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# FROBENIUS STRUCTURES IN STAR-AUTONOMOUS CATEGORIES \*

CÉDRIC DE LACROIX AND LUIGI SANTOCANALE

ABSTRACT. It is known that the quantale of sup-preserving maps from a complete lattice to itself is a Frobenius quantale if and only if the lattice is completely distributive. Since completely distributive lattices are the nuclear objects in the autonomous category of complete lattices and sup-preserving maps, we study the above statement in a categorical setting. We introduce the notion of Frobenius structure in an arbitrary autonomous category, generalizing that of Frobenius quantale. We prove that the monoid of endomorphisms of a nuclear object has a Frobenius structure. If the environment category is star-autonomous and has epi-mono factorizations, a variant of this theorem allows to develop an abstract phase semantics and to generalise the previous statement. Conversely, we argue that, in a star-autonomous category where the monoidal unit is a dualizing object, if the monoid of endomorphisms of an object has a Frobenius structure and the monoidal unit embeds into this object as a retract, then the object is nuclear.

**Keywords.** Quantale, Frobenius quantale, Girard quantale, associative algebra, star-autonomous category, nuclear object, adjoint.

## 1. INTRODUCTION

**Context.** A major motivation for giving birth to linear logic was to restore a classical negation in a constructive setting. This was achieved on the proof-theoretic side, by controlling the structural rules of weakening and contraction. As a byproduct, the logical connectives of conjunction and disjunction have been split into their multiplicative and additive versions, and the additive connectives no longer have a classical behaviour, that is, they no longer distribute over each other.

We might tackle the same problem—restoring a classical negation in a constructive setting—using algebraic and model theoretic approaches and considering variants of linear logic. A standard complete provability semantics of (possibly non-commutative) linear logic is on the class of structures known as quantales, see *e.g.* [39]. A quantale  $(Q, *)$  is a complete lattice with a multiplication which distributes over sups in both variables (*i.e.*, it a semigroup in the category  $\mathbf{SLatt}$  of complete lattices and join-preserving maps). Restoring classical negation means, in this context, considering Frobenius or Girard quantales. A quantale is Frobenius if it also comes with two suitable antitone maps  ${}^\perp(-)$  and  $(-)^\perp$  representing negations, and it is Girard if these two maps coincide. Among quantales, probably the most interesting are those on the set  $[L, L]$  of sup-preserving endomaps of a

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complete lattice  $L$ , with multiplication given by composition and the ordering computed pointwise. These quantales naturally arise in logic, computer science, and quantum mechanics. For example, for a set  $X$  and its powerset lattice  $P(X)$ , the quantale  $[P(X), P(X)]$  is isomorphic to the quantale  $P(X \times X)$  of relations on  $X$  and it is the main object of study in relation algebra [28]. Use of these quantales to model concurrency and quantum mechanics/computation abound in the literature, see *e.g.* [3, 4, 30]. Our interest for quantales of the form  $[L, L]$  stems from generalized linear orders [18, 33, 19, 36], which are sort of fuzzy relations whose set of truth values is a quantale. These relations need to satisfy constraints which are expressed using the negation, hence the quantale must be a Frobenius quantale. While developing a theory of these structures we came across (and contributed to) the following result:

**Theorem 1** (See [25, 13, 15, 35, 34]). *The quantale  $[L, L]$  of sup-preserving endomaps of a complete lattice  $L$  is a Frobenius quantale if and only if  $L$  is completely distributive.*

Completely distributivity is an infinitary generalization of the usual distributive law. Basic examples of completely distributive lattices are the complete linear orders, the powerset lattices and, of course, the finite distributive lattices. As logicians, the above theorem struck us, since it shows that a model theoretic approach to restoring a classical negation fundamentally diverges from the proof-theoretic one. Indeed, the theorem has the following reading: *if a classical negation is enforced on the most standard class of models of non-commutative intuitionistic linear logic (the sup-preserving endomaps of a complete lattice), then also the additive connectives of the logic (the suprema and infima of the lattice) have a classical behaviour (that is, they distribute over each other).*

**Contribution.** Due to its strength, the above statement deserves to be studied from the largest number of perspectives. We take here a categorical approach and give a proof of the statement that relies on the  $*$ -autonomous structure of  $\mathbf{SLatt}$ , the category of complete lattices and sup-preserving maps. In doing so, we generalise the statement to  $*$ -autonomous categories. This is possible due to a fundamental characterization of complete distributivity in the categorical language, stating that *the completely distributive lattices are exactly the nuclear objects in  $\mathbf{SLatt}$* , see [29, 21]. We recall that an object  $A$  of a symmetric monoidal closed category  $\mathcal{V} = (V, I, \otimes, \alpha, \lambda, \rho, [-, -], ev)$  is *nuclear* if the canonical map  $\text{mix} : A^* \otimes A \longrightarrow [A, A]$  is an isomorphism, where  $A^*$ , the dual of  $A$ , is the internal hom  $[A, I]$  (see *e.g.* [32]).

To achieve our goal, we define Frobenius structures in a symmetric monoidal closed category by strictly mimicking the definition of (possibly unitless) Frobenius quantales given in [26]. Definitions of similar structures appear in the literature [24, 22, 38, 14] and differ from our mainly w.r.t. the environment category and the presence of units. We consider the choice of an axiomatization only as a tool for our goal of generalizing Theorem 1 to a categorical setting. We insist, however, that the theorem is about characterizing when a canonical quantale on the tensor product  $L^* \otimes L$  of an object  $L$  in  $\mathbf{SLatt}$  has a unit. To define Frobenius structures, we use the notion of *dual pairing* in a monoidal category  $\mathcal{V}$ . This is a map  $\epsilon : A \otimes B \longrightarrow I$  with two given universal properties. It turns out that in a  $*$ -autonomous category such a dual pairing exists if and only if  $B$  is isomorphic to  $A^*$ , if and only if  $A$  is

isomorphic to  $B^*$ . This notion provides the framework by which to study objects that are dual to each other only up to isomorphism: for example  $(A^* \otimes A, [A, A])$  is part of a dual pairing in any  $*$ -autonomous category and, for any complete lattice  $L$ ,  $(L, L^{op})$  is part of a dual pairing in  $\mathbf{SLatt}$ . If  $\epsilon : A \otimes B \longrightarrow I$  is a dual pairing and  $A$  is a semigroup, then  $A$  acts on  $B$  on the left and on the right. The left and right actions, noted  $\alpha_A^\ell$  and  $\alpha_A^\rho$ , correspond, in the category  $\mathbf{SLatt}$ , to the two implications of a quantale. We define generalized Frobenius quantales in arbitrary symmetric monoidal category as follows:

**Definition.** A *Frobenius structure* is a tuple  $(A, B, \epsilon, \mu_A, l, r)$  where  $(A, B, \epsilon)$  is a dual pairing,  $(A, \mu_A)$  is a semigroup,  $l$  and  $r$  are invertible maps from  $A$  to  $B$  such that

$$\epsilon \circ (A \otimes l) = \epsilon \circ (A \otimes r) \circ \sigma_{A,A} \quad \text{and} \quad \alpha_A^\ell \circ (A \otimes r) = \alpha_A^\rho \circ (l \otimes A). \quad (1)$$

Almost by definition, in  $\mathbf{SLatt}$ , a Frobenius structure of the form  $(Q, Q^{op}, \epsilon, *, \perp(-), (-)^\perp)$  amounts to a Frobenius quantale, as defined in [26], that is a quantale  $(Q, *)$  coming with antitone negations satisfying

$$x \leq \perp y \text{ iff } y \leq x^\perp \quad \text{and} \quad y^\perp/x = y \setminus^\perp x.$$

We prove then the following result, generalizing the direct implication of Theorem 1.

**Theorem A.** *If  $A$  is nuclear, then there is a map  $l$  such that  $([A, A], [A, A]^*, ev_{[A,A],I}, \circ, l, l)$  is a Frobenius structure.*

The proof of this theorem is almost straightforward from the definition of Frobenius structure. Let us mention that Theorem A is strictly related to (and may be considered an instance of) Corollary 3.3 in [38]. The connection, however, depends on the fact that adjoints in a symmetric monoidal closed category, or dualisable objects, are exactly the nuclear objects, as argued in Section 6. The statement can be further generalised: if  $\mathbf{mix}$  is not invertible but the underlying  $*$ -autonomous category has some nice factorization system, then the image of  $\mathbf{mix}$  is the support of a Frobenius structure. In  $\mathbf{SLatt}$ , this construction yields the Girard quantale of tight endomaps of a complete lattice studied in [26]. The generalization is a consequence of a double negation construction that we described in [26] for Frobenius quantales, and that we generalise here to a categorical setting. The statement sounds as follows:

**Theorem B.** *Let  $\mathcal{V}$  be a  $*$ -autonomous category with an epi-mono factorization system. Let  $(A, \mu_A)$  be a semigroup in  $\mathcal{V}$  and let  $\epsilon : A \otimes B \longrightarrow I$  be a dual pairing. If  $f : A \longrightarrow B$  is a map such that  $\epsilon \circ (A \otimes f) = \epsilon \circ (A \otimes f) \circ \sigma_{A,A}$ , then the image of  $f$  is the carrier of a Frobenius structure.*

We also demonstrate that the converse of Theorem A actually holds if we add another condition, which we identify now as a key ingredient of the proof in [34] of Theorem 1. We say that an object  $A$  of a monoidal category is *pseudo-affine* if the tensor unit  $I$  embeds into  $A$  as a retract. For example, every complete lattice which is not a singleton is pseudo-affine in  $\mathbf{SLatt}$ . We prove:

**Theorem C.** *If  $A$  is a pseudo-affine object of a  $*$ -autonomous category and the canonical monoid  $([A, A], \circ)$  is part of a Frobenius structure, then  $A$  is nuclear.*

**Related Work.** Besides the works that we already mentioned, either those on the theory of quantales [25, 13, 15] or those generalizing the notion of Frobenius quantale to some kind monoidal setting [24, 22, 38, 14], we also wish to mention that the notion of nuclearity, originally conceived for Banach spaces [20, 32], has been by now generalised in several directions [1, 7]. For example, this notion is generalised in [1] to the category of Hilbert spaces, with motivations from quantum mechanics. While our initial motivations for developing this research were of a purely logical and order-theoretic nature, we could recognize, via the formalization in a categorical language and the notion of nuclearity, the similarity of our questionings with those arising in this line of research. In particular, we could construct in [26] a unitless Frobenius quantale from trace class operators (*i.e.*, nuclear endomaps) of an infinite dimensional Hilbert space. We noticed that the questioning about unitless structures is pervasive in this line of research [2, 17, 12]. We describe in the last section our first non-conclusive remarks on the scope of our results within this family of monoidal categories. We are convinced that connecting with existing research on nuclearity and categorical quantum mechanics is a research direction worth to be fully explored.

**Structure of the paper.** After introducing elementary notions in Section 2, we study the notion of dual pairing in Section 3 and give an overview of actions in dual pairs in Section 4. This makes it possible to introduce Frobenius structures in Section 5. We then state in Section 6 the equivalence between the notion of adjoint and that of nuclear object in autonomous categories. We finally prove Theorems A and B in Section 7, and Theorem C in Section 8. We add a discussion of the results obtained and sketch future research in Section 9.

## 2. BACKGROUND

Let us recall that a *quantale* is a semigroup in the  $*$ -autonomous category  $\mathbf{SLatt}$ , see [15]. Otherwise said, it is a pair  $(Q, *)$  with  $Q$  a complete lattice and  $*$  a semigroup operation which distributes in each place with suprema. An essential feature of a quantale  $(Q, *)$  are the two implication maps  $(-\backslash-) : Q \otimes Q^{\text{op}} \longrightarrow Q^{\text{op}}$  and  $(-/ -) : Q^{\text{op}} \otimes Q \longrightarrow Q^{\text{op}}$  satisfying the adjointness relations

$$x * y \leq z \text{ iff } x \leq z/y \text{ iff } y \leq x \backslash z.$$

We rely on the standard monograph [27] for elementary facts and notation concerning categories. We shall say that a category is *autonomous* if it is symmetric monoidal closed. Thus, an autonomous category is of the form  $\mathcal{V} = (\mathbf{V}, I, \otimes, \alpha, \lambda, \rho, \sigma, [-, -], ev)$ , where  $\mathbf{V}$  is a category,  $\otimes : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbf{V}$  is a bifunctor,  $[-, Z] : \mathbf{V}^{\text{op}} \longrightarrow \mathbf{V}$  is the right adjoint to  $(X \otimes -)$ ,  $ev$  is the counit of this adjunction and  $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ ,  $\rho_X : X \otimes I \cong X$ ,  $\lambda_X : I \otimes X \cong X$ ,  $\sigma_{X,Y} : X \otimes Y \cong Y \otimes X$  are all natural isomorphisms satisfying well-known constraints. We shall work with strict monoidal categories  $\mathcal{V}$ . If  $\mathcal{V}$  is autonomous and  $0$  is a fixed object, then we get a contravariant functor  $(-)^* := [-, 0]$ . We define the map  $j_X : X \longrightarrow X^{**}$  as the transpose of  $ev_{X,0} \circ \sigma_{X^*,X}$ . We say an object is *reflexive* if  $j_X$  is an isomorphism. If every object is reflexive, then  $0$  is said to be dualizing and  $\mathcal{V}$  is said to be a *\*-autonomous category*. If  $0 = I$  then, we can define the morphism  $\text{mix}_{X,Y} : Y^* \otimes X \longrightarrow [Y, X]$  as the transpose of  $\lambda_X \circ ev_{Y,I}$ . An object  $X$  is *nuclear* if  $\text{mix}_{X,X}$  is invertible. A nuclear object is necessarily reflexive.

**Definition 2.** A *magma* in a monoidal category  $\mathcal{V}$  is a pair  $(A, \mu_A)$  with  $\mu_A : A \otimes A \longrightarrow A$  a morphism in  $\mathcal{V}$ . A *bracketed magma* in  $\mathcal{V}$  is a triple  $(A, \mu_A, \pi_A)$  with  $(A, \mu_A)$  a magma and  $\pi_A : A \otimes A \longrightarrow I$ . A bracketed magma is *associative* if  $(A, \mu_A)$  is a semigroup in  $\mathcal{V}$  (that is, if  $\mu_A$  is associative) and  $\pi_A$  is associative, meaning that the following two horizontal arrows are equal:

$$A \otimes A \otimes A \begin{array}{c} \xrightarrow{\mu_A \otimes A} \\ \xrightarrow{A \otimes \mu_A} \end{array} A \otimes A \xrightarrow{\pi_A} I.$$

If  $\mathcal{V}$  is symmetric monoidal and  $(A, \mu_A)$  is a magma, a pairing  $\pi_A$  is said to be *co-associative* if  $(A, \mu_A \circ \sigma_{A,A}, \pi_A)$  is an associative bracketed magma.

**Fact 3.** A pairing  $\pi_A$  is associative if and only if the pairing  $\pi_A \circ \sigma_{A,A}$  is co-associative.

*Proof.* We show that if  $\pi_A$  is associative, then  $\pi_A \circ \sigma_{A,A}$  is co-associative.

$$\begin{aligned} \pi_A \circ \sigma_{A,A} \circ ((\mu_A \circ \sigma_{A,A}) \otimes A) &= \pi_A \circ (A \otimes (\mu_A \circ \sigma_{A,A})) \circ \sigma_{A \otimes A, A} \\ &= \pi_A \circ (A \otimes \mu_A) \circ (A \otimes \sigma_{A,A}) \circ \sigma_{A \otimes A, A} \\ \pi_A \circ \sigma_{A,A} \circ (A \otimes (\mu_A \circ \sigma_{A,A})) &= \pi_A \circ ((\mu_A \circ \sigma_{A,A}) \otimes A) \circ \sigma_{A, A \otimes A} \\ &= \pi_A \circ (\mu_A \otimes A) \circ (\sigma_{A,A} \otimes A) \circ \sigma_{A, A \otimes A}, \end{aligned}$$

thus the statement follows from the equality

$$(A \otimes \sigma_{A,A}) \circ \sigma_{A \otimes A, A} = (\sigma_{A,A} \otimes A) \circ \sigma_{A, A \otimes A}$$

and associativity of  $\pi_A$ .  $\square$

Notice that if  $(A, \mu_A)$  is a semigroup and  $\text{tr} : A \longrightarrow I$  is any arrow, then  $\text{tr} \circ \mu_A : A \otimes A \longrightarrow I$  is associative.

**Example 4.** In every autonomous category,  $([A, A], \circ)$  is a well-known semigroup where  $\circ$  is the internal composition defined as the transpose of  $ev_{A,A} \circ (ev_{A,A} \otimes [A, A])$ . Also, if  $0 = I$ , then  $(A^* \otimes A, \mu_{A^* \otimes A}, \pi_{A^* \otimes A})$  is an associative bracketed magma with  $\mu_{A^* \otimes A} := (A^* \otimes \lambda_A) \circ (A^* \otimes ev_{A,I} \otimes A)$  and  $\pi_{A^* \otimes A} := ev_{A,I} \circ \sigma_{A^*, A} \circ \mu_{A^* \otimes A}$ , see e.g. [25].

**Definition 5.** Let  $(A, \mu_A)$  be magma. A map  $\alpha^\ell : A \otimes X \longrightarrow X$  is a *left action* if the two parallel arrows

$$A \otimes A \otimes X \begin{array}{c} \xrightarrow{\mu_A \otimes X} \\ \xrightarrow{A \otimes \alpha^\ell} \end{array} A \otimes X \xrightarrow{\alpha^\ell} X$$

are equal. A map  $\alpha^r : X \otimes A \longrightarrow X$  is defined to be a *right action* in a similar way.

**Definition 6.** Let  $(A, \mu_A), (B, \mu_B)$  be two magmas. A *magma homomorphism* from  $(A, \mu_A)$  to  $(B, \mu_B)$  is a map  $f : A \longrightarrow B$  making the diagram below on the left commutative. If  $(A, \mu_A, \pi_A)$  and  $(B, \mu_B, \pi_B)$  are bracketed magmas, a *bracketed magma homomorphism* from  $(A, \mu_A, \pi_A)$  to  $(B, \mu_B, \pi_B)$  is a magma homomorphism  $f : (A, \mu_A) \longrightarrow (B, \mu_B)$  making the diagram below in the middle commutative.

$$\begin{array}{ccc} A \otimes A \xrightarrow{f \otimes f} B \otimes B & A \otimes A \xrightarrow{f \otimes f} B \otimes B & A \xrightarrow{\pi_A^\#} A^* \\ \downarrow \mu_A & \searrow \pi_A \quad \downarrow \pi_B & \downarrow f \quad f^* \uparrow \\ A \xrightarrow{f} B & I & B \xrightarrow{\pi_B^\#} B^* \end{array}$$

Let us notice that, when the environment category is autonomous, the diagram above in the middle commutes if and only if the diagram on the right does.

**Fact 7.** The arrow  $\text{mix} : A^* \otimes A \longrightarrow [A, A]$  is a semigroup homomorphism.

*Proof.* We must check the commutativity of

$$\begin{array}{ccc} A^* \otimes A \otimes A^* \otimes A & \xrightarrow{\text{mix} \otimes \text{mix}} & [A, A] \otimes [A, A] \\ \downarrow \mu_{A^* \otimes A} & & \downarrow \circ \\ A^* \otimes A & \xrightarrow{\text{mix}} & [A, A]. \end{array}$$

Transposing and developing this diagram, we obtain

$$\begin{array}{ccc} A \otimes A^* \otimes A \otimes A^* \otimes A & \xrightarrow{ev_{A,I} \otimes A \otimes A^* \otimes A} & I \otimes A \otimes A^* \otimes A \\ \downarrow A \otimes A^* \otimes ev_{A,I} \otimes A & & \downarrow \lambda_A \otimes A^* \otimes A \\ A \otimes A^* \otimes I \otimes A & & A \otimes A^* \otimes A \\ \downarrow A \otimes \lambda_A \otimes A & & \downarrow ev_{A,I} \otimes A \\ A \otimes A^* \otimes A & & I \otimes A \\ \downarrow ev_{A,I} \otimes A & & \downarrow \lambda_A \\ I \otimes A & \xrightarrow{\lambda_A} & A \end{array}$$

which reduces to the commutativity of

$$\begin{array}{ccc} A \otimes A^* \otimes A \otimes A^* & \xrightarrow{ev_{A,I} \otimes A \otimes A^*} & I \otimes A \otimes A^* \\ \downarrow A \otimes A^* \otimes ev_{A,I} & & \downarrow \lambda_A \otimes A^* \\ A \otimes A^* \otimes I & & A \otimes A^* \\ \downarrow A \otimes \lambda_{A^*} & & \downarrow ev_{A,I} \\ A \otimes A^* & \xrightarrow{ev_{A,I}} & I. \end{array}$$

□

**Lemma 8.** Let  $\mathcal{V}$  be autonomous and let  $e : (A, \mu_A, \pi_A) \longrightarrow (B, \mu_B, \pi_B)$  be an epimorphism of bracketed magmas. If  $(A, \mu_A, \pi_A)$  is associative, then so is  $(B, \mu_B, \pi_B)$ .

*Proof.* Let us recall that since  $\mathcal{V}$  is biclosed, then  $X \otimes e$  (resp.  $e \otimes X$ ) is epi if  $e$  is epi. From this, it follows that an arrow which is a tensor of epic arrows is epi. We argue below that  $\pi_B$  is associative.

$$\begin{array}{ccccc}
 B \otimes B \otimes B & \xleftarrow{e \otimes e \otimes e} & A \otimes A \otimes A & \xrightarrow{e \otimes e \otimes e} & B \otimes B \otimes B \\
 \mu_B \otimes B \downarrow & & \mu_A \otimes A \downarrow & & B \otimes \mu_B \downarrow \\
 B \otimes B & \xleftarrow{e \otimes e} & A \otimes A & \xrightarrow{e \otimes e} & B \otimes B \\
 & \searrow \pi_B & \downarrow \pi_A & \swarrow \pi_B & \\
 & & I & & 
 \end{array}$$

Since the two vertical parallel arrows from  $A \otimes A \otimes A$  are equal and all the squares and triangles commute, the left leg of the above diagram equals its right leg. Since  $e \otimes e \otimes e$  is epi, we deduce that  $\pi_B \circ (\mu_B \otimes B) = \pi_B \circ (B \otimes \mu_B)$ .

The reader will immediately adapt this argument to show that  $\mu_B$  is associative.  $\square$

### 3. DUAL PAIRINGS

The goal of this section is to introduce the technical notion of dual pairing, needed later to define Frobenius structures. We also present a few properties of dual pairings. Throughout the section we let  $0$  be a fixed object of an environment monoidal category  $\mathcal{V}$ .

**Definition 9.** A map  $\epsilon : A \otimes B \longrightarrow 0$  in  $\mathcal{V}$  is said to be a *dual pairing* (w.r.t. the object  $0$ ) if the induced natural transformations

$$\mathrm{hom}(X, B) \longrightarrow \mathrm{hom}(A \otimes X, 0), \quad \mathrm{hom}(X, A) \longrightarrow \mathrm{hom}(X \otimes B, 0),$$

are isomorphisms.

Clearly, the above definition is equivalent to requiring that  $\epsilon$  has two universal properties: 1. *for each  $f : A \otimes X \longrightarrow 0$  there exists a unique map  $f^\sharp : X \longrightarrow B$  such that  $\epsilon \circ (A \otimes f^\sharp) = f$* , and 2. *for each  $g : X \otimes B \longrightarrow 0$  there exists a unique map  $g^\sharp : X \longrightarrow A$  such that  $\epsilon \circ (g^\sharp \otimes B) = g$* . We call  $f^\sharp$  (and  $g^\sharp$ ) the *transpose* of  $f$  (resp.  $g$ ). For  $f : X \longrightarrow B$  (resp.  $g : X \longrightarrow A$ ), we let  $f^\flat = \epsilon \circ (A \otimes f)$  (resp.  $g^\flat = \epsilon \circ (g \otimes B)$ ). If  $\epsilon : A \otimes B \longrightarrow 0$  is a dual pairing, then we shall also say that the triple  $(A, B, \epsilon)$  is a dual pairing, thus emphasizing the typing. We shall also informally say that  $(A, B)$  is a *dual pair*, leaving aside the arrow  $\epsilon$ . A dual pairing  $\epsilon : A \otimes B \longrightarrow 0$  is unique up to unique isomorphism, as stated next.

**Proposition 10.** *If  $(A, B_1, \epsilon_{A, B_1})$  is a dual pairing and  $\psi : B_0 \longrightarrow B_1$  is iso, then  $(A, B_0, \epsilon_{A, B_1} \circ (A \otimes \psi))$  is a dual pairing. Conversely, if  $(A, B_0, \epsilon_{A, B_0})$  and  $(A, B_1, \epsilon_{A, B_1})$  are two dual pairings, then there exists a unique iso  $\psi : B_0 \longrightarrow B_1$  such that  $\epsilon_{A, B_0} = \epsilon_{A, B_1} \circ (A \otimes \psi)$ .*

*Proof.* We have to check that  $\epsilon_{A, B_0} := \epsilon_{A, B_1} \circ (A \otimes \psi)$  is a dual pairing. Indeed, we have two natural isomorphisms

$$\begin{aligned}
 \mathrm{hom}(X, B_0) &\xrightarrow{\mathrm{hom}(X, \psi)} \mathrm{hom}(X, B_1) \xrightarrow{(-)^\flat} \mathrm{hom}(A \otimes X, 0), \\
 \mathrm{hom}(X, A) &\xrightarrow{(-)^\flat} \mathrm{hom}(X \otimes B_1, 0) \xrightarrow{\mathrm{hom}(X \otimes \psi, 0)} \mathrm{hom}(X \otimes B_0, 0),
 \end{aligned}$$

sending both  $\mathrm{id}_{B_0}$  and, respectively,  $\mathrm{id}_A$  to  $\epsilon_{A, B_0}$ .

Conversely, if  $(A, B_0, \epsilon_{A, B_0})$  and  $(A, B_1, \epsilon_{A, B_1})$  are two dual pairings, we have by definition two natural isomorphisms

$$\mathrm{hom}(X, B_0) \xrightarrow{(-)^\flat} \mathrm{hom}(A \otimes X, I) \xrightarrow{(-)^\sharp} \mathrm{hom}(X, B_1),$$

whose composition, using the Yoneda Lemma, is induced by post-composition with an invertible map  $\psi : B_0 \longrightarrow B_1$ . Taking the image of  $\text{id}_{B_0}$ , we see that  $\epsilon_{A,B_0}^\sharp = \psi$ , that is, we have  $\epsilon_{A,B_0} = \epsilon_{A,B_1} \circ (A \otimes \psi)$ .  $\square$

Obviously, similar statements also hold with respect to the object on the left, for example: if  $(A_0, B, \epsilon_{A_0,B})$  and  $(A_1, B, \epsilon_{A_1,B})$  are two dual pairings, then there exists a unique iso  $\chi : A_0 \longrightarrow A_1$  such that  $\epsilon_{A_0,B} = \epsilon_{A_1,B} \circ (\chi \otimes B)$ . While our presentation of the notion of dual pairing is symmetry-free, we shall work later within symmetric and autonomous categories. This imposes a few remarks.

**Lemma 11.** *If  $\mathcal{V}$  is symmetric and  $(A, B, \epsilon)$  is a dual pairing, then so is  $(B, A, \epsilon \circ \sigma_{B,A})$ .*

*Proof.* Clearly we have natural isomorphisms

$$\begin{aligned} \text{hom}(X, A) &\longrightarrow \text{hom}(X \otimes B, 0) \xrightarrow{\text{hom}(\sigma_{B,X}, 0)} \text{hom}(B \otimes X, 0), \\ \text{hom}(X, B) &\longrightarrow \text{hom}(A \otimes X, 0) \xrightarrow{\text{hom}(\sigma_{X,A}, 0)} \text{hom}(X \otimes A, 0), \end{aligned}$$

with universal arrows equal to  $\epsilon \circ \sigma_{B,A} : B \otimes A \longrightarrow 0$ .  $\square$

For  $\mathcal{V}$  an autonomous category and an arrow  $f : X \longrightarrow Y^*$  in  $\mathcal{V}$ , we let  $f^\perp : Y \longrightarrow X^*$  be the transpose of the map  $X \otimes Y \xrightarrow{\sigma_{X,Y}} Y \otimes X \xrightarrow{Y \otimes f} Y \otimes Y^* \xrightarrow{ev_{Y,0}} 0$ .

Notice that  $(-)^{\perp} : \text{hom}(X, Y^*) \longrightarrow \text{hom}(Y, X^*)$  is natural in  $X$  and  $Y$ . If  $g = f^\perp$ , then we say that  $f$  and  $g$  are *mates*. Let us remark that the canonical arrow  $j_X : X \longrightarrow X^{**}$  is the mate of  $\text{id}_{X^*}$ , from which the following statement easily follows:

**Lemma 12.** *For  $f : X \longrightarrow Y^*$ ,  $f^\perp = f^* \circ j_Y$ .*

*Proof.* Indeed, we have  $(f^* \circ j_Y)^\perp = (f^* \circ \text{id}_{Y^*}^\perp j_Y)^\perp = (\text{id}_{Y^*})^{\perp\perp} \circ f = \text{id}_{Y^*} \circ f = f$ .  $\square$

Mates allow to precisely characterize dual pairs in autonomous categories, as follows.

**Proposition 13.** *If  $\mathcal{V}$  is autonomous, then*

- (1)  *$(A, B)$  is a dual pair if and only if there are invertible mates  $\phi : A \rightarrow B^*$  and  $\psi : B \rightarrow A^*$ .*
- (2) *If  $(A, B)$  is a dual pair, then  $(A^*, B^*)$  is also a dual pair.*
- (3) *If  $(A, B)$  is a dual pair, then both  $A$  and  $B$  are reflexive objects of  $\mathcal{V}$ .*
- (4) *If  $A$  is reflexive, then  $(A, A^*, ev_{A,0})$  is a dual pairing.*
- (5)  *$(A, B)$  is a dual pair if and only if  $A$  is reflexive and there is some iso  $\psi : B \longrightarrow A^*$ .*

*Proof.* 1. If  $(A, B)$  is a dual pair, then we have natural isomorphisms

$$\text{hom}(X, B) \longrightarrow \text{hom}(A \otimes X, 0) \longrightarrow \text{hom}(X, A^*),$$

whose composition is induced, by the Yoneda Lemma, by post-composition with an invertible map  $\psi : B \longrightarrow A^*$ . Taking the image of  $\text{id}_B$ , we see that  $(\epsilon)^\sharp = \psi$ . Similarly, the chain of natural isomorphisms

$$\text{hom}(X, A) \longrightarrow \text{hom}(X \otimes B, 0) \longrightarrow \text{hom}(B \otimes X, 0) \longrightarrow \text{hom}(X, B^*)$$

yields an invertible map  $\phi : A \longrightarrow B^*$  satisfying  $(\epsilon \circ \sigma_{B,A})^\sharp = \phi$ . We have therefore  $\phi = \psi^\perp$ .

Conversely, given those isomorphisms  $\phi$  and  $\psi$ , we obtain the two natural isomorphisms as follows:

$$\begin{aligned} \text{hom}(X, B) &\longrightarrow \text{hom}(X, A^*) \longrightarrow \text{hom}(A \otimes X, 0), \\ \text{hom}(X, A) &\longrightarrow \text{hom}(X, B^*) \longrightarrow \text{hom}(B \otimes X, 0) \longrightarrow \text{hom}(X \otimes B, 0). \end{aligned}$$

The two universal arrows are, respectively,  $\psi^b$  and  $\phi^b \circ \sigma_{A,B}$  which are equal since we suppose  $\phi = \psi^\perp$ .

2. If  $(A, B)$  is dual, then  $(B, A)$  is dual, by Lemma 11. By Proposition 10, dual pairs are closed under isos. Since  $B \simeq A^*$  and  $A \simeq B^*$ , the pair  $(A^*, B^*)$  is dual.

3. Let  $\phi : A \longrightarrow B^*$  and  $\psi : B \longrightarrow A^*$  be the two isomorphisms stated in 1. Using Lemma 12, we obtain  $\phi = \psi^\perp = \psi^* \circ j_A$ . Since  $\psi$  is invertible,  $\psi^*$  is invertible by functoriality, therefore  $j_A$  is also an isomorphism and  $A$  is a reflexive object. By Lemma 11,  $B$  is reflexive as well.

4. By definition,  $j_A$  is the transpose of  $ev_{A,0} \circ \sigma_{A^*,A}$ , that is,  $j_A = 1_{A^*}^\perp$ . Therefore, if  $j_A$  is invertible then  $(A, A^*, ev_{A,0})$  is a dual pair.

5. The direct implication is induced By 1 and 3. If  $A$  is reflexive, then  $(A, A^*)$  is a dual pair and if  $\psi : B \longrightarrow A^*$  is invertible then by lemma 10,  $(A, B)$  is also a dual pair.  $\square$

We next give some examples where the notion of dual pair naturally arises.

**Example 14.** Proposition 13.5 shows that in a  $*$ -autonomous category  $\mathcal{V}$ , where all the objects are reflexive, any isomorphism  $B \longrightarrow A^*$  is enough to build a dual pair  $(A, B)$ . This is a key observation for the coherence theorem in [9]. For example, in  $\mathbf{SLatt}$ , taking as object 0 the two-element Boolean algebra  $\mathbf{2} := \{\perp, \top\}$  (which is also the unit of the tensor), the canonical isomorphism  $L^{\text{op}} \simeq L^*$  yields the dual pair  $(L, L^{\text{op}})$ . The pairing  $\epsilon : L \otimes L^{\text{op}} \longrightarrow \mathbf{2}$  is given by  $\epsilon(x, y) = \perp$ , if  $x \leq y$ , and  $\epsilon(x, y) = \top$ , otherwise.

**Example 15.** For any object  $L$  of a  $*$ -autonomous category  $\mathcal{V}$ , the pair  $(L^* \otimes L, [L, L])$  is dual with pairing given by  $\epsilon := ev_{L,0} \circ \sigma_{L^*,L} \circ (L^* \otimes ev_{L,L})$ . Indeed, every object of  $\mathcal{V}$  is reflexive, so  $(L^* \otimes L, (L^* \otimes L)^*, ev_{L^* \otimes L,0})$  is a dual pairing by Proposition 13.4. It is well known that in a  $*$ -autonomous category the transpose of  $ev_{L,0} \circ \sigma_{L^*,L} \circ (L^* \otimes ev_{L,L})$  is an isomorphism  $\psi : [L, L] \longrightarrow (L^* \otimes L)^*$ . Then, by Proposition 10,  $ev_{L^* \otimes L,0} \circ (L^* \otimes L \otimes \psi) = ev_{L,0} \circ \sigma_{L^*,L} \circ (L^* \otimes ev_{L,L})$  is a dual pairing.

**Example 16.** It is also possible to understand dual pairs as a generalization of the well-known notion of *adjunction*. Recall that a tuple  $(A, B, \epsilon, \eta)$  with  $\epsilon : A \otimes B \longrightarrow I$  and  $\eta : I \longrightarrow B \otimes A$  is said to be an adjunction if the two diagrams below commute.

$$\begin{array}{ccc} A \otimes B \otimes A & \xleftarrow{A \otimes \eta} & A \otimes I \\ \downarrow \epsilon \otimes A & & \downarrow \rho_A \\ I \otimes A & \xrightarrow{\ell_A} & A \end{array} \quad \begin{array}{ccc} I \otimes B & \xrightarrow{\eta \otimes B} & B \otimes A \otimes B \\ \downarrow \ell_B & & \downarrow B \otimes \epsilon \\ B & \xleftarrow{\rho_B} & B \otimes I \end{array}$$

$A$  is said to be left adjoint to  $B$ ,  $B$  right adjoint to  $A$ ,  $\eta$  is the unit of the adjunction and  $\epsilon$  is the counit of the adjunction. In an autonomous category, if

$(A, B, \eta, \epsilon)$  is an adjunction, then so is  $(B, A, \sigma_{A,B} \circ \eta, \epsilon \circ \sigma_{B,A})$ , and therefore we won't distinguish between left and right adjoints, we just say that  $A$  (or  $B$ ) is part

of an adjunction.<sup>1</sup> It is a standard exercise in monoidal categories to verify the following statement.

**Fact 17.** If  $(A, B, \eta, \epsilon)$  is an adjunction, then  $(A, B, \epsilon)$  is a dual pairing.

*Proof.* Then  $(A, B, \epsilon)$  is a dual pairing (where the object 0 is  $I$ ). Indeed, given  $f : X \otimes B \longrightarrow I$ , we get a map  $f^\sharp : X \longrightarrow A$  defined by

$$f^\sharp := X \simeq X \otimes I \xrightarrow{X \otimes \eta} X \otimes B \otimes A \xrightarrow{f \otimes A} I \otimes A \simeq A.$$

Clearly,  $\epsilon \circ (f^\sharp \otimes B) = f$  and if  $\epsilon \circ (g \otimes B) = f$ , then the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\rho_X^{-1}} & X \otimes I & \xrightarrow{X \otimes \eta} & X \otimes B \otimes A \\ \downarrow g & & \downarrow g \otimes I & & \downarrow g \otimes B \otimes A \\ A & \xrightarrow{\rho_A^{-1}} & A \otimes I & \xrightarrow{A \otimes \eta} & A \otimes B \otimes A \\ & \searrow \text{id}_A & \downarrow \rho_A & & \downarrow \epsilon \otimes A \\ & & A & \xrightarrow{\lambda_A^{-1}} & I \otimes A \end{array}$$

shows that  $g = f^\sharp$ . Thus, we have that  $\text{hom}(X \otimes B, I) \simeq \text{hom}(X, A)$ . Observe also that  $\epsilon^\sharp = \text{id}_A$ , by the first of the two diagrams defining an adjunction. With a similar argument, we show that the transformation sending  $f : A \otimes X \longrightarrow I$  to  $\rho_B \circ (B \otimes f) \circ (\eta \otimes X) \circ \lambda_X^{-1} : X \longrightarrow B$  is a natural isomorphism and that  $\epsilon$  corresponds to  $\text{id}_B$  under the isomorphism.  $\square$

Not all the dual pairs arise from adjunctions. This can be simply observed by looking at the  $*$ -autonomous category  $\mathbf{SLatt}$ , where an object is part of an adjunction if and only if it is a nuclear object, by the content of Section 6, if and only if it is completely distributive, by [29, 21]. Yet,  $(L, L^{op}, \epsilon)$ , described in Example 14, is always a dual pairing, even when  $L$  is not completely distributive.

In [10], the authors introduce the notion of *linear adjunction*. Recall that a linearly distributive category, see e.g. [11], is a category with distinct monoidal structures  $(I, \otimes)$  and  $(0, \oplus)$  coming with natural maps  $\kappa^\ell : X \otimes (Y \oplus Z) \longrightarrow (X \otimes Y) \oplus X$  and  $\kappa^\rho : (X \oplus Y) \otimes Z \longrightarrow X \oplus (Y \otimes Z)$  satisfying some constraints. A linear adjunction is then a tuple  $(A, B, \epsilon, \eta)$  with  $\epsilon : A \otimes B \longrightarrow 0$  and  $\eta : I \longrightarrow B \oplus A$  making the diagrams on the

$$\begin{array}{ccc} A \otimes (B \oplus A) & \xleftarrow{A \otimes \eta} & A \otimes I \\ \downarrow \kappa_{A,B,A}^\ell & & \downarrow \rho_A^\otimes \\ (A \otimes B) \oplus A & & A \\ \downarrow \epsilon \oplus A & \xrightarrow{\ell_A^\oplus} & A \\ 0 \oplus A & & A \end{array} \quad \begin{array}{ccc} I \otimes B & \xrightarrow{\eta \otimes B} & (B \oplus A) \otimes B \\ \downarrow \ell_B^\otimes & & \downarrow \kappa_{B,A,B}^\rho \\ B \oplus (A \otimes B) & & B \oplus (A \otimes B) \\ \downarrow B \oplus \epsilon & & \downarrow B \oplus \epsilon \\ B & \xleftarrow{\rho_B^\oplus} & B \oplus 0 \end{array}$$

right commute.

A  $*$ -autonomous category is, canonically, a linearly distributive category when we let  $X \oplus Y := (Y^* \otimes X^*)^*$ . In view of Example 14 and Theorem 1.2 in [14], if the

<sup>1</sup>This also to avoid the naming conflict with the notion of adjoint, presented below in this section. An object  $A$  which is part of an adjunction is often called *dualizable*.

environment category is  $*$ -autonomous, then  $(A, B, \epsilon)$  is a dual pairing if and only if we can find  $\eta : I \longrightarrow A \oplus B$  such that  $(A, B, \eta, \epsilon)$  is a linear adjunction.

**More on mates, adjoints.** In  $\mathbf{SLatt}$ , for every sup-preserving map  $f : L \longrightarrow M$  between complete lattices, there exists a unique sup-preserving map  $\rho(f) : M^{\text{op}} \longrightarrow L^{\text{op}}$  such that, for each  $x \in L$  and  $y \in M$ ,  $f(x) \leq y$  iff  $x \leq \rho(f)(y)$ . The map  $\rho(f)$  is called the (right) *adjoint* of  $f$ . The equivalences defining the adjoint can be expressed diagrammatically, by means of the equation  $\epsilon_{M, M^{\text{op}}} \circ (f \otimes M^{\text{op}}) = \epsilon_{L, L^{\text{op}}} \circ (L \otimes \rho(f))$ . This suggests that a similar notion can be defined for dual pairings in an arbitrary monoidal category.

$$\frac{\frac{A_0 \longrightarrow A_1}{A_0 \otimes B_1 \longrightarrow 0}}{B_1 \longrightarrow B_0}$$

Let  $(A_0, B_0, \epsilon_0)$  and  $(A_1, B_1, \epsilon_1)$  be two dual pairings. For a morphism  $f : A_0 \longrightarrow A_1$ , we denote by  $\tilde{f} : B_1 \longrightarrow B_0$  the arrow given by the two transposes on the right. We say that  $f : A_0 \longrightarrow A_1$  is *adjoint* to  $g : B_1 \longrightarrow B_0$  if  $\tilde{f} = g$ .

$$\begin{array}{ccc} X_0 \otimes Y_1 & \xrightarrow{f \otimes Y_1} & Y_0 \otimes Y_1 \\ \downarrow X_0 \otimes g & & \downarrow p_Y \\ X_0 \otimes X_1 & \xrightarrow{p_X} & 0. \end{array}$$

A dual pairing  $(A, B, \epsilon)$  is a special object of the category  $\text{Chu}_{\mathcal{V}, 0}$ , see [6]. Recall that a morphism between two objects  $(X_0, X_1, p_X)$  and  $(Y_0, Y_1, p_Y)$  of  $\text{Chu}_{\mathcal{V}, 0}$  is a pair of  $\mathcal{V}$  morphisms  $(f : X_0 \longrightarrow Y_0, g : Y_1 \longrightarrow X_1)$  such the square on the right commutes. If  $(A_0, B_0, \epsilon_0)$  and  $(A_1, B_1, \epsilon_1)$  are dual pairings, then a  $\text{Chu}_{\mathcal{V}, 0}$  morphism between them is necessarily of the form  $(f, \tilde{f})$  and, also, a pair  $(f, \tilde{f})$  is obviously a morphism in  $\text{Chu}_{\mathcal{V}, 0}$ . Denoting by  $\text{Pair}_{\mathcal{V}, 0}$  the full subcategory of  $\text{Chu}_{\mathcal{V}, 0}$  whose objects are dual pairings, these remarks amount to the following statement, whose most relevant consequences is presented immediately after.

**Lemma 18.** *The obvious projection functors,  $\mathcal{P}_0 : \text{Pair}_{\mathcal{V}, 0} \longrightarrow \mathcal{V}$  and  $\mathcal{P}_1 : \text{Pair}_{\mathcal{V}, 0} \longrightarrow \mathcal{V}^{\text{op}}$ , are full and faithful.*

**Corollary 19.** *Let  $(A_i, B_i, \epsilon_i)$ ,  $i = 0, 1, 2$ , be dual pairings. If  $f : A_0 \longrightarrow A_1$  and  $g : A_1 \longrightarrow A_2$ , then  $\widetilde{\text{id}_{A_i}} = \text{id}_{B_i}$  and  $\tilde{g} \circ \tilde{f} = \widetilde{(f \circ g)}$ . In particular, if  $f : A_0 \longrightarrow A_1$  is inverted by  $g : A_1 \longrightarrow A_0$ , then  $\tilde{f}$  is inverted by  $\tilde{g}$ .*

*Proof.* The two equalities  $\widetilde{\text{id}_{A_i}} = \text{id}_{B_i}$  and  $\tilde{g} \circ \tilde{f} = \widetilde{(f \circ g)}$  are direct consequence of the functoriality of  $\mathcal{P}_0$  and  $\mathcal{P}_1$ . We directly deduce the last statement by  $\tilde{f} \circ \tilde{g} = \widetilde{(g \circ f)} = \widetilde{\text{id}_{A_0}} = \text{id}_{B_0}$ .  $\square$

Let  $X, Y$  be reflexive, so  $(X, X^*, ev_{X, 0})$ ,  $(Y, Y_{Y, 0}^*)$ , and  $(Y^*, Y, ev_{Y, 0} \circ \sigma_{Y^*, Y})$  are dual pairings. Notice the following relations: for  $f : X \longrightarrow Y$ ,  $\tilde{f} = f^*$ , that is, the adjoint of  $f$  is the result of the functorial action on the map  $f$ ; for  $g : X \longrightarrow Y^*$ ,  $\tilde{g} = g^\perp$ , that is, the adjoint of  $g$  is just its mate. This yields a number of relations. For example, if also  $Z$  is reflexive,  $g : Z \longrightarrow Y$  and  $f : X \longrightarrow Y^*$ , then using Corollary 19 we have

$$(g^* \circ f)^\perp = f^\perp \circ g : Z \longrightarrow X^*. \quad (2)$$

Notice that equation (2) amounts to the naturality of  $(-)^{\perp} : \text{hom}(X, Y^*) \longrightarrow \text{hom}(Y, X^*)$ .

Finally, let us mention that we shall focus on maps  $f : A \longrightarrow B$ , where  $(A, B, \epsilon)$  is a dual pairing and, consequently,  $(B, A, \epsilon \circ \sigma_{B,A})$  as well. The following statement is then easily verified.

**Lemma 20.** *For  $(A, B, \epsilon)$  a dual pairing and  $f : A \longrightarrow B$ , the transposes of  $f$  and  $\tilde{f}$  differ by a symmetry:  $\epsilon \circ (A \otimes \tilde{f}) = \epsilon \circ (A \otimes f) \circ \sigma_{A,A}$ .*

#### 4. ACTIONS IN DUAL PAIRS

The implications of a quantale validate the two equations

$$z/(y * x) = (z/y)/x \quad \text{and} \quad (x * y) \setminus z = x \setminus (y \setminus z).$$

These two equations can be understood by saying that the implications are, respectively, left and right actions of  $Q$  over  $Q^{\text{op}}$ , see Example 22. In this section we show that actions arise from dual pairs when one of the objects of a dual pair is a semigroup. From now on,  $\mathcal{V}$  will be a symmetric monoidal category. Let  $(A, B, \epsilon)$  be a dual pairing in  $\mathcal{V}$  such that  $(A, \mu_A)$  is a magma. We define two morphisms

$$\alpha_A^{\ell} : A \otimes B \longrightarrow B \quad \text{and} \quad \alpha_A^{\rho} : B \otimes A \longrightarrow B,$$

as the only morphisms making these two diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes B & \xrightarrow{A \otimes \alpha_A^{\ell}} & A \otimes B \\ \downarrow \mu_{A \otimes B} & & \downarrow \epsilon \\ A \otimes B & \xrightarrow{\epsilon} & 0 \end{array} \quad \begin{array}{ccc} B \otimes A \otimes A & \xrightarrow{B \otimes \mu_A} & B \otimes A \xrightarrow{\sigma_{B,A}} A \otimes B \\ \downarrow \alpha_A^{\rho} \otimes A & & \downarrow \epsilon \\ B \otimes A & \xrightarrow{\sigma_{B,A}} & A \otimes B \xrightarrow{\epsilon} 0 \end{array} \quad (3)$$

**Fact 21.** Letting  $A^{\text{op}}$  be the magma  $(A, \mu_A \circ \sigma_{A,A})$ , the relation  $\alpha_{A^{\text{op}}}^{\ell} = \alpha_A^{\rho} \circ \sigma_{A,B}$  holds.

*Proof.* The fact is then an immediate cosequence. The diagram on the right of (3) commutes if and only if the diagram below commutes:

$$\begin{array}{ccc} A \otimes A \otimes B & \xrightarrow{\sigma_{A,A} \otimes B} & A \otimes A \otimes B \xrightarrow{\mu_{A \otimes B}} A \otimes B \\ \downarrow A \otimes \sigma_{A,B} & & \downarrow \epsilon \\ A \otimes B \otimes A & \xrightarrow{A \otimes \alpha_A^{\rho}} & A \otimes B \xrightarrow{\epsilon} 0 \end{array}$$

commutes. □

**Example 22.** As mentioned in Example 14,  $(Q, Q^{\text{op}})$  is a dual pair in  $\text{SLatt}$ . If  $(Q, *)$  is a quantale, commutativity of the left (resp. right) diagram in (3) amounts to the relation

$$x * y \leq z \text{ iff } x \leq \alpha_Q^{\ell}(y, z) \quad (\text{resp.}, \quad x * y \leq z \text{ iff } y \leq \alpha_Q^{\rho}(z, x)).$$

Therefore, by the uniqueness of the adjoint, we have  $\alpha_Q^{\ell}(y, z) = z/y$  and  $\alpha_Q^{\rho}(z, x) = x \setminus z$ .

**Example 23.** Let  $A$  be a finite dimensional  $k$ -algebra,  $x, y \in A$ ,  $l \in A^* = [A, k]$ . It is direct to verify that  $\alpha_A^{\ell}(y, l) : x \mapsto l(xy)$  and  $\alpha_A^{\rho}(l, x) : y \mapsto l(xy)$ .

Let us give some elementary properties of  $\alpha_A^{\ell}$  and  $\alpha_A^{\rho}$ .

**Lemma 24.** *Let  $(A, B, \epsilon)$  be a dual pairing and  $(A, \mu_A)$  be a magma.*

(1) *The following equation, relating  $\alpha_A^\ell$  and  $\alpha_A^\rho$ , holds:*

$$\epsilon \circ (A \otimes \alpha_A^\rho) = \epsilon \circ \sigma_{B,A} \circ (\alpha_A^\ell \otimes A). \quad (4)$$

(2) *If  $(A, \mu_A)$  is a semigroup, then the map  $\alpha_A^\ell$  is a left action (see Definition 5) and the map  $\alpha_A^\rho$  is a right action.*

(3) *If  $(A, \mu_A)$  is a semigroup, then the two actions are equivariant to each other, that is*

$$\alpha_A^\ell \circ (A \otimes \alpha_A^\rho) = \alpha_A^\rho \circ (\alpha_A^\ell \otimes A).$$

*Proof.* Proof of 1. The equation is given by the commutativity of the diagram

$$\begin{array}{ccccc}
 A \otimes B \otimes A & \xrightarrow{A \otimes \alpha_A^\rho} & A \otimes B & & \\
 \downarrow \sigma_{A,B \otimes A} & & \downarrow \sigma_{A,B} & \searrow \epsilon & \\
 B \otimes A \otimes A & \xrightarrow{\alpha_A^\rho \otimes B} & B \otimes A & & 0 \\
 \downarrow \sigma_{B,A \otimes A} & \searrow B \otimes \mu_A & \downarrow \sigma_{B,A} & \nearrow \epsilon & \\
 A \otimes A \otimes B & \xrightarrow{\mu_A \otimes B} & A \otimes B & & \\
 \downarrow \sigma_{A,A \otimes B} & \searrow A \otimes \alpha_A^\ell & \uparrow \sigma_{B,A} & & \\
 A \otimes B \otimes A & \xrightarrow{\alpha_A^\ell \otimes A} & B \otimes A & & 
 \end{array}$$

Proof of 2. In the following diagram

$$\begin{array}{ccccc}
 A^3 \otimes B & \xrightarrow{A \otimes \mu_A \otimes B} & A^2 \otimes B & \xrightarrow{A \otimes \alpha_A^\ell} & A \otimes B \\
 \downarrow \mu_A \otimes A \otimes B & & \downarrow \mu_A \otimes B & & \downarrow \epsilon \\
 A^2 \otimes B & \xrightarrow{\mu_A \otimes B} & A \otimes B & & 0 \\
 \downarrow A^2 \otimes \alpha_A^\ell & \searrow A \otimes \alpha_A^\ell & \downarrow \epsilon & \nearrow \epsilon & \\
 A^2 \otimes B & \xrightarrow{\mu_A \otimes B} & A \otimes B & & \\
 \downarrow A \otimes \alpha_A^\ell & & \downarrow \epsilon & & \\
 A \otimes B & & A \otimes B & & 
 \end{array}$$

the upper leg and lower leg are the transposes of the two parallel maps that are required to be equal for  $\alpha_A^\ell$  to be a left action. If  $(A, \mu_A)$  is a semigroup, then the above diagram is commutative, and so these two maps are indeed equal.

The second statement follows since  $(B, A, \epsilon)$  is a dual pair in the monoidal category  $\mathcal{V}'$  with same objects and arrows, where however we have  $X \otimes' Y = Y \otimes X$ . Proof of 3. By 2,  $\alpha_A^\rho$  and  $\alpha_A^\ell$  are two actions, therefore the diagram

$$\begin{array}{ccccc}
A \otimes A \otimes B \otimes A & \xrightarrow{A \otimes A \otimes \alpha_A^\rho} & A \otimes A \otimes B & \xrightarrow{A \otimes \alpha_A^\ell} & A \otimes B \\
A \otimes \alpha_A^\ell \otimes B \downarrow \downarrow \mu_A \otimes B \otimes A & & \downarrow \mu_A \otimes B & & \downarrow \epsilon \\
A \otimes B \otimes A & \xrightarrow{A \otimes \alpha_A^\rho} & A \otimes B & \xrightarrow{\epsilon} & 0 \\
& \searrow \alpha_A^\ell \otimes A & \nearrow \sigma_{B,A} & & \\
& & B \otimes A & & 
\end{array}$$

commutes.  $\square$

As the dual pair  $([A, A], A^* \otimes A)$  is the main object studied in this paper, we characterize next the actions of  $([A, A], \circ)$  over  $A^* \otimes A$ , and of  $(A^* \otimes A, \mu_{A^* \otimes A})$  over  $[A, A]$ , see Examples 4 and 15.

**Proposition 25.** *In a  $*$ -autonomous category  $\mathcal{V}$ , we have*

$$\alpha_{[A,A]}^\rho = A^* \otimes ev_{A,A}, \quad \alpha_{[A,A]}^\ell = \mu_{A,A,0} \otimes A.$$

*Proof.* Clearly, the following diagram commutes:

$$\begin{array}{ccc}
A \otimes [A, A] \otimes [A, A] & \xrightarrow{A \otimes \mu_{A,A,A}} & A \otimes [A, A] \\
\downarrow ev_{A,A} \otimes [A, A] & & \downarrow ev_{A,A} \\
A \otimes [A, A] & \xrightarrow{ev_{A,A}} & A
\end{array}$$

By tensoring on the left with  $A^*$  and then by postcomposing with  $tr := ev_{A,0} \circ \sigma_{A^*,A}$ , we obtain commutativity of

$$\begin{array}{ccc}
A \otimes B \otimes B & \xrightarrow{A \otimes \mu_B} & A \otimes B \\
\downarrow A^* \otimes ev_{A,A} \otimes B & & \downarrow \epsilon_{A,B} \\
A \otimes B & \xrightarrow{\epsilon_{A,B}} & 0
\end{array}$$

showing that  $\alpha_{[A,A]}^\rho = A^* \otimes ev_{A,A}$ . In similar way, the following diagram commutes

$$\begin{array}{ccc}
[A, A] \otimes [A, A] \otimes A^* & \xrightarrow{[A,A] \otimes \mu_{A,A,0}} & [A, A] \otimes A^* \\
\downarrow \mu_{A,A,A} \otimes A^* & & \downarrow \mu_{A,A,0} \\
[A, A] \otimes A^* & \xrightarrow{\mu_{A,A,0}} & A^*
\end{array}$$

and (by tensoring on the right with  $A$ ) implies that

$$\begin{array}{ccc}
B \otimes B \otimes A & \xrightarrow{B \otimes \mu_{A,A,0} \otimes A} & B \otimes A \\
\downarrow \mu_{A,A,A} \otimes A & & \downarrow \epsilon_{B,A} \\
B \otimes A & \xrightarrow{\epsilon_{B,A}} & 0
\end{array} \tag{5}$$

commutes, thus yielding  $\alpha_{[A,A]}^\ell = \mu_{A,A,0} \otimes A$ . Indeed, we have

$$\epsilon_{B,A} = \epsilon_{A,B} \circ \sigma_{B,A} = ev_{A,0} \circ \sigma_{A^*,A} \circ (\mu_{A,A,0} \otimes A^*)$$

as witnessed by the diagram below:

$$\begin{array}{ccccc}
 [A, A] \otimes A^* \otimes A & \xrightarrow{\mu_{A,A,0} \otimes A} & & & A^* \otimes A \\
 \downarrow \sigma_{[A,A], A^* \otimes A} & \searrow \sigma_{[A,A] \otimes A^*, A} & & & \downarrow \sigma_{A^*, A} \\
 A^* \otimes A \otimes [A, A] & \xrightarrow{\sigma_{A^*, A} \otimes [A, A]} & A \otimes [A, A] \otimes A^* & \xrightarrow{A \otimes \mu_{A,A,0}} & A \otimes A^* \\
 \downarrow A^* \otimes ev_{A,0} & & \downarrow ev_{A,A} \otimes A^* & & \downarrow ev_{A,0} \\
 A^* \otimes A & \xrightarrow{\sigma_{A^*, A}} & A \otimes A^* & \xrightarrow{ev_{A,0}} & 0
 \end{array} \quad \square$$

## 5. FROBENIUS STRUCTURES

We finally define in this section the notion of Frobenius structure in a symmetric monoidal category  $\mathcal{V}$  and give some elementary properties of this structure. While other approaches are possible, notably [38, 14], they are not strictly equivalent, starting from the assumptions on the environment category.

$$\begin{array}{ccc}
 A \otimes A & \xrightarrow{A \otimes r} & A \otimes B \\
 \downarrow l \otimes A & & \downarrow \alpha_A^\ell \\
 B \otimes A & \xrightarrow{\alpha_A^\rho} & B.
 \end{array} \quad (6)$$

**Definition 26.** A *Frobenius structure* is a tuple  $(A, B, \epsilon, \mu_A, l, r)$  where  $(A, B, \epsilon)$  is a dual pairing,  $(A, \mu_A)$  is a semigroup, and  $l$  and  $r$  are adjoint invertible maps from  $A$  to  $B$  such that diagram (6) commutes.

Notice that, by Lemma 20, the above definition coincides with the one given in the Introduction.

**Example 27.** Definition 26 strictly mimics the definition of Frobenius quantales in [26]. A Frobenius quantale is defined there as a tuple  $(Q, *, {}^\perp(-), (-)^\perp)$  such that  $(Q, *)$  is a quantale, and where  $({}^\perp(-), (-)^\perp)$  form an invertible Galois connection satisfying the equation

$$y \setminus {}^\perp x = y^\perp / x. \quad (7)$$

The commutative diagram (6) is just a direct translation of equation (7), called the contraposition law in [16].

Let us recall a few considerations developed in [26]. If a quantale has a dualizing element 0, then the two maps  ${}^\perp(-) := 0/(-)$  and  $(-)^{\perp} := (-)\setminus 0$  are inverse to each other, form a Galois connection, and satisfy equation (7). And if a Frobenius quantale has a unit 1 then  $0 := {}^\perp 1 = 1^\perp$  is a dualizing element. That is, contrarily to the usual definition (see for instance [39, 31, 25, 15]), a Frobenius quantale does

not need a unit.<sup>2</sup> Now it is immediate to see that Frobenius quantales as defined in [26] are exactly the Frobenius structures  $(A, B, \epsilon, \mu, l, r)$  in  $\mathbf{SLatt}$  for which  $B = A^{\text{op}}$ .

Before giving other examples of Frobenius structures, let us give some characterization of these structures. In a quantale, equation (7) is equivalent to the shift/associative relations:

$$x * y \leq {}^\perp z \quad \text{iff} \quad x^\perp \geq y * z \quad \text{iff} \quad x \leq {}^\perp (y * z).$$

This also holds in the general setting, simply by transposing diagram (6) we have:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu_A \otimes l} & A \otimes B \\ \downarrow r \otimes \mu_A & & \downarrow \epsilon \\ B \otimes A & \xrightarrow{\sigma_{B,A}} & A \otimes B \xrightarrow{\epsilon} 0 \end{array} \quad (8)$$

**Lemma 28.** *A pair of morphisms  $(l, r)$  makes diagram (6) commute if and only if the shift diagram (8) commutes.*

*Proof.* Diagram (6) commutes if and only if its transpose commutes:

$$\begin{array}{ccccc} A \otimes A \otimes A & \xrightarrow{A \otimes A \otimes r} & A \otimes A \otimes B & \xrightarrow{A \otimes \alpha_A^\ell} & A \otimes B \\ A \otimes l \otimes A \downarrow & & & & \downarrow \epsilon \\ A \otimes B \otimes A & \xrightarrow{A \otimes \alpha_A^\rho} & A \otimes B & \xrightarrow{\epsilon} & 0 \end{array}$$

By (3), the upper leg above equals to  $\epsilon \circ (\mu_A \otimes r)$ :

$$\begin{array}{ccccc} A \otimes A \otimes A & \xrightarrow{A \otimes A \otimes r} & A \otimes A \otimes B & \xrightarrow{A \otimes \alpha_A^\ell} & A \otimes B \\ & \searrow \mu_A \otimes r & \downarrow \mu_A \otimes B & & \downarrow \epsilon \\ & & A \otimes B & \xrightarrow{\epsilon} & 0 \end{array}$$

By equation (4), the lower leg above equals to  $\epsilon \circ (\mu_A \otimes l) \circ \sigma_{A \otimes A, A}$ :

$$\begin{array}{ccccc} A \otimes A \otimes A & \xrightarrow{\sigma_{A \otimes A, A}} & A \otimes A \otimes A & & \\ A \otimes l \otimes A \downarrow & & \downarrow A \otimes A \otimes l & & \\ A \otimes B \otimes A & \xrightarrow{\sigma_{A \otimes B, A}} & A \otimes A \otimes B & & \\ & \searrow \alpha_A^\ell \otimes A & \downarrow A \otimes \alpha_A^\ell & \downarrow \mu_A \otimes B & \\ A \otimes \alpha_A^\rho \downarrow & & B \otimes A & \xrightarrow{\sigma_{B,A}} & A \otimes B \\ & & \downarrow \epsilon & & \\ A \otimes B & \xrightarrow{\epsilon} & 0 & & \end{array}$$

Thus, diagram (6) commutes if and only if

$$\epsilon \circ (\mu_A \otimes r) = \epsilon \circ (\mu_A \otimes l) \circ \sigma_{A \otimes A, A}.$$

□

<sup>2</sup>As a matter of fact, the theorems that we shall present in the next two sections can be understood as characterizing the existence of units in specific cases.

Our next goal is to show that, for a Frobenius structure  $(A, B, \epsilon, \mu_A, l, r)$ , the dual pair  $(B, A)$  also carries a Frobenius structure, in particular  $B$  carries a canonical semigroup structure.

$$\begin{array}{ccc}
 B \otimes B & \xrightarrow{B \otimes r^{-1}} & B \otimes A \\
 l^{-1} \otimes B \downarrow & \searrow^{\mu_B} & \downarrow \alpha_A^p \\
 A \otimes B & \xrightarrow{\alpha_A^\ell} & B
 \end{array} \quad (9)$$

**Lemma 29.** *Let  $(A, B, \epsilon)$  be a dual pair,  $(A, \mu_A)$  a semigroup, and  $(l, r)$  invertible adjoint maps from  $A$  to  $B$ . The tuple  $(A, B, \epsilon, \mu_A, l, r)$  is a Frobenius structure if and only if diagram (9) on the right commutes.*

*Proof.* By Lemma 19,  $l$  and  $r$  are adjoint maps iff  $l^{-1}$  and  $r^{-1}$  are adjoints. Assuming that diagram (6) commutes, we prove commutation of diagram (9) as follows:

$$\begin{array}{ccccc}
 B \otimes B & & & & B \otimes A \\
 & \searrow^{B \otimes r^{-1}} & & & \downarrow \alpha_A^p \\
 & & A \otimes A & \xrightarrow{l \otimes A} & B \otimes A \\
 & \searrow^{l^{-1} \otimes r^{-1}} & \downarrow A \otimes r & & \downarrow \alpha_A^p \\
 & & A \otimes B & \xrightarrow{\alpha_A^\ell} & B \\
 & \searrow^{l^{-1} \otimes B} & & & \\
 & & & & 
 \end{array}$$

A similar argument shows that commutation of diagram (9) implies that of diagram (6).  $\square$

For a Frobenius structure  $(A, B, \epsilon, \mu_A, l, r)$ , let us denote by  $\mu_B$  the diagonal of diagram (9). We next consider the magma  $(B, \mu_B)$ . Note that from the definition of  $\mu_B$  and of the actions of a magma over its dual, we obtain two pairs of parallel equal arrows

$$\begin{array}{c}
 \begin{array}{ccccccc}
 & & B \otimes \alpha_B^\ell & & & & \\
 & \curvearrowright & & \curvearrowleft & & & \\
 B \otimes B \otimes A & \xrightarrow{B \otimes r^{-1} \otimes A} & B \otimes A \otimes A & \xrightarrow[\alpha_A^p \otimes A]{B \otimes \mu_A} & B \otimes A & \xrightarrow{\sigma_{B,A}} & A \otimes B \xrightarrow{\epsilon} 0, \\
 & \curvearrowleft & & \curvearrowright & & & \\
 & & \mu_B \otimes A & & & & 
 \end{array} \\
 \begin{array}{ccccccc}
 & & \alpha_B^p \otimes B & & & & \\
 & \curvearrowright & & \curvearrowleft & & & \\
 A \otimes B \otimes B & \xrightarrow{A \otimes l^{-1} \otimes B} & A \otimes A \otimes B & \xrightarrow[\alpha_A^\ell \otimes B]{\mu_A \otimes B} & A \otimes B & \xrightarrow{\epsilon} & 0, \\
 & \curvearrowleft & & \curvearrowright & & & \\
 & & A \otimes \mu_B & & & & 
 \end{array}
 \end{array}$$

showing that the actions of the magma  $(B, \mu_B)$  on the dual  $A$  depends on  $\mu_A$  by:

$$\alpha_B^\ell = \mu_A \circ (r^{-1} \otimes A), \quad \alpha_B^p = \mu_A \circ (A \otimes l^{-1}), \quad (10)$$

and that  $\mu_A$  is obtained from the action of  $(B, \mu_B)$  over  $A$  by:

$$\begin{array}{ccc}
 A \otimes A & \xrightarrow{A \otimes l} & A \otimes B \\
 r \otimes A \downarrow & \searrow^{\mu_A} & \downarrow \alpha_B^p \\
 B \otimes A & \xrightarrow{\alpha_B^\ell} & A.
 \end{array} \quad (11)$$

We collect the properties of Frobenius structures that we want to emphasize next.

**Proposition 30.** *For a Frobenius structure  $(A, B, \epsilon, \mu_A, l, r)$ , the following statements hold:*

- (1)  $(A, B, \epsilon, \mu_A \circ \sigma_{A,A}, r, l)$  is a Frobenius structure.
- (2) The magma  $(B, \mu_B)$  is a semigroup.
- (3) The following diagrams commute:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu_A} & A \\ \downarrow A \otimes l & & \downarrow l \\ A \otimes B & \xrightarrow{\alpha_A^\ell} & B \end{array} \quad \begin{array}{ccc} A \otimes A & \xrightarrow{\mu_A} & A \\ \downarrow r \otimes A & & \downarrow r \\ A \otimes B & \xrightarrow{\alpha_A^\rho} & B \end{array} \quad (12)$$

- (4) The maps  $l$  and  $r$  are both semigroup homomorphisms from  $(A, \mu_A)$  to  $(B, \mu_B)$ .
- (5) The tuple  $(B, A, \epsilon \circ \sigma_A, \mu_B, r^{-1}, l^{-1})$  is a Frobenius structure.

*Proof.* 1 This immediately follows from the fact that the pairing  $\epsilon \circ (A \otimes r)$  is co-associative, see Lemma 28.

2. In view of Lemma 24, the following diagram commutes:

$$\begin{array}{ccccc} B \otimes B \otimes B & \xrightarrow{B \otimes (\alpha_A^\rho \circ (B \otimes r^{-1}))} & & & B \otimes B \\ & \searrow l^{-1} \otimes B \otimes r^{-1} & & & \downarrow l^{-1} \otimes B \\ & & A \otimes B \otimes A & \xrightarrow{A \otimes \alpha_A^\rho} & A \otimes B \\ (\alpha_A^\ell \circ (l^{-1} \otimes B)) \otimes B & & \downarrow \alpha_A^\ell \otimes A & & \downarrow \alpha_A^\ell \\ B \otimes B & \xrightarrow{B \otimes l^{-1}} & B \otimes A & \xrightarrow{\alpha_A^\rho} & B \end{array}$$

3. Using Lemma 28, we remark that the following diagram commutes:

$$\begin{array}{ccccc} A \otimes A \otimes A & \xrightarrow{\begin{matrix} A \otimes \mu_A \\ \mu_A \otimes A \end{matrix}} & A \otimes A & \xrightarrow{A \otimes l} & A \otimes B \\ \downarrow A \otimes A \otimes l & & & & \downarrow \epsilon \\ A \otimes A \otimes A & \xrightarrow{\begin{matrix} \mu_A \otimes A \\ A \otimes \alpha_A^\ell \end{matrix}} & A \otimes B & \xrightarrow{\epsilon} & 0 \end{array}$$

We obtain therefore the equality  $\epsilon \circ (A \otimes (\alpha_A^\ell \circ (A \otimes l))) = \epsilon \circ (A \otimes (l \circ \mu_A))$ , which, by uniqueness, implies  $\alpha_A^\ell \circ (A \otimes l) = l \circ \mu_A$ .

In a similar way we obtain that  $\alpha_{A^{op}}^\ell \circ (A \otimes r) = r \circ \mu_{A^{op}}$ . Considering that  $\alpha_{A^{op}}^\ell = \alpha_B^\rho \circ \sigma_{A,B}$ , we infer  $\alpha_B^\rho \circ (r \otimes A) \circ \sigma_{A,A} = r \circ \mu_{A^{op}} = r \circ \mu_A \circ \sigma_{A,A}$ , whence  $\alpha_B^\rho \circ (r \otimes A) = r \circ \mu_A$ .

4. Immediate from diagrams (12). For example:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu_A} & A \\ \downarrow A \otimes l & & \downarrow l \\ A \otimes B & \xrightarrow{\alpha_A^\ell} & B \\ \downarrow l \otimes B & \nearrow \mu_B & \\ A \otimes B & & \end{array}$$

5. By Lemma 11  $(B, A, \epsilon \circ \sigma_{B,A})$  is a dual pair. By item 2. of this proposition  $(B, \mu_B)$  is a semigroup. By Lemma 19  $(r^{-1}, l^{-1})$  are adjoint. By diagram (11) and Lemma 29, the tuple  $(B, A, \epsilon \circ \sigma_{B,A}, \mu_B, r^{-1}, l^{-1})$  is a Frobenius structure.  $\square$

The last item of the Proposition exhibits the expected duality. Indeed, using diagram (11), it is easily seen that the dual of  $(B, A, \epsilon \circ \sigma_{B,A}, \mu_B, r^{-1}, l^{-1})$  is  $(A, B, \epsilon, \mu_A, l, r)$  itself. Notice that having such duality is not an obvious result, since our definition of Frobenius structure is not self dual. *E.g.*, the dual  $B$  of  $A$  is not required to be a semigroup itself and the requirement that diagram (6) commutes is a condition on the semigroup  $(A, \mu_A)$ .

Finally, let us establish the connection with associative bracketed magmas.

**Proposition 31.** *Let  $(A, B, \epsilon, \mu_A, l, r)$  be a Frobenius structure and define  $\pi_A^l := \epsilon \circ (A \otimes l)$  and  $\pi_A^r := \epsilon \circ (A \otimes r)$ . Then  $(A, \mu_A, \pi_A^l)$  is an associative bracketed magma,  $(A, \mu_A, \pi_A^r)$  is a co-associative bracketed magma and  $\pi_A^l$  and  $\pi_A^r$  are dual pairings. Conversely, given an associative bracketed magma  $(A, \mu_A, \pi_A)$  for which  $\pi_A$  is a dual pairing, and given another dual pairing  $(A, B, \epsilon)$ , define  $l$  so that  $\epsilon \circ (A \otimes l) = \pi_A$ ,  $r = \bar{l}$ , then  $(A, B, \epsilon, \mu_A, l, r)$  is a Frobenius structure.*

*Proof.* By Lemma 28, the diagram (6) commutes if and only if

$$\epsilon \circ (\mu_A \otimes r) = \epsilon \circ (\mu_A \otimes l) \circ \sigma_{A \otimes A, A}.$$

By precomposing the equality above with the invertible  $\sigma_{A, A \otimes A}$ , diagram (6) commutes if and only if

$$\epsilon \circ (\mu_A \otimes l) = \epsilon \circ (\mu_A \otimes r) \circ \sigma_{A, A \otimes A} = \epsilon \circ \sigma_{B, A} \circ (r \otimes \mu_A).$$

Assume now that  $(l, r)$  are adjoint. By the previous remark, the diagram below commutes:

$$\begin{array}{ccccccc} A \otimes A \otimes A & \xrightarrow{\mu_A \otimes A} & A \otimes A & \xrightarrow{A \otimes l} & A \otimes B & & \\ \downarrow \sigma_{A, A \otimes A} & \searrow r \otimes \mu_A & & & \downarrow \epsilon & & \\ & & B \otimes A & & & & \\ & & \downarrow \sigma_{B, A} & & & & \\ A \otimes A \otimes A & \xrightarrow{\mu_A \otimes A} & A \otimes A & \xrightarrow{A \otimes r} & A \otimes B & \xrightarrow{\epsilon} & 0 \\ \downarrow \sigma_{A \otimes A, A} & & \downarrow \sigma_{A, A} & & \downarrow \epsilon & & \\ A \otimes A \otimes A & \xrightarrow{A \otimes \mu_A} & A \otimes A & \xrightarrow{A \otimes l} & A \otimes B & \xrightarrow{\epsilon} & 0 \end{array}$$

making  $\pi_A = \epsilon \circ (A \otimes l)$  into an associative pairing. By Fact 3,  $\pi_A^r$  is a co-associative pairing. Also, as  $\epsilon$  is a dual pairing and  $l$  and  $r$  are two isomorphisms, by Lemma 10,  $\pi_A^l$  and  $\pi_A^r$  are dual pairing.

Conversely, using Lemma 10,  $l$  and  $r$  are two isomorphisms. Notice also that if the outside diagram above commutes, then the rightmost commutes as well, showing that associativity of  $\epsilon \circ (A \otimes l)$  implies commutativity of diagram (8).  $\square$

From this correspondence between Frobenius structures and associative bracketed magmas, it is easy to find some examples from the literature.

**Example 32.** A Frobenius algebra is a finite-dimensional  $k$ -algebra  $(A, \mu_A)$  coming with a non degenerate form  $\langle -, - \rangle$  such that  $\langle xy, z \rangle = \langle x, yz \rangle$ . That is, it is a bracketed semigroup  $(A, \mu_A, \langle -, - \rangle)$  and  $\langle -, - \rangle$  is a dual pairing as its transpose is an isomorphism between  $A$  and  $A^*$ . Hence,  $(A, A^*, ev, \mu_A, r, l)$  is a Frobenius structure where  $l$  and  $r$  are defined as stated in Proposition 31.

**Example 33.** A Frobenius algebra  $(A, t, \delta, e, \mu)$  in a symmetric monoidal category, as defined in [22], yields an associative bracketed magma  $(A, \mu, t \circ \mu)$ . Moreover, in [22], it is noticed that  $(A, A, t \circ \mu, \delta \circ e)$  is an adjunction which implies that  $(A, A, t \circ \mu)$  is a dual pairing (see Example 16). Whence, a Frobenius structure on the semigroup  $(A, \mu)$  is obtained as stated in Proposition 31.

**Example 34.** A similar argument shows that a Frobenius monoid  $(A, t, \delta, e, \mu)$  in a linear distributive category as defined in [14] yields a Frobenius structure, when the category is actually  $*$ -autonomous. Indeed, one obtains an associative bracketed magma with the same construction as in the last example. Moreover, Theorem 3.1 of [14] ensures that the pairing is the co-unit of a linear adjunction. That is  $(A, A, t \circ \mu)$  is a dual pairing (see Example 16) and we obtain a Frobenius structure just like before.

**Example 35.** A Frobenius structure in the category of sets and relation is easily seen to amount to a tuple  $(X, R, l, r)$  with  $R$  a ternary relation on  $X$  which is associative—*i.e.* such that, for all  $x, y, z, w \in X$ ,  $xyRu$  and  $uzRw$  (for some  $u \in X$ ) if and only if  $xvRw$  and  $yzRv$  (for some  $v \in X$ )—and with  $l, r$  inverse bijections on  $X$  satisfying  $xyRl(z)$  if and only if  $yzRr(x)$ , for all  $x, y, z \in X$ .

## 6. NUCLEAR OBJECTS AND ADJUNCTIONS

For completeness, we state here that, in an autonomous category, an object is nuclear if and only if it is part of an adjunction. This statement is actually a folklore result in the literature. For example, it is explicitly mentioned in [32] that compact closed categories are those autonomous categories for which every object is nuclear, while the same categories are defined in [23] by the property that every object is part of an adjunction. We assume from now on that the object  $0 = I$  is the unit of the autonomous category  $\mathcal{V}$ .

The next statement shows that, in an adjunction of the form  $(A, A^*, \eta, \epsilon)$ , we can assume that the counit  $\epsilon$  is the evaluation map and, moreover, require commutativity of only one of the two diagrams defining an adjunction.

**Lemma 36.** *In an autonomous category the following conditions are equivalent:*

$$\begin{array}{ccc} A \otimes I & \xrightarrow{A \otimes \eta} & A \otimes A^* \otimes A \\ \downarrow \rho_A & & \downarrow ev_{A, I \otimes A} \\ A & \xrightarrow{\lambda_A^{-1}} & I \otimes A \end{array} \quad (13)$$

- (1)  $A$  is part of an adjunction,
- (2) there exists a map  $\eta : I \longrightarrow A^* \otimes A$  such that diagram (13) commutes,
- (3) there exists a map  $\eta : I \longrightarrow A^* \otimes A$  such that  $(A, A^*, \eta, ev_{A, I})$  is an adjunction.

*Proof.* If  $(A, B, v, \epsilon)$  is an adjunction, then  $(A, B, \epsilon)$  is a dual pairing. By Lemma 10, there exists an isomorphism  $\psi : B \longrightarrow A^*$  such that  $\epsilon = ev_{A, I} \circ (A \otimes \psi)$ . Define

$\eta := (\psi \otimes A) \circ v$ , we derive

$$\begin{aligned} (ev_{A,I} \otimes A) \circ (A \otimes \eta) &= (ev_{A,I} \otimes A) \circ (A \otimes \psi \otimes A) \circ (A \otimes v) \\ &= (\epsilon \otimes A) \circ (A \otimes v) = \lambda_A^{-1} \circ \rho_A. \end{aligned}$$

Suppose now that diagram (13) commutes. We claim that the diagram

$$\begin{array}{ccc} I \otimes A^* & \xrightarrow{\eta \otimes A^*} & A^* \otimes A \otimes A^* \\ \downarrow \lambda_{A^*} & & \downarrow A^* \otimes ev_{A,I} \\ A^* & \xleftarrow{\rho_{A^*}} & A^* \otimes I \end{array} \quad (14)$$

commutes as well, thus making  $(A, A^*, \eta, ev_{A,I})$  into an adjunction. Indeed, the transpose of  $\lambda_{A^*}$  is the canonical map  $A \otimes I \otimes A^* \longrightarrow I$  arising, up to unital isomorphisms, from evaluation. The commutative diagram

$$\begin{array}{ccccc} A \otimes I \otimes A^* & \xrightarrow{A \otimes \eta \otimes A^*} & A \otimes A^* \otimes A \otimes A^* & \xrightarrow{A \otimes A^* \otimes ev_{A,I}} & A \otimes A^* \otimes I \\ \rho_{A \otimes A^*} \downarrow & & \downarrow ev_{A,I} \otimes A \otimes A^* & & \downarrow ev_{A,I} \otimes I \\ A \otimes A^* & \xrightarrow{\lambda_A^{-1} \otimes A^*} & I \otimes A \otimes A^* & \xrightarrow{I \otimes ev_{A,I}} & I \otimes I \\ & & \downarrow \lambda_{A \otimes A^*} & & \downarrow \lambda_I \\ & & A \otimes A^* & \xrightarrow{ev_{A,I}} & I \end{array}$$

shows that the transpose of the rightmost path in (14) is the same canonical map.

Finally, if  $(A, A^*, \eta, ev_{A,I})$  is an adjunction, then obviously  $A$  is part of an adjunction.  $\square$

Recall now that an object  $A$  of an autonomous category is *nuclear* if the canonical map  $\text{mix} : A^* \otimes A \longrightarrow [A, A]$  is an isomorphism.

**Theorem 37.** *In an autonomous category, the following are equivalent:*

- (1)  $A$  is nuclear,
- (2) there exists a map  $\eta : I \longrightarrow A^* \otimes A$  such that  $\text{mix} \circ \eta = \rho_A^\sharp : I \longrightarrow [A, A]$ ,
- (3)  $A$  is part of an adjunction.

*Proof.* If  $\text{mix}$  is invertible then define  $\eta := \text{mix}^{-1} \circ \rho_A^\sharp$ . That is, 1 implies 2.

To see that 2 implies 3, observe that the relation  $\text{mix} \circ \eta = \rho_A^\sharp$  is equivalent (under transposition) to commutativity of (13). Therefore, by Lemma 36,  $A$  is an adjoint.

Let us argue that 3 implies 1. Suppose that  $A$  is an adjoint. By Lemma 36, there exists  $\eta$  such that  $(A, A^*, \eta, ev_{A,I})$  is an adjunction. Then, we define the map  $\psi : [A, A] \longrightarrow A^* \otimes A$  by

$$\psi := [A, A] \xrightarrow{\lambda_{[A,A]}^{-1}} I \otimes [A, A] \xrightarrow{\eta \otimes [A,A]} A^* \otimes A \otimes [A, A] \xrightarrow{A^* \otimes ev_{A,A}} A^* \otimes A.$$

To see that  $\text{mix} \circ \psi$  is the identity of  $[A, A]$  observe that its transpose is the evaluation map, as witnessed by the commutativity of the diagram

$$\begin{array}{ccccc}
A \otimes I \otimes [A, A] & \xrightarrow{A \otimes \eta \otimes [A, A]} & A \otimes A^* \otimes A \otimes [A, A] & \xrightarrow{A \otimes A^* \otimes ev_{A, A}} & A \otimes A^* \otimes A \\
\uparrow \rho_A^{-1} \otimes [A, A] = A \otimes \lambda_{[A, A]}^{-1} & & \downarrow ev_{A, I} \otimes A \otimes [A, A] & & \downarrow ev_{A, I} \otimes A \\
A \otimes [A, A] & \xrightarrow{\lambda_A^{-1} \otimes [A, A]} & I \otimes A \otimes [A, A] & \xrightarrow{I \otimes ev_{A, A}} & I \otimes A \\
& & \downarrow \lambda_{A \otimes [A, A]} & & \downarrow \lambda_A \\
& & A \otimes [A, A] & \xrightarrow{ev_{A, A}} & A
\end{array}
\quad \begin{array}{l} \\ \\ \\ \\ \end{array} \left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right) ev_{A, A} \circ (A \otimes \text{mix})$$

Also, commutativity of the diagram

$$\begin{array}{ccccc}
A^* \otimes A & \xrightarrow{\text{mix}} & [A, A] & & \\
\downarrow \lambda_{A^* \otimes A}^{-1} = \lambda_{A^*}^{-1} \otimes A & & \downarrow \lambda_{[A, A]}^{-1} & & \\
I \otimes A^* \otimes A & \xrightarrow{I \otimes \text{mix}} & I \otimes [A, A] & & \\
\downarrow \lambda_{A^* \otimes A} & \searrow \eta \otimes A^* \otimes A & \downarrow \eta \otimes [A, A] & & \\
A^* \otimes A & \xrightarrow{\rho_{A^*}^{-1} \otimes A} & A^* \otimes I \otimes A & \xrightarrow{\rho_{A^*} \otimes A = A^* \otimes \lambda_A} & A^* \otimes A \\
& & \downarrow A \otimes ev_{A, I} \otimes A^* & & \downarrow A^* \otimes ev_{A, A} \\
& & A^* \otimes A \otimes A^* \otimes A & \xrightarrow{A^* \otimes A \otimes \text{mix}} & A^* \otimes A \otimes [A, A]
\end{array}$$

shows that  $\psi \circ \text{mix} = \text{id}_{A^* \otimes A}$ .  $\square$

## 7. FROM NUCLEARITY TO FROBENIUS STRUCTURES

The next two sections present our main results. First we show how nuclearity yields Frobenius structures on the objects on the internal hom  $[A, A]$ .

**Theorem 38.** *If  $A$  is a nuclear object in a  $*$ -autonomous category  $\mathcal{V}$ , then there is a map  $l$  such that  $([A, A], [A, A]^*, ev_{[A, A], I}, \circ, l, l)$  is a Frobenius structure.*

*Proof.* As mentioned in Example 4, the object  $A^* \otimes A$  has the semigroup structure  $\mu_{A^* \otimes A} = \rho_{A^*} \circ (A^* \otimes \lambda_A) \circ (A^* \otimes ev_{A, I} \otimes A)$ , see e.g. [25]. It is easily seen that  $\epsilon \circ (A^* \otimes A \otimes \text{mix}) = ev_{A, I} \circ \sigma_{A^*, A} \circ \mu_{A^* \otimes A}$ , which immediately ensures that the map  $\epsilon \circ (A^* \otimes A \otimes \text{mix})$ , that is, the transpose of  $\text{mix}$ , is associative. It is also verified that  $\epsilon \circ (A^* \otimes A \otimes \text{mix}) \circ \sigma_{A^* \otimes A, A^* \otimes A} = \epsilon \circ (A^* \otimes A \otimes \text{mix})$ , whence  $\text{mix}$  is self-adjoint.

It follows that, if  $\text{mix}$  is invertible, then  $(A^* \otimes A, [A, A], \epsilon, \mu_{A^* \otimes A}, \text{mix}, \text{mix})$  and  $([A, A], A^* \otimes A, \epsilon \circ \sigma_{[A, A], A^* \otimes A}, \mu_{[A, A]}, \text{mix}^{-1}, \text{mix}^{-1})$  are Frobenius structures. Since  $\text{mix}$  is a semigroup homomorphism from  $(A^* \otimes A, \mu_{A^* \otimes A})$  to  $([A, A], \circ)$ , then  $\mu_{[A, A]}$  is the standard monoid structure  $\circ$  induced from composition in  $[A, A]$ . There is a canonical isomorphism  $\psi : A^* \otimes A \cong [A, A]^*$  such that  $\epsilon \circ ([A, A] \circ \psi) = ev_{[A, A], I}$ , see Example 15. Then, it is easily seen that  $([A, A], [A, A]^*, ev_{[A, A], I}, \circ, \psi \circ \text{mix}^{-1}, \psi \circ \text{mix}^{-1})$  is a Frobenius structure.  $\square$

Let us notice that, in view of Theorem 37 identifying nuclear objects and adjoints, the previous statement can be seen as an instance of [38, Corollary 3.3]. This statement admits a sort of generalization. Since the category  $\mathbf{SLatt}$  has an epimono factorization system which lifts to the category of quantales, we argued in [26] that, by taking the appropriate quantic nucleus, one can always obtain a Girard quantale from the image of  $\text{mix}$ . The general situation is abstracted as follows:

**Theorem 39.** *Let  $\mathcal{V}$  be a  $*$ -autonomous category with an epi-mono factorization system,  $(A, B, \epsilon)$  be a dual pairing, and  $(A, \mu_A)$  be a semigroup in  $\mathcal{V}$ . Let  $f : A \longrightarrow B$  be a map, put  $\psi_A := \epsilon \circ (A \otimes f)$  and suppose that  $\psi_A = \psi_A \circ \sigma_{A,A}$ . Factor  $f$  as  $f = m \circ e$  with  $e : A \longrightarrow C$  epi and  $m : C \longrightarrow B$  mono. If  $C$  is a magma with multiplication  $\mu_C$  and  $e$  is a magma homomorphism, then there exists a map  $g : C \longrightarrow C^*$  making  $(C, C^*, ev_{C,I}, \mu_C, g, g)$  into a Frobenius structure.*

To prove the theorem, we need a Lemma relating factorization systems to dual pairs. If  $f = m \circ e$  with  $e$  epi and  $m$  mono, then we denote by  $Im(f)$  the codomain of  $e$ .

**Lemma 40.** *Let  $\mathcal{V}$  be  $*$ -autonomous with an epi-mono factorization system. If  $(A, B)$  is a dual pair and  $f : A \longrightarrow B$  then  $(Im(f), Im(\tilde{f}))$  is a dual pair.*

*Proof.* As from Proposition 13.1, let  $\phi : A \longrightarrow B^*$  and  $\psi : B \longrightarrow A^*$  be isomorphisms such that  $\phi = \psi^\perp$ . We pretend that, up to these isomorphisms,  $\tilde{f}$  equals  $f^*$ , as stated below.

*Claim 1.* We have  $\psi \circ \tilde{f} = f^* \circ \phi$ .

Let now  $(e, m)$  and  $(\tilde{e}, \tilde{m})$  be epi-mono factorizations of  $f$  and  $\tilde{f}$ , respectively. By applying the duality functor  $(-)^*$ ,  $(m^*, e^*)$  (resp.  $(\tilde{m}^*, \tilde{e}^*)$ ) is an epi-mono factorization of  $f^*$  (resp.  $\tilde{f}^*$ ). Therefore, we obtain two different epi-mono factorizations of  $\psi \circ \tilde{f} = f^* \circ \phi$  (resp.  $\psi \circ f = \tilde{f}^* \circ \phi$ ), as follows:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & \xrightarrow{\tilde{f}} & B \\
 \searrow \tilde{e} & & \nearrow \tilde{m} \\
 & Im(\tilde{f}) & \\
 & \downarrow \chi & \\
 & Im(f)^* & \\
 \nearrow m^* & & \searrow e^* \\
 B^* & \xrightarrow{f^*} & A^*
 \end{array}
 &
 \begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \searrow e & & \nearrow m \\
 & Im(f) & \\
 & \downarrow \xi & \\
 & Im(\tilde{f})^* & \\
 \nearrow \tilde{m}^* & & \searrow \tilde{e}^* \\
 B^* & \xrightarrow{\tilde{f}^*} & A^*
 \end{array}
 \end{array}$$

and therefore isomorphisms  $\chi : Im(\tilde{f}) \longrightarrow Im(f)^*$  and  $\xi : Im(f) \longrightarrow Im(\tilde{f})^*$ . In order to conclude that we have a dual pair, we need to argue that  $\xi = \chi^\perp$ . To this goal, compute as follows:

$$(\chi \circ \tilde{e})^\perp \circ e = (e^* \circ \chi \circ \tilde{e})^\perp = (f^* \circ \phi)^\perp = \phi^\perp \circ f = \psi \circ f = \tilde{e}^* \circ \xi \circ e.$$

From this, considering that  $e$  is epi, it follows that  $(\chi \circ \tilde{e})^\perp = \tilde{e}^* \circ \xi$ . By applying the operator  $(-)^{\perp}$  to both sides of this equality, we obtain  $\chi \circ \tilde{e} = (\tilde{e}^* \circ \xi)^\perp = \xi^\perp \circ \tilde{e}$ , from which the desired equality  $\chi = \xi^\perp$  follows.  $\square$

*Proof of Theorem 39.* Let  $A, B$  be dual and let  $f : A \longrightarrow B$  be such that  $\tilde{f} = f$ . Let  $e : A \longrightarrow C$  and  $m : C \longrightarrow B$  be a factorization of  $f$ . Lemma 40 exhibits  $(C, C)$  as a dual pair with isomorphism  $\chi, \xi : C \longrightarrow C^*$  such that  $\chi^\perp = \xi = \chi$ . Define therefore the pairing  $\pi_C := ev_{C,I} \circ (C \otimes \chi) : C \otimes C \longrightarrow I$  and observe that it is symmetric. The diagram in the proof of Lemma 40 also exhibits the equality  $\psi \circ f = e^* \circ \chi \circ e : A \longrightarrow A^*$ . Transposing this relation, we obtain the pairing

$$\pi_A = (\psi \circ f)^\flat = (e^* \circ \chi \circ e)^\flat = \chi^\flat \circ (e \otimes e) = \pi_C \circ (e \otimes e).$$

Thus, assuming that  $f : A \longrightarrow B$  is a semigroup homomorphism factoring as  $f = m \circ e$  with  $e : (A, \mu_A) \longrightarrow (C, \mu_C)$  a magma homomorphisms, we have that  $e : (A, \mu_A, \pi_A) \longrightarrow (C, \mu_C, \pi_C)$  is a bracketed magma homomorphism. As argued in Proposition 8,  $(C, \mu_C, \pi_C)$  is an associative bracketed magma, and, as mentioned in Proposition 31, this is enough to have a Frobenius structure on the dual pair  $(C, C^*)$ .  $\square$

## 8. FROM FROBENIUS STRUCTURES TO NUCLEAR OBJECTS

We finally give the converse of Theorem 38. However, to do so we impose additional conditions to the ambient  $*$ -autonomous category  $\mathcal{V}$ .

**Definition 41.** We say that an object  $A$  of a monoidal category  $\mathcal{V}$  is *pseudo-affine* if the tensor unit  $I$  embeds into  $A$  as a retract. If every object of  $\mathcal{V}$  which is not terminal nor initial is pseudo-affine, then  $\mathcal{V}$  is said to be *pseudo-affine*.

For instance, the category  $\mathbf{SLatt}$  is pseudo-affine. This property was actually used in [34] to prove that if  $[L, L]$  is endowed with a Frobenius quantale structure, then  $L$  is completely distributive. Before going there, let us introduce some useful observations.

**Lemma 42.** *Tensoring with a pseudo-affine object yields a faithful functor.*

*Proof.* Let  $A$  be pseudo-affine and suppose that  $f, g : X \longrightarrow Y$  and consider the following diagram:

$$\begin{array}{ccc}
 & X & \\
 p_X^\ell \swarrow & & \searrow p_X^\rho \\
 A \otimes X & & X \otimes C \\
 p_{A \otimes X}^\rho = A \otimes p_X^\rho \searrow & & \swarrow p_{X \otimes C}^\ell = p_X^\ell \otimes C \\
 & A \otimes X \otimes C &
 \end{array}$$

If  $f \otimes A = g \otimes A : X \otimes A \longrightarrow Y \otimes A$ , then also  $p_Y^\rho \circ f = p_Y^\rho \circ g$  and then  $f = g$ , since  $p_Y^\rho$  is monic.  $\square$

**Proposition 43.** *Identity morphisms of pseudo-affine objects are orthogonal to each other, in the following sense: if  $A, C$  are pseudo-affine,  $f : A \otimes B_0 \longrightarrow A \otimes B_1$ ,  $g : B_0 \otimes C \longrightarrow B_1 \otimes C$ , and  $f \otimes C = A \otimes g : A \otimes B_0 \otimes C \longrightarrow A \otimes B_1 \otimes C$ , then there exists  $h : B_0 \longrightarrow B_1$  such that  $f = A \otimes h$  and  $g = h \otimes C$ .*

*Proof.* First, using that  $C$  is pseudo-affine, we have

$$\begin{aligned}
 f &= c_{A \otimes B_0}^\rho \circ (f \otimes C) \circ p_{A \otimes B_0}^\rho = c_{A \otimes B_1}^\rho \circ (A \otimes g) \circ p_{A \otimes B_0}^\rho \\
 &= (A \otimes c_{B_1}^\rho) \circ (A \otimes g) \circ A \otimes (p_{B_0}^\rho) = A \otimes (c_{B_0}^\rho \circ g \circ p_{B_0}^\rho)
 \end{aligned}$$

Using this equation and various properties of naturality, we obtain

$$\begin{aligned}
 c_{B_1}^\ell \circ f \circ p_{B_0}^\ell &= c_{B_1}^\ell \circ (A \otimes (c_{B_0}^\rho \circ g \circ p_{B_0}^\rho)) \circ p_{B_0}^\ell \\
 &= c_{B_1}^\ell \circ (A \otimes c_{B_1}^\rho) \circ (A \otimes g) \circ (A \otimes p_{B_0}^\rho) \circ p_{B_0}^\ell \\
 &= c_{B_1}^\ell \circ c_{A \otimes B_1}^\rho \circ (A \otimes g) \circ p_{A \otimes B_0}^\rho \circ p_{B_0}^\ell \\
 &= c_{B_1}^\rho \circ c_{B_1 \otimes C}^\ell \circ (A \otimes g) \circ p_{B_0 \otimes C}^\ell \circ p_{B_0}^\rho \\
 &= c_{B_1}^\rho \circ g \circ p_{B_0}^\rho.
 \end{aligned}$$

Therefore, we can define  $h := c_{B_1}^\ell \circ f \circ p_{B_0}^\ell = c_{B_1}^\rho \circ g \circ p_{B_0}^\rho$  which complete the proof of the statement.  $\square$

The following is the main result in this section.

**Theorem 44.** *If  $A$  is pseudo-affine and the semigroup  $([A, A], \circ)$  can be completed to a Frobenius structure, then  $A$  is part of an adjunction.*

*Proof.* We can suppose that the dual of  $[A, A]$  is  $A^* \otimes A$ . The dual multiplication  $\mu_{A^* \otimes A} : A^* \otimes A \otimes A^* \otimes A \longrightarrow A^* \otimes A$  arises from a map  $f : A^* \otimes A \longrightarrow [A, A]$  as the diagonal of

$$\begin{array}{ccc} A^* \otimes A \otimes A^* \otimes A & \xrightarrow{A^* \otimes A \otimes \tilde{f}} & A^* \otimes A \otimes [A, A] \\ f \otimes A^* \otimes A \downarrow & \searrow \mu_{A^* \otimes A} & \downarrow A^* \otimes \epsilon v_{A, A} \\ [A, A] \otimes A^* \otimes A & \xrightarrow{\mu_{A, A, I} \otimes A} & A^* \otimes A \end{array}$$

the diagram on the right. Since  $A$  is pseudo-affine, then, by Proposition 43, we necessarily have

$$\mu_{A^* \otimes A} = (A^* \otimes \lambda_A) \circ (A^* \otimes \epsilon \otimes A)$$

for a map  $\epsilon : A \otimes A^* \longrightarrow I$ .

Moreover, since  $([A, A], \circ)$  is unital and  $f$  is an isomorphism,  $\mu_{A^* \otimes A}$  has a unit  $\eta : I \longrightarrow A^* \otimes A$ , which means that the following diagrams commute:

$$\begin{array}{ccccccc} A^* \otimes A & & A^* \otimes A & \xrightarrow{\lambda_{A^*}^{-1} \otimes A} & I \otimes A^* \otimes A & \xrightarrow{\eta \otimes A^* \otimes A} & A^* \otimes A \otimes A^* \otimes A \\ \downarrow A^* \otimes \rho_A^{-1} & & & \searrow A^* \otimes A & & & \downarrow A^* \otimes \epsilon \otimes A \\ A^* \otimes A \otimes I & & & & & & A^* \otimes I \otimes A \\ \downarrow A^* \otimes A \otimes \eta & & & & & & \downarrow \rho_{A^* \otimes A} = A^* \otimes \lambda_A \\ A^* \otimes A \otimes A^* \otimes A & \xrightarrow{A^* \otimes \epsilon \otimes A} & A^* \otimes I \otimes A & \xrightarrow{A^* \otimes \lambda_A} & A^* \otimes A & & A^* \otimes A \end{array}$$

Since tensoring with  $A$  and  $A^*$  is faithful, commutativity of the above diagrams is equivalent to the equalities

$$\lambda_A \circ (\epsilon \otimes A) \circ (A \otimes \eta) \circ \rho_A^{-1} = id_A, \quad id_{A^*} = \rho_A \circ (A^* \otimes \epsilon) \circ (\eta \otimes A^*) \circ \lambda_A^{-1},$$

showing that  $(A, A^*, \eta, \epsilon)$  is an adjunction.  $\square$

In view of Theorem 37, we can restate Theorem 44 as follows:

**Theorem 45.** *If  $A$  is pseudo-affine and  $([A, A], \circ)$  carries a Frobenius structure, then  $A$  is nuclear.*

## 9. DISCUSSION AND FUTURE WORK

The work [37] provides a wide playground where to test the strength of the results presented in this paper. For example, for a commutative Girard quantale  $Q$ , the category  $Q\text{Set}$ , whose objects are the pairs  $(\alpha, X)$  with  $\alpha : X \longrightarrow Q$  and whose morphisms  $(\alpha, X) \longrightarrow (\beta, Y)$  are the relations  $R \subseteq X \times Y$  such that  $xRy$  implies  $\alpha(x) \leq \beta(y)$ , is  $*$ -autonomous. If the relation  $1 = 1^\perp$  holds in  $Q$ , then the unit of the monoidal structure is a dualizing object. It is not difficult to characterize

pseudo-affine and nuclear objects in this category. It turns out that if a monoid on  $[(\alpha, X), (\alpha, X)]$  can be endowed with a Frobenius structure, then  $(\alpha, X)$  is nuclear, even when  $(\alpha, X)$  is not pseudo-affine. That is, this sort of model shows that, in Theorem 44, the assumption that an object is pseudo-affine is not necessary. We are leaving it open whether this assumption can be eliminated.

The notion of nuclearity was originally conceived for Banach spaces [20]. In [26] we have shown how to construct unitless Girard quantales from nuclear endomaps (trace class operators) of an infinite dimensional Hilbert space. It is natural to ask how the results presented in this paper may be extended to categories of Banach spaces and other similar categories (*e.g.* the category of Hilbert spaces). Most of these categories are symmetric monoidal; however, they are not  $*$ -autonomous and, in the case of Hilbert spaces and continuous linear maps, not even autonomous. Even though the definitions of dual pair and of Frobenius structure only require the environment category to be symmetric monoidal, a closer look at our results exhibits their dependency on  $*$ -autonomy. For example, we have used in Theorem 39 the fact that, for  $m$  mono,  $m^*$  is epi. In categories of Banach spaces, due to the Hanh-Banach theorem, the theorem might still hold if  $m$  is a *regular* mono, that is, an isometry. However, when considering the algebra of trace class operators—which, given the analogy with the tight endomaps of [26], is the natural candidate where to apply Theorem 39—the relevant map  $m$  is not an isometry. We also have used in Theorem 45 and elsewhere that  $(A^* \otimes A, [A, A])$  is a dual pair. This fails in autonomous categories, for example, if  $A$  is the reflexive Banach space  $\ell_p$  with  $1 < p < \infty$ , then the tensor  $A^* \otimes A$  is no longer reflexive, see [5]. One might restrict to categories of finite dimensional Banach space, which are  $*$ -autonomous. However, these categories turn out to be uninteresting: if bounded linear maps are taken as morphisms, then all object are nuclear, and if we restrict maps to linear contractions (those linear maps for which  $\|f\| \leq 1$ ), then the only nuclear objects are isomorphic to the monoidal unit [32, §4].

Future research will continue investigating applications of these theorems in concrete  $*$ -autonomous categories. A main difficulty here will be finding adequate characterizations of nuclear and pseudo-affine objects. We shall also investigate how to relax  $*$ -autonomy, which might yield results on categories appealing for a wider community of computer scientists and mathematicians—and a step, on our side, towards categorical models of quantum computing.

The considerations just developed suggest, on the other hand, the existence of a close interdependence between provability models of classical linear logic (Frobenius quantales and, in a wider sense, Frobenius structures) and models of proofs of this logic (the  $*$ -autonomous categories). It is our desire to investigate further whether there is any relation between these structures, similarly to what happens for intuitionistic logic with the Heyting algebra of truth values of a topos. Steps in this direction have already been taken, see *e.g.* [37, 8].

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