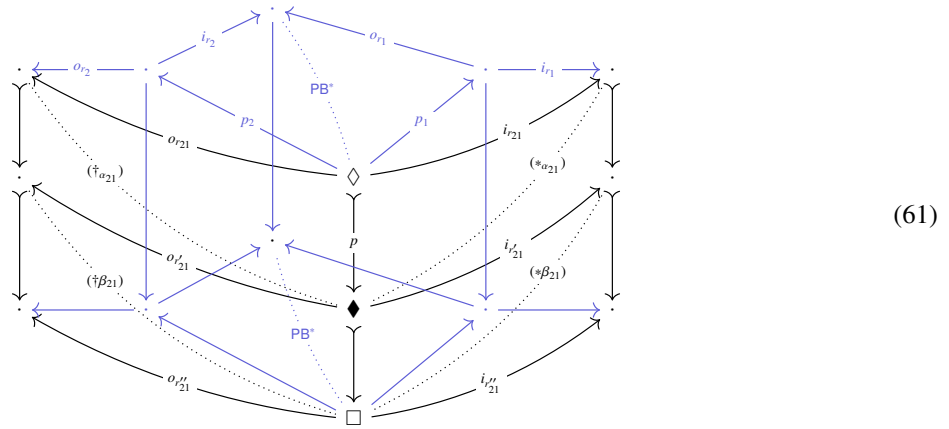
1284 We have thus proved for **DPO-semantics** and for **SqPO-semantics** that the functors  $\diamond_h$  of the corresponding **crDCs**  
 1285 possess **residual op-Cartesian lifts**, whose residues are indeed **globular isomorphisms**. Concretely, we found that from  
 1286 each diagram as in (55), one may obtain a diagram of the following shape:



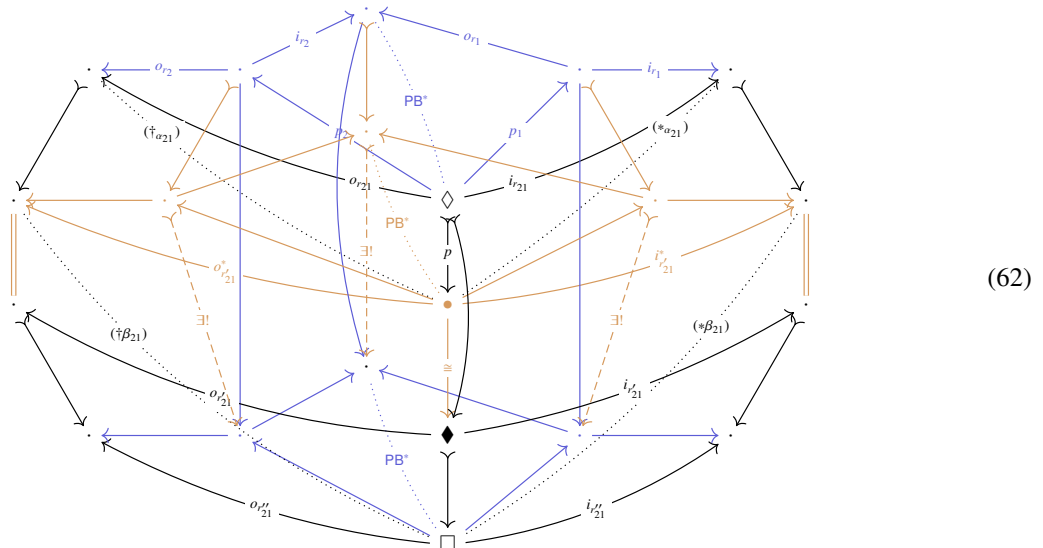
1287 Moreover, it follows from the universal properties of pushouts, pullbacks and FPCs that the two constructions are  
 1288 essentially unique.

1289 It remains to prove the **complex decomposition property**, which for **crDCs** of either DPO- or SqPO-semantics  
 1290

1291 takes as its premise a diagram of the following shape:



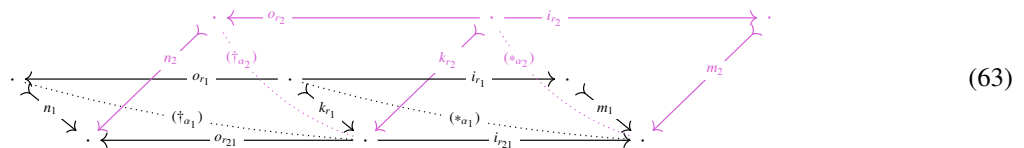
1292 The claim then follows by constructing the following diagram, where the top half is constructed analogously to the  
 1293 DPO- or SqPO-variants of the **horizontal decomposition property**:



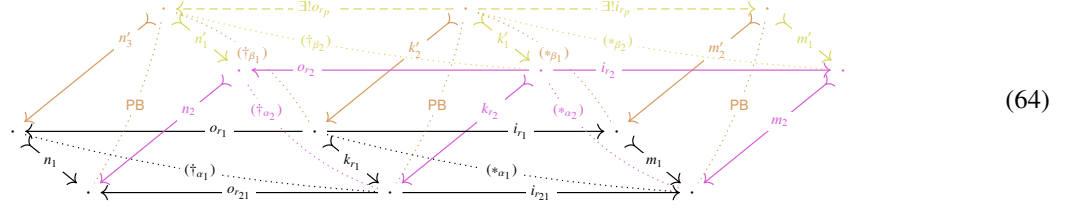
1294 The existence of the dashed vertical arrows in the bottom half of the diagram in (62) may be derived from the respective  
 1295 universal properties of the pushout and FPC squares present. Finally, the various splitting lemmata for pushouts and  
 1296 FPCs then permit to demonstrate that the back squares of the middle and bottom half of the diagram (i.e., the vertical  
 1297 squares adjacent to the dashed arrows) indeed constitutes a horizontal composition of two DPO- or SqPO-type direct  
 1298 derivations, which concludes the proof.  $\square$

1299 **Proposition 6.6.** *Let  $\mathbf{C}$  be a category with a **stable system of monics**  $\mathcal{M}$ , and such that  $\mathbf{C}$  is suitable for the chosen  
 1300 rewriting semantics according to Definition 6.1. Let  $\mathbb{D}_1$  denote the category of **rules** as objects and direct derivations  
 1301 of the chosen semantics as morphisms. Then if  $\mathbf{C}$  is a **vertical weak adhesive HLR category**,  $\mathbb{D}_1$  has pullbacks.*

1302 **PROOF.** Consider a cospan in  $\mathbb{D}_1$ , which amounts to a diagram of the form below:



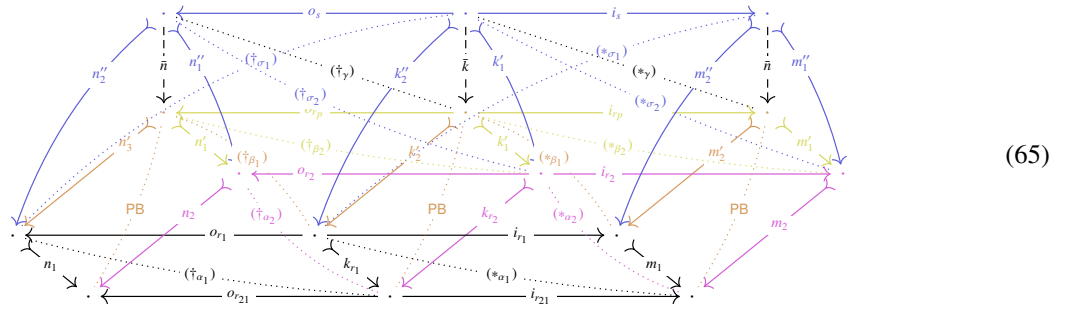
1303 By assumption on the underlying category,  $\mathbf{C}$  has pullbacks along  $\mathcal{M}$ -morphisms, which permits us to construct the  
 1304 diagram below from the one in (63) via taking three pullbacks:



1305 By the **universal property of pullbacks**, there exist unique morphisms  $o_{r_p}$  and  $i_{r_p}$  that make the diagram commute, and  
 1306 thus by **pullback-pullback decomposition**, we find that the squares marked  $(\dagger_{\beta_1})$ ,  $(\dagger_{\beta_2})$ ,  $(*_\beta_1)$  and  $(*_\beta_2)$  are pullbacks.

- 1307 • Since axiom (V-iii-a) of the definition of **vertical weak adhesive HLR categories** holds in  $\mathbf{C}$  (i.e., if **pushouts**  
 1308 **along  $\mathcal{M}$ -morphisms are stable under  $\mathcal{M}$ -pullbacks**), the squares marked  $(\dagger_{\beta_1})$  and  $(\dagger_{\beta_2})$  are pushouts.
- 1309 • For **DPO-semantics**, since (V-iii-a) holds in  $\mathbf{C}$ , the squares marked  $(*_\beta_1)$  and  $(*_\beta_2)$  are pushouts.
- 1310 • For **SqPO-semantics**, by **stability of FPCs under pullbacks**,  $(*_\beta_1)$  and  $(*_\beta_2)$  are FPCs.

1311 It then remains to demonstrate that the construction provided indeed yields a pullback in  $\mathbb{D}_1$ . To this end, consider  
 1312 a diagram as below, where the upper blue squares encode a span in  $\mathbb{D}_1$  that together with the cospan in  $\mathbb{D}_1$  that was  
 1313 already depicted in (63) yields a commutative square in  $\mathbb{D}_1$ :



1314 We have to prove that there exists a unique mediating morphism in  $\mathbb{D}_1$  (i.e., the dashed  $\mathbb{D}_0$ -morphisms that make the  
 1315 diagram commute, and such that the squares marked  $(\dagger_\gamma)$  and  $(*_\gamma)$  are of the correct kinds for the given semantics:

- 1316 • By the **universal property of pullbacks**, there exist uniquely the morphisms  $\bar{n}$ ,  $\bar{k}$  and  $\bar{m}$  marked with dashed  
 1317 arrows in (65), which make the diagram commute. By the **decomposition property of  $\mathcal{M}$ -morphisms**, these  
 1318 morphisms are moreover in  $\mathcal{M}$ .
- 1319 • By **pushout-pullback decomposition**, the square marked  $(\dagger_\gamma)$  is a pushout.
- 1320 • For the case of **DPO-semantics**, yet again by **pushout-pullback decomposition**, the square marked  $(*_\gamma)$  is a  
 1321 pushout.
- 1322 • For the case of **SqPO-semantics**, by **vertical FPC-pullback decomposition**, the square marked  $(*_\gamma)$  is an FPC.

1323 In summary, we have thus demonstrated the unique existence of a  $\mathbb{D}_1$ -morphism consisting of the squares marked  $(\dagger_\gamma)$   
 1324 and  $(*_\gamma)$  that make the diagram in  $\mathbb{D}_1$  commute, which concludes the proof.  $\square$

1325 Finally, taking full advantage of the results presented in Section 2, we will investigate the existence of the requisite  
 1326 fibrational structures for the source and target functors on the **double categories** for all of the categorical rewriting  
 1327 semantics of Definition 6.1. Let us first recall the properties that have to be satisfied by a category  $\mathbf{C}$  to be suitable to  
 1328 carry **DPO-semantics** or **SqPO-semantics**:

- For DPO-semantics, it is required that  $\mathbf{C}$  has  $\mathcal{M}$ -multi-IPCs, that it has pushouts along  $\mathcal{M}$ -morphisms, and that  $\mathcal{M}$ -morphisms are stable under pushout (i.e., the latter two points amount to axiom (V-ii)).
- For SqPO-semantics, it is required that  $\mathbf{C}$  has FPCs along  $\mathcal{M}$ -morphisms, that it has pushouts along  $\mathcal{M}$ -morphisms, and that  $\mathcal{M}$ -morphisms are stable under pushout (i.e., the latter two points amount to axiom (V-ii)).

For DPO-semantics, recall from Lemma 4.11 that a sufficient condition to ensure that  $\mathbf{C}$  has  $\mathcal{M}$ -multi-IPCs is that  $\mathbf{C}$  has pullbacks along  $\mathcal{M}$ -morphisms (i.e., axiom (V-i)), that pushouts along  $\mathcal{M}$ -morphisms are stable under  $\mathcal{M}$ -pullbacks (i.e., axiom (V-iii-a)), and that pushouts along  $\mathcal{M}$ -morphisms are pullbacks (cf. Theorem 5.4); hence, in summary, it is sufficient to require that  $\mathbf{C}$  is a vertical weak adhesive HLR category. For SqPO-semantics, in addition to asking that  $\mathbf{C}$  be a vertical weak adhesive HLR category, we must further ask that it has FPCs along  $\mathcal{M}$ -morphisms.

We can now state the theorem:

**Theorem 6.7.** *Let  $\mathbf{C}$  be a category that is finitary and a vertical weak adhesive HLR category with respect to a stable system of monics  $\mathcal{M}$ . For the case of SqPO-semantics, we assume further that  $\mathbf{C}$  has FPCs along  $\mathcal{M}$ -morphisms. Let  $\mathbb{D}$  denote the double category based upon  $\mathbf{C}$  and direct derivations of the respective kind as introduced in Section 6.2. Then the following fibrational properties hold:*

- (i) *The functor  $S : \mathbb{D}_1 \rightarrow \mathbb{D}_0$  is a multi-opfibration.*
- (ii) *The functor  $T : \mathbb{D}_1 \rightarrow \mathbb{D}_0$  is a residual multi-opfibration.*

PROOF. In the case of DPO-semantics, the category  $\mathbf{C}$  therefore supports the following fibrational structures:

- By Theorem 4.12, since  $\mathbf{C}$  has pullbacks along  $\mathcal{M}$ -morphisms, and since pushouts along  $\mathcal{M}$ -morphisms are stable under  $\mathcal{M}$ -pullbacks in  $\mathbf{C}$ , the target functor  $T_{\text{PO}} : \text{PO}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  is a multi-opfibration.
- By Theorem 4.13, since  $\mathbf{C}$  has pushouts along  $\mathcal{M}$ -morphisms, and since  $\mathcal{M}$ -morphisms are stable under pushout, the source functor  $S_{\text{PO}} : \text{PO}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  is a Grothendieck opfibration.

As for SqPO-semantics, according to Theorem 5.20(iii) that since  $\mathbf{C}$  is finitary and a vertical weak adhesive HLR category with respect to the stable system of monics  $\mathcal{M}$ ,  $\mathbf{C}$  is  $(\mathcal{E}, \mathcal{M})$ -structured, for  $\mathcal{E}$  the class of extremal morphisms (w.r.t.  $\mathcal{M}$ ). We thus find the following results from Section 2:

- By Theorem 4.8, since  $\mathbf{C}$  has pullbacks along  $\mathcal{M}$ -morphisms and it has FPCs along  $\mathcal{M}$ -morphisms, the target functor  $T_{\text{FPC}} : \text{FPC}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  is a Grothendieck opfibration.
- By Theorem 4.20, since  $\mathbf{C}$  is  $(\mathcal{E}, \mathcal{M})$ -structured, has pullbacks, pushouts and FPCs along  $\mathcal{M}$ -morphisms, such that  $\mathcal{M}$ -morphisms are stable under pushout, and such that pushouts along  $\mathcal{M}$ -morphisms are stable under  $\mathcal{M}$ -pullbacks, the source functor  $S_{\text{FPC}} : \text{FPC}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  is a residual multi-opfibration.

With these preparations, it then remains to prove that indeed the functors  $S, T : \mathbb{D}_1 \rightarrow \mathbb{D}_0$  from  $\mathbb{D}_1$  (i.e., the category with rules as objects, and direct derivations as morphisms) to  $\mathbb{D}_0$  (i.e., the category  $\mathbf{C}|_{\mathcal{M}}$ ) are a residual multi-opfibration in the case of  $S$ , and a multi-opfibration in the case of  $T$ , respectively.

As for the functor  $S : \mathbb{D}_1 \rightarrow \mathbb{D}_0$ , the existence of multi-op-Cartesian liftings is induced from the property that the functor  $T_{\text{PO}} : \text{PO}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  is a multi-opfibration, and that both  $S_{\text{PO}} : \text{PO}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  and  $T_{\text{FPC}} : \text{FPC}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  are Grothendieck opfibrations:

- Consider the following diagram:

$$\begin{array}{ccc}
 O \longleftarrow r \longrightarrow I & & O \longleftarrow o_r \longrightarrow K_r \longrightarrow i_r \longrightarrow I \\
 \downarrow i & := & \downarrow i \\
 I' & & I'
 \end{array} \tag{66}$$

1365 Since  $T_{\text{PO}} : \text{PO}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  is a **multi-opfibration**, while  $T_{\text{FPC}} : \text{FPC}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  is a **Grothendieck**  
 1366 **opfibration** (i.e., a special case of a **multi-opfibration** where each family is a singleton), this entails the existence  
 1367 of a family of **multi-op-Cartesian liftings** (the  $\kappa'_j$ s, in blue in the diagram below), each of whose elements via  
 1368 the **Grothendieck opfibration** property of  $S_{\text{PO}} : \text{PO}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  (the  $\omega_j$ s and  $o'_{r'_j}$ s, cf. orange part of the  
 1369 diagram below) lifts into a **DPO-type direct derivation** or an **SqPO-type direct derivation**, respectively, i.e., to  
 1370 an element of  $\mathbb{D}_1$ :

$$\begin{array}{ccccc}
 O & \xleftarrow{o_r} & K_r & \xrightarrow{i_r} & I \\
 \downarrow \omega_j & & \downarrow \kappa'_j & & \downarrow i \\
 O'_{r'_j} & \xleftarrow{o'_{r'_j}} & K'_{r'_j} & \xrightarrow{i'_{r'_j}} & I'
 \end{array}
 \quad \begin{array}{l}
 \text{via } S_{\text{PO}} \\
 \text{via } T_X
 \end{array}
 \quad (67)$$

1371 Moreover, consider a diagram as the one marked (i) below:

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 O & \xleftarrow{\quad} & K & \xrightarrow{\quad} & I \\
 \downarrow \omega'' & & \downarrow \kappa'' & & \downarrow i \\
 O'' & \xleftarrow{\quad} & K'' & \xrightarrow{\quad} & I''
 \end{array} & \xrightarrow{\text{via } T_X} & \begin{array}{ccccc}
 O & \xleftarrow{\quad} & K & \xrightarrow{\quad} & I \\
 \downarrow \omega'' & & \downarrow \kappa'' & & \downarrow i \\
 O'' & \xleftarrow{\quad} & K'' & \xrightarrow{\quad} & I''
 \end{array} \\
 (i) & & (ii)
 \end{array}
 \quad \xrightarrow{\text{via } S_{\text{PO}}} \quad
 \begin{array}{ccccc}
 O & \xleftarrow{\quad} & K & \xrightarrow{\quad} & I \\
 \downarrow \omega & & \downarrow \kappa'' & & \downarrow i \\
 O' & \xleftarrow{\quad} & K' & \xrightarrow{\quad} & I' \\
 \downarrow \omega' & & \downarrow \kappa' & & \downarrow i' \\
 O'' & \xleftarrow{\quad} & K'' & \xrightarrow{\quad} & I''
 \end{array}
 \quad (iii)
 \quad (68)$$

- 1372 – Diagram (ii) is obtained via invoking the fact that  $T_{\text{PO}}$  is a **multi-opfibration** in the **DPO-semantics** case,
- 1373 or via the fact that  $T_{\text{FPC}}$  is a **Grothendieck opfibration** in the **SqPO-semantics** case.
- 1374 – Diagram (iii) is obtained via using that  $S_{\text{PO}}$  is a **Grothendieck opfibration**.
- 1375 • Since evidently the above constructions are essentially unique, we have thus proved that  $S : \mathbb{D}_1 \rightarrow \mathbb{D}_0$  “inherits”
- 1376 a **multi-opfibration** structure from the properties of  $S_{\text{PO}}$  and  $T_{\text{PO}}$  or  $T_{\text{FPC}}$ , respectively.

1377 Next, for the case of the target functor  $T : \mathbb{D}_1 \rightarrow \mathbb{D}_0$  **DPO-semantics**, due to the symmetry in the definition  
 1378 of **DPO-semantics**, the derivation that  $T$  is a **multi-opfibration** follows the same line of arguments as the one for  
 1379  $S : \mathbb{D}_1 \rightarrow \mathbb{D}_0$  for this semantics. Moreover, since a **multi-opfibration** is a special case of a **residual multi-opfibration**,  
 1380 namely the case when each **residue** is an identity morphism, this demonstrates that indeed  $T : \mathbb{D}_1 \rightarrow \mathbb{D}_0$  also carries  
 1381 the structure of a **residual multi-opfibration**.

1382 Finally, the proof for the fibrational property of  $T : \mathbb{D}_1 \rightarrow \mathbb{D}_0$  for the case of **SqPO-semantics** is considerably more  
 1383 involved.

- 1384 • Consider a diagram as below:

$$\begin{array}{ccccc}
 O & \xleftarrow{o_r} & K_r & \xrightarrow{i_r} & I \\
 \downarrow \omega & & \downarrow \kappa_j & & \downarrow i_{jk} \\
 O' & \xleftarrow{o'_{r'_j}} & K'_{r'_j} & \xrightarrow{i'_{r'_j}} & I' \\
 \downarrow (\omega_j)_{*k} & & \downarrow (\kappa_j)_{*k} & & \downarrow i'_{jk} \\
 O''_{j,k} & \xleftarrow{o'_{r'_{j,k}}} & K''_{j,k} & \xrightarrow{i'_{r'_{j,k}}} & I''_{j,k}
 \end{array}
 \quad \begin{array}{l}
 \text{via } T_{\text{PO}} \\
 \text{via } S_{\text{FPC}} \\
 \text{via } S_{\text{PO}}
 \end{array}
 \quad (69)$$

- 1385 – By the **multi-opfibration** property of  $T_{\text{PO}}$ , there exists a family of **multi-op-Cartesian liftings** (blue part of  
 1386 the diagram in (69)).
- 1387 – By the **residual multi-opfibration** property of  $S_{\text{FPC}}$ , for each element of the aforementioned **multi-op-**  
 1388 **Cartesian lifting** (indexed by  $j$ ), there exists a family of **residual multi-op-Cartesian liftings** (orange part





1408

in yellow.

- 1409 • Essential uniqueness of the fibrational properties of  $S : \mathbb{D}_1 \rightarrow \mathbb{D}_0$  are inherited from the essential uniqueness of  
 1410 the functors  $S_{PO}$ ,  $S_{FPC}$  and  $T_{PO}$ .

1411 In summary, we have thus succeeded in demonstrating that  $S : \mathbb{D}_1 \rightarrow \mathbb{D}_0$  for **SqPO-semantics** carries a **residual**  
 1412 **multi-opfibration** structure, which concludes the proof.  $\square$

1413 We conclude this discussion of theoretical results with the following observations:

1414 **Remark 6.8.** A folklore result of categorical rewriting theory, and especially in the tradition of Ehrig et al. [47] has  
 1415 been that the notion of **vertical weak adhesive HLR categories** is a reasonably general characterization of categories  
 1416 with sufficient properties to support some form of compositional semantics. While previous works did not consider  
 1417 the validity of an **associativity theorem** as a prerequisite for a rewriting theory to be compositional, the main criterion  
 1418 was indeed the existence of a **concurrency theorem** for the given theory. As our analysis demonstrates, **vertical weak**  
 1419 **adhesive HLR categories** are *almost* the main type of categories to support compositional rewriting, were it not for  
 1420 the additional properties required as presented in Table 3 for the various generalizations of **linear semantics** (which in  
 1421 effect was the only kind of semantics fully analyzed in the traditional literature [3]). Indeed, the discriminating factors  
 1422 in this regard are the **horizontal composition** (Proposition 6.3) and the **horizontal decomposition** (Proposition 6.5)  
 1423 properties required for a given semantics to yield a **compositional rewriting double category**, which for instance  
 1424 disqualifies the category **SGraph** of **directed simple graphs** to support compositional **generic DPO-semantics** (i.e., due  
 1425 to failure of axiom (L-iii) in **SGraph**; cf. also the discussion in Example 6.10). On the other hand, referring to Table 2  
 1426 for a list of practically relevant examples of categories with **adhesivity properties**, in many cases properties beyond  
 1427 weak adhesivity such as the existence of all pullbacks are indeed verified, which raises the interesting theoretical  
 1428 question of whether it might be possible to find a more general classification of categories that takes the additional  
 1429 properties presented in Table 3 as its basis, and that would permit an easier access to determining the kind of semantics  
 1430 a given category supports. Moreover, since many examples provided in Table 2 are indeed obtained as some form  
 1431 of **comma category** construction based upon **adhesive categories** such as **Set**, one might envision an extension of  
 1432 Theorem 5.9 that would permit to also determine whether a given comma category possesses additional structures  
 1433 such as an **M-partial map classifier**, existence of pullbacks or an **M-initial object**. We leave these open questions to  
 1434 future work.

#### 1435 6.4. Illustration: compositional non-linear double- and sesqui-pushout rewriting

1436 In this final part of the paper, we will present in some further detail the quintessential examples of compositional  
 1437 rewriting theories in the sense of our novel framework, i.e., the “non-linear” variants of double- and sesqui-pushout  
 1438 rewriting over suitable categories. By suitably restricting the formulae provided in the following to the relevant notion  
 1439 of linearity, one may moreover obtain explicit formulae also for the remaining six types of semantics according to  
 1440 Definition 6.1.

1441

1442 The aim of the ensuing results consists in providing explicit formulae for both the notion of **direct derivations** and  
 1443 of **rule compositions**, i.e., in a formulation perhaps somewhat more familiar to experts in graph rewriting theory. This  
 1444 involves in particular extracting the important notion of **rule compositions** from the **concurrency theorems**:

1445 **Lemma 6.9.** *Let  $\mathbf{C}$  be a category that **has pullbacks**, that is **finitary** and that is an **adhesive HLR category** with respect  
 1446 to a **stable system of monics**  $\mathcal{M}$ . Consider **generic Double-Pushout (DPO) semantics** over  $\mathbf{C}$ , where **direct derivations**  
 1447 are defined more explicitly as follows (compare Definition 6.1):*

- 1448 • The **set of DPO-admissible matches** of rule  $r = (O \xleftarrow{o_r} K_r \xrightarrow{i_r} I) \in \text{span}(\mathbf{C})$  into object  $X \in \text{obj}(\mathbf{C})$  is defined as

$$1449 \quad \mathcal{M}_r^{\text{DPO}}(X) := \{(m, k_\alpha, i_\alpha) \mid m \in \mathcal{M}, (k_\alpha, i_\alpha) \in \mathcal{P}(i_r, m)\} / \sim, \quad (73)$$

1449 where equivalence  $\sim$  is defined as equivalence up to **universal isomorphisms** of  $\mathcal{M}$ -**multi-IPCs**.

- 1450 • A **DPO-type direct derivation** of  $X \in \text{obj}(\mathbf{C})$  with rule  $r$  along  $m \in \mathcal{M}_r^{\text{DPO}}(X)$  is defined as a diagram in (74)  
 1451 below, where (1) is the  $\mathcal{M}$ -**multi-IPC** element chosen as part of the data of the **admissible match**, while (2) is  
 1452 formed as a **pushout**.

$$1453 \quad \begin{array}{ccccc} O & \xleftarrow{o_r} & K_r & \xrightarrow{i_r} & I \\ \downarrow m_\alpha^* & & \downarrow k_\alpha & & \downarrow m \\ r_\alpha(X) & \xleftarrow{o_{r_\alpha}} & K_{r_\alpha} & \xrightarrow{i_{r_\alpha}} & X \end{array} \quad (74)$$

1453 Moreover, the **synthesis** part of the **concurrency theorem** (cf. in particular (13)) yields the following notions:

- 1454 • Given  $r_2, r_1 \in \text{span}(\mathbf{C})$ , let the **set of DPO-type admissible matches** of rule  $r_2$  into  $r_1$  (also referred to as the  
 1455 dependency relation [3]) be defined as follows:

$$1456 \quad \mathcal{M}_{r_2}^{\text{DPO}}(r_1) := \{(j_2, j_1, \kappa_2, i_{r_2}, \kappa_1, o_{r_1}) \mid (j_2, j_1) \in \sum_{\mathcal{M}} (I_2, O_1), (\kappa_2, i_{r_2}) \in \mathcal{P}(i_{r_2}, j_2), (\kappa_1, o_{r_1}) \in \mathcal{P}(o_{r_1}, j_1)\} / \sim \quad (75)$$

1456 Here, the equivalence  $\sim$  by which we quotient is defined via the compatible universal isomorphisms of  $\mathcal{M}$ -**multi-**  
 1457 **sums** and  $\mathcal{M}$ -**multi-IPCs** (i.e., “compatible” relative to the diagram in (76) below).

$$1458 \quad \begin{array}{ccccccc} O_2 \xleftarrow{o_{r_2}} K_{r_2} \xrightarrow{i_{r_2}} I_2 & & & & O_1 \xleftarrow{o_{r_1}} K_{r_1} \xrightarrow{i_{r_1}} I_1 \\ \downarrow \omega_2 & (2_2) & \downarrow \kappa_2 & (1_2) & \downarrow \kappa_1 & (1_1) & \downarrow \iota_1 \\ O_{21} \xleftarrow{o_{r_2}} \bar{K}_2 \xrightarrow{i_{r_2}} J_{21} & \xrightarrow{j_2} & J_{21} & \xleftarrow{j_1} & \bar{K}_1 \xrightarrow{i_{r_1}} I_{21} \\ \text{PO} & \text{mIPC} & & \text{mIPC} & \text{PO} & & \end{array} \quad (76)$$

- 1458 • A **DPO-type rule composition** of two general rules  $r_1, r_2 \in \text{span}(\mathbf{C})$  along an **admissible match**  $\mu \in \mathcal{M}_{r_2}^{\text{DPO}}(r_1)$  is  
 1459 defined via a diagram as in (76) above, where (1<sub>2</sub>) and (1<sub>1</sub>) are the  $\mathcal{M}$ -**multi-IPC** elements chosen as part of the  
 1460 data of the match, while (2<sub>2</sub>) and (2<sub>1</sub>) are **pushouts**. We then define the **composite rule** via **span composition**:

$$1461 \quad r_2 \stackrel{\mu}{\triangleleft} r_1 := (O_{21} \leftarrow \bar{K}_2 \rightarrow J_{21}) \circ (J_{21} \leftarrow \bar{K}_1 \rightarrow I_{21}) \quad (77)$$

1461 With these definitions, one recovers a variant of the **concurrency theorem** whereby the statement of (13) is expressed  
 1462 as follows:

- 1463 • **Synthesis:** given an object  $X_0 \in \text{obj}(\mathbf{C})$ , for every pair  $((r_2, v_2), (r_1, v_1))$  of **rules** and **admissible matches**, where  
 1464  $v_1 \in \mathcal{M}_{r_1}^{\text{DPO}}(X_0)$  and  $v_2 \in \mathcal{M}_{r_2}^{\text{DPO}}(X_1)$  with  $X_1 := r_{1v_1}(X_0)$ , there exists an **admissible match**  $\mu \in \mathcal{M}_{r_2}^{\text{DPO}}(r_1)$  of rule  
 1465  $r_2$  into rule  $r_1$  and an **admissible match**  $v_{21} \in \mathcal{M}_{r_{2\mu}}^{\text{DPO}}(X_0)$  of the **composite rule**  $r_{2\mu}$  defined as in (77) such that  
 1466  $(r_{2\mu})_{v_{21}}(X_0) \cong_\bullet r_{2v_2}(r_{1v_1}(X_0))$
- 1467 • **Analysis:** for every **admissible match**  $\mu \in \mathcal{M}_{r_2}^{\text{DPO}}(r_1)$  of rule  $r_2$  into rule  $r_1$  and for every **admissible match**  
 1468  $v_{21} \in \mathcal{M}_{r_{2\mu}}^{\text{DPO}}(X_0)$  of the **composite rule**  $r_{2\mu}$  into the object  $X_0$ , there exists a pair  $((r_2, v_2), (r_1, v_1))$  of **admissible**  
 1469 **matches**  $(v_1, v_2)$ , where  $v_1 \in \mathcal{M}_{r_1}^{\text{DPO}}(X_0)$  and  $v_2 \in \mathcal{M}_{r_2}^{\text{DPO}}(r_{1v_1}(X_0))$ , such that  $r_{2v_2}(r_{1v_1}(X_0)) \cong_\bullet (r_{2\mu})_{v_{21}}(X_0)$ .

1470 PROOF. According to Lemma 4.11,  $\mathbf{C}$  has  $\mathcal{M}$ -multi-IPCs, hence the notion of **direct derivations** is well-posed. The  
 1471 rest of the proof then follows by instantiating (13) for the case of **generic DPO-semantics**. In particular, the explicit  
 1472 formula for the **rule composition** is obtained by taking advantage of the results of Theorem 6.7, i.e., noting that the  
 1473 **source functor**  $S : \mathbb{D}_1 \rightarrow \mathbb{D}_0$  of the **compositional rewriting double category** is a **multi-opfibration**, and that the **target**  
 1474 **functor**  $T : \mathbb{D}_1 \rightarrow \mathbb{D}_0$  is a **multi-opfibration**, and hence a special type of **residual multi-opfibration**. The latter statement  
 1475 is illustrated in the diagram below, which makes explicit the “identity-residue”  $\overline{K}_1 - id_{\overline{K}_1} \rightarrow \overline{K}_1$ :

1476 □

1477 **Example 6.10.** One of the most striking outcomes of the analysis presented in this paper is the **failure** of the category **SGraph** of **directed simple graphs** to support **generic DPO-semantics**. It is worthwhile emphasizing that even  
 1478 though in many cases the full-fledged generality of this type of semantics (which, as discussed also in Section 1.1,  
 1479 supports cloning and fusing subobjects) might not be needed, the case of **SGraph** is indeed much more fundamental,  
 1480 and, in a certain sense, was one of the main motivations for the developments presented in this paper. To wit, it is  
 1481 well-known (cf. e.g. [7]) that **SGraph** is a **vertical weak adhesive HLR category** only with respect to the **stable system**  
 1482 **of monics**  $\mathcal{M}_{\mathbf{SGraph}} = \text{rm}(\mathbf{SGraph})$  of **regular monomorphisms**, which are the **edge-reflecting monomorphisms** (cf.  
 1483 Theorem 5.18(S-ii)), i.e., in particular *not* with respect to the class of all monomorphisms in **SGraph**. Moreover, it  
 1484 is well-known that **SGraph** is *not* an **adhesive HLR category** w.r.t. the class  $\text{rm}(\mathbf{SGraph})$  [3, 7, 8], and hence indeed  
 1485 strictly does **not** support **generic DPO-semantics** (cf. Table 3). This failure is particularly startling since it is perfectly  
 1486 well possible to define **DPO-type direct derivations** (with  $\mathcal{M} = \text{rm}(\mathbf{SGraph})$ ), as **SGraph** *does* possess the properties  
 1487 required according to Definition 6.1. In fact, the examples of **multi-initial pushout complements** depicted in the left  
 1488 diagram of (48) may be interpreted as **DPO-type direct derivations** along a rule with an identity output morphism,  
 1489 demonstrating that **generic DPO-semantics** is at least in principle definable in **SGraph**.

1491

1492 Finally, let us turn towards **SqPO-semantics**, presented in the remainder of this section in explicit detail for its  
 1493 **generic variant**<sup>14</sup> (from which explicit definitions for the other variants of **SqPO-semantics** may be obtained by re-  
 1494 stricting the horizontal morphisms to be in  $\mathcal{M}$  as appropriate for the given semantics).

1495 **Lemma 6.11.** *Let  $\mathbf{C}$  be a category that has pullbacks, that is finitary and that is an adhesive HLR category with*  
 1496 *respect to a stable system of monics  $\mathcal{M}$ . Assume further that  $\mathbf{C}$  has FPCs along  $\mathcal{M}$ -morphisms. Consider generic*  
 1497 *Sesqui-Pushout (SqPO) semantics over  $\mathbf{C}$ , where direct derivations are defined more explicitly as follows (compare*  
 1498 *Definition 6.1):*

- 1499 • The set of **SqPO-admissible matches** of a rule  $r = (O \leftarrow K \rightarrow I) \in \text{span}(\mathbf{C})$  into an object  $X \in \text{obj}(\mathbf{C})$  is  
 1500 defined as

$$M_r^{\text{SqPO}}(X) := \{I - m \rightarrow X \mid m \in \mathcal{M}\}. \quad (79)$$

- 1501 • A **SqPO-type direct derivation** [23] of  $X \in \text{obj}(\mathbf{C})$  with rule  $r$  along  $m \in M_r^{\text{SqPO}}(X)$  is defined as a diagram as

<sup>14</sup>Note that in the original conference version [1] of this paper, we had provided a variant of this definition for the case of the underlying category being a **quasi-topos**, yet the results of the present paper permit to formulate this definition for a more general class of categories.

1502

in (80) below, where (1) is formed as an FPC, while (2) is formed as a pushout.

$$\begin{array}{ccccc}
 O & \xleftarrow{o_r} & K_r & \xrightarrow{i_r} & I \\
 \downarrow m^* & & \downarrow k_r & & \downarrow m \\
 r_m(X) & \xleftarrow{o_m} & K_{r_m} & \xrightarrow{i_m} & X
 \end{array} \quad (80)$$

1503

Moreover, the *synthesis* part of the *concurrency theorem* (cf. in particular (13)) yields the following notions:

1504

- Given  $r_2, r_1 \in \text{span}(\mathbf{C})$ , the set of SqPO-type admissible matches of rule  $r_2$  into  $r_1$  is defined as<sup>15</sup>

$$\begin{aligned}
 \mathcal{M}_{r_2}^{\text{SqPO}}(r_1) := \{ & (j_2, j_1, o_{r'_1}, \kappa_1, (\kappa_1)_{\star k}, i_{r_{1k}}, e_{1_k}) \mid (j_2, j_1) \in \sum_{\mathcal{M}} (I_2, O_1), (o_{r'_1}, \kappa_1) \in \mathcal{P}(o_{r_1}, j_1), \\
 & ((\kappa_1)_{\star k}, i_{r_{1k}}, e_{1_k}) \in \text{FPA}(i_{r_1}, \iota_1, \kappa_1, i_{r'_1}), \\
 & (i_{r'_1}, \iota_1) \in \text{PO}(\kappa_1, i_{r_1}) \} / \sim,
 \end{aligned} \quad (81)$$

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where the notation  $(i_{r'_1}, \iota_1) \in \text{PO}(\kappa_1, i_{r_1})$  entails that the cospan  $(i_{r'_1}, \iota_1)$  is a pushout of the span  $(\kappa_1, i_{r_1})$ , and where equivalence is defined up to the compatible universal isomorphisms of  $\mathcal{M}$ -multi-sums,  $\mathcal{M}$ -multi-IPCs and FPAs (i.e., “compatibility” relative to the diagram in (82) below).

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- An SqPO-type rule composition of two general rules  $r_1, r_2 \in \text{span}(\mathbf{C})$  along an admissible match  $\mu \in \mathcal{M}_{r_2}^{\text{SqPO}}(r_1)$  is defined via a diagram as in (82) above, which may be constructed step-wise (going clockwise) by letting square (1) be an mlIPC, square (2) a pushout, square (3) an FPA, square (4) a pushout, square (5) an FPC, and finally square (6) a pushout. We then define the composite rule via span composition:

$$r_2 \triangleleft^{\mu} r_1 := (O_{21} \leftarrow \bar{K}_2 \rightarrow \bar{J}_{21}) \circ (\bar{J}_{21} \leftarrow \bar{K}_1 \rightarrow I_{21}) \quad (83)$$

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With these definitions, one recovers a variant of the *concurrency theorem* whereby the statement of (13) is expressed as follows:

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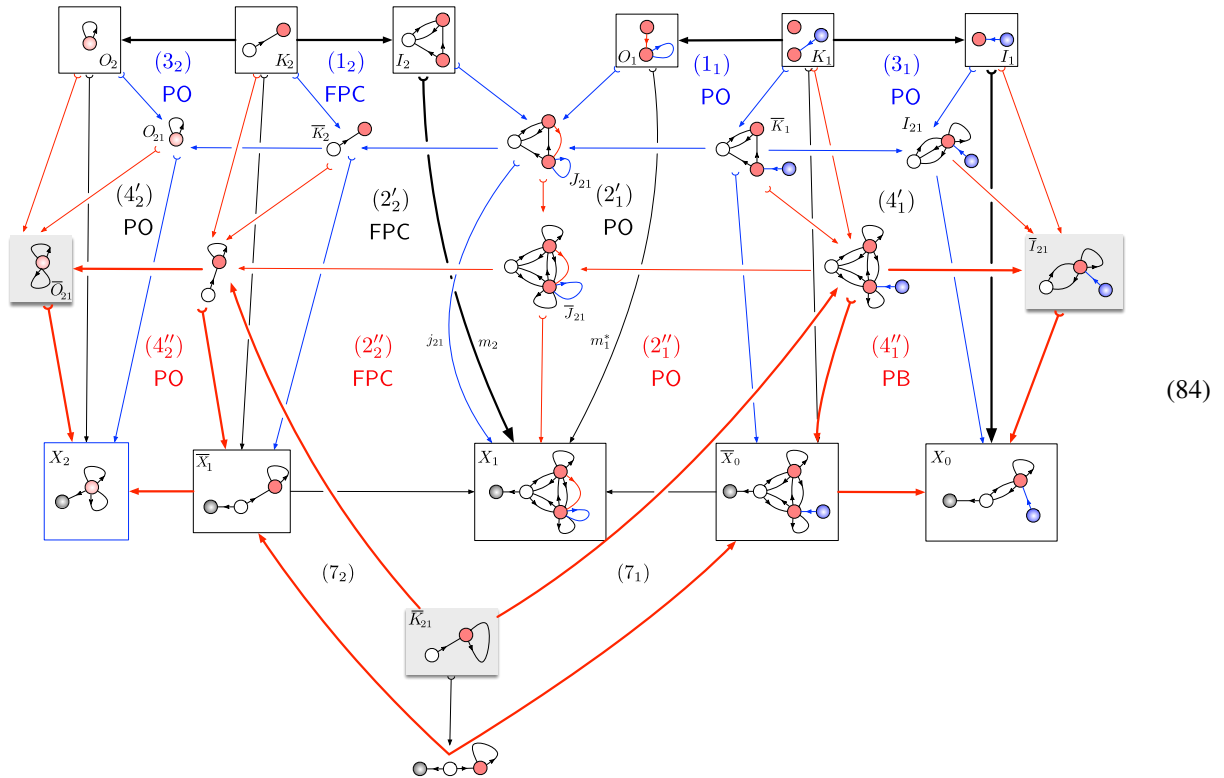
1520

- **Synthesis:** given an object  $X_0 \in \text{obj}(\mathbf{C})$ , for every pair  $((r_2, m_2), (r_1, m_1))$  of rules and admissible matches, where  $m_1 \in \mathcal{M}_{r_1}^{\text{SqPO}}(X_0)$  and  $m_2 \in \mathcal{M}_{r_2}^{\text{SqPO}}(X_1)$  with  $X_1 := r_{1m_1}(X_0)$ , there exists an admissible match  $\mu \in \mathcal{M}_{r_2}^{\text{SqPO}}(r_1)$  of rule  $r_2$  into rule  $r_1$  and an admissible match  $m_{2\mu} \in \mathcal{M}_{r_{2\mu}}^{\text{SqPO}}(X_0)$  of the composite rule  $r_{2\mu}$  defined as in (83) such that  $(r_{2\mu})_{m_{2\mu}}(X_0) \cong_{\bullet} r_{2m_2}(r_{1m_1}(X_0))$ .
- **Analysis:** for every admissible match  $\mu \in \mathcal{M}_{r_2}^{\text{SqPO}}(r_1)$  of rule  $r_2$  into rule  $r_1$  and for every admissible match  $m_{2\mu} \in \mathcal{M}_{r_{2\mu}}^{\text{SqPO}}(X_0)$  of the composite rule  $r_{2\mu}$  into an object  $X_0$ , there exists a pair  $(m_1, m_2)$  of admissible matches, where  $m_1 \in \mathcal{M}_{r_1}^{\text{SqPO}}(X_0)$  and  $m_2 \in \mathcal{M}_{r_2}^{\text{SqPO}}(r_{1m_1}(X_0))$ , such that  $(r_{2\mu})_{m_{2\mu}}(X_0) \cong_{\bullet} r_{2m_2}(r_{1m_1}(X_0))$ .

<sup>15</sup>In the conference version of this paper, we had opted for a slightly different variant of the definition of FPAs than in the current paper, i.e., where the pushout square was not explicitly mentioned; however, due to the nature of the equivalence relation  $\sim$  in (81), we in fact arrive at an equivalent notion of admissible SqPO-type matches of rules.

1521 **PROOF.** By assumption,  $\mathbf{C}$  has FPCs along  $\mathcal{M}$ -morphisms, hence the definition of SqPO-type direct derivations is well-  
 1522 posed. Moreover, according to Lemma 5.28, the assumptions on  $\mathbf{C}$  suffice to demonstrate that  $\mathbf{C}$  has FPAs, hence the  
 1523 notions of **admissible matches of rules** and of **composite rules** are also well-posed. The rest of the proof then follows by  
 1524 instantiating (13) for the case of **generic SqPO-semantics**. In particular, the explicit formula for the **rule composition**  
 1525 is obtained by taking advantage of the results of Theorem 6.7, i.e., noting that the **source functor**  $S : \mathbb{D}_1 \rightarrow \mathbb{D}_0$  of the  
 1526 **compositional rewriting double category** is a **multi-opfibration**, and that the **target functor**  $T : \mathbb{D}_1 \rightarrow \mathbb{D}_0$  is a **residual**  
 1527 **multi-opfibration**.  $\square$

1528 **Example 6.12.** Let us illustrate the notion of SqPO-type rule composition, as given in Lemma 6.11, with the following  
 1529 example in the setting of **directed multigraphs**.



1530 In order to provide the interested readers with some further intuitions for the relatively complex structure of **rule**  
 1531 **compositions** in **generic SqPO-semantics**, we will present in the following a heuristic explanation for the precise  
 1532 shape of the diagram in (84) (which is constructed via invoking Lemma 6.11).

1533 In this example, we have two rules. The first clones one node<sup>16</sup>, but not its incident edge, adds a new edge between  
 1534 that original node and its clone and then merges that original node with the other (blue) node of the input graph. The  
 1535 second rule deletes one node and then merges the two remaining nodes. The given applications to the graphs  $X_0$  and  
 1536  $X_1$  illustrate some of the idiosyncrasies of SqPO-rewriting:

- 1537 • Since the node of  $X_0$  that is being cloned possesses a self-loop, the result of cloning is two nodes, each with a  
 1538 self-loop, with one edge going each way between them.
- 1539 • In the application of the second rule to  $X_1$ , we see the side-effect whereby all edges incident to the deleted node  
 1540 are themselves deleted (as also occurs in SPO-, but not in DPO-rewriting).

<sup>16</sup>Note that the structure of the homomorphisms may be inferred from the node positions, with the exception of the vertex clonings that are explicitly mentioned in the text.

1541 The overall effect of the two rewrites can be seen in  $X_2$ ; as usual, this depends on the overlap between the images of  
1542  $O_1$  and  $I_2$  in  $X_1$ . This overlap is precisely the **multi-sum** element  $J_{21}$ . Since our example is set in an **adhesive category**,  
1543 this can be most easily computed by taking the **pullback** of  $m_1^*$  and  $m_2$  and then the **pushout** of the resulting span. The  
1544 **pushout** that defines the rewrite from  $\bar{X}_0$  to  $X_1$  can now be factorized by computing the **pullback** of the arrow  $j_{21}$  from  
1545  $J_{21}$  to  $X_1$  and the arrow from  $\bar{X}_0$  to  $X_1$ ; this determines  $\bar{K}_1$  and its universal arrow from  $K_1$  with the consequence that  
1546  $(1_1)$  and  $(2_1)$ , the vertical pasting of  $(2'_1) + (2''_1)$ , are both **pushouts**. Let us note that  $\bar{K}_1$  is the appropriate member of  
1547 the **multi-IPC**, as determined by the particular structure of  $\bar{X}_0$ .

1548 The **pushout**  $(3_1)$  induces a universal arrow from  $I_{21}$  to  $X_0$ ; but an immediate inspection reveals that this homomor-  
1549 phism is not a monomorphism (nor an epimorphism in this case). As such, we cannot hope to use  $I_{21}$  as the input side  
1550 of the composite rule. Furthermore, we find that the square  $(4_1)$ , the vertical pasting of  $(4'_1) + (4''_1)$ , is neither a **pullback**  
1551 nor a **pushout**. However, the **FPA**  $\bar{I}_{21}$  resolves these problems by enabling a factorization of this square, based upon  
1552 a factorization of the morphism  $I_{21} \rightarrow X_0$  into an epimorphism  $I_{21} \xrightarrow{e_{21}} \bar{I}_{21}$  and a monomorphism  $\bar{I}_{21} \xrightarrow{m_{21}} X_0$ . Note  
1553 also that  $(4'_1)$  and  $(3_1) + (4'_1)$  are **pullbacks** and indeed **FPCs**. This factorization, as determined by  $e_{21}$ , can now be  
1554 *back-propagated* to factorize  $(2_1)$  into **pushouts**  $(2'_1)$  and  $(2''_1)$ , which gives rise to an augmented version  $\bar{J}_{21}$  of the  
1555 **multi-sum** object  $J_{21}$ . Note moreover that the effect of back-propagation concerns also the contribution of the second  
1556 rule in the composition: the final output graph contains an extra self-loop (compared to the graph  $O_{21}$  defined by the  
1557 **pushout**  $(3_2)$ ), which is induced by the extra self-loop of  $\bar{J}_{21}$  that appears due to back-propagation.

1558 We may then compute the composite rule via taking a **pullback** to obtain  $\bar{K}_{21}$ , yielding in summary the rule  
1559  $\bar{O}_{21} \leftarrow \bar{K}_{21} \rightarrow \bar{I}_{21}$ . Performing the remaining steps of the **synthesis** part of the **concurrency theorem** then amounts to  
1560 constructing the commutative cube in the middle of the diagram, yielding the **FPC**  $(7_1)$  and the **pushout**  $(7_2)$ , and thus  
1561 finally the one-step **SqPO-type direct derivation** from  $X_0$  to  $X_2$  along the **composite rule**  $\bar{O}_{21} \leftarrow \bar{K}_{21} \rightarrow \bar{I}_{21}$ .

1562 Let us finally note, as a general remark, that if the first rule in an SqPO-type rule composition is **output-linear** then  
1563 the **mIPC** is uniquely determined; and if it is **input-linear** then the **pushout**  $(3_1)$  is also an **FPC** (cf. Lemma 5.24) and  
1564  $(4_1)$  is a **pullback**, by Lemma 2(h) of [19]. In this case, the **FPA** is trivial, and consequently so is the back-propagation  
1565 process. Our rule composition can thus be seen as a conservative extension of that defined for **linear** rules in [19].

## 1566 7. Conclusions

### 1567 7.1. Concluding remarks

1568 We have presented a novel formalism for graph transformation and exploited this to provide generic results—the  
1569 **concurrency** and **associativity theorems**—that characterize the categorical rewriting theories as compositional. These  
1570 results have been proved once and for all and in a *universal*, i.e., semantics-independent, fashion based upon our novel  
1571 notion of **compositional rewriting double categories** (crDCs). We have then investigated the conditions under which  
1572 a variety of variants of both **DPO-** and **SqPO-semantics** yield crDCs, and thus compositional categorical rewriting  
1573 theories. In the case of **SqPO-semantics**, we have in particular established that rewriting is compositional for fully  
1574 general rules in the concrete setting of simple graphs. In the case of **DPO-semantics**, our results also establish general  
1575 conditions under which rewriting is compositional; it is worthwhile emphasizing that these conditions exclude the  
1576 case of non-linear rules on **directed simple graphs**. This failure (due concretely to **SGraph** not satisfying axiom (L-  
1577 iii-b); cf. Table 3) appears to be indicative of the utility of our fine-grained analysis, in that the precise role played by  
1578 each of the assumptions found to be sufficient for a given compositional rewriting semantics are highlighted, such that  
1579 perhaps in future work alternative semantics or categorical constructions might be constructible in order to overcome  
1580 these limitations.

1581 Our results are based on a new formalism that expresses the required categorical structure in terms of certain  
1582 fibrational structures together with a small number of axioms specific to the case of rewriting. One particularly  
1583 significant aspect of our approach is that it provides an intrinsic, and cognitively convenient, structuring of the very  
1584 large number of lemmata used in graph transformation whose statements and proofs are scattered across the literature.  
1585 We hope that this higher-level structuring will aid the process of formalizing this area of mathematics via proof  
1586 assistants such as Coq [63], Isabelle/HOL [64], or Lean [65]. In this paper, the approach already enables more  
1587 compact proofs and, indeed, we feel that the **associativity theorem** would simply not be possible to express without  
1588 these means. We are convinced that additional benefit may be extracted from this general setting.

1589 7.2. Relation to the extended abstract

1590 In the extended abstract of this paper [1, 2], we showed that the important use case of transformation of **directed**  
1591 **simple graphs** under **SqPO-semantics** requires the use of a restricted notion of matching—to *edge-reflecting* injective  
1592 homomorphisms, abstractly characterized by the so-called *regular monos*—in order to prove the **concurrency theorem**  
1593 that provides a proper notion of rule composition. This led us to investigate more generally the categorical structure  
1594 required to support rule composition, under the SqPO semantics, for fully general non-linear rules, and we estab-  
1595 lished that **quasi-topoi** [6, 7, 44, 45, 57] naturally possess all the necessary structure. In particular, our theorems do  
1596 therefore apply to the category of **directed simple graphs**—which forms a **quasi-topos** [7, 45] but fails to satisfy the  
1597 axioms of **adhesive** or **quasi-adhesive** categories [7]—for which no satisfactory prior **concurrency theorem** has been  
1598 proved<sup>17</sup>. These results significantly generalized previous work on concurrency theorems for **linear SqPO rewriting**  
1599 over **adhesive categories** [19], and for **linear SqPO rewriting** with conditions in  **$\mathcal{M}$ -adhesive categories** [32, 62].

1600 This proof of the **concurrency theorem** under the **SqPO-semantics** relied on the existence of certain structures in  
1601 quasi-topoi that, to the best of our knowledge, had not been previously noted in the literature: restricted notions of  
1602 **multi-sum** and **multi-IPC** (mIPC) plus that of **FPC-pushout-augmentation** (FPA). The **multi-sum** construction provides  
1603 a generalization of the property of effective unions (in adhesive categories) that guarantees that all necessary monos are  
1604 regular. The notions of **mIPC** and **FPA** handle the “backward non-determinism” introduced by non-linear rules: given  
1605 a rule and a matching from its output graph, we cannot—unlike with linear or reversible non-linear rules—uniquely  
1606 determine a matching from the input graph of the rule.

1607 In the case of **DPO-semantics**, we established (again in [1, 2]) a generalization of the **concurrency theorem** to  
1608 **generic DPO-semantics**, and presented **rm-adhesive categories** as possessing sufficient properties in order to support  
1609 this semantics. However, unlike in the case of SqPO-rewriting, this does not capture the case of generic DPO-rewriting  
1610 in the category of simple graphs as this setting is well-known to satisfy only the weaker axioms of **rm-quasi-adhesive**  
1611 **categories**. In the present paper, we have demonstrated that, slightly more generally, **adhesive HLR categories** can  
1612 also support **generic DPO-semantics**, thereby extending the range of applications of this type of semantics, although  
1613 this still does not cover the important case of simple graphs.

1614 In comparison to the much more technically involved “direct” proofs found in [1, 2], the high-level abstraction  
1615 offered by the novel fibrational approach to compositional rewriting theories permits the modularization of the proofs  
1616 of the concurrency and associativity theorems in a very efficient manner. In particular, this relies upon making a clear  
1617 separation of the concrete definitions of compositional rewriting theories, i.e., proving that a certain semantics and  
1618 choice of base category gives rise to a **compositional rewriting double category** (crDC), from the universal structures  
1619 offered by a crDC.

1620 7.3. Related work

1621 Conditions under which **final pullback complements** (FPCs) are guaranteed to exist have been studied in [24]  
1622 and also in [30] which provides a direct construction assuming the existence of appropriate  **$\mathcal{M}$ -partial map classi-**  
1623 **fiers** [48, 57]. We make additional use of these  **$\mathcal{M}$ -partial map classifiers** in order to construct **multi-initial pushout**  
1624 **complements**. This construction is a mild, but necessary for our purposes, generalization of the notion of minimal  
1625 pushout complement defined in [26] that requires the universal property with respect to a larger class of encompassing  
1626 pushouts—precisely analogous to the definition of **FPC**—that additionally allows us to specify a family of solutions  
1627 that collectively satisfy this universal property so as to handle the backward non-determinism of non-linear rules.

1628 We also exploit the  **$\mathcal{E}$ - $\mathcal{M}$ -factorization** which every **finitary  $\mathcal{M}$ -adhesive category** possesses [8], where  $\mathcal{E}$  is the  
1629 class of **extremal morphisms** with respect to  $\mathcal{M}$  (cf. Theorem 5.20). Noting that this factorization coincides with the  
1630 well-known **epi-regular mono-factorization** [45] in **quasi-topoi**, we extended our approach from [1, 2] for constructing  
1631  **$\mathcal{M}$ -multi-sums** to the more general setting of **finitary  $\mathcal{M}$ -adhesive categories** (cf. Lemma 5.21). This makes it clear  
1632 that the notion of  **$\mathcal{M}$ -multi-sum** is in fact nothing other than the well-known notion of  **$(\mathcal{E}', \mathcal{M})$ -pair factorizations**  
1633 from the traditional literature on **DPO-semantics** for  **$\mathcal{M}$ -adhesive categories** [3, 8]. Moreover, we also find that the  
1634 notion of  **$\mathcal{E}$ - $\mathcal{M}$ -factorization** permits a more general construction of **FPC-pushout-augmentations** (FPAs) than the one  
1635 we originally presented in [1, 2], thereby opening up **generic SqPO-semantics** to categories other than **quasi-topoi** (cf.  
1636 Section 5.3.4).

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<sup>17</sup>The setting of  $\mathcal{M}$ -linear rules, where  $\mathcal{M}$  is the regular monos, prevents rules from deleting or adding edges and so has limited use in practice.

Overall, and as discussed throughout the paper, but in particular in full detail in Sections 5 and 6, our approach to determining suitable classes of categories supporting the various rewriting semantics relies heavily on the categorical rewriting and category theory literature, such as the traditional framework of Ehrig et al. that is based upon the notion of  $\mathcal{M}$ -adhesive categories [3], but also a large number of works on a variety of other categories with adhesivity properties [4, 5, 6, 7, 23, 44, 46, 47, 55, 57].

While traditional results mostly concerned linear or semi-linear DPO-semantics, for which  $\mathcal{M}$ -adhesive categories were found to pose a very general class of suitable categories [3, 47], and while every quasi-topos is indeed also an  $\mathcal{M}$ -adhesive category [48, Lem. 13] (for  $\mathcal{M}$  the class of regular monos),  $\mathcal{M}$ -adhesive categories or quasi-topoi are in general *not* adhesive HLR categories, and may thus fail to support generic DPO-semantics. In particular, as mentioned above, the category **SGraph** of directed simple graphs does *not* support generic DPO-semantics. This is highly significant in view of practical applications: since restricting to edge-reflecting monomorphisms for defining the rules would mean that no edges could be created or deleted, in many practical applications, even those with no need for cloning or merging, it would be necessary to work with generic monomorphisms, which consequently entails that one no longer has even a concurrency theorem available for analyzing the resulting rewriting systems.

A notable application example in this regard is the MØD framework [17], which aims to implement a rewriting system capable of modeling organo-chemical reaction systems. As explained in further detail in [32], since this framework uses typed simple graphs with additional complex type- and degree-constraints for its base category, the operations of rule compositions in MØD [17] would have to be considered strictly **not** mathematically consistent, since as demonstrated above **SGraph** does **not** support even a concurrency theorem in generic DPO-semantics (and thus the same holds for typed and constrained variants of directed simple graphs). This would be a technically severe failure, since ultimately organo-chemical rewriting is intended to faithfully model the continuous-time Markov chains (CTMCs) that encode organo-chemical reaction systems; failure of being a compositional rewriting theory more explicitly would entail the absence of a suitable rule algebra and stochastic mechanics formalism, rendering a mathematically consistent formulation of the CTMCs impossible.

In summary, the work presented in the present paper represents a clear warning regarding the imprecise interpretation of rewriting semantics, and, more positively, provides a first step towards streamlining the framework of compositional rewriting theory such that verifying the consistency of applications of rewriting theory in a more transparent way.

#### 7.4. Future work

The immediately preceding discussion about the failure of generic DPO-rewriting in certain concrete settings leads us naturally to the notion of  $\mathcal{M}, \mathcal{N}$ -adhesive categories [66, 67]: according to recent results of [53, Thm. 3.1], **SGraph** (referred to as “**DGraph**” in loc. cit.) is in fact  $(\text{rm}(\mathbf{SGraph}), \text{mono}(\mathbf{SGraph}))$ -adhesive and  $(\text{mono}(\mathbf{SGraph}), \text{rm}(\mathbf{SGraph}))$ -adhesive, i.e., carries two types of  $(\mathcal{M}, \mathcal{N})$ -adhesivity structures. Based upon this type of property, as demonstrated in [66, 67] one may modify the definition of generic DPO-semantics to a variant where the vertical morphisms in the definition of direct derivations (cf. (85)) are required to be in the class  $\mathcal{N}$ , while the horizontal morphisms (i.e., those from which the rules are constructed) to be in the class  $\mathcal{M} \subseteq \text{mono}(\mathbf{C})$ . For the concrete case of the category **SGraph**, one possible choice of this modified type of DPO-semantics would be to let  $\mathcal{N} = \text{rm}(\mathbf{SGraph})$  (depicted in the diagram below with  $\rightrightarrows$  arrows) and  $\mathcal{M} = \text{mono}(\mathbf{SGraph})$  (depicted below<sup>18</sup> with  $\leftrightarrow$  arrows):

$$\begin{array}{ccc}
 \begin{array}{ccc}
 O & \xleftrightarrow{r} & I \\
 \downarrow m' & \Downarrow \alpha & \downarrow m \\
 O' & \xleftrightarrow{r'} & I'
 \end{array} & := & \begin{array}{ccccc}
 O & \xleftrightarrow{o_r} & K_r & \xleftrightarrow{i_r} & I \\
 \downarrow m' & \Downarrow (\dagger_\alpha) & \downarrow k_\alpha & \Downarrow (\ast_\alpha) & \downarrow m \\
 O' & \xleftrightarrow{o_{r'}} & K_{r'} & \xleftrightarrow{i_{r'}} & I'
 \end{array}
 \end{array} \tag{85}$$

According to [67, Thm. 1.5], pushout complements in an  $(\mathcal{M}, \mathcal{N})$ -adhesive category are essentially unique if they exist, so the aforementioned notion of modified DPO-semantics lies in a certain sense “in between” linear and generic

<sup>18</sup>Coincidentally, this diagram is close in structure to the semantics of direct derivations in the aforementioned MØD framework, in that rules therein are in particular identities on the vertices of the graphs involved (albeit **SGraph** itself does of course not take into account the type- and degree-constraints relevant to organic chemistry).



1677 **DPO-semantics**. More importantly, it was demonstrated in [67] that this modified DPO-semantics over  $(\mathcal{M}, \mathcal{N})$ -  
1678 adhesive categories admits a concurrency theorem (and most of the other properties of adhesive HLR categories in a  
1679 suitably modified form), hence we believe it would be highly interesting to submit this notion of rewriting to our novel  
1680 analysis method, i.e., to determine if (or under which additional conditions) the modified DPO-rewriting semantics  
1681 yields a **compositional rewriting double category**.

1682 Clearly, it would also be interesting to study some other graph transformation semantics, such as PBPO(+) [68, 69]  
1683 or AGREE [30], from the new viewpoint of our fibrational approach. It would also be fruitful to investigate how the  
1684 proofs of additional key theorems, such as local confluence, might carry over to this framework as this would increase  
1685 our confidence in its general applicability. In general, it will be highly desirable to develop a curated collection of  
1686 mathematical techniques (ideally formalized in proof assistants) that will permit to efficiently construct and analyze  
1687 categories used in rewriting theories with suitable **adhesivity properties**, including the aforementioned generalized  
1688 notion thereof, as well as **quasi-topos** structures, such that for practical applications of compositional categorical  
1689 rewriting theories the entry barrier posed by the considerably technically intricate theoretical framework may be  
1690 considerably lowered. As already presented in Sections 5.1 and 5.2, at present there already exists a certain amount of  
1691 mathematical methodology in this regard, most notably **comma category** constructions of categories with **adhesivity**  
1692 **properties**, and a variant thereof, so-called *Artin gluing* [7], for constructing **quasi-topoi** (cf. also [53] for recent  
1693 advances in constructing  $(\mathcal{M}, \mathcal{N})$ -adhesive categories). On the other hand, as summarized in Table 3, we are left  
1694 to wonder whether it is indeed **adhesivity properties** that most generally characterize categories suitable for **SqPO-**  
1695 **semantics**, since this semantics does *not* require the fully-fledged variant of the relevant van Kampen square axioms  
1696 (i.e., only axioms (X-iii-a), but not axioms (X-iii-b)), while on the other hand in order to support compositional SqPO  
1697 semantics, it is required that the underlying category **has pullbacks**, and that it **has FPCs along  $\mathcal{M}$ -morphisms**. It  
1698 would thus be highly desirable to find a more fine-grained and better adapted classification scheme for categories  
1699 supporting compositional rewriting semantics of various kinds, for which in the present paper we have provided a first  
1700 stepping stone.

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1855 2. In a **regular mono (rm)-quasi-adhesive category** [6, Def. 1.1 and Cor. 4.7], all pushouts along regular monomor-  
 1856 phisms exist, these pushouts are also pullbacks, and in particular **pushouts along regular monomorphisms are**  
 1857 **stable under pullbacks**. A useful characterization of **rm-quasi-adhesive categories** is the following: a small  
 1858 category  $\mathbf{C}$  which **has pullbacks** and which **has pushouts along regular monomorphisms** is **rm-quasi-adhesive**  
 1859 **iff it has a full embedding into a quasi-topos** (preserving the aforementioned two properties).

1860 **Lemma Appendix A.4** ([28], Lem. 1). Let  $\mathbf{C}$  be a category that **has pullbacks**. Then **FPCs are stable under pull-**  
 1861 **backs**.

1862 **Proposition Appendix A.5**. In a **quasi-topos**  $\mathbf{C}$ , **unions of regular subobjects are effective** [7, Prop. 10], i.e., the  
 1863 union of two subobjects is computed as the pushout of their intersection, and moreover the following property holds:  
 1864 in a commutative diagram such as below, where  $(c, a)$  is the pullback of  $(h, p)$ ,  $(d, b)$  the pushout of  $(c, a)$ , where all  
 1865 morphisms (except possibly  $x$ ) are monomorphisms, and where either  $p \in \text{rm}(\mathbf{C})$  or  $h \in \text{rm}(\mathbf{C})$ , then the induced  
 1866 morphism  $x : D \rightarrow E$  is a monomorphism [6, Prop. 2.4]:

$$\begin{array}{ccccc}
 A & \xrightarrow{a} & B & & \\
 \downarrow c & & \downarrow b & \searrow p & \\
 \text{PO} & & & & \\
 C & \xrightarrow{d} & D & \xrightarrow{x} & E \\
 & \searrow h & & & \swarrow
 \end{array}
 \tag{A.2}$$

1867 Appendix A.3. Single-square lemmata specific to  $\mathcal{M}$ -adhesive categories

1868 **Lemma Appendix A.6**. Let  $\mathbf{C}$  be an  $\mathcal{M}$ -adhesive category.

$$\begin{array}{ccccc}
 A & \xrightarrow{a} & B & & \\
 \downarrow \gamma & & \downarrow \beta & \text{(*)} & \\
 C & \xrightarrow{d} & D & & 
 \end{array}
 \tag{A.3}$$

1869 1. **Pushouts along  $\mathcal{M}$ -morphisms are pullbacks**: if  $(*)$  is a pushout and  $\gamma \in \mathcal{M}$ , then  $(*)$  is also a pullback.

1870 2. **Stability of  $\mathcal{M}$ -morphisms under pushout**: if  $(*)$  is a pushout and  $\gamma \in \mathcal{M}$ , then  $\beta \in \mathcal{M}$ .

1871 Appendix A.4. Double-square lemmata

1872 **Lemma Appendix A.7**. Let  $\mathbf{C}$  be a category. If in the following statements a class of monics  $\mathcal{M}$  is mentioned, these  
 1873 statements require  $\mathbf{C}$  to possess a **stable system of monics**  $\mathcal{M}$ .

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 A & \xrightarrow{a} & B & \xrightarrow{b} & C \\
 \downarrow \alpha & & \downarrow \beta & & \downarrow \chi \\
 A' & \xrightarrow{a'} & B' & \xrightarrow{b'} & C' \\
 & \searrow & & \nearrow & \\
 & & A' & & 
 \end{array} & & 
 \begin{array}{ccccc}
 F & \xrightarrow{f} & F' & & \\
 \downarrow \varphi & & \downarrow \varphi' & & \\
 G & \xrightarrow{g} & G' & & F' \\
 \downarrow \gamma & & \downarrow \gamma' & & \\
 H & \xrightarrow{h} & H' & & 
 \end{array}
 \end{array}
 \tag{A.4}$$

1874 Given commutative diagrams as above, the following statements hold:

1875 1. **Pushout-pushout-(de-)composition**: if (1) is a pushout, (1) + (2) is a pushout iff (2) is a pushout.

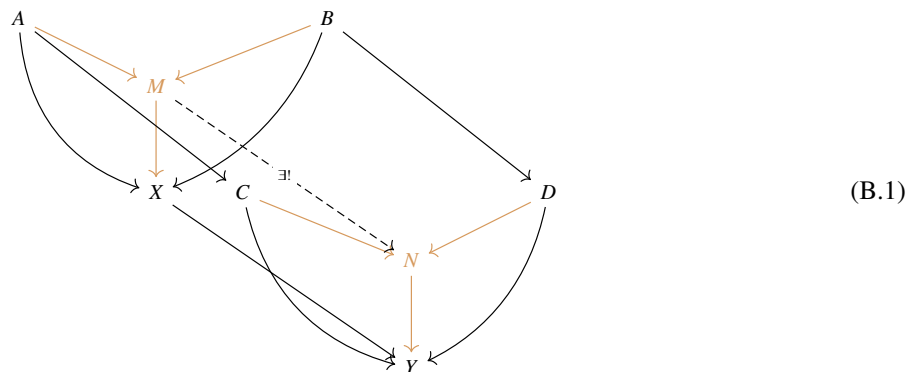
1876 2. **Pullback-pullback-(de-)composition**: if (2) is a pullback, (1) + (2) is a pullback iff (1) is a pullback.

- 1877 3. *Pushout-pullback-decomposition* [28, Lem. 4]: if (1) + (2) is a **stable pushout**<sup>19</sup> and (1), (2), (\*) are pullbacks,  
 1878 then (1) and (2) are both pushouts. (Note: If  $a'$  and  $b'$  are monomorphisms, the condition on (\*) is always  
 1879 satisfied.) Alternatively [3, Thm. 4.26.2], if  $\mathbf{C}$  is a *vertical weak adhesive HLR category* with respect to a *stable*  
 1880 *system of monics*  $\mathcal{M}$ , then if  $a'$  and  $b'$  are in  $\mathcal{M}$ , (1) + (2) is a pushout, and (2) is a pullback, then (1) and (2)  
 1881 are both pushouts. If the category is in fact a *weak adhesive HLR category* (i.e., if it also has *horizontal weak*  
 1882 *VK squares*), then the decomposition also holds if instead of  $a'$  the morphism  $\alpha$  is in  $\mathcal{M}$ .
- 1883 4. *Pullback-pushout-decomposition* [70, Lem. B.2]: if  $\mathbf{C}$  is a *vertical weak adhesive HLR category* with respect to  
 1884 a *stable system of monics*  $\mathcal{M}$ ,  $\chi$  is in  $\mathcal{M}$ , (1) + (2) is a pullback and (1) is **stable pushout**, then (1) and (2) are  
 1885 both pullbacks.
- 1886 5. *Horizontal FPC-FPC-(de-)composition* [23, Lem. 2 & 3], [71, Prop. 36]: if (2) is an FPC (i.e.,  $(\beta, b')$  is an  
 1887 FPC of  $(b, \chi)$ ), (1) + (2) is an FPC iff (1) is an FPC.
- 1888 6. *Vertical FPC-FPC-(de-)composition* [71, Prop. 36]: if (3) is an FPC (i.e.,  $(\varphi, g)$  is an FPC of  $(f, \varphi')$ ),  
 1889 (a) if (4) is an FPC (i.e.,  $(\gamma, h)$  is an FPC of  $(g, \gamma')$ ), then (3) + (4) is an FPC (i.e.,  $(\gamma \circ \varphi, h)$  is an FPC of  
 1890  $(f, \gamma' \circ \varphi')$ )  
 1891 (b) if (3) + (4) is an FPC (i.e.,  $(\gamma \circ \varphi, h)$  is an FPC of  $(f, \gamma' \circ \varphi')$ ) and if (4) is a pullback, then (4) is an FPC  
 1892 (i.e.,  $(\varphi, g)$  is an FPC of  $(f, \varphi')$ ).
- 1893 7. *Vertical FPC-pullback decomposition* [28, Lem. 3]: if (3)+(4) is an FPC (i.e.,  $(\gamma \circ \varphi, h)$  is an FPC of  $(f, \gamma' \circ \varphi')$ ),  
 1894 both (4) and  $(\dagger)$  are pullbacks, and if the diagram commutes, then (3) is an FPC (i.e.,  $(\varphi, g)$  is an FPC of  $(f, \varphi')$ )  
 1895 and (4) is an FPC (i.e.,  $(\gamma, h)$  is an FPC of  $(g, \gamma')$ ). (Note: If  $\gamma'$  and  $\varphi'$  are monomorphisms, the condition on  
 1896  $(\dagger)$  is always satisfied.)

1897 **Appendix B. Proofs not included in the main text**

1898 *Appendix B.1. Proofs of Section 2*

1899 **Lemma 2.2 (Multi-sum extension).** Let  $\mathbf{C}$  be a category that has **multi-sums** and that has **pullbacks**. Then for every  
 1900 commutative diagram such as in (B.1) below, where  $A \rightarrow M \leftarrow B$  and  $C \rightarrow N \leftarrow D$  are multi-sum elements, there  
 1901 exists a universal arrow  $M \rightarrow N$  that makes the diagram commute.



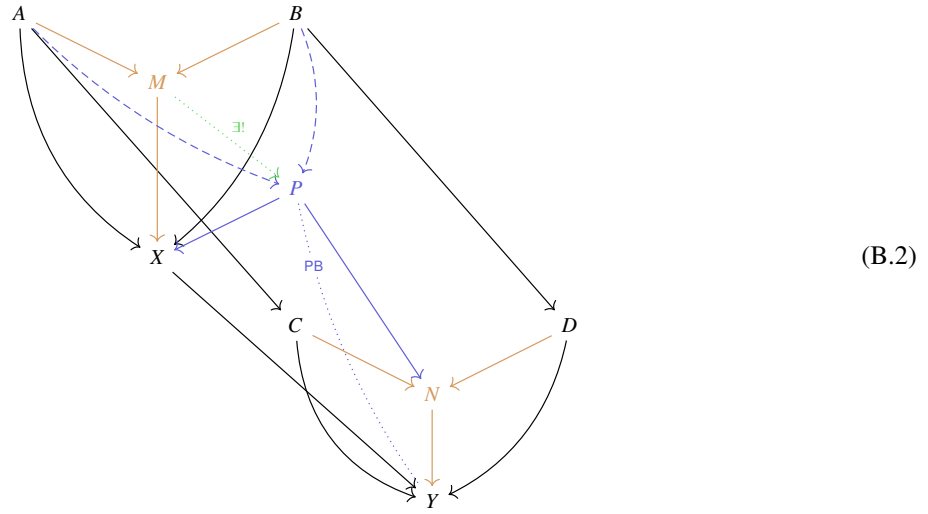
1902

1903 **PROOF.** Construct the diagram in (B.2) below as follows:

- 1904 • Since  $\mathbf{C}$  has pullbacks, take a pullback in order to obtain the span  $X \leftarrow P \rightarrow N$ . Then by the **universal property**  
 1905 **of pullbacks**, there exist morphisms  $A \rightarrow P$  and  $B \rightarrow P$ .

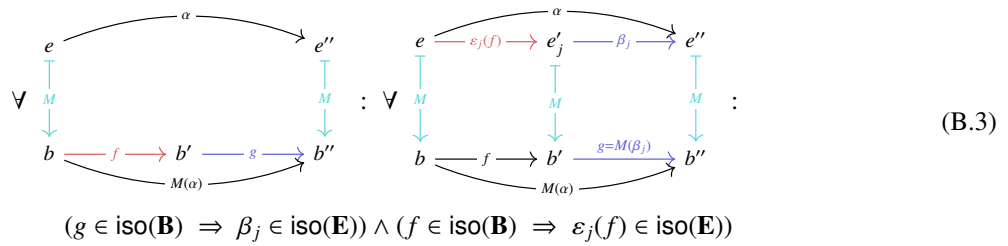
<sup>19</sup>Here, “stable” refers to stability under pullbacks.

- 1906 • Since  $\mathbf{C}$  has multi-sums, and since the cospan  $A \rightarrow P \leftarrow B$  and the morphism  $P \rightarrow X$  provide a factorization of  
 1907 the cospan  $A \rightarrow X \leftarrow B$ , by the [universal property of multi-sums](#) there exists a unique morphism  $M \rightarrow P$  that  
 1908 makes the diagram commute.
- 1909 • The morphism  $M \rightarrow N$  claimed to exist is then obtained as the composition of the morphisms  $M \rightarrow P$  and  
 1910  $P \rightarrow N$ . Moreover, if  $X \leftarrow P' \rightarrow N$  is another pullback of  $X \leftarrow Y \rightarrow N$ , by the [universal property of pullbacks](#)  
 1911 there exists a unique isomorphism  $P \rightarrow P'$ ; therefore, the composites of  $M \rightarrow P$  and  $P \rightarrow N$ , and of  $M \rightarrow P'$   
 1912 and  $P' \rightarrow N$ , respectively, yield the same morphism  $M \rightarrow N$ , hence demonstrating unique existence, which  
 1913 concludes the proof.



1914 □

1915 **Lemma 2.6.** Let  $M : \mathbf{E} \rightarrow \mathbf{B}$  be a [strong multi-opfibration](#). Then the following [lifting property of isomorphisms](#) is  
 1916 satisfied:



1917

1918 **PROOF.** Let us first consider the claim  $g \in \text{iso}(\mathbf{B}) \Rightarrow \beta_j \in \text{iso}(\mathbf{E})$ :

- 1919 • Since  $M$  is a functor, we have that  $M(e'') = b''$  implies  $M(\text{id}_{e''}) = \text{id}_{b''}$ .
- 1920 • Since  $b' - g \rightarrow b''$  is by assumption an isomorphism in  $\mathbf{B}$ , there exists a morphism  $b'' - g^{-1} \rightarrow b'$  such that  
 1921  $\text{id}_{b''} = g \circ g^{-1}$ .
- 1922 • By the defining properties of [multi-opfibrations](#), this in turn entails that there exist morphisms  $e'' - \varepsilon_k(g^{-1}) \rightarrow e'_k$   
 1923 and  $e'_k - \gamma_k \rightarrow e''$  in  $\mathbf{E}$  such that  $\gamma_k \circ \varepsilon_k(g^{-1}) = \text{id}_{e''}$ ,  $M(\varepsilon_k(g^{-1})) = g^{-1}$  and  $M(\gamma_k) = g$ , as depicted in diagram (B.4)  
 1924 below.

(B.4)

- 1925 • Since  $M$  is a functor,  $M(\varepsilon_k(g^{-1}) \circ \beta_j \circ \varepsilon_j(f)) = f$ . Thus the diagram in (B.4) encodes two different liftings of  
 1926  $g \circ f$  (one via  $\beta_j \circ \varepsilon_j(f)$ , and the other one via  $\gamma_k \circ (\varepsilon_k(g^{-1}) \circ \beta_j \circ \varepsilon_j(f))$ ), hence by **essential uniqueness** of  
 1927 **strong multi-opfibrations**, there exists an isomorphism  $e'_j - \varphi \rightarrow e'_k$  that makes the diagram commute, so that in  
 1928 particular  $\varepsilon_k(g^{-1}) \circ \beta_j$  is an isomorphism.
- 1929 • By standard category theory, (i) a morphism is an isomorphism iff it is both a section and a retraction [45,  
 1930 Prop. 7.36], (ii) if the composite  $y \circ x$  of two morphisms  $y$  and  $x$  is an isomorphism,  $x$  is a section and  $y$   
 1931 is a retraction [45, Prop. 7.21 & 7.27], and (iii) the composite of two isomorphisms is an isomorphism [45,  
 1932 Prop. 3.14].
  - 1933 – Since  $\varepsilon_k(g^{-1}) \circ \beta_j$  is an isomorphism,  $\varepsilon_k(g^{-1})$  is a retraction (and  $\beta_j$  a section).
  - 1934 – Since  $\gamma_k \circ \varepsilon_k(g^{-1})$  is an identity morphism and thus an isomorphism,  $\varepsilon_k(g^{-1})$  is a section (and  $\gamma_k$  a retraction).
  - 1935

1936 Since  $\varepsilon_k(g^{-1})$  is thus both a section and a retraction, it is an isomorphism, hence  $\beta_j = (\varepsilon_k(g^{-1}))^{-1} \circ \varphi$  is the  
 1937 composite of two isomorphisms and therefore an isomorphism, which proves the claim.

1938 The proof of the claim  $f \in \text{iso}(\mathbf{B}) \Rightarrow \varepsilon_j(f) \in \text{iso}(\mathbf{E})$  is completely analogous, with the salient steps summarized  
 1939 in diagram (B.5) below:

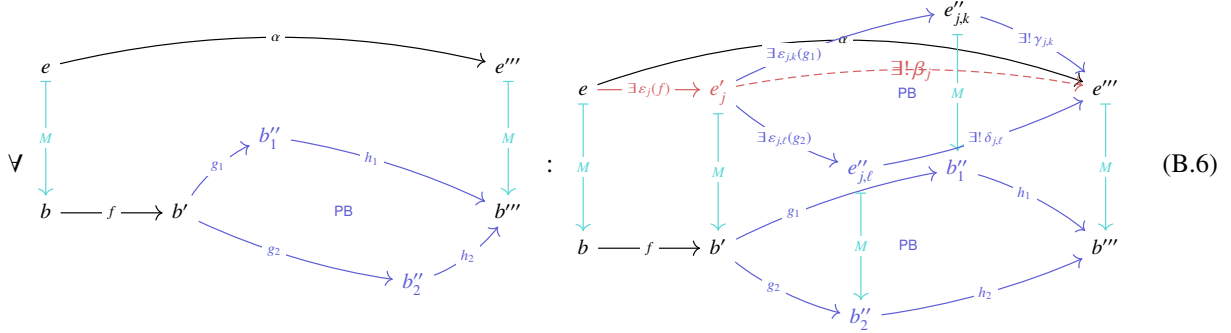
(B.5)

- 1940 •  $f$  being an isomorphism entails that  $f^{-1} \circ f = id_b$ , hence since  $M$  is a functor and  $M(e) = b$ ,  $M(id_e) = id_b =$   
 1941  $f^{-1} \circ f$ .
- 1942 • By the **universal property of multi-opfibrations**, there exists an  $\mathbf{E}$ -morphism  $e - \varepsilon_\ell(f) \rightarrow e'_\ell$  such that there exists  
 1943 a unique  $\mathbf{E}$ -morphism  $e'_\ell - \gamma_\ell \rightarrow e$  satisfying  $M(\varepsilon_\ell(f)) = f$ ,  $M(\gamma_\ell) = f^{-1}$ , and  $\gamma_\ell \circ \varepsilon_\ell(f) = id_e$ .
- 1944 • By the **essential uniqueness** property of **strong multi-opfibrations**, there exists a unique isomorphism  $e'_\ell - \psi \rightarrow e'_j$   
 1945 that makes the diagram commute, and so that in particular  $\varepsilon_j(f) \circ \gamma_\ell = \psi$ .
- 1946 • Since  $\varepsilon_j(f) \circ \gamma_\ell = \psi$  is an isomorphism,  $\gamma_\ell$  is a section; since  $\gamma_\ell \circ \varepsilon_\ell(f) = id_e$  is an identity morphism and thus  
 1947 an isomorphism,  $\gamma_\ell$  is a retraction; since thus  $\gamma_\ell$  is both a retraction and a section, it is an isomorphism.
- 1948 • Finally, since  $\varepsilon_j(f) = \psi \circ \gamma_\ell^{-1}$ ,  $\varepsilon_j(f)$  is an isomorphism, which concludes the proof.

1949 □



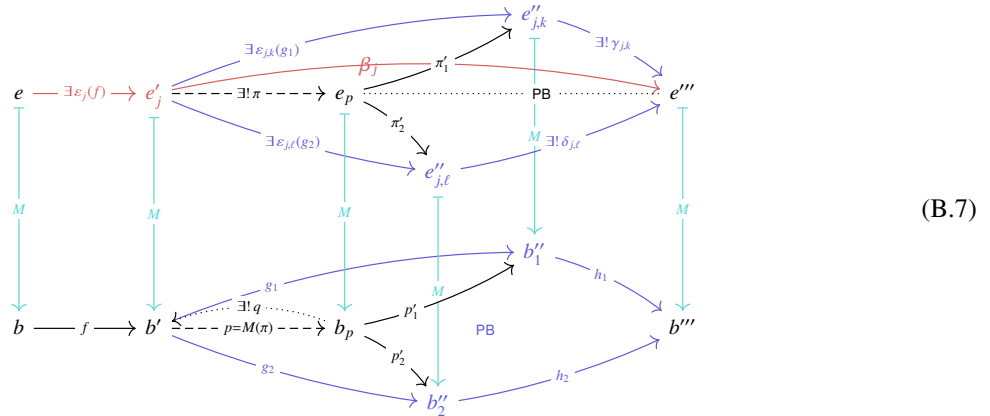
1950 **Lemma 2.7 (Pullback-lifting lemma for strong multi-opfibrations).** Let  $\mathbf{E}$  be a category that has pullbacks, and let  
 1951  $M : \mathbf{E} \rightarrow \mathbf{B}$  be a strong multi-opfibration. Then the following property holds:



1952 More explicitly, for every diagram such as on the left of (B.6), whose bottom part contains a pullback square in  $\mathbf{B}$ , the  
 1953 following properties hold true:

- 1954 (i) There exists an  $\mathbf{E}$ -morphism  $e - \varepsilon_j(f) \rightarrow e'_j$  such that there exists a unique  $\mathbf{E}$ -morphism  $e'_j - \beta_j \rightarrow e'''$  with  
 1955  $M(\varepsilon_j(f)) = f$  and  $M(\beta_j) = h_1 \circ g_1 = h_2 \circ g_2$ , and such that the diagram commutes.
- 1956 (ii) There then exist  $\mathbf{E}$ -morphisms  $e'_j - \varepsilon_{j,k}(g_1) \rightarrow e''_{j,k}$  and  $e'_j - \varepsilon_{j,\ell}(g_2) \rightarrow e''_{j,\ell}$  such that there exist unique  $\mathbf{E}$ -  
 1957 morphisms  $e''_{j,k} - \gamma_{j,k} \rightarrow e'''$  and  $e''_{j,\ell} - \delta_{j,\ell} \rightarrow e'''$  such that  $M(\varepsilon_{j,k}(g_1)) = g_1$ ,  $M(\varepsilon_{j,\ell}(g_2)) = g_2$ ,  $M(\gamma_{j,k}) = h_1$  and  
 1958  $M(\delta_{j,\ell}) = h_2$ , and such that the diagram commutes.
- 1959 (iii) Moreover, the square in  $\mathbf{E}$  into  $e'''$  is a pullback.

1960 **PROOF.** Claims (i) and (ii) follow directly from repeated applications of the universal property of multi-opfibrations. It  
 1961 thus remains to prove claim (iii), i.e., that the square in  $\mathbf{E}$  on the top right of the diagram in (B.6) is indeed a pullback.  
 1962 To this end, we construct the auxiliary diagram below by taking a pullback:

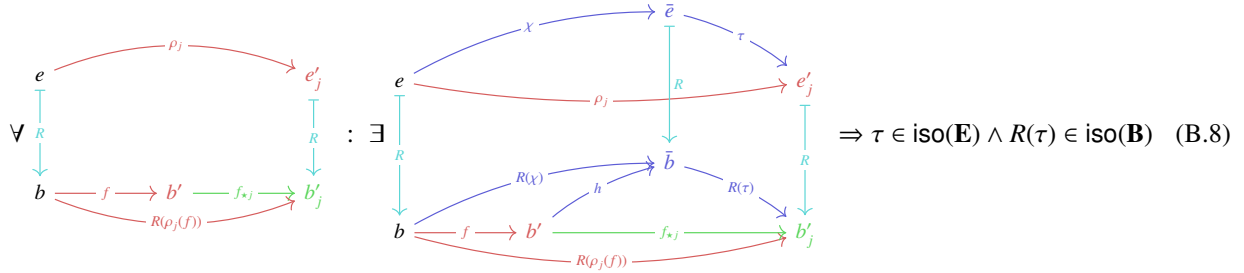


- 1963 • By the universal property of pullbacks, there exists an  $\mathbf{E}$ -morphism  $e'_j - \pi \rightarrow e_p$  (where  $e_p$  denotes the pullback  
 1964 object) that makes the diagram commute.
- 1965 • Since  $M$  is a functor, we also obtain  $\mathbf{B}$ -morphisms  $b' - p \rightarrow b_p$  (where  $b_p = M(e_p)$ ),  $p'_1 = M(\pi'_1)$  and  $p'_2 = M(\pi'_2)$   
 1966 that make the diagram commute.
- 1967 • By the universal property of pullbacks, there exists a unique morphism  $b_p - q \rightarrow b'$  that makes the diagram  
 1968 commute; since  $(p \circ q) \circ p = p \circ (q \circ p) = p$  and  $p$  is unique,  $(p \circ q) = id_{b'}$  and  $q \circ p = id_{b_p}$ , i.e.,  $p$  is both a  
 1969 section and a retraction, hence an isomorphism (and thus also  $q = p^{-1}$ ).

1970 • Finally, by applying Lemma 2.6 for  $f = M(\pi)$  and  $g = id_{b_p}$ , we may demonstrate that  $\pi$  is an isomorphism,  
 1971 hence indeed the claim that the square in  $\mathbf{E}$  marked in blue is a pullback.

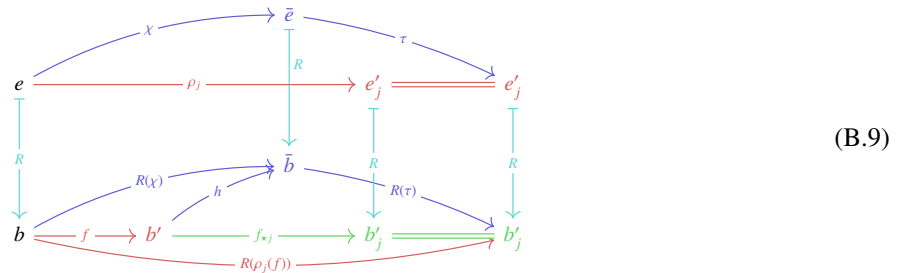
1972 □

1973 **Lemma 2.9.** Let  $R : \mathbf{E} \rightarrow \mathbf{B}$  be a residual multi-opfibration. Then residues have the following universal property:



1974 In particular, this property entails that if a residue  $f_{*k}$  factorizes a residue  $f_{*j}$  as  $f_{*j} = R(\beta_k) \circ f_{*k}$  for some  $\beta_k \in \mathbf{E}$ ,  
 1975 then the residues  $f_{*j}$  and  $f_{*k}$  (both of the same morphism  $f \in \mathbf{B}$ ) are related by an isomorphism  $R(\beta_k) \in \text{iso}(\mathbf{B})$ , as are  
 1976 their liftings  $\rho_j(f) = \beta_k \circ \rho_k(f)$  via  $\beta_k \in \text{iso}(\mathbf{E})$ .

1977 **PROOF.** It suffices to restate the second diagram in (B.8) in the following equivalent form:



1978 The claim then follows by essential uniqueness of residual multi-opfibrations. □

## 1979 Appendix B.2. Proofs of Section 4

1980 **Lemma 4.3.** The categories  $\mathsf{T}_h(\mathbf{C}, \mathcal{M})$  and  $\mathsf{T}_v(\mathbf{C}, \mathcal{M})$  for  $\mathsf{T} \in \{\text{PB}, \text{PO}, \text{FPC}\}$  as introduced in Definition 4.2 are  
 1981 well-defined, i.e., their composition operations are well-typed, associative and unital.

1982 **PROOF.** Well-definedness of the horizontal and vertical composition operations is a standard result for pullback and  
 1983 pushout squares, while for final pullback complements (FPCs) this is a slight generalization of Lemma Appendix  
 1984 A.7(5. & 6.):

1985 • *horizontal FPC composition:* in sub-diagram (a) of (B.10) below, given two horizontally composed FPC  
 1986 squares, an outer square which is a pullback, and morphisms  $X \rightarrow A$  and  $X \rightarrow C$  such that the diagram  
 1987 commutes, we have to prove that there exists a unique morphism  $Y \rightarrow A'$  such that the diagram commutes  
 1988 and such that the square over  $Y \rightarrow A'$  is a pullback.

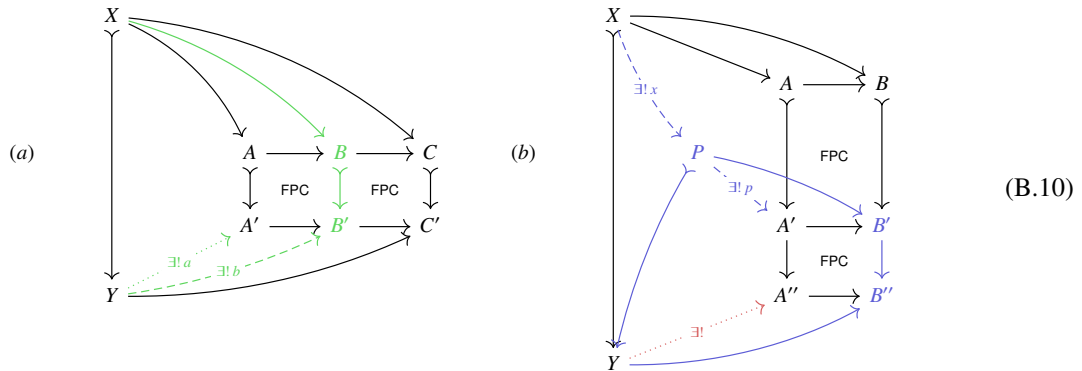
1989 1. Since the square  $XCC'Y$  is a pullback and the triangle  $XBC$  commutes, by the universal property of FPCs  
 1990 there exists a unique morphism  $Y - b \rightarrow B'$  that makes the diagram commute, and such that the square  
 1991  $XBB'Y$  is a pullback.

1992 2. Since the square  $XBB'Y$  is a pullback and the triangle  $XAB$  commutes, by the universal property of FPCs  
 1993 there exists a unique morphism  $Y - a \rightarrow A'$  that makes the diagram commute, and such that the square  
 1994  $XAA'Y$  is a pullback.

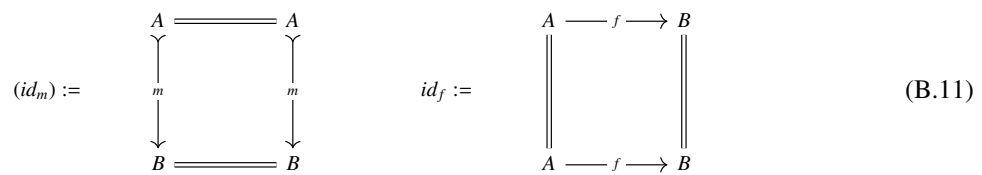
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• *vertical FPC composition*: in sub-diagram (b) of (B.10) below, given two vertically composed FPC squares, an outer square which is a pullback, and morphisms  $X \rightarrow A$  and  $X \rightarrow B$  such that the diagram commutes, we have to prove that there exists a unique morphism  $Y \rightarrow A''$  such that the diagram commutes and such that the square over  $Y \rightarrow A''$  is a pullback.

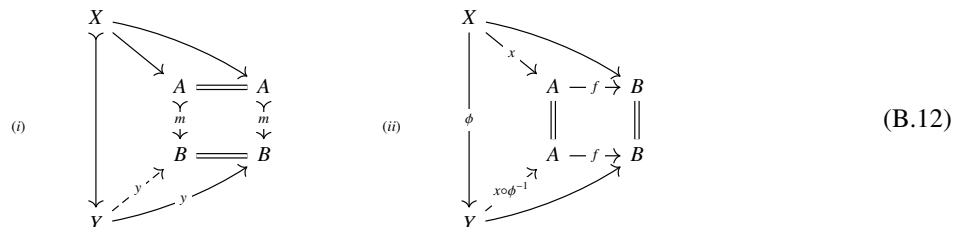
1. Take a pullback of  $Y \rightarrow B' \leftarrow B'$ , obtaining the span  $Y \leftarrow P \rightarrow B'$  (where  $Y \leftarrow P$  is in  $\mathcal{M}$  by [stability of  \$\mathcal{M}\$ -morphisms under pullback](#)). Since the square  $XBB'Y$  commutes, by the [universal property of pullbacks](#), there exists a unique morphism  $X \rightarrow P$  that makes the diagram commute.
2. By [pullback-pullback decomposition](#), the square  $XBB'P$  over  $P \rightarrow B'$  is a pullback. Since moreover the triangle  $XAB$  commutes, by the [universal property of FPCs](#), there exists a unique morphism  $P \rightarrow A'$  that makes the diagram commute, and such that the square  $XAA'P$  over  $P \rightarrow A'$  is a pullback.
3. Invoking the [universal property of FPCs](#) yet again, since the triangle  $PB'A'$  commutes and the square  $PB'B'Y$  is a pullback, there exists a unique morphism  $Y \rightarrow A''$  that makes the diagram commute, and such that the square  $PA'A''Y$  over this morphism is a pullback. By [pullback-pullback decomposition](#) (or, equivalently, by vertical pasting of pullback squares), the square  $XAA''Y$  is a pullback, which concludes the proof.



2010 Associativity of the horizontal and vertical pasting operations is manifest from the definition. Thus it remains to verify  
2011 that these compositions are unital. To this end, note first that the units of the horizontal and vertical compositions are  
2012 squares of the form  $id_m$  and  $id_f$ , respectively, as below:



2013 The only non-trivial part to prove is that squares of shapes  $id_m$  and  $id_f$  are simultaneously pullbacks, pushouts and  
2014 FPCs. The first two properties are a standard exercise to prove, yet the proof of the FPC property deserves a brief  
2015 clarification:



2016 As depicted in (B.12)(i), horizontal unitality trivially follows from [pullback-pullback decomposition](#). However, in  
2017 order to prove that vertical unitality holds, in a situation as depicted in (B.12)(ii), the additional observation necessary

2018 is that isomorphisms are stable under pullbacks, from which then together with [pullback-pullback decomposition](#) the  
 2019 claim follows.  $\square$

2020 **Theorem 4.4.** Let  $\mathbf{C}$  be a category with a [stable system of monics](#)  $\mathcal{M}$ , and with the following additional properties:

- 2021 1.  $\mathbf{C}$  has [pullbacks](#).
- 2022 2.  $\mathbf{C}$  has [pushouts and final pullback complements \(FPCs\) along  \$\mathcal{M}\$ -morphisms](#).
- 2023 3. Pushouts along  $\mathcal{M}$ -morphisms are [stable under pullbacks](#).
- 2024 4. [Pushouts along  \$\mathcal{M}\$ -morphisms are pullbacks](#).

2025 Then the domain functor  $dom : \mathbf{PB}_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  from the category of pullback squares along  $\mathcal{M}$ -morphisms and  
 2026 under horizontal composition to the underlying category  $\mathbf{C}$  satisfies the following properties:

- 2027 (i)  $dom : \mathbf{PB}_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  is a [Grothendieck fibration](#)  $dom : \mathbf{PB}_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  is a [Grothendieck fibration](#), with  
 2028 the [Cartesian liftings](#) given by FPCs.
- 2029 (ii)  $dom : \mathbf{PB}_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  is a [Grothendieck opfibration](#)  $dom : \mathbf{PB}_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  is a [Grothendieck opfibration](#),  
 2030 with the [op-Cartesian liftings](#) given by pushouts.
- 2031 (iii)  $dom : \mathbf{PB}_h(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}$  satisfies a [Beck-Chevalley condition \(BCC\)](#): adopting the notation  $m - (f, f') \rightarrow n$  for  
 2032 morphisms in  $\mathbf{PB}_h(\mathbf{C}, \mathcal{M})$  (cf. Figure 3), consider a commutative square in  $\mathbf{PB}_h(\mathbf{C}, \mathcal{M})$  that is mapped by  $dom$   
 2033 into a pullback square in  $\mathbf{C}$ :

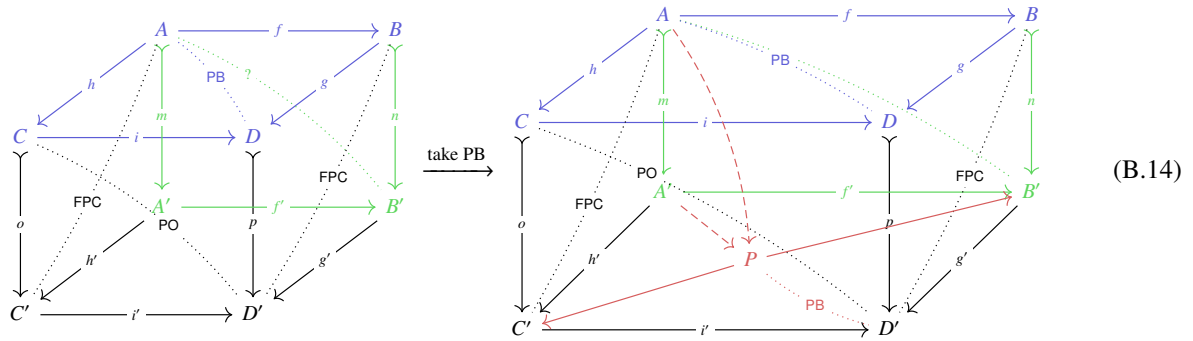
$$\begin{array}{ccc}
 \begin{array}{ccc}
 m & \xrightarrow{(f, f')} & n \\
 \downarrow (h, h') & & \downarrow (g, g') \\
 o & \xrightarrow{(i, i')} & p
 \end{array} & \xrightarrow{dom} & \begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow h & \text{PB} & \downarrow g \\
 C & \xrightarrow{i} & D
 \end{array}
 \end{array} \tag{B.13}$$

2034 Then the following two equivalent conditions hold:

- 2035 • **(BCC-1)**:  $(f, f')$  is [op-Cartesian](#) if  $(i, i')$  is [op-Cartesian](#) and  $(g, g')$  and  $(h, h')$  are [Cartesian](#).
- 2036 • **(BCC-2)**:  $(g, g')$  is [Cartesian](#) if  $(h, h')$  is [Cartesian](#) and  $(f, f')$  and  $(i, i')$  are [op-Cartesian](#).

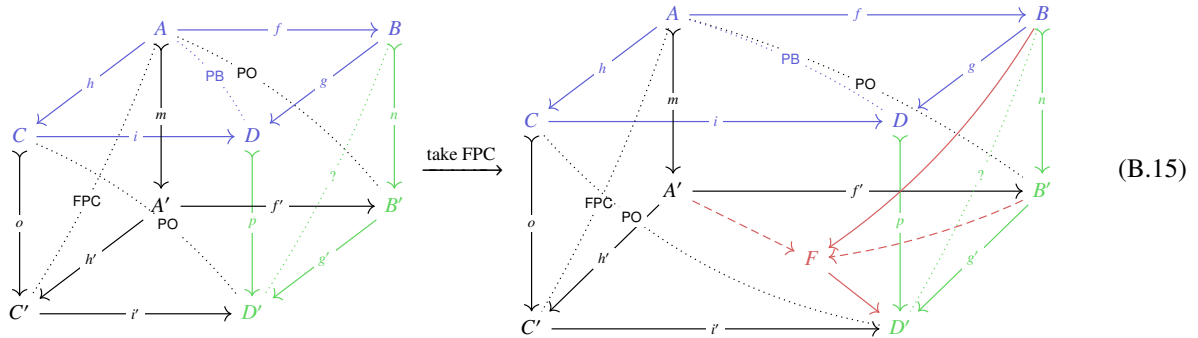
2037 **PROOF.** *Ad (iii) — Beck-Chevalley condition (BCC):*

- 2038 • **(BCC-1)**: the premise of this condition is explicitly depicted in (B.14), i.e., the top and back squares are pull-  
 2039 backs, the left and right squares are FPCs, and the front square is a pushout. In order to demonstrate that this  
 2040 entails that the back square is a pushout, we take a pullback of the cospan  $C' - i' \rightarrow D' \leftarrow g' - B'$ , obtaining a  
 2041 span  $C' \leftarrow P \rightarrow D'$ .
  - 2042 – By the [universal property of pullbacks](#), there exist unique morphisms  $A \rightarrow P$  and  $A' \rightarrow P$  as indicated  
 2043 with dashed arrows.
  - 2044 – By [pullback-pullback decomposition](#), the squares over  $C' \leftarrow P$  and over  $P \rightarrow B'$  are both pullbacks.
  - 2045 – Since by assumption pushouts are [stable under pullbacks](#), the square over  $P \rightarrow B'$  is a pushout.
  - 2046 – Since according to Lemma Appendix A.4 in a category such as  $\mathbf{C}$  which by assumption [has pullbacks](#),  
 2047 FPCs are [stable under pullbacks](#), the square over  $C' \leftarrow P$  is an FPC.
  - 2048 – By the [universal property of FPCs](#), the arrow  $A' \rightarrow P$  is an isomorphism, hence the back square is a  
 2049 pushout, which proves **(BCC-1)**.



2050 • (BCC-2): the premise of this condition is explicitly depicted in (B.15), i.e., the top and right squares are pull-  
 2051 backs, the left square is an FPC, and the front and back squares are pushouts. In order to demonstrate that this  
 2052 entails that the right square is an FPC, we take the final pullback complement of the sequence of morphisms  
 2053  $B - g \rightarrow D \twoheadrightarrow D'$  (which is admissible since by assumption  $\mathbf{C}$  has FPCs along  $\mathcal{M}$ -morphisms), obtaining a  
 2054 sequence of morphisms  $B \rightarrow F \rightarrow D'$ .

- 2055 – Since the front and left squares are morphisms in  $\text{PB}_h(\mathbf{C}, \mathcal{M})$  and thus pullbacks, by [pullback-pullback](#)  
 2056 [composition](#) so is the vertical diagonal square  $ADD'A'$  that arises as the composite of the front and left  
 2057 squares. Thus by the [universal property of FPCs](#), there exists a unique morphism  $A' \rightarrow F$  as indicated  
 2058 with a dashed arrow. Since the back square is a pushout, by the [universal property of pushouts](#) there exists  
 2059 a unique morphism  $B' \rightarrow F$ , again indicated with a dashed arrow.
- 2060 – Noting that the resulting configuration corresponds precisely to the precondition of (BCC-1), we find that  
 2061 the square  $ABFA'$  over  $A' \rightarrow F$  is a pushout. Thus by the [universal property of pushouts](#),  $B' \rightarrow F$  is an  
 2062 isomorphism, and thus the right square is an FPC, which proves (BCC-2).



2063 □

2064 **Theorem 4.8.** Let  $\mathbf{C}$  be a category with a [stable system of monics](#)  $\mathcal{M}$  and that [has FPCs along  \$\mathcal{M}\$ -morphisms](#). Then  
 2065 the target functor  $T : \text{FPC}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  is a [Grothendieck opfibration](#).

2066 **PROOF.** Let us first provide the claim in more explicit form, i.e., by instantiating the defining properties of a [Grothendieck](#)

2067 **opfibration** (cf. equation (5)) to the case at hand, where we use the shorthand notation  $T$  for the target functor:

$$\forall \begin{array}{c} f \\ \downarrow T \\ B \end{array} \xrightarrow{\beta} B' \quad : \exists \begin{array}{ccc} f & \xrightarrow{(\alpha, \beta)} & f' \\ \downarrow T & & \downarrow T \\ B & \xrightarrow{\beta} & B' \end{array} \quad : \\
 \forall \begin{array}{ccc} f & \xrightarrow{(\alpha, \beta)} & f' \\ \downarrow T & & \downarrow T \\ B & \xrightarrow{\beta} & B' \end{array} \xrightarrow{(\alpha'', \beta'')} \begin{array}{ccc} f & \xrightarrow{(\alpha, \beta)} & f'' \\ \downarrow T & & \downarrow T \\ B & \xrightarrow{\beta} & B'' \end{array} \quad : \begin{array}{ccc} f & \xrightarrow{(\alpha, \beta)} & f' \\ \downarrow T & & \downarrow T \\ B & \xrightarrow{\beta} & B' \end{array} \xrightarrow{\exists! (\alpha', \beta')} \begin{array}{ccc} f & \xrightarrow{(\alpha, \beta)} & f'' \\ \downarrow T & & \downarrow T \\ B & \xrightarrow{\beta} & B'' \end{array}$$

(B.16)

2068 Recalling the definition of the category  $\text{FPC}_h(\mathbf{C}, \mathcal{M})$  (Definition 4.2), we may further expand the claim into the  
 2069 following equivalent form:

$$\forall \begin{array}{ccc} A & \xrightarrow{f} & B \\ & & \downarrow \beta \\ & & B' \end{array} \quad : \exists \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \alpha & \text{FPC} & \downarrow \beta \\ A' & \xrightarrow{f'} & B' \end{array} \quad : \\
 \forall \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \alpha & \text{FPC} & \downarrow \beta \\ A' & \xrightarrow{f'} & B' \end{array} \xrightarrow{(\alpha'', \beta'')} \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \alpha & \text{FPC} & \downarrow \beta \\ A'' & \xrightarrow{f''} & B'' \end{array} \quad : \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \alpha & \text{FPC} & \downarrow \beta \\ A' & \xrightarrow{f'} & B' \end{array} \xrightarrow{\exists! \alpha'} \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \alpha & \text{FPC} & \downarrow \beta \\ A'' & \xrightarrow{f''} & B'' \end{array}$$

(B.17)

2070 The first part of the claim, i.e., the existence of suitable liftings follows since  $\mathbf{C}$  by assumption has **FPCs along  $\mathcal{M}$ -**  
 2071 **morphisms**. In order to prove the claim that these liftings have the **op-Cartesianity** property, we take a pullback  
 2072 arrive at the diagram below:

$$\begin{array}{ccc} \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \alpha & \text{FPC} & \downarrow \beta \\ A' & \xrightarrow{f'} & B' \end{array} & \xrightarrow{\text{take PB}} & \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \alpha & \text{FPC} & \downarrow \beta \\ A' & \xrightarrow{f'} & B' \end{array} \\ \downarrow \alpha'' & & \downarrow \alpha'' \\ A'' & \xrightarrow{f''} & B'' \end{array}$$

(B.18)

- 2073 • By **pullback-pullback decomposition**, the square over  $g'$  is a pullback.
- 2074 • By **stability of  $\mathcal{M}$ -morphisms under pullback**,  $P - \gamma' \rightarrow A''$  is an  $\mathcal{M}$ -morphism. Since  $A - \alpha'' \rightarrow A''$  is in  $\mathcal{M}$   
 2075 and  $\alpha'' = \gamma' \circ \gamma$ , by **decomposition property of  $\mathcal{M}$ -morphisms**, we find that  $\gamma \in \mathcal{M}$ .

- Invoking [vertical FPC-pullback decomposition](#), the square under  $P$  and the square over  $g'$  are FPCs.
- By the [universal property of FPCs](#), there exists thus a morphism  $A' - \eta \rightarrow P$ , which by the [universal property of FPCs](#) is an isomorphism.

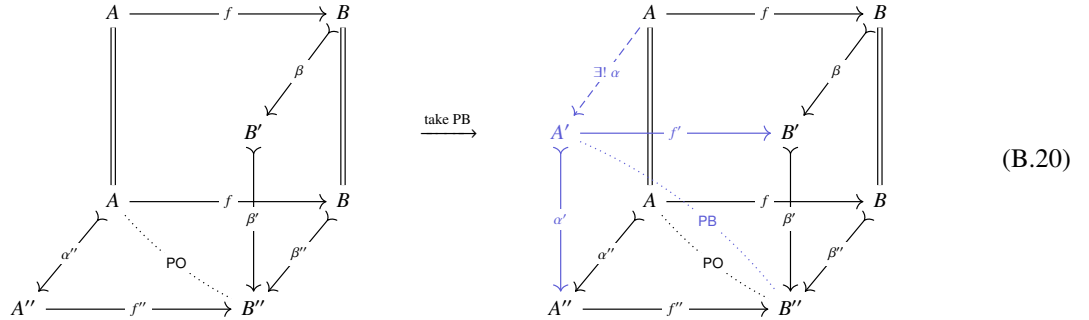
Up until this point, we have proved that there exists a morphism  $\alpha' = \gamma' \circ \eta$ , and that the square under  $\alpha'$  is an FPC. It remains to prove uniqueness of  $\alpha'$ . To this end, upon closer inspection of the second diagram in (B.18), since  $\beta'$  is in  $\mathcal{M}$  and thus in particular a monomorphism, the right vertical square is a pullback, and thus by [pullback-pullback composition](#), the composite of the right and back vertical squares is a pullback. Therefore, we may identify  $\alpha'$  as the morphism that according to the [universal property of FPCs](#) is guaranteed to exist (mediating before the aforementioned pullback square and the FPC in the front vertical square), and that is moreover unique as per the universal property.  $\square$

**Lemma 4.11.** Let  $\mathbf{C}$  be a category with a [stable system of monics](#)  $\mathcal{M}$ . If [pushouts along  \$\mathcal{M}\$ -morphisms are stable under  \$\mathcal{M}\$ -pullbacks](#), and if [pushouts along  \$\mathcal{M}\$ -morphisms are pullbacks](#), then  $\mathbf{C}$  has [multi-initial pushout complements \(mIPCs\) along  \$\mathcal{M}\$ -morphisms](#).

PROOF. By definition, for every composable sequence of morphisms  $A - f \rightarrow B - \beta \rightarrow B'$  (i.e., with  $\beta \in \mathcal{M}$ ), the [multi-initial pushout complement](#)  $\mathcal{P}(f, \beta)$  consists of all composable sequences of morphisms  $A - \alpha \rightarrow A' - f' \rightarrow B'$  such that the resulting commutative square is a pushout:

$$\mathcal{P}(f, \beta) := \{(A - \alpha \rightarrow A', A' - f' \rightarrow B') \in \text{mor}(\mathbf{C}) \times \text{mor}(\mathbf{C}) \mid \alpha \in \mathcal{M} \wedge (f', \beta) = \text{PO}(\alpha, f)\}. \quad (\text{B.19})$$

Since this class may in general be empty, it is non-trivial to prove the [universal property](#) of mIPCs. To this end, let us construct the diagrams below:

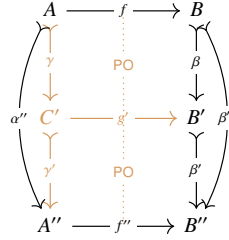


Here, the left diagram encodes the premise of the [universal property of mIPCs](#). The *existence* part of the universal property may be demonstrated as follows:

- Taking a pullback as indicated to obtain the right diagram (which is admissible since by assumption  $\mathbf{C}$  has [pullbacks along  \$\mathcal{M}\$ -morphisms](#)), we obtain morphisms  $f'$ ,  $\alpha$  and  $\alpha'$ .
- By [stability of  \$\mathcal{M}\$ -morphisms under pullback](#),  $\alpha'$  is in  $\mathcal{M}$ . Since  $\alpha'' = \alpha' \circ \alpha$  is in  $\mathcal{M}$  as well, by the [decomposition property of  \$\mathcal{M}\$ -morphisms](#), we find that  $\alpha \in \mathcal{M}$ .
- Since  $\alpha$  and  $\beta$  are in  $\mathcal{M}$ , the right and left vertical squares are pullbacks. The back vertical square is a pullback, since all squares of this form are such.
- By assumption, [pushouts along  \$\mathcal{M}\$ -morphisms are stable under  \$\mathcal{M}\$ -pullbacks](#), hence the top square is a pushout. Thus by [pushout-pushout decomposition](#), so is the front square.

We have thus exhibited an element of  $\mathcal{P}(f, \beta)$ .

It remains to prove the *essential uniqueness* property of mIPCs. Suppose we were given another pair of vertically composable pushouts as follows:



2104 By assumption, [pushouts along  \$\mathcal{M}\$ -morphisms are pullbacks](#), hence the pushout square  $C' B' B'' A''$  is also a pullback,  
 2105 which by the [universal property of pullbacks](#) entails the existence of a unique isomorphism  $C' \rightarrow A'$ .  $\square$

2106 **Theorem 4.19.** Let  $\mathbf{C}$  be a category with a [stable system of monics  \$\mathcal{M}\$](#) , that is  $(\mathcal{E}, \mathcal{M})$ -structured, that has [pushouts](#)  
 2107 [and FPCs along  \$\mathcal{M}\$ -morphisms](#), such that [mathcal{M}-morphisms are stable under pushout](#), and such that [pushouts along](#)  
 2108 [mathcal{M}-morphisms are stable under  \$\mathcal{M}\$ -pullbacks. Then the category  \$\text{FPC}\_v\(\mathbf{C}, \mathcal{M}\)\$  is \[\\(auto-augmented, inert\\)-structured\]\(#\).  
 2109 Here, the class of \[auto-augmented FPCs\]\(#\) is defined as](#)

$$\begin{array}{ccc}
 \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \alpha & & \downarrow \beta \\ A' & \xrightarrow{f'} & B' \end{array} & \text{FPC} & \in \text{mor}(\text{FPC}_v(\mathbf{C}, \mathcal{M}))|_{\text{auto-augmented}} \\
 \downarrow \alpha & & \downarrow \beta \\
 A' & \xrightarrow{f'} & B'
 \end{array} \Leftrightarrow \exists \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \alpha & & \downarrow \beta \\ A' & \xrightarrow{f'} & B' \end{array} \begin{array}{c} \text{PO} \\ \eta \\ P \\ p \\ e \end{array} \quad (\text{B.21})$$

2110 In words: an FPC square along an  $\mathcal{M}$ -morphism (seen as a morphism in  $\text{FPC}_v(\mathbf{C}, \mathcal{M})$ ) is auto-augmented iff when  
 2111 taking a pushout of the span within the FPC, the mediating morphism into the cospan object of the FPC is a morphism  
 2112 in  $\mathcal{E}$ .<sup>20</sup> Moreover, the class of [inert FPCs](#) is defined as

$$\text{mor}(\text{FPC}_v(\mathbf{C}, \mathcal{M}))|_{\text{inert}} := \left\{ \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \alpha & & \downarrow \beta \\ A' & \xrightarrow{f'} & B' \end{array} \text{FPC} \left| \alpha \in \mathcal{E} \cap \mathcal{M} = \text{iso}(\mathbf{C}) \right. \right\} \quad (\text{B.22})$$

2113  
 2114 **PROOF.** In order to demonstrate that the two classes of morphisms are both [closed under composition with isomor-](#)  
 2115 [phisms](#), note first that this is true by definition for the [inert FPCs](#). For the [auto-augmented FPCs](#), it is useful to observe  
 2116 the following auxiliary fact about isomorphisms in  $\text{FPC}_v(\mathbf{C}, \mathcal{M})$ :

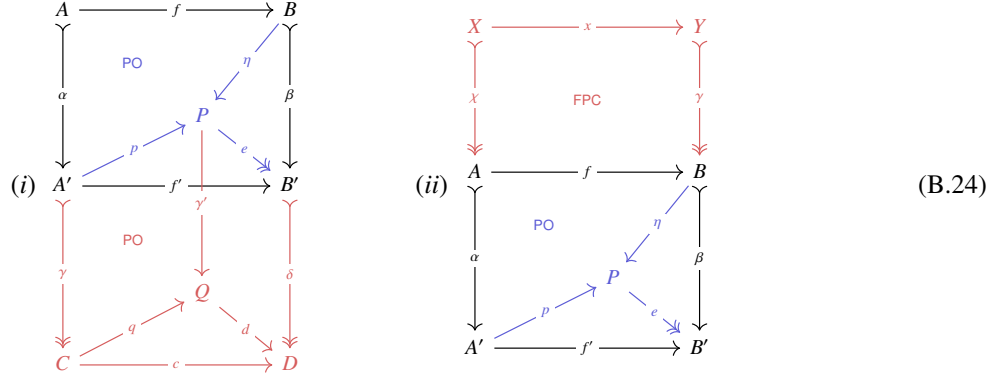
$$\forall \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \alpha & & \downarrow \beta \\ A' & \xrightarrow{f'} & B' \end{array} \text{FPC} \in \text{iso}(\text{FPC}_v(\mathbf{C}, \mathcal{M})) : \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \alpha & & \downarrow \beta \\ A' & \xrightarrow{f'} & B' \end{array} \begin{array}{c} \text{PO} \\ \eta \\ P \\ p \\ q \end{array} \Rightarrow \eta, q \in \mathcal{M} \cap \mathcal{E} = \text{iso}(\mathbf{C}) \quad (\text{B.23})$$

2117 In words: for every isomorphism in  $\text{FPC}_v(\mathbf{C}, \mathcal{M})$ , which is an FPC square where the vertical morphisms are in  
 2118  $\mathcal{M} \cap \mathcal{E} = \text{iso}(\mathbf{C})$ , if we take a pushout  $A' \rightarrow P \leftarrow \eta - B$  of its span  $A' \leftarrow \alpha - A \rightarrow B$ , which by the [universal property](#)  
 2119 [of pushouts](#) yields a mediating morphism  $P - q \rightarrow B'$ , then  $\eta$  is in  $\mathcal{M} \cap \mathcal{E} = \text{iso}(\mathbf{C})$  since isomorphisms are stable under

<sup>20</sup>Note that since we admit arbitrary morphisms of  $\mathbf{C}$  for the horizontal morphisms, the mediating morphism would in general be a morphism with a non-trivial  $\mathcal{E}$ - $\mathcal{M}$ -factorization, hence for this morphism to be an  $\mathcal{E}$ -morphism is indeed a non-trivial requirement.



2120 pushout, hence  $q = \beta \circ \eta^{-1}$  is the composite of two isomorphisms, and thus itself an isomorphism. Consequently, we  
 2121 find that isomorphisms in  $\text{FPC}_v(\mathcal{C}, \mathcal{M})$  are also pushout squares, and they are moreover both **auto-augmented FPCs**  
 2122 and **inert FPCs**. To conclude that the class of **auto-augmented FPCs** is closed under composition with isomorphisms,  
 2123 it suffices then to consider the following diagrams:

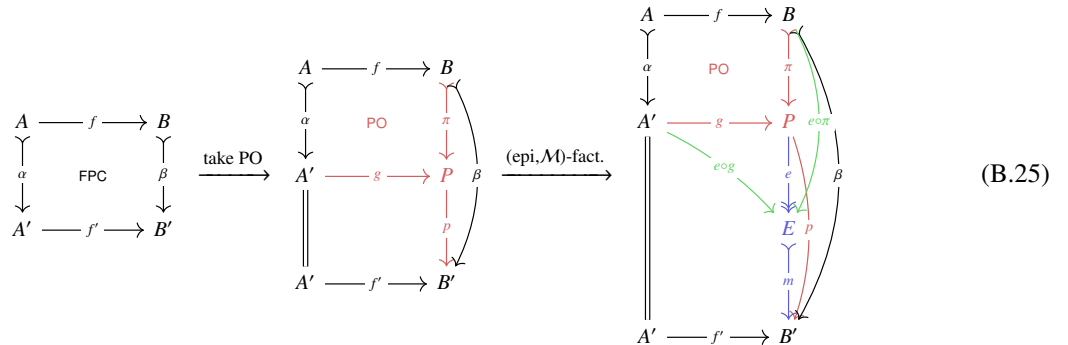


2124 Diagram (i) above demonstrates that post-composing an isomorphism with an **auto-augmented FPC**, and taking the  
 2125 indicated pushouts, the morphism  $\gamma'$  in (B.24)(i) is an isomorphism by stability of isomorphisms under pushout, hence  
 2126 in particular also an  $\mathcal{E}$ -morphism; thus the morphism  $d$  which satisfies  $d \circ \gamma' = \delta \circ e$  is an  $\mathcal{E}$ -morphism, which proves that  
 2127 the composite square is indeed an **auto-augmented FPC**. For diagram (B.24)(ii), which illustrates the pre-composition  
 2128 of an **auto-augmented FPC** with an isomorphism, since isomorphisms in  $\text{FPC}_v(\mathcal{C}, \mathcal{M})$  are as demonstrated above also  
 2129 pushout squares, we find that the vertical composition of the pushout squares in (B.24)(ii) yields a pushout square with  
 2130 mediating morphism  $e$  that is an  $\mathcal{E}$ -morphism, which demonstrates that the pre-composition of an **auto-augmented FPC**  
 2131 with an isomorphism yields an **auto-augmented FPC**.

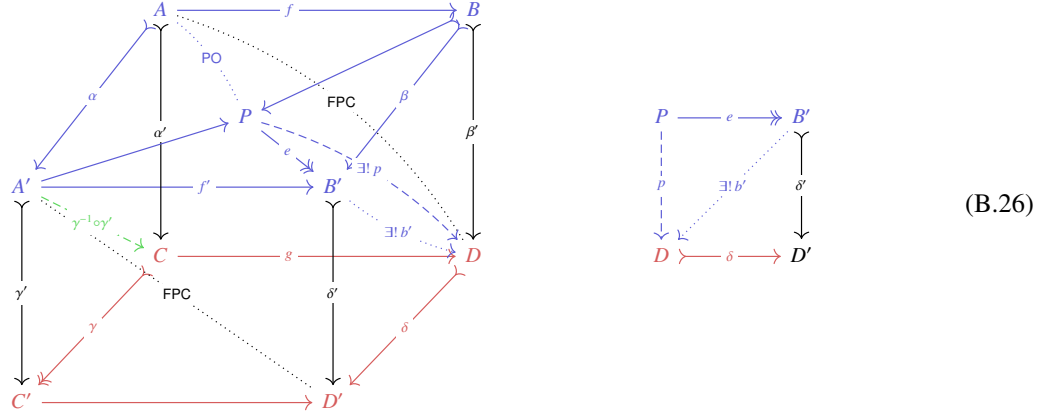
2132 The next part of the proof amounts to showing that  $\text{FPC}_v(\mathcal{C}, \mathcal{M})$  has **(auto-augmented, inert)-factorizations of**  
 2133 **morphisms**. To this end, for every morphism in  $\text{FPC}_v(\mathcal{C}, \mathcal{M})$ , i.e., for an FPC square as in diagram B.25 below, we  
 2134 exhibit a factorization into an **auto-augmented FPC** and an **inert FPC** as follows:

- 2135 • Take a pushout, thus obtaining a mediating morphism  $P - p \rightarrow B'$ .
- 2136 • Applying the  $\mathcal{E}$ - $\mathcal{M}$ -factorization to  $p$  yields an  $\mathcal{E}$ -morphism  $P - e \rightarrow E$  and an  $\mathcal{M}$ -morphism  $E - m \rightarrow B'$ .
  - 2137 – Since every square of the form as the square under  $A' - e \circ g \rightarrow E$  is a pullback, by **pullback-pullback**  
 2138 **decomposition** also the square over  $e \circ g$  is a pullback. Thus by **vertical FPC-pullback decomposition**,  
 2139 both the squares over and under  $e \circ g$  are FPC squares.
  - 2140 – By the **decomposition property of  $\mathcal{M}$ -morphisms**, since  $\beta = m \circ (e \circ \pi)$  and  $m$  are in  $\mathcal{M}$ ,  $(e \circ \pi)$  is in  $\mathcal{M}$ .

2141 We thus confirm that the top subdiagram over  $e \circ g$  manifestly has the structure of an **auto-augmented FPC**,  
 2142 while the bottom subdiagram over  $e \circ g$  encodes an **inert FPC**.



2143 It thus remains to prove that  $\text{FPC}_v(\mathbf{C}, \mathcal{M})$  has a **unique (auto-augmented, inert) diagonalization property**. More explic-  
 2144 itly, considering a diagram as in the left of (B.26) below, where the top square is an **auto-augmented FPC**, while the  
 2145 bottom square is an **inert FPC**:



- 2146 • Since by definition of **inert FPCs**  $\gamma$  is an isomorphism, we obtain a morphism  $\gamma^{-1} \circ \gamma'$ , which is moreover in  $\mathcal{M}$   
 2147 (since also  $\gamma'$  is in  $\mathcal{M}$  by definition of morphisms in  $\text{FPC}_v(\mathbf{C}, \mathcal{M})$ ).
- 2148 • The existence of the morphism  $\gamma^{-1} \circ \gamma'$  in turn reveals that there exists a cospan  $A' \xrightarrow{g \circ \gamma^{-1} \circ \gamma'} D \xleftarrow{\beta'} B$ , which by  
 2149 the **universal property of pushouts** entails the existence of a morphism  $P - p \rightarrow D$ .
- 2150 • Extracting the subdiagram as in the right of (B.26) above, we find that by the **unique  $(\mathcal{E}, \mathcal{M})$ -diagonalization**  
 2151 **property** there exists a unique morphism  $B' - b' \rightarrow D$  such that  $\delta \circ b' = \delta'$ ; the latter then entails by the  
 2152 **decomposition property of  $\mathcal{M}$ -morphisms** that  $b'$  is in  $\mathcal{M}$ .
- 2153 • Since the front and bottom squares are FPCs and thus in particular pullbacks, and since the diagram commutes,  
 2154 by **pullback-pullback decomposition** the diagonal square containing the morphisms  $\gamma^{-1} \circ \gamma'$  and  $b'$  is a pullback.
- 2155 • Applying the two variants of **vertical FPC-FPC decomposition**, we may then conclude that the diagonal square  
 2156 containing the morphisms  $\gamma^{-1} \circ \gamma'$  and  $b'$  is the unique FPC square that simultaneously decomposes both the  
 2157 back and the front FPCs, thus concluding the proof.

2158 □

2159 **Theorem 4.20.** Let  $\mathbf{C}$  be a category with a **stable system of monics**  $\mathcal{M}$ , that is  $(\mathcal{E}, \mathcal{M})$ -structured, that has **pullbacks,**  
 2160 **pushouts and FPCs along  $\mathcal{M}$ -morphisms**, such that  **$\mathcal{M}$ -morphisms are stable under pushout**, and such that **pushouts**  
 2161 **along  $\mathcal{M}$ -morphisms are stable under  $\mathcal{M}$ -pullbacks**. Then  $S : \text{FPC}_v(\mathbf{C}, \mathcal{M}) \rightarrow \mathbf{C}|_{\mathcal{M}}$  is a **residual multi-opfibration**.

2162 **PROOF.** Let us first utilize the assumptions on the underlying category  $\mathbf{C}$  in order to provide the following construction  
 2163 on FPC squares that are morphisms in  $\text{FPC}_v(\mathbf{C}, \mathcal{M})$  as described in (B.27) below:

- 2164 • Taking a pushout of the span  $A' \leftarrow \alpha - A - f \rightarrow B$ , we obtain a cospan  $A' - p \rightarrow P \leftarrow \beta - B$  where  $\beta \in \mathcal{M}$  by  
 2165 **stability of  $\mathcal{M}$ -morphisms under pushout**, as well as a unique mediating morphism  $P - \beta' \rightarrow B'$ .
- 2166 • Applying  **$\mathcal{E}$ - $\mathcal{M}$ -factorization** to  $\beta'$ , we obtain an  $\mathcal{E}$ -morphism  $P - e \rightarrow E$  and an  $\mathcal{M}$ -morphism  $E - m \rightarrow B'$   
 2167 such that  $\beta' = m \circ e$ .
- 2168 • Taking a pullback of the cospan  $A'' \leftarrow f'' \rightarrow B'' \leftarrow m - E$ , we obtain a span  $A'' \leftarrow l' - I - i \rightarrow E$ , where by  
 2169 **stability of  $\mathcal{M}$ -morphisms under pullback** we find that  $l' \in \mathcal{M}$ , and a unique mediating morphism  $A' - \iota \rightarrow I$ ,  
 2170 which by the **decomposition property of  $\mathcal{M}$ -morphisms** is also in  $\mathcal{M}$ .

- 2171 • Finally, applying **pullback-pullback decomposition** followed by **vertical FPC-FPC decomposition**, we may conclude that the bottom square is an FPC, and so is the vertical composite of the top and middle square. Moreover, the middle square is a **FPC-pushout-augmentation** for the top (pushout) square.

(B.27)

2174 It is clear that the vertical decomposition of the original FPC square provided via the above construction is essentially  
 2175 unique, in the sense that both the  $\mathcal{E}\text{-}\mathcal{M}$ -factorization as well as the pullback taken in the last two steps are unique  
 2176 only up to isomorphisms.<sup>21</sup> More explicitly, we have the following chain of arguments demonstrating the existence of  
 2177 unique isomorphisms mediating between any two vertical FPC decompositions obtained via the above procedure:

- 2178 • If as in (B.28) below  $A' \leftarrow q \rightarrow Q \leftarrow \gamma \rightarrow B$  is another pushout of  $A' \leftarrow \alpha \rightarrow A \xrightarrow{f} B$ , yielding also a unique  
 2179 mediating morphism  $Q \leftarrow \gamma' \rightarrow B''$ , by the **universal property of pushouts** there exists a unique isomorphism  
 2180  $P \leftarrow \pi \rightarrow Q$  such that  $\gamma \circ f = \pi \circ \beta \circ f = q \circ \alpha = \pi \circ p \circ \alpha$ .
- 2181 • If  $Q \leftarrow \tilde{e} \rightarrow \tilde{E} \leftarrow \tilde{m} \rightarrow B''$  is an  $\mathcal{E}\text{-}\mathcal{M}$ -factorization of  $Q \leftarrow \gamma' \rightarrow B''$ , since  $Q \leftarrow e \circ \pi^{-1} \rightarrow E \leftarrow m \rightarrow B''$  is another  
 2182  $\mathcal{E}\text{-}\mathcal{M}$ -factorization of  $Q \leftarrow \gamma' \rightarrow B''$ , and since  $\mathcal{E}\text{-}\mathcal{M}$ -factorizations are **essentially unique**, there exists a unique  
 2183 isomorphism  $E \leftarrow \varepsilon \rightarrow \tilde{E}$  such that  $\varepsilon \circ e = \tilde{e} \circ \pi$  and  $\tilde{m} \circ \varepsilon = m$ .
- 2184 • Finally, if  $A'' \leftarrow \tilde{\nu} \rightarrow \tilde{I} \leftarrow \tilde{i} \rightarrow \tilde{E}$  is a pullback of  $A'' \leftarrow f'' \rightarrow B'' \leftarrow \tilde{m} \rightarrow \tilde{E}$ , with  $A' \leftarrow \tilde{\nu} \rightarrow \tilde{I}$  the unique mediating  
 2185 morphism, by the **universal property of pullbacks**, there exists a unique isomorphism  $I \leftarrow \varphi \rightarrow \tilde{I}$  that makes the  
 2186 diagram commute.

(B.28)

2187 In summary, we have proved that for a given span of the form  $A' \leftarrow \alpha \rightarrow A \xrightarrow{f} B$  with  $\alpha$  in  $\mathcal{M}$ , the class of **FPCs** for  
 2188 any pushout over  $(\alpha, f)$  provides an explicit construction of **residual multi-op-Cartesian liftings** (with **residues** realized

<sup>21</sup>Note that also the pushout taken in the first step is unique only up to isomorphisms, yet the pushout itself is not retained as part of the data of the vertical decomposition into two FPC squares, hence in this sense does not contribute to the effective “degrees of freedom” of the construction.

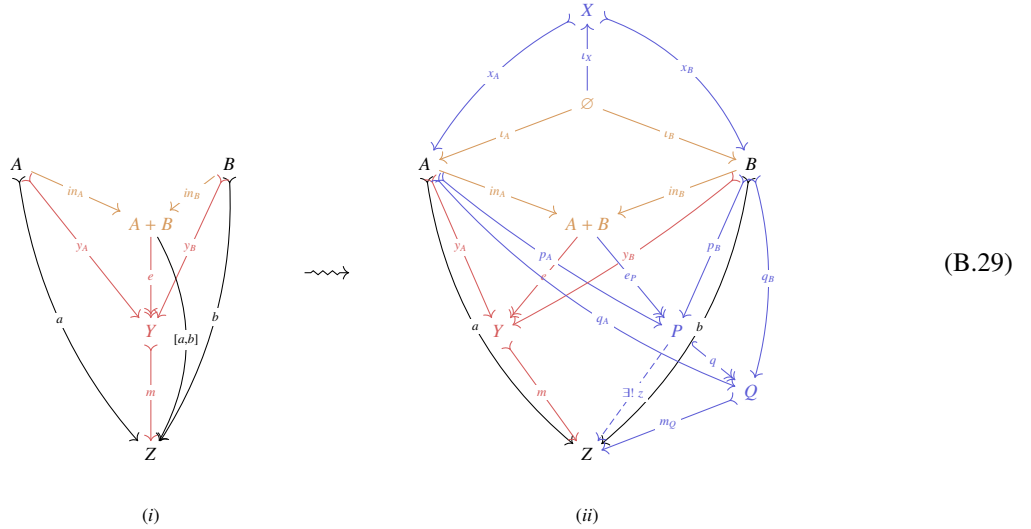
2189 via FPAs; compare (9)), while the above-mentioned arguments demonstrate that this construction indeed yields the  
 2190 requisite vertical decomposition property of FPC squares up to residues (thus realizing the **universal property of**  
 2191 **residual multi-opfibrations**) in an **essentially unique** form.  $\square$

2192 *Appendix B.3. Proofs of Section 5*

2193 **Lemma 5.21** ([3]; [8], **Fact A.3.7**). Let  $\mathbf{C}$  be a **finitary vertical weak adhesive HLR category** with respect to a **stable**  
 2194 **system of monics**  $\mathcal{M}$ , and denote by  $\mathcal{E}$  the class of **extremal morphisms** with respect to  $\mathcal{M}$ .

- 2195 (i) **Existence**: If  $\mathbf{C}$  has binary coproducts, then every cospan of  $\mathcal{M}$ -morphisms  $A \multimap a \rightarrow Z \leftarrow b \multimap B$  factors es-  
 2196 sentially uniquely through a cospan of  $\mathcal{M}$ -morphisms  $A \multimap y_A \rightarrow Y \leftarrow y_B \multimap B$  and an  $\mathcal{M}$ -morphism  $Y \multimap m \rightarrow Z$ ,  
 2197 where  $m$  is obtained via the  **$\mathcal{E}$ - $\mathcal{M}$ -factorization**  $A + B \multimap e \twoheadrightarrow Y \multimap m \rightarrow Z$  of the induced morphism  $A + B \multimap [a, b] \rightarrow$   
 2198  $Z$ , and where  $y_A = e \circ in_A$  and  $y_B = e \circ in_B$ .
- 2199 (ii) **Construction**: if  $\mathbf{C}$  in addition has an  **$\mathcal{M}$ -initial object**  $\emptyset$ , then  $\Sigma_{\mathcal{M}}(A, B)$  consists of cospans of  $\mathcal{M}$ -morphisms  
 2200 obtained as pushouts  $A \multimap p_A \rightarrow P \leftarrow p_B \multimap B$  of  $\mathcal{M}$ -spans  $A \leftarrow x_A \multimap X \multimap x_B \rightarrow B$  (i.e., “ $\mathcal{M}$ -partial overlaps”)  
 2201 extended by  $\mathcal{E}$ -morphisms  $P \multimap q \twoheadrightarrow Q$  such that  $q_A = q \circ p_A$  and  $q_B = q \circ p_B$  are in  $\mathcal{M}$ .
- 2202 (iii) **Refinements**: if  $\mathbf{C}$  in addition **has pullbacks**, and if **pushouts along  $\mathcal{M}$ -morphisms** in  $\mathbf{C}$  are **stable under pull-**  
 2203 **backs**, then the extension morphisms  $P \multimap q \twoheadrightarrow Q$  are morphisms in  $\mathcal{E} \cap \text{mono}(\mathbf{C})$  (so-called “refinements”).

2204 **PROOF.** Even though most of this proof is in principle “folklore” in the graph rewriting literature [3, 8], we provide full  
 2205 details here, since we wish to demonstrate the claims in the generality stated, plus the presented statement regarding  
 2206 **refinements** is a slight generalization of the corresponding statement in [1]. As depicted in the diagram below left, the  
 2207 **existence of  $\mathcal{M}$ -multi-sums** is guaranteed via  **$\mathcal{E}$ - $\mathcal{M}$ -factorization** of the induced morphism  $A + B \multimap [a, b] \rightarrow Z$ , where  
 2208  $y_A = e \circ in_A$  and  $y_B = e \circ in_B$  are in  $\mathcal{M}$  by the **decomposition property of  $\mathcal{M}$ -morphisms**:

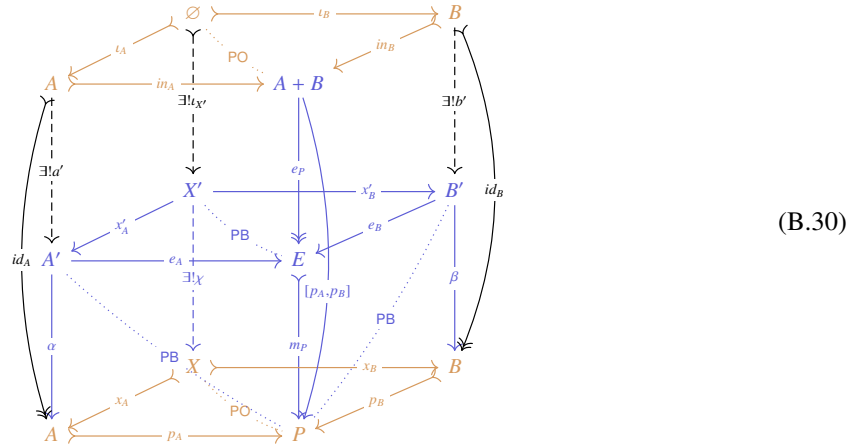


2209 In order to prove that the **construction** provided for the case that  $\mathbf{C}$  has an  **$\mathcal{M}$ -initial object**  $\emptyset$  (i.e., in addition to  
 2210 being a **vertical weak adhesive HLR category**) is sound and characterizes the  **$\mathcal{M}$ -multi-sums** in  $\mathbf{C}$  uniquely, consider  
 2211 diagram (b) in (B.29).

- 2212 • We first demonstrate that for every pushout  $A \multimap p_A \rightarrow P \leftarrow p_B \multimap B$  of an  $\mathcal{M}$ -span  $A \leftarrow x_A \multimap X \multimap x_B \rightarrow B$ , the  
 2213 induced morphism  $A + B \multimap [p_A, p_B] \rightarrow P$  is an  $\mathcal{E}$ -morphism. To this end, construct the commutative cube below,

2214

where the top square is a pushout (cf. e.g. [8, Fact 2.6]), the bottom square is a pushout:



2215

– Construct an  $\mathcal{E}$ - $\mathcal{M}$ -factorization  $A+B \xrightarrow{e_P} E \xrightarrow{m_P} P$  of the induced morphism  $A+B \xrightarrow{[p_A, p_B]} P$ .

2216

– Since  $\mathbf{C}$  has pullbacks along  $\mathcal{M}$ -morphisms, we can take pullbacks in order to obtain the bottom front and right vertical squares in (B.30), which by the universal property of pullbacks induces unique morphisms  $A \xrightarrow{a'} A'$  and  $B \xrightarrow{b'} B'$ , and by stability of  $\mathcal{M}$ -morphisms under pullback, all morphisms of the two pullback squares are in  $\mathcal{M}$ .

2217

– Since  $\mathbf{C}$  has an  $\mathcal{E}$ - $\mathcal{M}$ -factorization, isomorphisms such as in particular identity morphisms are both in  $\mathcal{M}$  and in  $\mathcal{E}$ ; since  $id_A = \alpha \circ a'$ ,  $id_B = \beta \circ b'$ , by extremality  $\alpha$  and  $\beta$  being  $\mathcal{M}$ -morphisms implies that they are isomorphisms, and hence so are  $a'$  and  $b'$ .

2220

– Take another pullback to obtain the middle horizontal square in (B.30), which induces the unique  $\mathcal{M}$ -morphism  $\emptyset \xrightarrow{\iota_{X'}} X$  (since  $\emptyset$  is an  $\mathcal{M}$ -initial object). Since according to Corollary 5.4 pushouts along  $\mathcal{M}$ -morphisms are pullbacks, we also obtain a unique morphism  $X' \xrightarrow{x} X$  via the universal property of pullbacks.

2224

– By pullback-pullback decomposition, the bottom back and bottom left vertical squares are pullbacks. By stability of isomorphisms under pullback,  $X' \xrightarrow{x} X$  is an isomorphism.

2226

– Since the bottom square is a vertical weak VK square, and since all four vertical squares in the bottom half of the diagram are pullbacks along  $\mathcal{M}$ -morphisms, the middle horizontal square is a pushout. Thus by the universal property of pushouts,  $m_P$  is an isomorphism, which proves that  $A+B \xrightarrow{[p_A, p_B]} P$  is an  $\mathcal{E}$ -morphism.

2227

• It remains to demonstrate that any factorization of a cospan of  $\mathcal{M}$ -morphisms  $A \xrightarrow{a} Z \leftarrow b \xrightarrow{B}$  obtained via  $\mathcal{E}$ - $\mathcal{M}$ -factorization of the induced morphism  $A+B \xrightarrow{[a, b]} Z$  may be equivalently obtained via extending a pushout  $A \xrightarrow{p_A} P \leftarrow p_B \xrightarrow{B}$  of some  $\mathcal{M}$ -span  $A \leftarrow x_A \xrightarrow{X} x_B \xrightarrow{B}$  with an  $\mathcal{E}$ -morphism  $P \xrightarrow{e} Q$ . To this end, consider yet again diagram (b) in (B.29):

2233

– Take a pullback of  $A \xrightarrow{a} Z \leftarrow b \xrightarrow{B}$  to obtain a span  $A \leftarrow x_A \xrightarrow{X} x_B \xrightarrow{B}$ , which by stability of  $\mathcal{M}$ -morphisms under pullback is a span of  $\mathcal{M}$ -morphisms.

2238

– Take a pushout  $A \xrightarrow{p_A} P \leftarrow p_B \xrightarrow{B}$  of  $A \leftarrow x_A \xrightarrow{X} x_B \xrightarrow{B}$ , which by the universal property of pushouts yields a unique morphism  $P \xrightarrow{z} Z$ . Moreover, since  $\mathbf{C}$  is a vertical weak adhesive HLR category, both  $p_A$  and  $p_B$  are  $\mathcal{M}$ -morphisms.

2239

– Take an  $\mathcal{E}$ - $\mathcal{M}$ -factorization  $P \xrightarrow{q} Q \xrightarrow{m_Q} Z$ .

2242

\* Let  $q_A = q \circ p_A$  and  $q_B = q \circ p_B$ ; since both  $\mathcal{M}$  and  $\mathcal{E}$  are closed under composition,  $q_A$  and  $q_B$  are in  $\mathcal{M}$ , while  $q \circ e_P$  is in  $\mathcal{E}$ .

2244

\* By essential uniqueness of  $\mathcal{E}$ - $\mathcal{M}$ -factorizations, there exists a unique isomorphism  $Y \rightarrow Q$ .

2246

This concludes the proof of the soundness and completeness of our [construction](#) for  $\mathcal{M}$ -multi-sums.

2247

Finally, let us consider the claim regarding [refinements](#), whereby if  $\mathbf{C}$  in addition to being a [vertical weak adhesive HLR category](#) also [has pullbacks](#), and moreover satisfies the property that pushouts along  $\mathcal{M}$ -morphisms are [stable under pullbacks](#), then the morphism  $P - q \rightarrow Q$  of the above [construction](#) of  $\mathcal{M}$ -multi-sums is both in  $\mathcal{E}$  and a monomorphism. To this end, first consider diagram (i) in equation (B.31) below:

2251

- By [pullback-pullback decomposition](#), the left and back vertical squares in the bottom of diagram (i) are pullbacks, thus by stability of isomorphisms under pullback,  $Q'$  is isomorphic to  $X$ , hence also the upper left and back vertical squares are pullbacks.

2252

2253

2254

- By [pullback-pushout decomposition](#), the upper front and right vertical squares are pullbacks.

2255

To finish the proof, construct diagram (ii) in (B.31) via taking a pullback (i.e., the square under  $R$ ):

2256

- As the span  $\langle id_A, p_A \rangle$  is a pullback of the cospan  $\langle q_A, q \rangle$ , by [pullback-pullback decomposition](#)  $\langle id_A, r_A \rangle$  is a pullback of  $\langle a, p_R \rangle$ . Analogously, as  $\langle id_B, p_B \rangle$  is a pullback of the cospan  $\langle q_B, q \rangle$ , by [pullback-pullback decomposition](#)  $\langle id_B, r_B \rangle$  is a pullback of  $\langle b, p_R \rangle$ .

2257

2258

2259

- Since the inner bottom horizontal square (i.e., the square marked PO into  $P$ ) is a pushout of  $\mathcal{M}$ -morphisms, and the vertical squares over its boundary are all pullbacks, by the assumed [stability under pullbacks](#) the inner horizontal middle square is a pushout.

2260

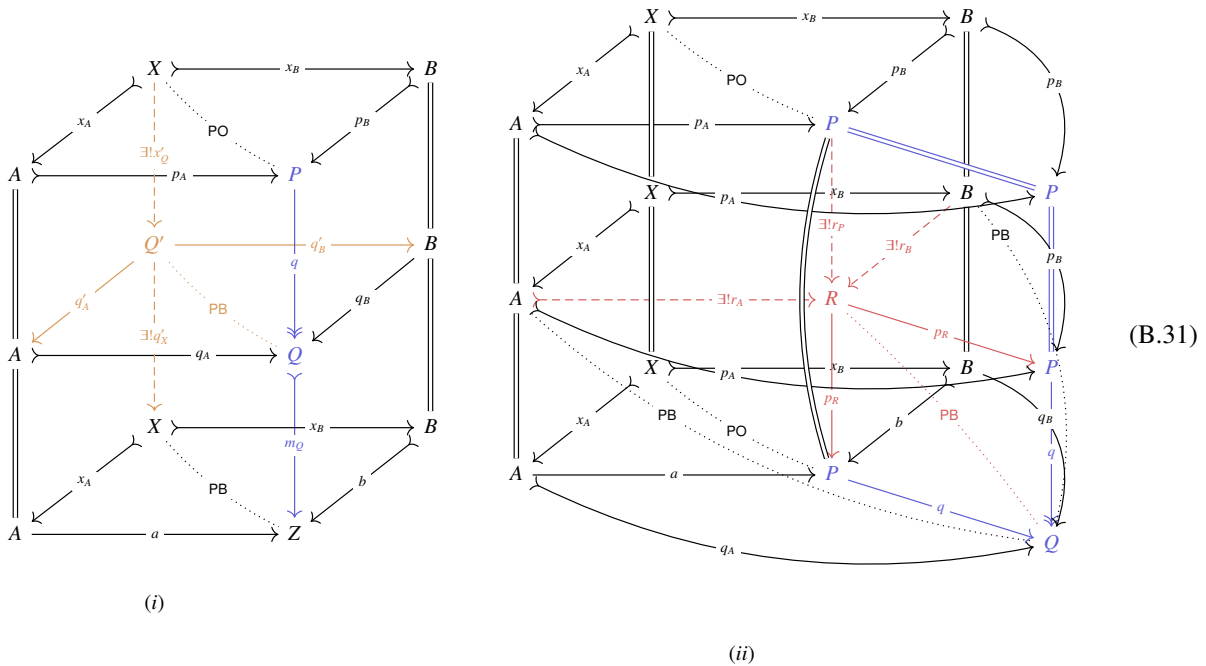
2261

2262

- By the [universal property of pullbacks](#), we find that  $R - p_R \rightarrow P$  is an isomorphism (and thus also  $P - r_P \rightarrow R$ ).

2263

We have thus proved that the span  $\langle id_P, id_P \rangle$  is a pullback of the cospan  $\langle q, q \rangle$ , which entails that  $q$  is a monomorphism.



(B.31)

2264

□