Effect of saturation structure on finite-time stabilization of nonlinear systems with impulsive disturbance

Shuchen Wu, Xiaodi Li, Emmanuel Moulay

Effect of saturation structure on finite-time stabilization of nonlinear systems with impulsive disturbance ☆

Shuchen Wu[a], Xiaodi Li[a,b,*], Emmanuel Moulay[c]

[a]School of Mathematics and Statistics, Shandong Normal University, Ji'nan 250014, PR China
[b]School of Automation and Electrical Engineering, Linyi University, Linyi 276005, PR China
[c]XLIM (UMR CNRS 7252), Université de Poitiers, 11 bd Marie et Pierre Curie, 86073 Poitiers Cedex 9, France

Abstract

This paper studies the finite-time stabilization (*FTS*) of nonlinear systems with impulsive disturbance, where the effect of saturation structure on the control input is fully considered. By establishing a potential relationship between saturation structure and impulsive disturbance, some *FTS* criteria are presented via a class of saturated control input, where estimations of domain of attraction and settling-time are derived. Compared with the control input without saturation structure, the constraints on impulse time sequence are strengthened to guarantee that the saturated control input has an effective stabilizing effect on the system. Based on such constraints, the system subjected to impulsive disturbance can still achieve *FTS*. Then theoretical results are applied to finite-time synchronization for a class of impulsive dynamical networks. Finally, two examples and their simulations are given to illustrate the effectiveness of the proposed results.

Keywords: Saturated structure; Finite-time stabilization; Impulsive disturbance; Domain of attraction; Settling-time.

1. Introduction

Impulsive system consists of continuous dynamics and discrete dynamics, which exhibits the discontinuities or jumps in the state trajectories of the system at some certain instants (usually called impulse instants), see [1–6]. In past years, these results were extended to stochastic impulsive systems [7, 8], impulsive switching systems [9], and impulsive systems with delays [10–12]. With the development of the theory of impulsive systems, the related results are applied to various fields, such as secure communication in image encryption [13, 14], satellite rendezvous in aerospace industry [15], and data sampling in high-speed trains [16], and so on. In accordance with the effect of impulses on the system, the system involving impulses can be roughly divided into three classes: system with impulsive control, system with impulsive disturbance, and the combination of both of them, usually called system with hybrid impulses. In this paper, we study a class of nonlinear systems with impulsive disturbance.

Notice that there are many interesting results of nonlinear systems with impulsive disturbance, such as almost sure stability [17], asymptotical stability [18], exponential stability [19, 20], and fixed-time stability [21, 22], and so on. In particular, [23] and [24] studied general nonlinear systems with state-dependent impulses and state-independent impulses, respectively. In [24], some *FTS* results were established by a class of constrained impulse time sequences and the estimation of the settling-time was derived. Later, these results were not only extended to time-varying nonlinear impulsive systems by constructing a time-varying differential inequality of Lyapunov function, such as [25], but also applied to finite-time synchronization of complex dynamical networks, such as [26] and [27]. However, it is worth noting that the existing results in [24–27] are all based on the idealized environment that the control input

☆This work was supported by National Natural Science Foundation of China (62173215), Major Basic Research Program of the Natural Science Foundation of Shandong Province in China (ZR2021ZD04, ZR2020ZD24), and the Support Plan for Outstanding Youth Innovation Team in Shandong Higher Education Institutions (2019KJI008).

*Corresponding Author.
Email address:lxd@sdnu.edu.cn. (X. Li), emmanuel.moulay@univ-poitiers.fr (E. Moulay).

in transmission is not limited by the physical constraints and always available. Indeed, in practical applications, the actual control input is inevitably constrained by saturation structure, see [28–30]. Sometimes, saturation structure can degrade the performances of a system and even lead to the system instability. Recently, in the framework of saturation structure, there are a few results of nonlinear systems with impulsive disturbances, see input-to-state stability in [31] and exponential stabilization in [32]. Except the above mentioned work, to our best knowledge, in the framework of saturation structure, there is almost no *FTS* results for nonlinear systems with impulsive disturbances. In fact, compared with the case without saturation structure, there exist some difficulties and challenges to achieve *FTS*. Firstly, when the effect of saturation structure on systems is fully considered, the conventional controller is usually in form of strictly state feedback, which can be easily handled by linear differential inclusion approach [29–31] and anti-windup approach [32, 33]. Unfortunately, it is difficult to obtain the *FTS* in virtue of such controllers and moreover, naturally, such two approaches above cannot be applied to the design of saturated control input in sense of finite time. Secondly, when impulsive disturbance and saturation structure are both involved, due to the uncertainty of both impulse time sequences and the amplitude of impulsive disturbance, the saturated control input may have no effective stabilizing effect on the system, which causes that the performance of the system may degrade or become unstable. Hence, it is hard to guarantee the validity of saturated control input for systems involving impulsive disturbances due to the uncertainty and complexity of the relationship between saturation structure and impulsive disturbance. This is an interesting but challenging work.

Motivated by the discussion above, this paper studies the *FTS* of nonlinear systems with impulsive disturbances, where the potential effect of saturation structure is fully considered. By imposing constraints on impulse time sequence and then making full use of them to establish the potential relationship between saturation structure and impulsive disturbance, some *FTS* criteria are presented via a class of saturated control input. Moreover, we apply the theoretical results to finite-time synchronization of impulsive dynamical networks. The outline of this paper is as follows. The preliminaries are given in Section 2. The theoretical results are presented in Section 3 and application to finite-time synchronization of impulsive dynamical networks in Section 4. Two examples are given in Section 5. Finally, the conclusion is presented in Section 6.

Notations: Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{Z}$ denote the set of integers. $\mathbb{R}_+ := \{a \in \mathbb{R} : a \geq 0\}$ and $\mathbb{Z}_+ := \{a \in \mathbb{Z} : a > 0\}$. $\mathbb{R}^{n\times m}$ denotes the $n \times m$-dimensional matrix space equipped with Frobenious Norm $|\cdot|$. When $m = 1$, the notation $\mathbb{R}^n$ is an abbreviation of $\mathbb{R}^{n\times 1}$. Denote $\Omega = \{1, 2, \cdots, n\}$, $\mathcal{N} = \{1, 2, \cdots, N\}$, and $\mathcal{M} = \{1, 2, \cdots, M\}$. Let $E_i^m \in \mathbb{R}^{1\times m}$ denote an m-dimensional row vector that the i-th element is 1 and the others are 0, $i = 1, 2, \cdots, p$. I_n denotes the unit matrix with dimension n. $\otimes$ denotes the Kronecker product operator of two matrices. For any $\epsilon > 0$, $\mathcal{S}_\epsilon := \{x \in \mathbb{R}^n : |x| \leq \epsilon\}$. The minimum value of a and b is denoted by $a \wedge b$. For a locally Lipschitz continuous function $H : \mathbb{R}^n \to \mathbb{R}_+$, D^+H denotes the upper right-hand Dini derivative.

2. Preliminaries

Consider a class of nonlinear systems with impulsive disturbance

$$\begin{cases} \dot{x}(t) = Cx(t) + Af(t, x(t)) + u(t), \ \ t \neq t_k, \ t \geq 0, \\ \Delta x(t) = Dx(t^-), \ \ t = t_k, \ k \in \mathbb{Z}_+, \\ x(0) = x_0 \in \mathcal{S} \subseteq \mathbb{R}^n, \end{cases} \tag{1}$$

where $C, A, D \in \mathbb{R}^{n\times n}$, $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$ is the system state, and $u = (u_1, u_2, \cdots, u_n)^T \in \mathbb{R}^n$ is the control input. $\Delta x(t) = x(t^+) - x(t^-)$ denotes the change of the system state when subjecting to impulsive disturbance, where $x(t^+)$ and $x(t^-)$ are the right and left limits at instant t, respectively. $\mathcal{S}$ denotes the initial state domain containing the origin. The nonlinear function $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is bounded, i.e., there exists a constant $\omega > 0$ such that $|f(t, x)| \leq \omega$, for all $t \geq 0$, $x \in \mathbb{R}^n$. In this paper, we assume that $x(t^+) = x(t)$, i.e., the solutions of system (1) are right continuous. The impulse time sequences $\{t_k\}_{k\in\mathbb{Z}_+}$ is strictly increasing on $\mathbb{R}_+$ and either finite or infinite and unbounded, short for $\{t_k\}$. We denote such kind of impulse time sequences as the set $\mathscr{F}_0$ for later use. In particular, the subset of $\mathscr{F}_0$ containing N impulse instants is denoted by $\mathscr{F}_N$, where the impulse time sequence $\{t_k\}^N$ satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_N < +\infty$ (t_1 is the first impulse instant). For any given impulse time sequence $\{t_k\}$, let $x(t) = x(t, 0, x_0, \{t_k\})$ denote the solution of system (1) through $(0, x_0)$ with the initial state $x_0 \in \mathcal{S}$. And assume that the solution $x(t)$ uniquely exists in forward time within relevant time intervals, see [34] for related information.

Definition 1. [24] Consider system (1) without control input u. System (1) is said to be *locally finite-time stable* with $\mathcal{S}$ over the class $\mathscr{F}_0$ if there exists a function $T(x_0, \{t_k\}) : \mathcal{S} \times \mathscr{F}_0 \to \mathbb{R}_+$ such that the solution of system (1) starting from the initial state $x_0 \in \mathcal{S}$ and subjecting to the impulses $\{t_k\} \in \mathscr{F}_0$ is Lyapunov stable and finite-time convergent, i.e.,

$$x_0 \in \mathcal{S} \Rightarrow x(t, 0, x_0, \{t_k\}) \equiv 0, \quad \forall t \geq T(x_0, \{t_k\}). \tag{2}$$

Moreover, the *settling-time* of system (1) is defined as

$$T_{\inf}(x_0, \{t_k\}) := \inf_{t \geq T(x_0)} \{T(x_0, \{t_k\}) \geq 0 : x(t, 0, x_0, \{t_k\}) = 0\}.$$

Definition 2. System (1) is *locally finite-time stabilized* (*locally FTS*) with $\mathcal{S}$ over the class $\mathscr{F}_0$ if there exists an admissible control input u such that system (1) is *locally finite-time stable* with $\mathcal{S}$ over the class $\mathscr{F}_0$.

In the sense of finite time, there are some interesting results of *FTS* criteria for nonlinear systems with impulsive disturbance, such as [23] for state-dependent impulses and [24] for state-independent impulses. Subsequently, these results were extended to time-varying nonlinear impulsive systems [25], impulsive memristor-based neural networks [26], and delayed complex dynamical networks [27], and so on. It is worth noting that these existing results are all based on the desired control input without saturation structure. In fact, due to the existence of the physical constraints, the control input may be limited during the transmission of the actuator, which leads to a fact that the actual control input is of saturation structure. To show the effect of saturation structure more clearly, consider the dynamic of the following simple example

$$\begin{cases} \dot{x}(t) = 0.5x(t) + \sin(x(t)) + u(t), & t \neq k, \\ \Delta x(t) = 0.3x(t^-), & t = k, \end{cases} \tag{3}$$

where $k \in \mathbb{Z}_+$, $t \geq 0$, and $x \in \mathbb{R}$ is the state of system (3). We consider the following control input

$$u(t) = -0.5x(t) - 3\mathrm{sgn}(x(t)).$$

When there is no saturation structure in system (3), it follows from the *FTS* results of impulsive systems in [24] that system (3) with control input u is *FTS*, see Figure 1, where the initial state $x_0 = 8$. While saturation structure is involved in system (3), the control input u is updated to

$$\overline{u}(t) = \mathrm{sgn}(u(t))(u(t) \wedge 5) = \begin{cases} 5, & u(t) \in (5, +\infty), \\ u(t), & u(t) \in [-5, 5], \\ -5, & u(t) \in (-\infty, -5). \end{cases}$$

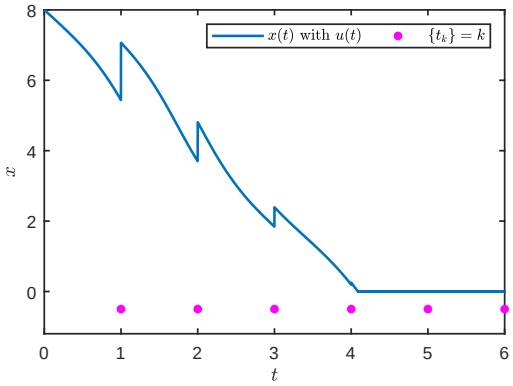

Figure 1. The trajectory of system (3) with $u(t)$.

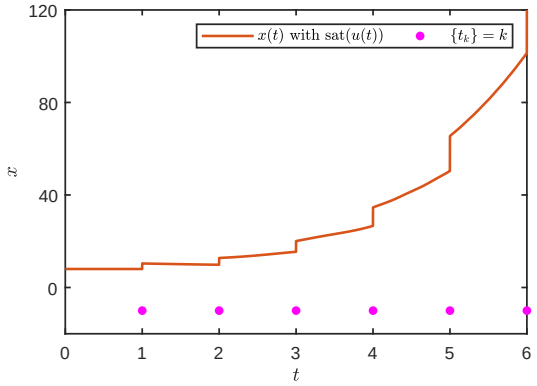

Figure 2. The trajectory of system (3) with $\overline{u}(t)$.

Under the same initial state $x_0 = 8$, one may observe that due to the effect of saturation structure, system (3) becomes unstable, not to mentioned *FTS*, see Figure 2. Inspired by the above insight, it is natural to ask, if in the framework of saturation structure, it is possible to establish some *FTS* criteria for nonlinear systems with impulsive disturbance.

Note that in finite-time sense, in view of a fact that when saturation input and finite-time control are both involved in the impulsive system, it may increase the complexity of the control input. Here we consider a special case of system (1) with the unit control input matrix. Motivated by the discussions above, our aim is that in the framework of saturation structure, we explore a potential relationship between saturation structure and impulsive disturbance to guarantee the *FTS*. To this end, system (1) can be rewritten as

$$\begin{cases} \dot{x}(t) = Cx(t) + Af(t, x(t)) + \mathrm{sat}_\rho(u(t)), \ \ t \neq t_k, \ t \geq 0, \\ x(t) = (I + D)x(t^-), \ \ t = t_k, \ k \in \mathbb{Z}_+, \\ x(0) = x_0 \in \mathcal{S} \subseteq \mathbb{R}^n, \end{cases} \tag{4}$$

where $\rho > 0$ is a prior given saturation level. The saturation function $\mathrm{sat}_\rho : \mathbb{R}^n \to \mathbb{R}^n$ is defined as $\mathrm{sat}_\rho(u) = (\mathrm{sat}_\rho(u_1), \mathrm{sat}_\rho(u_2), \cdots, \mathrm{sat}_\rho(u_n))^T$, where $\mathrm{sat}_\rho(u_i) := \mathrm{sgn}(u_i)(u_i \wedge \rho)$, for all $i \in \Omega$. In the sense of finite time, the initial state domain $\mathcal{S}$ is called the domain of attraction of system (1), i.e.,

$$\mathcal{S} := \Big\{ x_0 \in \mathbb{R}^n : x(t, 0, x_0, \{t_k\}) \text{ satisfies (2) in Definition 1} \Big\}.$$

Lemma 1. *For any vector $v = (v_1, v_2, \cdots, v_l)^T \in \mathbb{R}^l$, define $S(v) = (\mathrm{sgn}(v_1), \mathrm{sgn}(v_2), \cdots, \mathrm{sgn}(v_l))^T$. Then it holds that $v^T v \leq (v^T S(v))^2$.*

3. Theoretical results

In this section, the *FTS* criterion is proposed by designing a class of saturated control input. For a prior given saturation level ρ and a constant $\alpha \in (0, \rho)$, we consider the control input $u(t)$ of system (4) given by

$$u(t) = -Cx(t) - \alpha \cdot S(x(t)). \tag{5}$$

Theorem 1. *For a prior given saturation level $\rho > 0$, assume that there exist some positive constants α_i, $i = 1, 2, 3$, and $\beta \in [1, \infty)$, such that $AA^T \leq \alpha_1 I_n$, $(I + D)^T (I + D) \leq \beta^2 I_n$, and*

$$\alpha := \frac{1}{2}\Big(\frac{\alpha_1}{\alpha_2} + \alpha_2 \omega^2 + \alpha_3\Big) < \rho. \tag{6}$$

Then system (4) via saturated control input (5) is locally FTS with $\mathcal{S}_\epsilon$ over the class $\mathscr{F}^\star$, where $\mathcal{S}_\epsilon$ is contained in the domain of attraction satisfying

$$\epsilon = \frac{\rho - \alpha}{\max_{i \in \Omega} |E_i C|}, \tag{7}$$

and $\mathscr{F}^\star$ denotes a class of impulse time sequences $\{t_k\} \in \mathscr{F}_0$ satisfying

$$N := \min\Big\{ k \in \mathbb{Z}_+ : t_k \geq \beta^{k-1} \cdot \frac{2|x_0|}{\alpha_3} \Big\} < \infty, \tag{8a}$$

$$t_k > \beta^{k-1} \cdot \frac{2|x_0|}{\alpha_3} - \frac{2}{\alpha_3} \cdot \frac{\epsilon}{\beta}, \ \ \forall k \leq N - 1. \tag{8b}$$

In addition, the settling-time of system (4) though $(0, x_0)$ can be bounded by

$$T_{\inf}(x_0, \{t_k\}) \leq \beta^{N-1} \cdot \frac{2|x_0|}{\alpha_3}. \tag{9}$$

PROOF. For any given $\{t_k\} \in \mathscr{F}^\star$ and $x_0 \in \mathcal{S}_\epsilon$, let $x(t) = x(t, 0, x_0, \{t_k\})$ be the solution of system (4) though initial state $(0, x_0)$. Firstly, we show that under the impulse time sequence $\{t_k\} \in \mathscr{F}^\star$, $\mathcal{S}_\epsilon$ is an invariant set, i.e., $x(t) \le \epsilon$, $\forall t \ge 0$. Suppose that the statement is not true, then there exist some instants $t \ge 0$ such $x(t) \notin \mathcal{S}_\epsilon$. Let $t^\star := \inf\{t \ge 0 : x(t) \notin \mathcal{S}_\epsilon\}$. It is obvious that the definition of $t^\star$ is nonempty. Then we show that $t^\star$ is not within the continuous intervals. Without loss of generality, assume that $0 < t_1 < t_2 < \cdots < t_l \le t^\star < t_{l+1} \cdots < t_N < \cdots < +\infty$. Denote $\mathcal{L} = \{1, 2, \cdots, l\}$ for later use. It follows from $\{t_k\} \in \mathscr{F}^\star$ that impulse instants satisfy (8a) and (8b), that is,

$$\beta^{k-1} \cdot \frac{2|x_0|}{\alpha_3} - \frac{2}{\alpha_3} \cdot \frac{\epsilon}{\beta} < t_k < \beta^{k-1} \cdot \frac{2|x_0|}{\alpha_3}, \ \ \forall k \in \mathcal{L}. \tag{10}$$

Consider an auxiliary function $H(t) = x^T(t)x(t)$, one can obtain that $|x(t^\star)| = \epsilon$, $D^+H(t^\star) \ge 0$, and $|x(t)| \le \epsilon$, $\forall t \in [0, t^\star]$. It follows from the definition of ϵ that

$$|u_i(t)| = |E_i^n Cx(t) + \alpha \cdot S(x_i(t))| \le \max_{i\in\Omega} |E_i^n C| \cdot \epsilon + \alpha = \rho, \ \ \forall t \in [0, t^\star],$$

which implies that $\mathrm{sat}_\rho(u_i(t)) = u_i(t)$, $\forall t \in [0, t^\star]$, and $\mathrm{sat}_\rho(u(t)) = u(t)$, $\forall t \in [0, t^\star]$. Moreover, the derivative of $H(t)$ along system (4) satisfies

$$D^+H(t) = 2x^T(t)\dot{x}(t) = 2x^T(t)(Cx(t) + Af(t, x(t)) + \mathrm{sat}_\rho(u(t)), \ \ \forall t \in [t_{k-1}, t_k), \ \ k \in \mathcal{L}.$$

When $|x(t)| \neq 0$, according to $AA^T \le \alpha_1 I_n$ and Lemma 1, one derives that

$$\begin{aligned} 2x^T(t)Af(t, x(t)) &\le \frac{1}{\alpha_2} x^T(t)AA^T x(t) \frac{1}{x^T(t)S(x(t))} + \alpha_2 f^T(t, x(t)) f(t, x(t)) x^T(t)S(x(t)) \\ &\le \frac{\alpha_1}{\alpha_2} x^T(t)x(t) \frac{1}{x^T(t)S(x(t))} + \alpha_2\omega^2 x^T(t)S(x(t)) \\ &\le \frac{\alpha_1}{\alpha_2} x^T(t)S(x(t)) + \alpha_2\omega^2 x^T(t)S(x(t)). \end{aligned} \tag{11}$$

When $|x(t)| = 0$, it can be deduced that

$$0 = 2x^T(t)Af(t, x(t)) = \frac{\alpha_1}{\alpha_2} x^T(t)S(x(t)) + \alpha_2\omega^2 x^T(t)S(x(t)),$$

which indicates that in the both above two cases, (11) always holds. In view of Lemma 1 and (5), it yields that

$$\begin{aligned} D^+H(t) &= 2x^T(t)Cx(t) + 2x^T(t)Af(t, x(t)) + 2x^T(t)u(t) \\ &\le \frac{\alpha_1}{\alpha_2} x^T(t)S(x(t)) + \alpha_2\omega^2 x^T(t)S(x(t)) - 2\alpha \cdot x^T(t)S(x(t)) \\ &\le -\alpha_3 x^T(t)S(x(t)) \\ &\le -\alpha_3 H^{\frac{1}{2}}(t), \ \ \forall t \in [t_{k-1}, t_k), \ \ k \in \mathcal{L}. \end{aligned} \tag{12}$$

One can observe that $H(t)$ is decreasing within every continuous interval. Hence, $t^\star$ is not within the continuous intervals. Then $t^\star$ is an impulse instant, i.e., $t^\star = t_l$. Considering $(I + D)^T(I + D) \le \beta^2 I_n$, it leads to

$$H(t_k) = x(t_k)^T x(t_k) \le x(t_k^-)^T (I + D)^T (I + D) x(t_k^-) \le \beta^2 H(t_k^-), \ \ \forall k \in \mathcal{L}. \tag{13}$$

Based on inequality (12) and $t_1 < 2|x_0|/\alpha_3$, one can obtain that

$$H^{\frac{1}{2}}(t) \le H^{\frac{1}{2}}(0) - \frac{1}{2}\alpha_3 t, \ \ \forall t \in [0, t_1).$$

It follows from (13) and (10) that

$$H^{\frac{1}{2}}(t_1) = \beta H^{\frac{1}{2}}(t_1^-) \leq \beta\left(H^{\frac{1}{2}}(0) - \frac{1}{2}\alpha_3 t_1\right) < \beta\left(H^{\frac{1}{2}}(0) - \frac{1}{2}\alpha_3\left(\frac{2|x_0|}{\alpha_3} - \frac{2}{\alpha_3}\cdot\frac{\epsilon}{\beta}\right)\right) = \epsilon.$$

Similarly, one can derive that

$$H^{\frac{1}{2}}(t) \leq H^{\frac{1}{2}}(t_1) - \frac{1}{2}\alpha_3(t-t_1) \leq \beta H^{\frac{1}{2}}(0) - \frac{1}{2}\alpha_3\beta t_1 - \frac{1}{2}\alpha_3(t-t_1) \leq \beta H^{\frac{1}{2}}(0) - \frac{1}{2}\alpha_3 t, \ \ \forall t \in [t_1, t_2),$$

and

$$H^{\frac{1}{2}}(t_2) = \beta H^{\frac{1}{2}}(t_2^-) \leq \beta\left(\beta H^{\frac{1}{2}}(0) - \frac{1}{2}\alpha_3 t_2\right) < \beta\left(\beta H^{\frac{1}{2}}(0) - \frac{1}{2}\alpha_3\left(\beta\cdot\frac{2|x_0|}{\alpha_3} - \frac{2}{\alpha_3}\cdot\frac{\epsilon}{\beta}\right)\right) = \epsilon.$$

By mathematical induction, it can be finally deduced that

$$H^{\frac{1}{2}}(t_k) = \beta H^{\frac{1}{2}}(t_k^-) \leq \beta\left(\beta^{k-1}H^{\frac{1}{2}}(0) - \frac{1}{2}\alpha_3 t_k\right) < \beta\left(\beta^{k-1}H^{\frac{1}{2}}(0) - \frac{1}{2}\alpha_3\left(\beta^{k-1}\frac{2|x_0|}{\alpha_3} - \frac{2}{\alpha_3}\cdot\frac{\epsilon}{\beta}\right)\right) = \epsilon, \ \ \forall k \in \mathcal{L}.$$

It implies that $|x(t_k)| < \epsilon$, $\forall k \in \mathcal{L}$. When $t^\star = t_l$, it holds that $|x(t^\star)| = |x(t_l)| < \epsilon$, which contradicts with $|x(t^\star)| = \epsilon$. Hence, it always holds that $x(t) \in \mathcal{S}_\epsilon$, $\forall t \geq 0$.

Next, we show that for any $x_0 \in \mathcal{S}_\epsilon$, system (4) via saturated control input (5) is *locally FTS* over the class $\mathscr{F}^\star$ and the estimation of settling-time can be bounded by (9). Based on the discussion above, one can conclude that

$$H^{\frac{1}{2}}(t) \leq \beta^{k-1}H^{\frac{1}{2}}(0) - \frac{1}{2}\alpha_3 t, \ \ \forall t \in [t_{k-1}, t_k), \ \ k \in \mathcal{N},$$

that is,

$$|x(t)| \leq \beta^{k-1}|x_0| - \frac{1}{2}\alpha_3 t, \ \ \forall t \in [t_{k-1}, t_k), \ \ k \in \mathcal{N}.$$

Moreover, note that $\{t_k\} \in \mathscr{F}^\star$, then it holds that

$$t_k < \beta^{k-1}\cdot\frac{2|x_0|}{\alpha_3}, \ \ \forall k \leq N-1,$$

which indicates that the settling-time of system (4) cannot be estimated on the interval $[0, t_{N-1})$. In view of the definition of t_N, we have

$$\begin{cases} |x(t)| \leq |x_0| - \frac{1}{2}\alpha_3 t, \ \ t \in [0, t_1), \\ |x(t)| \leq \beta|x_0| - \frac{1}{2}\alpha_3 t, \ \ t \in [t_1, t_2), \\ \ \ \vdots \\ |x(t)| \leq \beta^{N-2}|x_0| - \frac{1}{2}\alpha_3 t, \ \ t \in [t_{N-2}, t_{N-1}), \\ |x(t)| \leq \beta^{N-1}|x_0| - \frac{1}{2}\alpha_3 t, \ \ t \in [t_{N-1}, \beta^{N-1}\cdot 2|x_0|/\alpha_3), \\ |x(t)| \equiv 0, \ \ t \in [\beta^{N-1}\cdot 2|x_0|/\alpha_3, +\infty), \end{cases}$$

which indicates that system (4) via saturated control input (5) is *locally FTS* with $\mathcal{S}_\epsilon$ over the class $\mathscr{F}^\star$ and the settling-time can be bounded by (9). The proof of Theorem 1 is completed. □

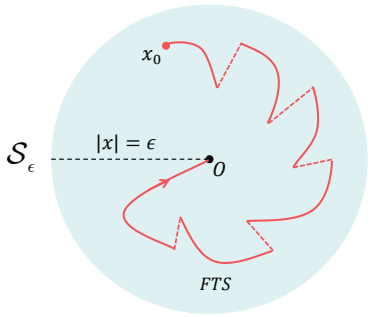

Figure 3. *FTS* of system via saturated control input.

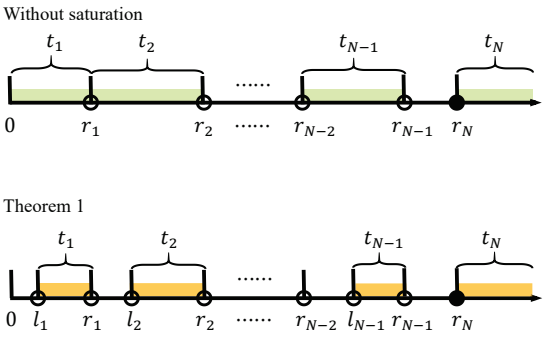

Figure 4. The illustration of impulse time sequence $\{t_k\}$.

Remark 1. In the framework of saturation structure, Theorem 1 presents the *FTS* criterion, where the effect of impulsive disturbance is fully considered. The idea behind Theorem 1 can be simply described in Figure 3, where the red curve represents the trajectory of system (4) with impulsive disturbance via saturated control input. In essence, the key point of Theorem 1 is to ensure that the domain of attraction $\mathcal{S}_\epsilon$ is an invariant set. In other words, even if system (4) is subjected to the dual influence of saturation structure and impulsive disturbance, with the help of an effective control input, not only system (4) cannot exceed the domain of attraction $\mathcal{S}_\epsilon$, but also the *FTS* can be guaranteed. In order to achieve such two objectives, firstly, a relationship between the domain of attraction, system structure, and the controller parameter is established (i.e., (7)), which ensures that saturated control input (5) is valid to stabilize system (4). Then some constraints on impulse time sequence $\{t_k\}$ are proposed (i.e., (8a) and (8b)), where (8a) plays an important role in estimating the settling-time and (8b) can guarantee that the amplitude of impulsive disturbance is not so large that system (4) would exceed the domain of attraction $\mathcal{S}_\epsilon$. Compared with the case without saturation structure, as pointed in [24], impulse time sequence $\{t_k\}$ only needs to satisfy (8a), under which *FTS* for system (4) with impulsive disturbance can be guaranteed. Such two kinds of impulse time sequences are intuitively illustrated in Figure 4, where the yellow region (i.e., $l_k < t_k < r_k$, $\forall k \leq N-1$, and $t_N \geq r_N$) represents the impulse time sequence satisfying (8a) and (8b) in Theorem 1 and the green region (i.e., $t_k < r_k$, $\forall k \leq N-1$, and $t_N \geq r_N$) represents the impulse time sequence satisfying (8a) in [24]. The time sequences $\{l_k\}$ and $\{r_k\}$ mentioned above are defined as

$$l_k = \beta^{k-1} \cdot \frac{2|x_0|}{\alpha_3} - \frac{2}{\alpha_3} \cdot \frac{\epsilon}{\beta} \ \text{ and } \ r_k = \beta^{k-1} \cdot \frac{2|x_0|}{\alpha_3}, \ \ \forall k \in \mathbb{Z}_+.$$

Based on the analysis above, it is worth mentioning that in the framework of saturation structure, (8b) is crucial and indispensable to achieve *FTS*. This assertion will be illustrated in Section 5. In particular, when only involving N impulse instants in system (4), the following corollary can be directly obtained.

Corollary 1. *Under conditions in Theorem 1, system* (4) *via saturated control input* (5) *is locally FTS with* $\mathcal{S}_\epsilon$ *over the class* $\mathcal{F}_N$*, where* $\mathcal{S}_\epsilon$ *is contained in the domain of attraction satisfying* (7) *and* $\mathcal{F}_N$ *denotes a class of impulse time sequences* $\{t_k\}^N$ *satisfying*

$$\beta^{k-1} \cdot \frac{2|x_0|}{\alpha_3} - \frac{2}{\alpha_3} \cdot \frac{\epsilon}{\beta} < t_k < \beta^{k-1} \cdot \frac{2|x_0|}{\alpha_3}, \ \ \forall k \in \mathcal{N}.$$

Moreover, the settling-time of system (4) *though* $(0, x_0)$ *can be bounded by* (9).

Remark 2. Notice that when involving saturation structure, there are only a few results for nonlinear systems with impulsive disturbance, see [32] and [31]. One may observe from these existing results in [32] and [31] that only exponential stabilization and input-to-state stability can be achieved. Moreover, to our best knowledge, in the framework of saturation structure, there is almost no results of *FTS* criteria for nonlinear systems with impulsive disturbance. In the present paper, by strengthening constraints on impulse time sequence and then making full use of them to establish the potential relationship between saturation structure and impulsive disturbance, Theorem 1 and its corollary present the *FTS* criteria, where the estimations of the domain of attraction and the settling-time are derived.

4. Application to finite-time synchronization of impulsive dynamical networks

Consider a class of impulsive dynamical networks consisting of $M(\geq 2)$ nodes via saturated control input

$$\begin{cases} \dot{x}_i(t) = g(x_i(t)) + \sigma \sum_{j=1}^{M} l_{ij}\Gamma x_j(t) + \mathrm{sat}_\rho(u_i(t)), & t \neq t_k, \\ \Delta x_i(t) = D_i x_i(t^-), & t = t_k,\ k \in \mathbb{Z}_+, \end{cases} \tag{14}$$

where $i \in \mathcal{M}$, $D_i \in \mathbb{R}^{n\times n}$, $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in \mathbb{R}^n$ is the state vector of the i-th dynamical node, and $g(x_i(t)) = (g_1(x_{i1}(t)), g_2(x_{i2}(t)), \cdots, g_n(x_{in}(t)))^T \in \mathbb{R}^n$ is the activation function satisfying $|g(\varsigma)| \leq \omega$, $\forall \varsigma \in \mathbb{R}^n$, where $\omega > 0$ is a known constant. $\sigma > 0$ is the coupling strengthen, $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \cdots, \gamma_n\} > 0$ is the inner coupling matrix, and $L = (l_{ij})_{M\times M}$ is the outer coupling matrix, which is defined as: if there is an edge from node i to node j ($j \neq i$), then $l_{ij} > 0$, otherwise, $l_{ij} = 0$, the diagonal elements are defined as $l_{ii} = -\sum_{j=1, j\neq i}^{M} l_{ij}$, $\forall i \in \mathcal{M}$. $u_i = (u_{i1}, u_{i2}, \cdots, u_{in})^T \in \mathbb{R}^n$ is the control input without saturation limitation. In terms of kronecker product, impulsive dynamical network (14) can be rewritten as

$$\begin{cases} \dot{x}(t) = G(x(t)) + \sigma(L \otimes \Gamma)x(t) + \overline{\mathrm{sat}}_\rho(u(t)), & t \neq t_k, \\ x(t) = \overline{D}x(t^-), & t = t_k,\ k \in \mathbb{Z}_+, \end{cases} \tag{15}$$

where $\overline{C} = I_M \otimes C$, $\overline{A} = I_M \otimes A$, $\overline{D} = \mathrm{diag}\{I_n + D_1, I_n + D_2, \cdots, I_n + D_M\}$, and

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_M(t) \end{pmatrix}, \quad G(x(t)) = \begin{pmatrix} g(x_1(t)) \\ g(x_2(t)) \\ \vdots \\ g(x_M(t)) \end{pmatrix}, \quad \overline{\mathrm{sat}}_\rho(u(t)) = \begin{pmatrix} \mathrm{sat}_\rho(u_1(t)) \\ \mathrm{sat}_\rho(u_2(t)) \\ \vdots \\ \mathrm{sat}_\rho(u_M(t)) \end{pmatrix}.$$

In the following, in the framework of saturation, we consider the leader-follower synchronization in a finite time, where impulsive dynamical network (15) is regarded as the follower and the leader $y(t) \in \mathbb{R}^n$ is of the form

$$\dot{y}(t) = g(y(t)). \tag{16}$$

Define the synchronization error between impulsive dynamical network (15) and system (16) as

$$e(t) = (e_1^T(t), e_2^T(t), \cdots, e_M^T(t))^T \in \mathbb{R}^{Mn},$$

where $e_i(t) = x_i(t) - y(t) \in \mathbb{R}^n$ is the synchronization error of i-th node. Then the error system is given by

$$\dot{e}(t) = G_e(e(t)) + \sigma(L \otimes \Gamma)e(t) + \overline{\mathrm{sat}}_\rho(u(t)), \tag{17}$$

where $G_e(e(t)) = ((g(x_1(t)) - g(y(t)))^T, (g(x_2(t)) - g(y(t)))^T, \cdots, (g(x_M(t)) - g(y(t)))^T)^T$.

Definition 3. Impulsive dynamical network (15) via an admissible control input u and system (16) can achieve *locally finite-time synchronization* with $\mathcal{S}$ over the class $\mathscr{F}_0$ if the error system (17) via an admissible control input u is *locally FTS* with $\mathcal{S}$ over the class $\mathscr{F}_0$.

For a prior given saturation level ρ and a constant $\alpha \in (0, \rho)$, we consider the control input $u(t)$ of error system (17) as follows

$$u(t) = -\sigma(L \otimes \Gamma)e(t) - \alpha S(e(t)). \tag{18}$$

Theorem 2. *For a prior given saturation level $\rho > 0$, suppose that there exist some positive constants α_1, α_2, and $\beta \in [1, \infty)$, such that $\overline{D}^T \overline{D} \le \beta^2 I_{Mn}$ and*

$$\alpha := \frac{1}{2}\left(\frac{1}{\alpha_1} + 4\alpha_1\omega^2 M + \alpha_2\right) < \rho.$$

Then impulsive dynamical network (15) via saturated control input (18) and system (16) can achieve locally finite-time synchronization with $\mathcal{S}_\epsilon$ over the class $\mathscr{F}^\star$, where $\mathcal{S}_\epsilon$ is contained in the domain of attraction satisfying

$$\epsilon = \frac{\rho - \alpha}{\sigma \max_{j\in\Omega} \gamma_j \cdot \max_{i\in\mathcal{M}} |L_i|},$$

and $\mathscr{F}^\star$ denotes a class of impulse time sequences $\{t_k\} \in \mathscr{F}_0$ satisfying

$$N := \min\left\{k \in \mathbb{Z}_+ : t_k \ge \beta^{k-1} \cdot \frac{2|e_0|}{\alpha_2}\right\} < \infty,$$

$$t_k > \beta^{k-1} \cdot \frac{2|e_0|}{\alpha_2} - \frac{2}{\alpha_2} \cdot \frac{\epsilon}{\beta}, \quad \forall k \le N-1.$$

In addition, the settling-time can be bounded by

$$T_{\inf}(e_0, \{t_k\}) \le \beta^{N-1} \cdot \frac{2|e_0|}{\alpha_2}.$$

PROOF. Different from Theorem 1, it follows from (18) that control input $u_{ij}(t)$ is expressed by

$$u_{ij}(t) = -\sigma(E_i^M L \otimes E_j^n \Gamma)e(t) - \alpha \cdot S(e_{ij}(t)), \quad \forall i \in \mathcal{M}, \quad j \in \Omega,$$

which indicated that it holds that for all $t \in [0, t^\star]$,

$$\begin{aligned}
|u_{ij}(t)| &\le |\sigma(E_i^M L \otimes E_j^n \Gamma)e(t) + \alpha \cdot S(e_{ij}(t))| \\
&\le |\sigma(E_i^M L \otimes E_j^n \Gamma))| \cdot |e(t)| + \alpha \\
&\le (\sigma \max_{j\in\Omega} \gamma_j \cdot \max_{i\in\mathcal{M}} |L_i|) \cdot \epsilon + \alpha \\
&\le \rho.
\end{aligned}$$

Choose $H(t) = e^T(t)e(t)$, then the derivative of $H(t)$ along system (17) satisfies

$$\begin{aligned}
D^+H|(t) &= 2e^T(t)\dot{e}(t) \\
&= 2e^T(t)\Big(G_e(e(t)) + c(L \otimes \Gamma)e(t) + \overline{\text{sat}}_\rho(u(t))\Big) \\
&= 2e^T(t)G_e(e(t)) - 2\alpha \cdot e^T(t)S(e(t)).
\end{aligned}$$

When $|e(t)| \neq 0$, it can be deduced that

$$\begin{aligned}
2e^T(t)G_e(e(t)) &\le \frac{1}{\alpha_1} e^T(t)e(t) \frac{1}{e^T(t)S(e(t))} + \alpha_1 G_e^T(e(t))G_e(e(t))e^T(t)S(e(t)) \\
&\le \frac{1}{\alpha_1} e^T(t)S(e(t)) + 4\alpha_1\omega^2 M e^T(t)S(e(t)).
\end{aligned}$$

When $|e(t)| = 0$, one can obtain that

$$0 = 2e^T(t)G_e(e(t)) = \frac{1}{\alpha_1} e^T(t)S(e(t)) + 4\alpha_1\omega^2 M e^T(t)S(e(t)).$$

According to the definition of α in Theorem 2, it yields that

$$D^+H(t) \leq -\alpha_2 e^T(t)S(e(t)) \leq -\alpha_2 H^{\frac{1}{2}}(t).$$

The rest of the proof is similar to Theorem 1, and omitted here. The proof of Theorem 2 is completed. □

5. Examples

In this section, two examples are given to illustrate our presented results.

Example 1. In the framework of saturation structure, consider system (3) again. In section 2, it has been shown that for a prior given saturation level $\rho = 5$, when the initial state $x_0 = 8$, system (3) with impulse time sequence $\{k\}$ is not *FTS*. In fact, when the effect of saturation structure is fully considered, one may observe from simulations that the actual control input is saturated state all the time, see Figure 5. On the contrary, when there is no limitation of saturation structure on control input, system (3) can achieve *FTS* and the corresponding control input is shown in Figure 6. Rather, we guess that such two insights arise from the fact that the initial state is not in the domain of attraction, i.e., $x_0 \notin \mathcal{S}$. In the following, we firstly verify the conjecture and then study when $\rho = 5$, what constraints on impulse time sequence $\{t_k\}$ are required to guarantee *FTS* and obtain the estimation of settling-time. To this end, we consider the following system

$$\begin{cases} \dot{x}(t) = 0.5x(t) + \sin(x(t)) + \text{sat}_5(u(t)), \ \ t \neq t_k, \ t \geq 0, \\ x(t) = 1.3x(t^-), \ \ t = t_k, \ k \in \mathbb{Z}_+. \end{cases} \tag{19}$$

In view of (19), we know that $C = 0.5$, $A = 1$, $D = 0.3$, and $f(x) = \sin(x)$, which indicates that we can choose $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = 4$, $\omega = 1$, and $\beta = 1.3$ satisfying conditions in Theorem 1. It yields that $\alpha = 3$, $\epsilon = 4$, and the control input u can be designed by

$$u(t) = -0.5x(t) - 3\text{sgn}(x(t)). \tag{20}$$

Then system (19) via saturated control input (20) is *locally FTS* with $\mathcal{S}_4$ over the class $\mathscr{F}^\star$, where $\mathcal{S}_4$ is contained in the domain of attraction and $\mathscr{F}^\star$ denotes a class of impulse time sequences $\{t_k\} \in \mathscr{F}_0$ satisfying

$$N := \min\left\{k \in \mathbb{Z}_+ : t_k \geq 1.3^{k-3.6419} \cdot |x_0|\right\} < \infty, \tag{21a}$$

$$t_k > 1.3^{k-3.6419} \cdot |x_0| - 1.5385, \ \ \forall k \leq N-1. \tag{21b}$$

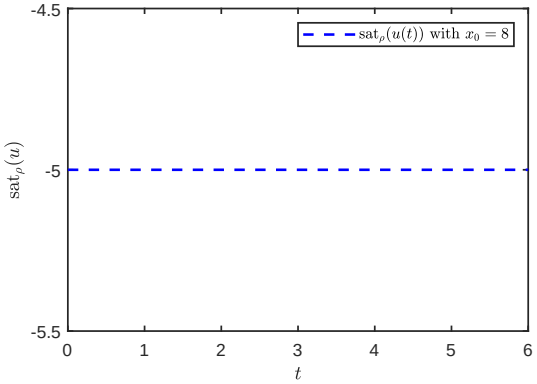

Figure 5. Saturated control input $\text{sat}(u(t))$ with $x_0 = 8$.

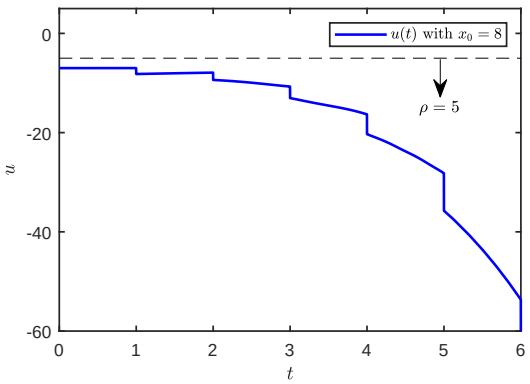

Figure 6. Control input $u(t)$ with $x_0 = 8$.

In addition, the settling-time of system (4) though $(0, x_0)$ can be bounded by

$$T_{\inf}(x_0, \{t_k\}) \leq 1.3^{N-3.6419} \cdot |x_0|.$$

It is easy to see that $8 = x_0 \notin \mathcal{S}_4$, which confirms our conjecture. In simulations, we choose the initial state $x_0^* = 4$ and consider $\{t_k\} = \{0.5, 1.2, 1.85, 4.4, \cdots\} \in \mathscr{F}^\star$ with $N = 4$ satisfying both (21a) and (21b). Figure 7 blue curve shows the trajectory of system (19) with $\{t_k\}$, which indicates that system (19) via saturated control input (20) is *locally FTS* with $\{t_k\}$ and the settling-time can be bounded by $T_{\inf}(4, \{t_k\}) \leq 4.394$. Under same conditions, if constraint (21b) is ignored, such as $\{t_k\}^\star = \{0.05, 0.11, 0.16, 4.4, \cdots\}$ only satisfying (21a) with $N = 4$ but not satisfying (21b), then one can observe that due to the effect of impulsive disturbance, the trajectory of system (19) would exceed the domain of attraction $\mathcal{S}_4$ at impulse instant $t_1 = 0.05$ and after that (see red curve in Figure 7) and moreover, the control input exceeds the saturation level, i.e., $u \leq -5$ (see Figure 8). It leads to the fact that system (19) via saturated control input (20) with $\{t_k\}^\star$ is not *FTS*, which shows the necessity of constraint (21b).

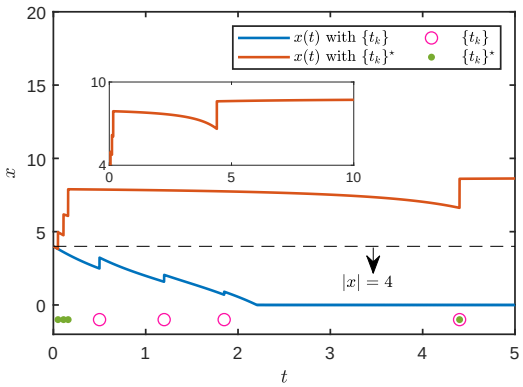

Figure 7. System (19) with different impulse time sequences.

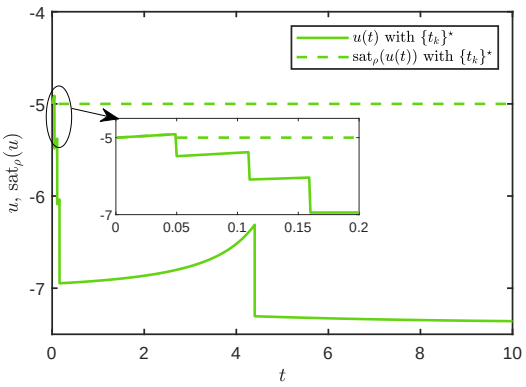

Figure 8. (Saturated) Control input with $\{t_k\}^\star$.

Example 2. Let us consider the *finite-time synchronization* of impulsive dynamical networks (14) in the framework of saturation structure, where impulsive dynamical networks (14) consists of three identical nodes (i.e., $M = 3$) and the corresponding parameters are given by $\sigma = 0.2$, $\rho = 10.5$, $D_1 = 0.3I_3$, $D_2 = 0.2I_3$, $D_3 = 0.15I_3$,

$$L = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0.375 & 0 & 0 \\ 0 & 0.125 & 0 \\ 0 & 0 & 0.25 \end{pmatrix},$$

and $g_i(s) = \tanh(s)$, $\forall s \in \mathbb{R}$, $i = 1, 2, 3$. It is easy to check that $\omega = \sqrt{3}$ such that the activation function g satisfies bounded. Then choose $\beta = 1.3$, $\alpha_1 = 1/6$, and $\alpha_2 = 8$ such that conditions in Theorem 2 hold, which yields that $\alpha = 0.5(1/\alpha_1 + 4\alpha_1\omega^2 M + \alpha_2) = 10 < 10.5 = \rho$ and

$$\epsilon = \frac{\rho - \alpha}{\sigma \max_{j\in\Omega} \gamma_j \cdot \max_{i\in\mathcal{M}} |L_i|} = 2.7212.$$

According to Theorem 2, one can obtain that impulsive dynamical networks (14) via saturated control input (18) and system (16) can achieve *locally finite-time synchronization* with $\mathcal{S}_{2.7212}$ over the class $\mathscr{F}^\star$, where $\mathcal{S}_{2.7212}$ is contained in the domain of attraction and $\mathscr{F}^\star$ denotes a class of impulse time sequences $\{t_k\} \in \mathscr{F}_0$ satisfying

$$N := \min\left\{k \in \mathbb{Z}_+ : t_k \geq 1.3^{k-6.2839} \cdot |e_0|\right\} < \infty, \tag{22a}$$

$$t_k > 1.3^{k-6.2839} \cdot |e_0| - 0.5234, \quad \forall k \leq N - 1. \tag{22b}$$

Moreover, the settling-time can be bounded by $T_{\inf}(e_0, \{t_k\}) \leq 1.3^{N-6.2839} \cdot |e_0|$. In simulations, let $N = 3$ and $\{t_k\} = \{0.01, 0.04, 0.73, \cdots\} \in \mathscr{F}^\star$ satisfying (22a) and (22b). Choose $e_0 = (e_{01}^T, e_{02}^T, e_{03}^T)^T$ with $e_{01} = (-0.4, 0.2, -0.6)^T$, $e_{02} = (0.5, -1, 0.2)^T$, and $e_{03} = (-0.2, 1, 0.3)^T$, where e_{0i} is the initial state of i-th node of error system (17), $i = 1, 2, 3$. It is obvious that $|e_0| = 1.7263 < 2.7212$, i.e., $e_0 \in \mathcal{S}_{2.7212}$. When there is no control input, impulsive dynamical networks (14) cannot achieve *finite-time synchronization*, see Figure 9. While involving saturated control input (18), the trajectories of three nodes of impulsive dynamical networks (14) are shown in Figure 10. One may observe from Figure 10 that impulsive disturbance occurs twice at instants $t_1 = 0.01$ and $t_2 = 0.04$, and moreover, impulsive dynamical networks (14) can achieve *finite-time synchronization*.

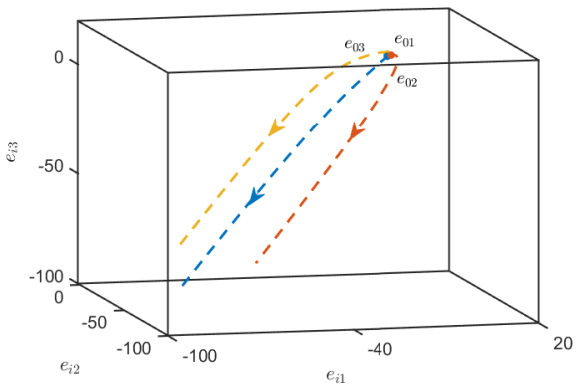

Figure 9. The trajectories of three nodes without control.

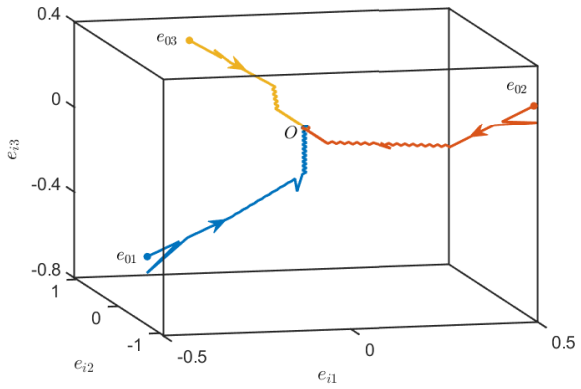

Figure 10. The trajectories of three nodes with control (18).

6. Conclusion

In this paper, in the framework of saturation structure, we presented some *FTS* criteria of nonlinear systems with impulsive disturbance. In order to achieve *FTS*, some constraints on impulse time sequence are proposed, based on which a potential relationship between saturation structure and impulsive disturbance is established. Moreover, the corresponding estimations of the domain of attraction and the settling-time are derived. In terms of the basic idea of dealing with the dual influence of saturation structure and impulsive disturbance, the future work focuses on finding another way to further expand the estimation of the domain of attraction and relax the constraints on the impulse time sequence to further improve our presented results.

References

[1] Vangipuram Lakshmikantham, Pavel S Simeonov, et al. *Theory of Impulsive Differential Equations*, volume 6. World scientific, 1989.

[2] Tao Yang. *Impulsive Control Theory*, volume 272. NewYork, USA: Springer, 2001.

[3] Jianquan Lu, Daniel WC Ho, and Jinde Cao. A unified synchronization criterion for impulsive dynamical networks. *Automatica*, 46(7):1215–1221, 2010.

[4] Ivanka Stamova and Gani T Stamov. *Applied Impulsive Mathematical Models*, volume 318. Springer, 2016.

[5] Bin Liu, David J Hill, and Zhijie Sun. Stabilisation to input-to-state stability for continuous-time dynamical systems via event-triggered impulsive control with three levels of events. *IET Control Theory & Applications*, 12(9):1167–1179, 2018.

[6] M. Syed Ali, R. Vadivel, and O.M. Kwon. Decentralised event-triggered impulsive synchronisation for semi-markovian jump delayed neural networks with leakage delay and randomly occurring uncertainties. *International Journal of Systems Science*, 50(8):1636–1660, 2019.

[7] Shuwen Zhang and Dejun Tan. Dynamics of a stochastic predator–prey system in a polluted environment with pulse toxicant input and impulsive perturbations. *Applied Mathematical Modelling*, 39(20):6319–6331, 2015.

[8] Yu Zhao, Sanling Yuan, and Qimin Zhang. The effect of lévy noise on the survival of a stochastic competitive model in an impulsive polluted environment. *Applied Mathematical Modelling*, 40(17-18):7583–7600, 2016.

[9] Xiangxiang Wang, Yongbin Yu, Shouming Zhong, Kaibo Shi, Nijing Yang, Dingfa Zhang, Jingye Cai, and Nyima Tashi. Novel heterogeneous mode-dependent impulsive synchronization for piecewise ts fuzzy probabilistic coupled delayed neural networks. *IEEE Transactions on Fuzzy Systems*, doi: 10.1109/TFUZZ.2021.3076525, 2021.

[10] Sergey Dashkovskiy, Michael Kosmykov, Andrii Mironchenko, and Lars Naujok. Stability of interconnected impulsive systems with and without time delays, using lyapunov methods. *Nonlinear Analysis: Hybrid Systems*, 6(3):899–915, 2012.

[11] Xinzhi Liu and Kexue Zhang. Synchronization of linear dynamical networks on time scales: Pinning control via delayed impulses. *Automatica*, 72:147–152, 2016.
[12] Xiaodi Li and Shiji Song. *Impulsive Systems with Delays: Stability and Control*. Springer, 2022.
[13] Deqiang Ouyang, Jie Shao, Haijun Jiang, Sing Kiong Nguang, and Heng Tao Shen. Impulsive synchronization of coupled delayed neural networks with actuator saturation and its application to image encryption. *Neural Networks*, 128:158–171, 2020.
[14] Liruo Zhang, Sing Kiong Nguang, Deqiang Ouyang, and Shen Yan. Synchronization of delayed neural networks via integral-based event-triggered scheme. *IEEE Transactions on Neural Networks and Learning Systems*, 31(12):5092–5102, 2020.
[15] Yang Li, Kwok Wo Wong, Xiaofeng Liao, and Chuandong Li. On impulsive control for synchronization and its application to the nuclear spin generator system. *Nonlinear Analysis: Real World Applications*, 10(3):1712–1716, 2009.
[16] Xiao Cai, Kaibo Shi, Shouming Zhong, Jun Wang, and Yiqian Tang. Dissipative analysis for high speed train systems via looped-functional and relaxed condition methods. *Applied Mathematical Modelling*, 96:570–583, 2021.
[17] Wangli He, Feng Qian, Qing-Long Han, and Guanrong Chen. Almost sure stability of nonlinear systems under random and impulsive sequential attacks. *IEEE Transactions on Automatic Control*, 65(9):3879–3886, 2020.
[18] Yuqian Guo, Yawen Shen, and Weihua Gui. Asymptotical stability of logic dynamical systems with random impulsive disturbances. *IEEE Transactions on Automatic Control*, 66(2):513–525, 2020.
[19] Huamin Wang, Guoliang Wei, Shiping Wen, and Tingwen Huang. Impulsive disturbance on stability analysis of delayed quaternion-valued neural networks. *Applied Mathematics and Computation*, 390:125680, 2021.
[20] Jinsen Zhang, Wu-Hua Chen, and Xiaomei Lu. Robust fuzzy stabilization of nonlinear time-delay systems subject to impulsive perturbations. *Communications in Nonlinear Science and Numerical Simulation*, 80:104953, 2020.
[21] Xinsong Yang, James Lam, Daniel WC Ho, and Zhiguo Feng. Fixed-time synchronization of complex networks with impulsive effects via nonchattering control. *IEEE Transactions on Automatic Control*, 62(11):5511–5521, 2017.
[22] Jingting Hu, Guixia Sui, Xiaoxiao Lu, and Xiaodi Li. Fixed-time control of delayed neural networks with impulsive perturbations. *Nonlinear Analysis: Modelling and Control*, 23(6):904–920, 2018.
[23] Sergey G Nersesov and Wassim M Haddad. Finite-time stabilization of nonlinear impulsive dynamical systems. *Nonlinear Analysis: Hybrid Systems*, 2(3):832–845, 2008.
[24] Xiaodi Li, Daniel WC Ho, and Jinde Cao. Finite-time stability and settling-time estimation of nonlinear impulsive systems. *Automatica*, 99:361–368, 2019.
[25] Jie Wu, Xiaodi Li, and Xiang Xie. Finite-time stability for time-varying nonlinear impulsive systems. *Mathematical Methods in the Applied Sciences*, doi: 10.1002/mma.7573, 2021.
[26] Xin Wang, Ju H Park, Huilan Yang, and Shouming Zhong. A new settling-time estimation protocol to finite-time synchronization of impulsive memristor-based neural networks. *IEEE Transactions on Cybernetics*, doi: 10.1109/TCYB.2020.3025932, 2020.
[27] Dan Yang, Xiaodi Li, and Shiji Song. Finite-time synchronization for delayed complex dynamical networks with synchronizing or desynchronizing impulses. *IEEE Transactions on Neural Networks and Learning Systems*, 33(2):736–746, 2022.
[28] AT Fuller. In-the-large stability of relay and saturating control systems with linear controllers. *International Journal of Control*, 10(4):457–480, 1969.
[29] Tingshu Hu and Zongli Lin. *Control Systems with Actuator Saturation: Analysis and Design*. Springer Science & Business Media, 2001.
[30] Bin Zhou, Wei Xing Zheng, and Guang-Ren Duan. An improved treatment of saturation nonlinearity with its application to control of systems subject to nested saturation. *Automatica*, 47(2):306–315, 2011.
[31] Haitao Zhu, Xiaodi Li, and Shiji Song. Input-to-state stability of nonlinear impulsive systems subjects to actuator saturation and external disturbance. *IEEE Transactions on Cybernetics*, doi:10.1109/TCYB.2021.3090803, 2021.
[32] Chenhong Zhu, Xiaodi Li, and Kening Wang. An anti-windup approach for nonlinear impulsive system subject to actuator saturation. *Chaos, Solitons & Fractals*, 133:109658, 2020.
[33] JM Gomes Da Silva and Sophie Tarbouriech. Antiwindup design with guaranteed regions of stability: an LMI-based approach. *IEEE Transactions on Automatic Control*, 50(1):106–111, 2005.
[34] Wassim M Haddad and Andrea L'Afflitto. Finite-time stabilization and optimal feedback control. *IEEE Transactions on Automatic Control*, 61(4):1069–1074, 2016.