



HAL
open science

On symmetries of singular foliations

Ruben Louis

► **To cite this version:**

| Ruben Louis. On symmetries of singular foliations. 2023. hal-03738952v2

HAL Id: hal-03738952

<https://hal.science/hal-03738952v2>

Preprint submitted on 20 Apr 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON SYMMETRIES OF SINGULAR FOLIATIONS

RUBEN LOUIS

ABSTRACT. This paper shows that a weak symmetry action of a Lie algebra \mathfrak{g} on a singular foliation \mathcal{F} induces a unique up to homotopy Lie ∞ -morphism from \mathfrak{g} to the DGLA of vector fields on a universal Lie ∞ -algebroid of \mathcal{F} . Such a morphism is known as L_∞ -algebra action in [24]. We deduce from this general result several geometrical consequences. For instance, we give an example of a Lie algebra action on an affine sub-variety which cannot be extended on the ambient space. Last, we introduce the notion of bi-submersion towers over a singular foliation and lift symmetries to those.

CONTENTS

Introduction	1
Acknowledgement	3
1. Definitions and examples of weak and strict symmetry actions	3
2. A Lie ∞ -morphism lifting a weak symmetry of a foliation	6
2.1. Cohomology of longitudinal graded vector fields	8
2.2. Proof of the main results	11
2.3. Particular examples	15
3. Lifts of weak symmetry actions and Lie ∞ -algebroids	15
4. On weak and strict symmetries: an obstruction theory	17
5. Bi-submersion towers and symmetries	21
5.1. Definitions and existence	22
5.2. Lift of a symmetry to the bi-submersion tower	26
5.3. Lifts of actions of a Lie algebra on a bi-submersion tower	30
Appendix A. Universal Lie ∞ -algebroids	31
Appendix B. Lie ∞ -morphisms of DGLA and homotopies	32
Appendix C. Proof of Theorem 3.3	37
References	38

INTRODUCTION

Singular foliations arise frequently in differential or algebraic geometry. Here, following [1, 5, 8, 9, 19], we define a singular foliation on a smooth, complex, algebraic, real analytic manifold M with sheaf of functions \mathcal{O} to be a subsheaf $\mathcal{F}: U \rightarrow \mathcal{F}(U)$ of the sheaf of vector fields \mathfrak{X} , which is closed under the Lie bracket and locally finitely generated as a \mathcal{O} -module. By Hermann's theorem [15], this is enough to induce a partition of the manifold M into immersed submanifolds of possibly different dimensions, called *leaves* of the singular foliation. Singular foliations appear for instance as orbits of Lie group actions with possibly different dimensions

Date: April 20, 2023.

The author acknowledges support of the CNRS Project Miti 80Prime *Granum*.

or as symplectic leaves of a Poisson structure. When all the leaves have the same dimension, we recover the usual “regular foliations” [10, 25].

The purpose of this paper is to look at symmetries of singular foliations. Let (M, \mathcal{F}) be a foliated manifold. A *global symmetry* of a singular foliation \mathcal{F} on M is a diffeomorphism $\phi: M \rightarrow M$ which preserves \mathcal{F} , that is, $\phi_*(\mathcal{F}) = \mathcal{F}$. The image of a leaf through a global symmetry is again a leaf (not necessarily the same leaf).

For G a Lie group, a *strict symmetry action* of G on a foliated manifold (M, \mathcal{F}) is a smooth action $G \times M \rightarrow M$ that acts by global symmetries [14]. Infinitesimally, it corresponds to a Lie algebra morphism $\mathfrak{g} \rightarrow \mathfrak{X}(M)$ between the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ of G and the Lie algebra of symmetries of \mathcal{F} , i.e., vector fields $X \in \mathfrak{X}(M)$ such that $[X, \mathcal{F}] \subseteq \mathcal{F}$. A strict symmetry action of G on M goes down to the leaf space M/\mathcal{F} , even though the latter space is not a manifold. The opposite direction is more sophisticated, since a strict symmetry action of G on M/\mathcal{F} does not induce a strict action over M in general. However, it makes sense to consider linear maps $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ that satisfy $[\varrho(x), \mathcal{F}] \subseteq \mathcal{F}$ for all $x \in \mathfrak{g}$, and which are Lie algebra morphisms up to \mathcal{F} , namely, $\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] \in \mathcal{F}$ for all $x, y \in \mathfrak{g}$. The latter linear maps are called “*weak symmetry actions*”. These actions induce a “strict action” on the leaf space i.e., a Lie algebra morphism $\mathfrak{g} \rightarrow \mathfrak{X}(M/\mathcal{F})$, whenever M/\mathcal{F} is a manifold, and an action of G on M/\mathcal{F} , at least if G is connected.

Now, on a priori different subject. Let us emphasize on the following observation: An infinitesimal action of a Lie algebra \mathfrak{g} on a manifold M is a Lie algebra morphism $\mathfrak{g} \rightarrow \mathfrak{X}(M)$. Replacing M by a Lie ∞ -algebroid (E, Q) seen as a Q -manifold, one expects to define Lie algebra actions on (E, Q) as Lie ∞ -algebra morphisms $\mathfrak{g}[1] \rightarrow \mathfrak{X}(E, Q)[1]$, the latter space being a DGLA of vector fields on that Q -manifold [17]. Such Lie ∞ -morphisms were studied by Mehta and Zambon [24] as “ L_{∞} -algebra actions”, and various results about those are given. In particular, these authors give several equivalent definitions and interpretations of those. It is easy to check that such a Lie ∞ -morphism induces a weak symmetry action of \mathfrak{g} on the singular foliation induced by Q .

In [19, 22], it is shown that behind every singular foliation or more generally any Lie-Rinehart algebra [20] there exists a Lie ∞ -algebroid structure which is unique up to homotopy called the *universal* Lie ∞ -algebroid. Here is a natural question: what does a symmetry of a singular foliation \mathcal{F} induce on a universal Lie ∞ -algebroid of \mathcal{F} ? Theorem 2.4 of this paper gives an answer to that question. It states that any weak symmetry action of a Lie algebra on a singular foliation \mathcal{F} can be lifted to a Lie ∞ -morphism valued in the DGLA of vector fields on a universal Lie ∞ -algebroid of \mathcal{F} . Furthermore, Theorem 2.4 says this lift is unique modulo homotopy equivalence. This goes in the same direction as [14] which already underlined Lie-2-group structures associated to strict symmetry action of Lie groups.

This result gives several geometric consequences. Here is an elementary question: can a Lie algebra action $\mathfrak{g} \rightarrow \mathfrak{X}(W)$ on an affine variety $W \subset \mathbb{C}^d$ be extended to a Lie algebra action $\mathfrak{g} \rightarrow \mathfrak{X}(\mathbb{C}^d)$ on \mathbb{C}^d ? Said differently: it is trivial that any vector field on W extends to \mathbb{C}^d , but can this extension be done in such a manner that it preserves the Lie bracket? Although no “ ∞ -oids” appears in this question, which seems to be a pure algebraic geometry question, we claim that the answer goes through Lie ∞ -algebroids and singular foliations. More precisely, the idea is then to say that any \mathfrak{g} -action on W induces a weak symmetry action on the singular foliation $\mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$ of all vector fields vanishing on W (here \mathcal{I}_W is the ideal that defines W). By Theorem 2.4, we know that it is possible to lift any weak symmetry action of singular foliation

into a Lie ∞ -morphism. The second order Taylor coefficient of that Lie ∞ -morphism, composed with the projection on vector fields of arity -1 , is of the form $\iota_{\eta(x,y)}$ where $\eta: \wedge^2 \mathfrak{g} \rightarrow \Gamma(E_{-1})$ satisfies $\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] = \rho(\eta(x, y))$, (here $\rho: E_{-1} \rightarrow TM$ is the anchor map of a universal Lie ∞ -algebroid (E, Q) of \mathcal{F} and ι_e stands for the vertical vector field associated to a section $e \in \Gamma(E_{-1})$). But is it possible to build such a Lie ∞ -morphism where the arity -1 of the second order Taylor coefficient is zero? There are cohomological obstructions. In some specific cases, obstruction classes appear on some cohomology, although in general the obstruction is rather a Maurer-Cartan-like equations that may or may not have solutions. We show if this class is non-zero, then we cannot manage to have $\eta = 0$, and then no strict action exists.

The outline of this paper is as follows: In Section 1 we present some definitions and facts on weak symmetry actions of Lie algebras on singular foliations and give some examples. In Section 2 we state the main results of this paper and present their proofs. In Section 3 we describe the relation between weak symmetry actions and Lie ∞ -algebroids that have some special properties. In Section 4 we define an obstruction class for extending a Lie algebra action on an affine variety to ambient space. In the last section of the paper, we introduce the notion of “bi-submersion tower” over singular foliations that we denote by \mathcal{T}_B . The latter notion, as the name suggests, is a family of “bi-submersions” which are built one over the other. The concept of bi-submersion over singular foliations has been introduced in [1] and it is used in K -theory [3] or differential geometry [2, 5, 13]. We show that such a bi-submersion tower over a singular foliation \mathcal{F} exists if and only if \mathcal{F} admits a geometric resolution. Provided that it exists, we show in Theorem 5.25 that any infinitesimal action of a Lie algebra \mathfrak{g} on the singular foliation \mathcal{F} lifts to the bi-submersion tower \mathcal{T}_B .

In Appendix A and B, we review Lie ∞ -algebroid structures and their morphisms and homotopies in order to fix notations.

ACKNOWLEDGEMENT

I am very grateful to my supervisor, Professor Camille Laurent-Gengoux, for guiding and giving me useful ideas all along of this paper. I would also like to acknowledge the full financial support for this PhD from Région Grand Est. Also, I thank the CNRS MITI 80Prime project GRANUM, and the Institut Henri Poincaré for hosting me in November 2021. I would like to thank the referee for his/her careful reading, and his/her advices that helped to improve the presentation.

1. DEFINITIONS AND EXAMPLES OF WEAK AND STRICT SYMMETRY ACTIONS

Convention 1.1. *Throughout this paper, M stands for a smooth or complex manifold, or an affine variety over \mathbb{C} . We will denote the sheaf of smooth or complex, or regular functions on M by \mathcal{O} and the sheaf of vector fields on M by $\mathfrak{X}(M)$, and $X[f]$ stands for a vector field $X \in \mathfrak{X}(M)$ applied to $f \in \mathcal{O}$. Also, \mathbb{K} stands for \mathbb{R} or \mathbb{C} .*

Definition 1.2. Let $\mathcal{F} \subset \mathfrak{X}(M)$ be a singular foliation on M .

- A diffeomorphism $\phi: M \rightarrow M$ is said to be a *symmetry* of \mathcal{F} , if $\phi_*(\mathcal{F}) = \mathcal{F}$.
- A vector field $X \in \mathfrak{X}(M)$ is said to be an *infinitesimal symmetry* of \mathcal{F} , if $[X, \mathcal{F}] \subset \mathcal{F}$.

The Lie algebra of infinitesimal symmetries of \mathcal{F} is denoted by $\mathfrak{s}(\mathcal{F})$.

In particular, $\mathcal{F} \subset \mathfrak{s}(\mathcal{F})$, since $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$. The latter are called *internal symmetries* of \mathcal{F} .

Proposition 1.3. [1, 12] *Let M be a smooth or complex manifold. The flow of an infinitesimal symmetry of \mathcal{F} , if it exists, is a symmetry of \mathcal{F} .*

As we will see in Section 2, one of the consequences of our future results is that any symmetry $X \in \mathfrak{s}(\mathcal{F})$ of a singular foliation \mathcal{F} admits a lift to a degree zero vector field on any universal NQ -manifold over \mathcal{F} that commutes with the homological vector field Q . This allows us to have an alternative proof and interpretation of Proposition 1.3.

Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a Lie algebra over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , depending on the context.

Definition 1.4. A *weak symmetry action* of the Lie algebra \mathfrak{g} on a singular foliation \mathcal{F} on M is a \mathbb{K} -linear map $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ that satisfies:

- $\forall x \in \mathfrak{g}, [\varrho(x), \mathcal{F}] \subseteq \mathcal{F}$,
- $\forall x, y \in \mathfrak{g}, \varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] \in \mathcal{F}$.

When $x \mapsto \varrho(x)$ is a Lie algebra morphism, we speak of *strict symmetry action* of \mathfrak{g} on \mathcal{F} . There is an equivalence relation on the set of weak symmetry actions which is defined as follows: two weak symmetry actions, $\varrho, \varrho': \mathfrak{g} \rightarrow \mathfrak{X}(M)$ are said to be *equivalent* if there exists a linear map $\varphi: \mathfrak{g} \rightarrow \mathcal{F}$ such that $\varrho - \varrho' = \varphi$.

Remark 1.5. It is important to notice that when \mathcal{F} is a regular foliation and M/\mathcal{F} is a manifold, any weak symmetry action of a Lie algebra \mathfrak{g} on \mathcal{F} induces a strict action of \mathfrak{g} over M/\mathcal{F} . Definition 1.4 is a way of extending this idea to all singular foliations.

Here is a list of examples.

Example 1.6. Let $\pi: M \rightarrow N$ be a submersion. Since any vector field on N comes from a π -projectable vector field on M , any Lie algebra morphism $\mathfrak{g} \rightarrow \mathfrak{X}(N)$ can be lifted to a weak symmetry action $\mathfrak{g} \rightarrow \mathfrak{X}(M)$ on the regular foliation $\Gamma(\ker d\pi)$, and any two such lifts are equivalent.

Furthermore, any weak action of a Lie algebra \mathfrak{g} on a singular foliation \mathcal{F} on N can be lifted to a class of weak symmetry actions on the pull-back foliation $\pi^{-1}(\mathcal{F})$, (see Definition 1.9 in [1]).

Example 1.7. Let \mathcal{F} be a singular foliation on M . For any point $m \in M$, the set $\mathcal{F}(m) = \{X \in \mathcal{F} \mid X(m) = 0\}$ is a Lie subalgebra of \mathcal{F} . Put $\mathcal{I}_m = \{f \in C^\infty(M) \mid f(m) = 0\}$. The quotient space $\mathfrak{g}_m = \frac{\mathcal{F}(m)}{\mathcal{I}_m \mathcal{F}}$ is a Lie algebra, since $\mathcal{I}_m \mathcal{F} \subseteq \mathcal{F}(m)$ is a Lie ideal. The Lie algebra \mathfrak{g}_m is called the *isotropy Lie algebra of \mathcal{F} at m* (see [4]). Let us denote, by $[\cdot, \cdot]_{\mathfrak{g}_m}$, its Lie bracket.

- (1) Consider $\varrho: \mathfrak{g}_m \rightarrow \mathcal{F}(m) \subset \mathfrak{X}(M)$ a section of the projection map,

$$\mathcal{I}_m \mathcal{F} \hookrightarrow \mathcal{F}(m) \xrightleftharpoons{\varrho} \mathfrak{g}_m \quad (1)$$

Then, $[\varrho(x), \mathcal{I}_m \mathcal{F}] \subset \mathcal{I}_m \mathcal{F}$ and $\varrho([x, y]_{\mathfrak{g}_m}) - [\varrho(x), \varrho(y)] \in \mathcal{I}_m \mathcal{F}$. Hence, the map $\varrho: \mathfrak{g}_m \rightarrow \mathfrak{X}(M)$ is a weak symmetry action of the singular foliation $\mathcal{I}_m \mathcal{F}$. A different section ϱ' of the projection map yields an equivalent weak symmetry action of \mathfrak{g}_m on $\mathcal{I}_m \mathcal{F}$. An obstruction class for having a strict symmetry action equivalent to ϱ will be given later in Section 4.

- (2) In particular, for $k \geq 1$, let us denote by \mathfrak{g}_m^k the isotropy Lie algebra of the singular foliation $\mathcal{I}_m^k \mathcal{F}$ at m . Every section $\varrho_k: \mathfrak{g}_m^k \rightarrow \mathfrak{X}(M)$ of the projection map

$$\mathcal{I}_m^{k+1} \mathcal{F} \hookrightarrow \mathcal{I}_m^k \mathcal{F} \xrightleftharpoons{\varrho_k} \mathfrak{g}_m^k \quad (2)$$

is a weak symmetry action of the Lie algebra \mathfrak{g}_m^k on the singular foliation $\mathcal{I}_m^{k+1} \mathcal{F}$.

Example 1.8. The following example comes from [21], and follows the same patterns as in Examples 1.6 and 1.7. Let (M, \mathcal{F}) be a singular foliation on a smooth manifold M and $L \subset M$ a leaf. Let $[L, M]$ be a neighborhood of L in M equipped with some projection $\pi: M \rightarrow L$. According to [21], upon replacing $[L, M]$ by a smaller neighborhood of L if necessary, there exists an Ehresmann connections (that is a vector sub-bundle $H \subset T[L, M]$ with $H \oplus \ker(d\pi) = T[L, M]$) which satisfies that $\Gamma(H) \subset \mathcal{F}$. Such an Ehresmann connection is called an *Ehresmann \mathcal{F} -connection* and induces a $C^\infty(L)$ -linear section $\varrho_H: \mathfrak{X}(L) \rightarrow \mathcal{F}^{\text{proj}}$ of the surjection $\mathcal{F}^{\text{proj}} \rightarrow \mathfrak{X}(L)$, where $\mathcal{F}^{\text{proj}}$ stands for vector fields of \mathcal{F} π -projectable on elements of $\mathfrak{X}(L)$. The section ϱ_H is a weak symmetry action of $\mathfrak{X}(L)$ on the *transverse* foliation $\mathcal{T} := \Gamma(\ker d\pi) \cap \mathcal{F}$. When the Ehresmann connection H is flat, ϱ_H is bracket-preserving, and defines a strict symmetry of $\mathfrak{X}(L)$ on the transverse foliation \mathcal{T} .

Example 1.9. Consider, for a fixed $k \in \mathbb{N}_0$, the singular foliation $\mathcal{F}_k := \mathcal{I}_0^k \mathfrak{X}(\mathbb{R}^d)$ of all vector fields on \mathbb{R}^d vanishing at order k at the origin. The action of the Lie algebra $\mathfrak{gl}(\mathbb{R}^d)$ on \mathbb{R}^d which is given by

$$\mathfrak{gl}(\mathbb{R}^d) \rightarrow \mathfrak{X}(\mathbb{R}^d), (a_{ij})_{1 \leq i, j \leq d} \mapsto \sum_{1 \leq i, j \leq d} a_{ij} x_i \frac{\partial}{\partial x_j}$$

is a strict symmetry of \mathfrak{g} on \mathcal{F}_k .

Example 1.10. Let $\varphi := (\varphi_1, \dots, \varphi_r)$ be a r -tuple of homogeneous polynomial functions in d variables over \mathbb{K} . Consider the singular foliation \mathcal{F}_φ (see [20] Section 3.2.1) which is generated by all polynomial vector fields $X \in \mathfrak{X}(\mathbb{K}^d)$ that satisfy $X[\varphi_i] = 0$ for all $i \in \{1, \dots, r\}$. The action $\mathbb{K} \rightarrow \mathfrak{X}(\mathbb{K}^d)$, $\lambda \mapsto \lambda \vec{E}$, is a strict symmetry of \mathbb{K} on \mathcal{F}_φ . Here, \vec{E} stands for the Euler vector field.

Example 1.11. Let W be an affine variety realized as a subvariety of \mathbb{C}^d and $\mathcal{I}_W \subset \mathbb{C}[x_1, \dots, x_d]$ its vanishing ideal. Let us denote by $\mathfrak{X}(W) := \text{Der}(\mathbb{C}[x_1, \dots, x_d]/\mathcal{I}_W)$ the Lie algebra of vector fields on W . Let $\mathcal{F}_W := \mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$ the singular foliation made of vector fields vanishing on W . Since every vector field on W can be extended to a vector field on \mathbb{C}^d tangent to W , every Lie algebra morphism $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(W)$ extends to a linear map $\tilde{\varrho}: \mathfrak{g} \rightarrow \mathfrak{X}(\mathbb{C}^d)$ that makes this diagram commutes,

$$\begin{array}{ccc} & & \mathfrak{X}(\mathbb{C}^d) \\ & \nearrow \tilde{\varrho} & \downarrow \\ \mathfrak{g} & \xrightarrow{\varrho} & \mathfrak{X}(W) \end{array}$$

For $x, y \in \mathfrak{g}$, the extension $\tilde{\varrho}$ satisfies: $\tilde{\varrho}(x)[\mathcal{I}_W] \subset \mathcal{I}_W$, so that $[\tilde{\varrho}(x), \mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)] \subset \mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$. We have $\tilde{\varrho}([x, y]_{\mathfrak{g}}) - [\tilde{\varrho}(x), \tilde{\varrho}(y)] \in \mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$ because $\tilde{\varrho}$ is a Lie algebra morphism when restricted to W . Hence, $\tilde{\varrho}$ is a weak symmetry action of \mathfrak{g} on \mathcal{F}_W . Two different extensions give equivalent symmetry actions. Here is a natural question: Can we extend the Lie algebra action of \mathfrak{g} on W to a Lie algebra action on \mathbb{C}^d ? This example shows that this question can be reformulated as:

is any extension $\tilde{\varrho}$ of ϱ equivalent to a strict symmetry action? Corollary 4.12 of Section 4 gives an obstruction class of extending this weak symmetry action to a strict one.

2. A LIE ∞ -MORPHISM LIFTING A WEAK SYMMETRY OF A FOLIATION

We refer the reader to Appendix A for the notion of universal Lie ∞ -algebroids of a singular foliation and for notations. We denote them by (E, Q) and their functions by \mathcal{E} . The triple $(\mathfrak{X}_\bullet(E), [\cdot, \cdot], \text{ad}_Q)$ is a differential graded Lie algebra, where $\mathfrak{X}_\bullet(E)$ stands for the module of graded vector fields (=graded derivations of \mathcal{E}) on E , the bracket $[\cdot, \cdot]$ is the graded commutator of derivations and $\text{ad}_Q := [Q, \cdot]$.

Also, see Appendix B for the notion of Lie ∞ -morphism of differential graded Lie algebras and for notations.

We now state the main theorem of the paper. In Appendix B, Proposition B.13 shows that a Lie ∞ -morphism between a Lie algebra \mathfrak{g} and the DGLA of graded vector fields of a Lie ∞ -algebroid (E, Q) , induces a weak symmetry action of \mathfrak{g} on the basic singular foliation $\mathcal{F} = \rho(\Gamma(E_{-1}))$ of (E, Q) . In this section, we show that any weak symmetry action of a Lie algebra \mathfrak{g} on a singular foliation \mathcal{F} arises this way.

Convention 2.1. *From now on and in the sequel, the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ (possibly of infinite dimension) is concentrated in degree 0 so that \mathfrak{g} shifted by 1, namely $\mathfrak{g}[1]$, is concentrated in degree -1 . The Lie bracket $[\cdot, \cdot]_{\mathfrak{g}} : \mathfrak{g}[1] \times \mathfrak{g}[1] \rightarrow \mathfrak{g}[1]$ of $\mathfrak{g}[1]$ is of degree $+1$.*

Convention 2.2. *In this paper, vector bundles are of finite rank. Lie ∞ -algebroids are of finite rank except in Theorem 3.3 we notice that the result holds true without this assumption.*

Definition 2.3. Let \mathcal{F} be a singular foliation on M and (E, Q) a Lie ∞ -algebroid over \mathcal{F} . Consider a weak symmetry action $\varrho : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ of \mathfrak{g} on \mathcal{F} .

- We say that a Lie ∞ -morphism of differential graded Lie algebras

$$\Phi : (\mathfrak{g}[1], [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q) \quad (3)$$

lifts the weak symmetry action ϱ to (E, Q) if for all $x \in \mathfrak{g}, f \in \mathcal{O}$, $\Phi_0(x)(f) = \varrho(x)[f]$.

- When Φ exists, we say then Φ is a lift of ϱ on (E, Q) .

We now state the main theorem of this paper.

Theorem 2.4. *Let \mathcal{F} be a singular foliation on a smooth manifold (or an affine variety) M and \mathfrak{g} a Lie algebra. Let $\varrho : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ be a weak symmetry action of \mathfrak{g} on \mathcal{F} . The following assertions hold:*

- (1) *for any universal Lie ∞ -algebroid (E, Q) of the singular foliation \mathcal{F} , there exists a Lie ∞ -morphism $\Phi : (\mathfrak{g}[1], [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q)$ that lifts ϱ to (E, Q) ,*
- (2) *any two such Lie ∞ -morphisms are homotopy equivalent over the identity of M ,*
- (3) *any two such lifts of any two equivalent weak symmetry actions of \mathfrak{g} on \mathcal{F} are homotopy equivalent over the identity of M .*

Remark 2.5. Lie ∞ -morphisms in item 1 of Theorem 2.4 are called \mathfrak{g} -actions on (E, Q) in [24].

Remark 2.6. Item 1 in Theorem 2.4 implies that

(1) there exists a linear map $\Phi_0: \mathfrak{g}[1] \longrightarrow \mathfrak{X}_0(E)[1]$ such that

$$\Phi_0(x)[f] = \varrho(x)[f], \text{ and } [Q, \Phi_0(x)] = 0, \quad \forall x \in \mathfrak{g}[1], f \in \mathcal{O}. \quad (4)$$

Φ_0 is not a graded Lie algebra morphism, but there exist a linear map $\Phi_1: \wedge^2 \mathfrak{g}[1] \longrightarrow \mathfrak{X}_{-1}(E)[1]$ such that for all $x, y, z \in \mathfrak{g}[1]$,

$$\Phi_0([x, y]_{\mathfrak{g}}) - [\Phi_0(x), \Phi_0(y)] = [Q, \Phi_1(x, y)].$$

Also,

$$\Phi_1([x, y]_{\mathfrak{g}}, z) - [\Phi_0(x), \Phi_1(y, z)] + \odot(x, y, z) = [Q, \Phi_2(x, y, z)]$$

for some linear map $\Phi_2: \wedge^3 \mathfrak{g}[1] \longrightarrow \mathfrak{X}_{-2}(E)[1]$. These compatibility conditions continue to higher multilinear maps.

(2) For every element $x \in \mathfrak{g}$ and $i \geq 1$, there is a degree zero map $\nabla_x \in \text{Der}(E)$ (i.e. $\nabla_x(fe) = f\nabla_x(e) + \varrho(x)[f]e$, for $f \in \mathcal{O}, e \in \Gamma(E)$) depending linearly on x , such that

$$\langle \Phi_0(x)^{(0)}(\alpha), e \rangle = \varrho(x)[\langle \alpha, e \rangle] - \langle \alpha, \nabla_x(e) \rangle, \quad \text{for all } \alpha \in \Gamma(E^*), e \in \Gamma(E) \quad (5)$$

where $\Phi_0(x)^{(0)}$ stands for the arity zero component of $\Phi_0(x)$. Therefore, by using Equations (4), (5) and the dual correspondence between Lie ∞ -algebroids and NQ -manifolds [6, 23, 28], we obtain these compatibility conditions:

$$\ell_1 \circ \nabla_x = \nabla_x \circ \ell_1 \quad \text{and} \quad \rho \circ \nabla_x = \text{ad}_{\varrho(x)} \circ \rho,$$

ℓ_1 stands for the corresponding unary bracket of (E, Q) . Also, for $X \in \mathfrak{X}(M)$, $\text{ad}_X := [X, \cdot]$. In general, the map $\mathfrak{g}[1] \longrightarrow \text{Der}(E)$, $x \mapsto \nabla_x$ is not a Lie algebra morphism even when the action ϱ is strict. In fact, there exists a bilinear map $\gamma: \wedge^2 \mathfrak{g}[1] \longrightarrow \text{End}(E)[1]$ of degree 0 that satisfies

$$\nabla_{[x, y]_{\mathfrak{g}}} - [\nabla_x, \nabla_y] = \gamma(x, y) \circ \ell_1 - \ell_1 \circ \gamma(x, y) + \ell_2(\eta(x, y), \cdot), \quad (6)$$

here ℓ_2 is the corresponding 2-ary bracket of (E, Q) , and $\eta: \wedge^2 \mathfrak{g} \longrightarrow \Gamma(E_{-1})$ is such that $\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] = \rho(\eta(x, y))$.

Corollary 2.7. *Let (E, Q) be a universal Lie ∞ -algebroid of a singular foliation \mathcal{F} . For any symmetry $X \in \mathfrak{X}(M)$ of \mathcal{F} , there exists a degree zero vector field $Z \in \mathfrak{X}_0(E)$*

- (1) *that commutes with Q , i.e., such that $[Z, Q] = 0$,*
- (2) *and that extends X in the sense that the following diagrams commute*

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{p^*} & \Gamma(S^\bullet(E^*)) & \text{and} & \Gamma(S^\bullet(E^*)) & \xrightarrow{\iota^*} & C^\infty(M) \\ X \downarrow & & \downarrow Z & & \downarrow Z & & \downarrow X \\ C^\infty(M) & \xrightarrow{p^*} & \Gamma(S^\bullet(E^*)) & & \Gamma(S^\bullet(E^*)) & \xrightarrow{\iota^*} & C^\infty(M) \end{array} \quad (7)$$

where $p: E \longrightarrow M$ is the projection map and $\iota: M \hookrightarrow E$ the zero section.

Remark 2.8. Geometrically, Equation (7) means that $p_*(Z) = X$ and $Z|_M = X$.

Proof. To construct Z , it suffices to apply Theorem 2.4 for $\mathfrak{g} = \mathbb{R}$ and take Z to be the image of 1 through $\Phi_0: \mathbb{R}[1] \longrightarrow \mathfrak{X}_0(E)[1]$. \square

Remark 2.9. In particular, Corollary 2.7 has the following consequences:

- (1) for any admissible t , the flow $\Phi_t^Z: \mathcal{E} \rightarrow \mathcal{E}$ of Z being an isomorphism of Lie ∞ -algebroids, it induces an isomorphism of vector bundles $E_{-1} \rightarrow E_{-1}$. Since $[Q, Z] = 0$, the following diagram commutes,

$$\begin{array}{ccc} \Gamma(E_{-1}) & \xrightarrow{(\Phi_t^Z)^{(0)}} & \Gamma(E_{-1}) \\ \rho \downarrow & & \downarrow \rho \\ \mathfrak{X}(M) & \xrightarrow{(\phi_t^X)_*} & \mathfrak{X}(M) \end{array}$$

where ϕ_t^X is the flow of X at t .

- (2) Consequently, for any open set $U \subset M$ which is stable under ϕ_t^X , there exists an invertible matrix \mathfrak{M}_X^t with coefficients in $\mathcal{O}(U)$ that satisfies

$$(\phi_t^X)_* \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \mathfrak{M}_X^t \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix},$$

for some generators X_1, \dots, X_n of \mathcal{F} over U . As announced earlier, we recover Proposition 1.3, that is, $(\phi_t^X)_*(\mathcal{F}) = \mathcal{F}$.

Let (E, Q) and (E', Q') be two universal Lie ∞ -algebroids of \mathcal{F} . A direct consequence of Ricardo Campos's Theorem 4.1 in [7] is that the differential graded Lie algebras $(\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q)$ and $(\mathfrak{X}_\bullet(E')[1], [\cdot, \cdot], \text{ad}_{Q'})$ are homotopy equivalent over the identity of M . This leads to the following statement.

Corollary 2.10. *Let $\rho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ be a weak symmetry action of a Lie algebra \mathfrak{g} on \mathcal{F} . Then, there exist Lie ∞ -morphisms, $\Phi: \mathfrak{g}[1] \rightsquigarrow (\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q)$ and $\Psi: \mathfrak{g}[1] \rightsquigarrow (\mathfrak{X}_\bullet(E')[1], [\cdot, \cdot], \text{ad}_{Q'})$ that lift ρ , and Φ, Ψ make the following diagram commute up to homotopy*

$$\begin{array}{ccc} & \mathfrak{g}[1] & \\ \Phi \swarrow & & \searrow \Psi \\ (\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q) & \overset{\sim}{\longleftrightarrow} & (\mathfrak{X}_\bullet(E')[1], [\cdot, \cdot], \text{ad}_{Q'}) \end{array} \quad (8)$$

Proof. The composition of Φ with the horizontal map in the diagram (8) is a lift of the action ρ . It is necessarily homotopy equivalent to Ψ by item 2 in Theorem 2.4. \square

2.1. Cohomology of longitudinal graded vector fields. In this section, we study the cohomology of longitudinal vector fields, which will help in proving the main results stated in the beginning of Section 2.

Let \mathcal{F} be a singular foliation on M .

Definition 2.11. Let E be a splitted graded manifold over M with sheaf of function $\mathcal{E} = \Gamma(S(E^*))$. A vector field $L \in \mathfrak{X}(E)$ is said to be a *longitudinal vector field for \mathcal{F}* if there exists vector fields $X_1, \dots, X_k \in \mathcal{F}$ and functions $\Theta_1, \dots, \Theta_k \in \mathcal{E}$ such that,

$$L(f) = \sum_{i=1}^k X_i[f]\Theta_i, \quad \forall f \in \mathcal{O}. \quad (9)$$

Example 2.12. Here are some examples.

- (1) Vertical¹ vector fields are longitudinal.
- (2) For any Q -manifold (E, Q) over a manifold M . The homological vector field $Q \in \mathfrak{X}(E)$ is a longitudinal vector field for its basic singular foliation $\mathcal{F} := \rho(\Gamma(E_{-1}))$.
- (3) Longitudinal vector fields are precisely of the form $\sum_{i=1}^k \Theta_i X_i + V$, for $X_1, \dots, X_k \in \mathcal{F}, \Theta_1, \dots, \Theta_k \in \mathcal{E}$ and $V \in \mathfrak{X}(E)$ a vertical vector field on E .
- (4) For (E, Q) a Q -manifold and $\mathcal{F} := \rho(\Gamma(E_{-1}))$ its basic singular foliation. For any extension of a symmetry $X \in \mathfrak{s}(\mathcal{F})$ of \mathcal{F} to a degree zero vector field $\widehat{X} \in \mathfrak{X}(E)$, the degree +1 vector field $[Q, \widehat{X}]$ is longitudinal for \mathcal{F} .

Let us show this last point using local coordinates (x_1, \dots, x_n) on M and a local trivialization ξ^1, ξ^2, \dots of graded sections in $\Gamma(E^*)$. The vector fields Q and \widehat{X} take the form:

$$\begin{aligned} Q &= \sum_j \sum_{k, |\xi^k|=1} Q_k^j(x) \xi^k \frac{\partial}{\partial x_j} + \sum_j \sum_{k, \iota_1, \dots, \iota_k} \frac{1}{k!} Q_{\iota_1, \dots, \iota_k}^j(x) \xi^1 \odot \dots \odot \xi^k \frac{\partial}{\partial \xi^j} \\ \widehat{X} &= X + \sum_j \sum_{k, \iota_1, \dots, \iota_k} \frac{1}{k!} X_{\iota_1, \dots, \iota_k}^j(x) \xi^1 \odot \dots \odot \xi^k \frac{\partial}{\partial \xi^j} \end{aligned} \quad (10)$$

where $X = \sum_{i=1}^n X_i(x) \frac{\partial}{\partial x_i}$. By using Equation (10) we note that all the terms of $[Q, \widehat{X}]$ are vertical except maybe for the ones where the vector field X appears. For $k \geq 1$, the vector field $[Q_{\iota_1, \dots, \iota_k}^j \xi^1 \odot \dots \odot \xi^k \frac{\partial}{\partial \xi^j}, X]$ is vertical; and for every fixed k , one has

$$\left[\sum_{j=1}^n Q_k^j \xi^k \frac{\partial}{\partial x_j}, X \right] = \xi^k \left[\sum_{j=1}^n Q_k^j \frac{\partial}{\partial x_j}, X \right].$$

$$\text{Now, } \left[\sum_{j=1}^n Q_k^j \frac{\partial}{\partial x_j}, X \right] \in \mathcal{F}, \text{ since } X \text{ is a symmetry for } \mathcal{F} \text{ and } \sum_{j=1}^n Q_k^j \frac{\partial}{\partial x_j} \in \mathcal{F}.$$

Remark 2.13. Longitudinal vector fields are stable under the graded Lie bracket.

Let us make two points on vector fields on E .

- (1) Sections of the graded vector bundle E are identified with derivations under the isomorphism mapping $e \in \Gamma(E) \mapsto \iota_e \in \mathfrak{X}(E)$. This allows us to identify a vertical vector field with (maybe infinite) sums of tensor products of the form $\Theta \otimes e$ with $\Theta \in \mathcal{E}, e \in \Gamma(E)$.
- (2) A TM -connection ∇ on the graded bundle E , i.e., a collection of TM -connections ∇^i on E_{-i} for $i \geq 1$, induces for $X \in \mathfrak{X}(M)$ a vector field of degree zero $\widetilde{\nabla}_X \in \mathfrak{X}(E)$ by setting for $f \in \mathcal{O}, \widetilde{\nabla}_X(f) := X[f]$ and $\widetilde{\nabla}_X(\xi) := \nabla_X^{i,*}(\xi)$ for every homogeneous element $\xi \in \Gamma(E_{-i}^*)$, where $\nabla_X^{i,*}$ is the dual TM -connection. Upon choosing a TM -connection on E as above, we give a $\mathbb{N}_0 \times \mathbb{Z}_-$ grading to vector fields on E by the identification below:

$$\begin{aligned} \mathfrak{X}_k(E) &\simeq \bigoplus_{j \geq 1} \mathcal{E}_{k+j} \otimes_{\mathcal{O}} \Gamma(E_{-j}) \oplus \mathcal{E}_k \otimes_{\mathcal{O}} \mathfrak{X}(M) \\ &\simeq \bigoplus_{j \geq 1} \Gamma(S(E^*)_{k+j} \otimes E_{-j}) \oplus \Gamma(S(E^*)_k \otimes TM) \end{aligned} \quad (11)$$

for all $k \in \mathbb{Z}$. Therefore, one can realize a vector field $P \in \mathfrak{X}_k(E)$ as a sequence $P = (p_0, p_1, \dots)$, where $p_0 \in \Gamma(S(E^*)_k \otimes TM)$ and $p_i \in \Gamma(S(E^*)_{k+i} \otimes E_{-i})$ for $i \geq 1$ are called *components* of P . In the diagram (13), $P = (p_0, p_1, \dots)$ is represented as an element of the anti-diagonal and p_i is on column i . We say that P is of *depth* $n \in \mathbb{N}$ if

¹We say a vector field on E is *vertical* if it is \mathcal{O} -linear.

$p_i = 0$ for all $i < n$. In particular, vector fields of depth greater or equal to 1 are vertical. Under the isomorphism (11), the differential map ad_Q takes the form

$$D = D^h + \sum_{s \geq 0} D^{v_s} \quad (12)$$

with $D^2 = 0$. Here, $D^h = \text{id} \otimes d$ or $\text{id} \otimes \rho$, and

$$D^{v_s} : \Gamma(S(E^*)_k \otimes E_{-i}) \rightarrow \Gamma(S(E^*)_{k+s+1} \otimes E_{-i-s})$$

for $i \geq 0$, $s \geq 0$, where $E_0 := TM$. We denote the latter complex by (\mathcal{L}, D) . The maps D^{v_s} , for $s \geq 1$, are represented as up-left-pointing arrows, and D^{v_0} by vertical arrows, in the following diagram, whose lines are complexes of \mathcal{O} -modules given by the differential map D^h :

$$\begin{array}{ccccc}
 & \vdots & & \vdots & & \vdots \\
 & \uparrow D^{v_0} & & \uparrow D^{v_0} & & \uparrow D^{v_0} \\
 \cdots & \longrightarrow & \Gamma(S(E^*)_{k+2} \otimes E_{-2}) & \xrightarrow{D^h = \text{id} \otimes d} & \Gamma(S(E^*)_{k+2} \otimes E_{-1}) & \xrightarrow{D^h = \text{id} \otimes \rho} & \Gamma(S(E^*)_{k+2} \otimes TM) \\
 & \nearrow D^{v_1} & \searrow D^{v_1} & & \nearrow D^{v_1} & \searrow D^{v_1} & \\
 & \vdots & & \vdots & & \vdots & \\
 \cdots & \longrightarrow & \Gamma(S(E^*)_{k+1} \otimes E_{-2}) & \xrightarrow{D^h = \text{id} \otimes d} & \Gamma(S(E^*)_{k+1} \otimes E_{-1}) & \xrightarrow{D^h = \text{id} \otimes \rho} & \Gamma(S(E^*)_{k+1} \otimes TM) \\
 & \nearrow D^{v_1} & \searrow D^{v_1} & & \nearrow D^{v_1} & \searrow D^{v_1} & \\
 & \vdots & & \vdots & & \vdots & \\
 \cdots & \longrightarrow & \Gamma(S(E^*)_k \otimes E_{-2}) & \xrightarrow{D^h = \text{id} \otimes d} & \Gamma(S(E^*)_k \otimes E_{-1}) & \xrightarrow{D^h = \text{id} \otimes \rho} & \Gamma(S(E^*)_k \otimes TM) \\
 & \nearrow D^{v_1} & \searrow D^{v_1} & & \nearrow D^{v_1} & \searrow D^{v_1} & \\
 & \vdots & & \vdots & & \vdots & \\
 & \uparrow D^{v_0} & & \uparrow D^{v_0} & & \uparrow D^{v_0} & \\
 & \vdots & & \vdots & & \vdots & \\
 & \text{column 2} & & \text{column 1} & & \text{column 0} & \\
 & & & & & & (13)
 \end{array}$$

Remark 2.14. For $j \geq 0$, $\Theta \in \mathcal{E}$ and $\xi \in \Gamma(E_{-j})$ one has $D^{v_0}(\Theta \otimes \xi) = Q(\Theta) \otimes \xi + (-1)^{|\Theta|} \Theta \odot D^{v_0}(1 \otimes \xi)$ and $D^{v_i}(\Theta \otimes \xi) + (-1)^{|\Theta|} D^{v_i}(1 \otimes \xi)$ for every $i \geq 1$. Here, $E_0 := TM$.

Under this correspondence, we understand longitudinal vector fields as the following.

Lemma 2.15. *A graded vector field $P = (p_0, p_1, \dots) \in \mathcal{L}$ is longitudinal if $p_0 \in \mathcal{E} \otimes_{\mathcal{O}} \mathcal{F}$.*

The following theorem is crucial for the rest of this paper.

Theorem 2.16. *Let (E, Q) be a universal Q -manifold of \mathcal{F} .*

- (1) *Longitudinal vector fields form an acyclic complex.*

More precisely, any longitudinal vector field on E which is an ad_Q -cocycle is the image through ad_Q of some vertical vector field on E .

- (2) *More generally, if a vector field on E of depth n is an ad_Q -cocycle, then it is the image through ad_Q of some vector field on E of depth $n + 1$.*

Proof. (E, Q) is a universal Q -manifold of \mathcal{F} implies that lines in (13) are exact when we restrict the 0-th column to sections in $\mathcal{E} \otimes_{\mathcal{O}} \mathcal{F}$. It is now a diagram chasing phenomena. Let $P = (p_0, p_1, \dots) \in \mathcal{L}$ be a longitudinal element which is a D -cocycle. By longitudinality there exists an element $b_1 \in \Gamma(S(E^*) \otimes E_{-1})$ such that $(\text{id} \otimes \rho)(b_1) = p_0$. Set $P_1 = (0, b_1, 0, \dots)$, that is,

we extend b_1 by zero on $\Gamma(S(E^*) \otimes E_{\leq -2})$ and $\Gamma(S(E^*) \otimes TM)$. It is clear that $P - D(P_1) = (0, p'_1, p'_2, \dots)$ is also a D -cocycle of depth 1. In particular, we have $D^h(p'_1) = 0$ by exactness there exists $b_2 \in \Gamma(S(E^*) \otimes E_{-2})$ such that $D^h(b_2) = p'_1$. As before put $P_2 = (0, 0, b_2, 0, \dots)$. Similarly, $P - D(P_1) - D(P_2) = (0, 0, p''_2, p''_3, \dots)$ is a D -cocycle. By recursion, we end up to construct P_1, P_2, \dots that satisfy $P - D(P_1) - D(P_2) + \dots = 0$, that is, there exists an element $B = (0, b_1, b_2, \dots) \in \mathfrak{L}$ such that $D(B) = P$. This proves item 1.

To prove item 2 it suffices to cross out in the diagram (13) the columns numbered $0, \dots, n-1$, which does not break exactness. The proof now follows as for item 1. \square

In particular, we deduce from item 1 of Theorem 2.16 the following exact subcomplex.

Corollary 2.17. *Let (E, Q) be a universal Q -manifold of \mathcal{F} . The subcomplex \mathfrak{V}_Q of $(\mathfrak{X}(E), \text{ad}_Q)$ made of vertical vector fields $P \in \mathfrak{X}(E)$ that satisfy $P \circ Q(f) = 0$ for all $f \in \mathcal{O}$ is acyclic.*

Proof. Let $P \in \mathfrak{X}(E)$ be a vertical vector field which is a ad_Q -cocycle. Notice that we have automatically $P \circ Q(f) = 0$ for all $f \in \mathcal{O}$: indeed, P is a ad_Q -cocycle implies $[Q, P](f) = 0$ for all $f \in \mathcal{O}$. Equivalently, $P \circ Q(f) = (-1)^{|P|} Q \circ P(f)$. Since P is vertical, $P(f) = 0$, which proves that $P \circ Q(f) = 0$. By Theorem 2.16 there exists a vertical vector field $\tilde{P} \in \mathfrak{X}(E)$ such that $[Q, \tilde{P}] = P$. Moreover, $\tilde{P} \in \mathfrak{V}_Q$, since for all $f \in \mathcal{O}$,

$$0 = [Q, \tilde{P}](f) = (-1)^{|\tilde{P}|} \tilde{P} \circ Q(f).$$

This completes the proof. \square

The following remark will be used in the proof of Theorem 2.4.

Remark 2.18. For a cocycle $P \in \mathfrak{V}_Q$ of degree 0 one has $P^{(-1)} = 0$ (for degree reason). By Corollary 2.17, P is the image by ad_Q of an element, $\tilde{P} \in \mathfrak{V}_Q$ i.e., such that $[Q, \tilde{P}] = P$. Also, one can choose $\tilde{P}^{(-1)} = 0$: we have

$$[Q^{(0)}, \tilde{P}^{(-1)}] = [Q, \tilde{P}]^{(-1)} = P^{(-1)} = 0.$$

By exactness of $\text{ad}_{Q^{(0)}}$ (see [19]), we have $\tilde{P}^{(-1)} = [Q^{(0)}, \vartheta]$ for some \mathcal{O} -linear map

$$\vartheta \in \text{Hom}(\Gamma(E^*), \Gamma(S^0(E^*)))$$

of degree -2 . Put $\bar{P} := \tilde{P} - [Q, \vartheta]$, where ϑ is extended to a derivation of arity -1 . Clearly,

$$[Q, \bar{P}] = P \quad \text{and} \quad \bar{P}^{(-1)} = \tilde{P}^{(-1)} - [Q, \vartheta]^{(-1)} = \tilde{P}^{(-1)} - [Q^{(0)}, \vartheta] = 0.$$

Therefore, $P = \text{ad}_Q(\bar{P})$ with $\bar{P}^{(-1)} = 0$.

2.2. Proof of the main results. This section is devoted to the proof of the main results stated in Section 2. For the notations, see Appendix A and B.

We start with the following lemma.

Lemma 2.19. *Assume (E, Q) is a universal Lie ∞ -algebroid over M . Let $\bar{\Phi}: (S_{\mathbb{K}}^{\bullet} \mathfrak{g}[1], Q_{\mathfrak{g}}) \rightarrow (S_{\mathbb{K}}^{\bullet} \mathfrak{X}(E)[1], \bar{Q})$ be a coalgebra morphism which is a Lie ∞ -morphism up to arity $n \geq 1$, i.e., $(\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi})^{(i)} = 0$ for all integer $i \in \{0, \dots, n\}$. Then, $\bar{\Phi}$ can be extended to a ∞ -morphism up to arity $n + 1$.*

Proof. For convenience, we omit the variables. The identity,

$$\bar{Q} \circ (\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi}) + (\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi}) \circ Q_{\mathfrak{g}} = 0$$

taken in arity $n + 1$ yields,

$$\begin{aligned} 0 &= (\bar{Q} \circ (\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi}))^{(n+1)} = \bar{Q}^{(0)} \circ (\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi})^{(n+1)} \\ &= [Q, (\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi})^{(n+1)}], \end{aligned}$$

since $Q_{\mathfrak{g}}^{(0)} = 0$ and $(\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi})^{(i)} = 0$ for $i \in \{0, \dots, n\}$. It is clear that for all $n \geq 0$ the map $(\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi})^{(n+1)} : S_{\mathbb{K}}^{n+2}(\mathfrak{g}[1]) \rightarrow \mathfrak{X}_{-n}(E)[1]$ takes values in vertical vector fields on E because vector fields of degree $n \geq 1$ are vertical for degree reasons. By virtue of Corollary 2.17 there exists a vector field $\zeta \in \mathfrak{X}_{-n-1}(E)[1]$ of degree $-n - 1$ such that

$$[Q, \bar{\Phi}^{(n+1)} + \zeta] = \bar{\Phi}^{(n)} \circ Q_{\mathfrak{g}}^{(1)} - \bar{Q}^{(1)} \circ \bar{\Phi}^{(n)}. \quad (14)$$

By replacing the arity $n + 1$ of $\bar{\Phi}$ by $\bar{\Phi}^{(n+1)} + \zeta$, and keeping the other arities fixed, one obtains a new map $\bar{\Psi} : (S_{\mathbb{K}}^{\bullet} \mathfrak{g}[1], Q_{\mathfrak{g}}) \rightarrow (S_{\mathbb{K}}^{\bullet} \mathfrak{X}(E)[1], \bar{Q})$ such that $\bar{\Psi}^{(j)} := \bar{\Phi}^{(j)}$ for $j \neq n + 1$ and $\bar{\Psi}^{(n+1)} := \bar{\Phi}^{(n+1)} + \zeta$. The map $\bar{\Psi}$ satisfies

$$[Q, \bar{\Psi}^{(n+1)}] = \bar{\Psi}^{(n)} \circ Q_{\mathfrak{g}}^{(1)} - \bar{Q}^{(1)} \circ \bar{\Psi}^{(n)}. \quad (15)$$

This implies that $(\bar{\Psi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Psi})^{(n+1)} = 0$. By construction, $\bar{\Psi}$ is a Lie ∞ -morphism up to arity $n + 1$, i.e., that satisfies $(\bar{\Psi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Psi})^{(i)} = 0$ for all integer $i \in \{0, \dots, n + 1\}$. The proof continues by recursion. \square

Let \mathcal{F} be a singular foliation, and (E, Q) a universal Lie ∞ -algebroid of \mathcal{F} . We start with the following lemma.

Lemma 2.20. *For every weak symmetry Lie algebra action of \mathfrak{g} on \mathcal{F} there exists a linear map, $\Phi_0 : \mathfrak{g}[1] \rightarrow \mathfrak{X}_0(E)[1]$, such that $[Q, \Phi_0(x)] = 0$ and $\Phi_0(x)[f] = \varrho(x)[f]$ for all $x \in \mathfrak{g}[1]$, $f \in \mathcal{O}$.*

Proof. For $x \in \mathfrak{g}$, let $\widehat{\varrho(x)} \in \mathfrak{X}_0(E)$ be any arbitrary extension of $\varrho(x) \in \mathfrak{s}(\mathcal{F})$ to a degree zero vector field on E . Since $\varrho(x)$ is a symmetry of \mathcal{F} , the degree +1 vector field $[\widehat{\varrho(x)}, Q]$ is also a longitudinal vector field on E , see Example 2.12 item 3. In addition, $[\widehat{\varrho(x)}, Q]$ is a ad_Q -cocycle. By item 1 of Theorem 2.16, there exists a vertical vector field $Y(x) \in \mathfrak{X}_0(E)$ of degree zero such that

$$[Q, Y(x) + \widehat{\varrho(x)}] = 0. \quad (16)$$

Let us set for $x \in \mathfrak{g}[1]$, $\Phi_0(x) := Y(x) + \widehat{\varrho(x)}$. By construction, we have, $[Q, \Phi_0(x)] = 0$ and $\Phi_0(x)[f] = \varrho(x)[f]$ for all $x \in \mathfrak{g}[1]$, $f \in \mathcal{O}$. \square

Proof of Theorem 2.4. Let us show item 1. Note that Lemma 2.20 gives the existence of a linear map $\Phi_0 : \mathfrak{g}[1] \rightarrow \mathfrak{X}_0(E)[1]$ such that, $[Q, \Phi_0(x)] = 0$ for all $x \in \mathfrak{g}[1]$. For $x, y \in \mathfrak{g}[1]$, consider

$$\Lambda(x, y) = \Phi_0([x, y]_{\mathfrak{g}}) - [\Phi_0(x), \Phi_0(y)]. \quad (17)$$

Since $\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] \in \mathcal{F}$ for all $x, y \in \mathfrak{g}[1]$, and since $\rho : \Gamma(E_{-1}) \rightarrow \mathcal{F}$ surjective, we have $\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] = \rho(\eta(x, y))$ for some element $\eta(x, y) \in \Gamma(E_{-1})$ depending linearly on x and y . Now we consider the vertical vector field of degree -1 , $\iota_{\eta(x, y)} \in \mathfrak{X}_{-1}(E)$ which is defined on $\Gamma(E^*)$ as:

$$\iota_{\eta(x, y)}(\alpha) := \langle \alpha, \eta(x, y) \rangle \text{ for all } \alpha \in \Gamma(E^*),$$

and extended it by derivation on the whole space. For every $f \in \mathcal{O}$,

$$\begin{aligned} (\Lambda(x, y) - [Q, \iota_{\eta(x, y)}])(f) &= (\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] - \rho(\eta(x, y)))(f) && \text{(by definition of } \Phi_0) \\ &= 0 && \text{(by definition of } \eta) \end{aligned}$$

It is clear that $\Lambda(x, y) + [Q, \iota_{\eta(x, y)}]$ is a ad_Q -cocycle. Also, $(\Lambda(x, y) + [Q, \iota_{\eta(x, y)}])^{(-1)} = 0$. Hence, by Corollary 2.17 and Remark 2.18, $\Lambda(x, y) + [Q, \iota_{\eta(x, y)}]$ is of the form $[Q, \Upsilon(x, y)]$ for some vertical vector field $\Upsilon(x, y) \in \mathfrak{X}_{-1}(E)$ of degree -1 with $\Upsilon(x, y)^{(-1)} = 0$. For all $x, y \in \mathfrak{g}[1]$, we define the Taylor coefficient $\Phi_1: \wedge^2 \mathfrak{g}[1] \rightarrow \mathfrak{X}_{-1}(E)[1]$ as $\Phi_1(x, y) := \Upsilon(x, y) + \iota_{\eta(x, y)}$. By construction, we have the following relation

$$\Phi_0([x, y]_{\mathfrak{g}}) - [\Phi_0(x), \Phi_0(y)] = [Q, \Phi_1(x, y)], \quad \forall x, y \in \mathfrak{g}[1]. \quad (18)$$

So far, in the construction of the Lie ∞ -morphism, we have shown the existence of a Lie ∞ -morphism $\bar{\Phi}: S_{\mathbb{K}}^{\bullet} \mathfrak{g}[1] \rightarrow S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1])$ up to arity 1, that is $(\bar{\Phi} \circ Q_{\mathfrak{g}})^{(i)} = (\bar{Q} \circ \bar{\Phi})^{(i)}$ for $i = 0, 1$. The proof continues by recursion by applying directly Lemma 2.19. This proves item 1 of the theorem. \square

Before proving item 2 of Theorem 2.4 we will need the following lemma. For convenience, we sometimes omit the variables in \mathfrak{g} . See Appendix B for the notations.

Lemma 2.21. *For any two Lie ∞ -morphisms $\Gamma, \Omega: (S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1]), Q_{\mathfrak{g}}) \rightarrow (S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1]), \bar{Q})$ which coincide up to arity $n \geq 0$, i.e. $\Gamma^{(i)} = \Omega^{(i)}$, for $0 \leq i \leq n$, their difference in arity $n+1$, namely,*

$$\Gamma^{(n+1)} - \Omega^{(n+1)}: S_{\mathbb{K}}^{n+2}(\mathfrak{g}[1]) \rightarrow \mathfrak{X}_{-n-1}(E)[1]$$

is valued in ad_Q -coboundary.

Proof. Indeed, a direct computation yields

$$\begin{aligned} \bar{Q} \circ (\Gamma - \Omega) &= (\Gamma - \Omega) \circ Q_{\mathfrak{g}} \implies \bar{Q}^{(0)} \circ (\Gamma - \Omega)^{(n+1)} - \underbrace{((\Gamma - \Omega) \circ Q_{\mathfrak{g}})^{(n+1)}}_{=0} = 0 \\ &\implies [Q, \Gamma^{(n+1)} - \Omega^{(n+1)}] = 0 \\ &\implies \Gamma^{(n+1)} - \Omega^{(n+1)} = [Q, H^{(n+1)}] \quad (\text{by item 1 of Theorem 2.16}) \end{aligned}$$

for some linear map $H^{(n+1)}: S_{\mathbb{K}}^{n+2}(\mathfrak{g}[1]) \rightarrow \mathfrak{X}_{-n-2}(E)[1]$. \square

Let us show item 2 of Theorem 2.4. Let $\Phi, \Psi: \mathfrak{g}[1] \rightsquigarrow \mathfrak{X}(E)[1]$ be two different lifts of the action $\mathfrak{g} \rightarrow \mathfrak{X}(M)$. We denote by $\bar{\Phi}, \bar{\Psi}: S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1]) \rightarrow S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1])$ the unique comorphisms given respectively by the Taylor's coefficients

$$\begin{cases} \bar{\Phi}^{(r)}: S_{\mathbb{K}}^{r+1}(\mathfrak{g}[1]) \xrightarrow{\Phi_r} \mathfrak{X}_{-r}(E)[1], & \text{for } r \geq 0 \\ \bar{\Psi}^{(r)}: S_{\mathbb{K}}^{r+1}(\mathfrak{g}[1]) \xrightarrow{\Psi_r} \mathfrak{X}_{-r}(E)[1] \end{cases} \quad (19)$$

For any $x \in \mathfrak{g}[1]$, the degree zero vector field $\Phi_0(x) - \Psi_0(x) \in \mathfrak{X}_0(E)$ is vertical. Moreover, we have, $[Q, \Phi_0(x) - \Psi_0(x)] = 0$. By Corollary 2.17 there exists a vector field $H_0 \in \mathfrak{X}_{-1}(E)$ of degree -1 , such that $\Psi_0(x) - \Phi_0(x) = [Q, H_0(x)]$

$$\begin{array}{ccc} & \mathfrak{g}[1] & \\ & \swarrow H_0 & \downarrow \Psi_0 - \Phi_0 \\ \mathfrak{X}_{-1}(E)[1] & \xrightarrow{\text{ad}_Q} & \mathfrak{X}_0(E)[1] \end{array} \quad (20)$$

Consider the following differential equation

$$\begin{cases} \frac{d\Xi_t}{dt} = \bar{Q} \circ H_t + H_t \circ Q_{\mathfrak{g}}, & t \in [0, 1] \\ \Xi_0 = \bar{\Phi} \end{cases} \quad (21)$$

where $(\Xi_t)_{t \in [0,1]}$ is as in Definition B.9, and for $t \in [0,1]$, H_t is the unique Ξ_t -coderivation where the only non-zero arity is $H^{(0)} = H_0$. Equation (21) gives a homotopy between $\bar{\Phi}$ and Ξ_1 . When we consider the arity zero component in Equation (21), one obtains

$$\begin{aligned} \frac{d\Xi_t^{(0)}}{dt} &= \bar{Q}^{(0)} \circ H_t^{(0)} + H_t^{(0)} \circ Q_{\mathfrak{g}}^{(0)} \\ &= [Q, H_0] \\ &= \Psi_0 - \Phi_0 = \bar{\Psi}^{(0)} - \bar{\Phi}^{(0)}. \end{aligned}$$

Therefore, $\Xi_t^{(0)} = \bar{\Phi}^{(0)} + t(\bar{\Psi}^{(0)} - \bar{\Phi}^{(0)})$, and $\bar{\Phi} \sim \Xi_1$ with $\bar{\Psi}^{(0)} = \Xi_1^{(0)}$. Using Lemma 2.21, the image of any element through the map $\bar{\Psi}^{(1)} - \Xi_1^{(1)}: S_{\mathbb{K}}^2(\mathfrak{g}[1]) \rightarrow \mathfrak{X}_{-1}(E)[1]$ is a ad_Q -coboundary. Thus, $\bar{\Psi}^{(1)} - \Xi_1^{(1)}$ can be written as

$$\bar{\Psi}^{(1)} - \Xi_1^{(1)} = [Q, H^{(1)}], \quad \text{with } H^{(1)}: S_{\mathbb{K}}^2(\mathfrak{g}[1]) \rightarrow \mathfrak{X}_{-2}(E)[1]. \quad (22)$$

Let us go one step further by considering the differential equation on $[0,1]$ given by

$$\begin{cases} \frac{d\Theta_t}{dt} = \bar{Q} \circ H_t + H_t \circ Q_{\mathfrak{g}} \\ \Xi_0 = \bar{\Xi}_1 \end{cases} \quad (23)$$

Here H_t is the extension of $H^{(1)}$ as the unique Θ_t -coderivation where all its arities vanish except the arity 1 which is given by $H^{(1)}$. In arity zero, $(\Theta_t^{(0)})_{t \in [0,1]}$ is constant and has value $\Theta_1^{(0)} = \bar{\Psi}^{(0)}$. In arity one we have,

$$\begin{aligned} \frac{d\Theta_t^{(1)}}{dt} &= \bar{Q}^{(0)} \circ H_t^{(1)} \\ &= [Q, H^{(1)}] = \bar{\Psi}^{(1)} - \bar{\Xi}_1^{(1)}. \end{aligned}$$

Hence, $\Theta_t^{(1)} = \bar{\Phi}^{(1)} + t(\bar{\Psi}^{(1)} - \bar{\Xi}_1^{(1)})$ with $\bar{\Psi}^{(i)} = \Theta_1^{(i)}$ for $i = 0, 1$. We then continue this procedure by gluing all these homotopies as in [20], p. 40-41. We will obtain at last a Lie ∞ -morphism Ω such that $\bar{\Phi} \sim \Omega$ and $\Omega^{(i)} = \bar{\Psi}^{(i)}$ for $i \geq 0$. That means $\Omega = \bar{\Psi}$, therefore $\bar{\Phi} \sim \bar{\Psi}$. This proves item 2. of Theorem 2.4.

Let us prove item 3 of Theorem 2.4. Given two equivalent weak symmetry actions ϱ, ϱ' of \mathfrak{g} on a singular foliation \mathcal{F} , i.e., ϱ, ϱ' differ by a linear map $\mathfrak{g} \rightarrow \mathfrak{X}(M)$ of the form $x \mapsto \rho(\beta(x))$ for some linear map $\beta: \mathfrak{g} \rightarrow \Gamma(E_{-1})$. Let $\Phi, \Phi': \mathfrak{g}[1] \rightsquigarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ be a lift into a Lie ∞ -morphism of the action ϱ and ϱ' respectively. One has for all $x \in \mathfrak{g}[1]$ and $f \in \mathcal{O}$,

$$\begin{aligned} (\Phi_0(x) - \Psi_0(x) - [Q, \iota_{\varphi(x)}])(f) &= \rho(\varphi(x))[f] - \langle Q(f), \varphi(x) \rangle \\ &= 0. \end{aligned}$$

Also,

$$\begin{aligned} [Q, \Phi_0(x) - \Psi_0(x) - [Q, \iota_{\varphi(x)}]] &= [Q, \Phi_0(x)] - [Q, \Psi_0(x)] - [Q, [Q, \iota_{\varphi(x)}]] \\ &= 0, \quad (\text{since } [Q, \Phi_0(x)] = [Q, \Psi_0(x)] = [Q, [Q, \iota_{\varphi(x)}]] = 0). \end{aligned}$$

By Corollary 2.17 there exists a vertical derivation $\hat{H}(x) \in \mathfrak{X}_{-1}(E)$ of degree -1 depending linearly on $x \in \mathfrak{g}[1]$ such that

$$\Phi_0(x) - \Psi_0(x) = [Q, \hat{H}(x) + \iota_{\varphi(x)}].$$

Let $H(x) := \hat{H}(x) + \iota_{\varphi(x)}$, for $x \in \mathfrak{g}[1]$. The proof continues the same as for item 2 of Theorem 2.4

2.3. Particular examples. We recall that for a regular foliation \mathcal{F} on a manifold M (i.e., $\mathcal{F} = \Gamma(F)$ for some involutive subvector bundle $F \subseteq TM$), the Lie algebroid $E_{-1} = F[1]$ whose sections form \mathcal{F} , is a universal Lie ∞ -algebroid of \mathcal{F} . In particular, $E_{-i} = 0$ for $i \geq 2$. Its corresponding Q -manifold is given by the leafwise De Rham differential on $\Gamma(\wedge^\bullet F^*)$. Also, for any symmetry $X \in \mathfrak{s}(\mathcal{F})$, i.e., any vector field $X \in \mathfrak{X}(M)$ such that $[X, \mathcal{F}] \subset \mathcal{F}$, the Lie derivative $\mathcal{L}_X: \Gamma(\wedge^k T^*M) \rightarrow \Gamma(\wedge^k T^*M)$ along X induces a vector field in $\mathfrak{X}_0(E)$ i.e., a degree zero derivation of $\Gamma(S^\bullet(E^*))$.

Example 2.22. Let \mathcal{F} be a regular foliation on a manifold M . Every weak symmetry action $\mathfrak{g} \rightarrow \mathfrak{X}(M)$, $x \mapsto \varrho(x)$, of \mathcal{F} , can be lifted to Lie ∞ -morphism $\Phi: \mathfrak{g}[1] \rightsquigarrow (\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q)$ given explicitly as follows:

$$x \in \mathfrak{g}[1] \mapsto \Phi_0(x) = \mathcal{L}_{\varrho(x)} \in \mathfrak{X}_0(E)[1] \quad (24)$$

$$x \wedge y \in \wedge^2 \mathfrak{g}[1] \mapsto \Phi_1(x, y) = \iota_{\chi(x, y)} \in \mathfrak{X}_{-1}(E)[1] \quad (25)$$

and $(\Phi_i: \wedge^{i+1} \mathfrak{g}[1] \rightarrow \mathfrak{X}_{-i}(E)) \equiv 0$, for all $i \geq 2$, where $\chi(x, y) := \varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)]$ for $x, y \in \mathfrak{g}$.

Example 2.23. Let \mathcal{F} be a singular foliation on a manifold M together with a strict symmetry action $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ such that $\varrho(\mathfrak{g}) \subset \mathcal{F}$. Hence, $C^\infty(M)\varrho(\mathfrak{g})$ is a singular foliation which is the image of the transformation Lie algebroid $\mathfrak{g} \times M$. The universality theorem (see [19, 20]) provides the existence of a Lie ∞ -morphism $\nu: \mathfrak{g}[1] \rightarrow E$. Let us call its Taylor coefficients $\nu_n: \wedge^{n+1} \mathfrak{g}[1] \rightarrow E_{-n-1}$, $n \geq 0$. We may take for example the 0-th and 1-th Taylor coefficients of a Lie ∞ -morphism that lifts ϱ as:

$$\Phi_0(x) := [Q, \iota_{\nu_0(x)}] \in \mathfrak{X}_0(E)[1], \text{ for } x \in \mathfrak{g}[1].$$

$$\Phi_1(x, y) := [Q, \iota_{\nu_1(x, y)}]^{(-1)} - \sum_{k \geq 0} [[Q, \iota_{\nu_0(x)}], \iota_{\nu_0(y)}]^{(k)} \in \mathfrak{X}_{-1}(E)[1], \text{ for } x, y \in \mathfrak{g}[1].$$

Note that in this case the action ϱ is equivalent to zero, therefore by item 3 of Theorem 2.4 the Lie ∞ -morphism Φ is homotopic to zero.

3. LIFTS OF WEAK SYMMETRY ACTIONS AND LIE ∞ -ALGEBROIDS

In this section, we consider the Lie algebra $\mathfrak{g}[1]$ as the trivial vector bundle over M with fiber $\mathfrak{g}[1]$.

The following proposition says that any lift of strict symmetry action of \mathfrak{g} on a singular foliation \mathcal{F} induces a Lie ∞ -algebroids with some special properties and vice versa. See [24], Proposition 3.3, for a proof of the following statement.

Proposition 3.1. *Let (E, Q) be a Lie ∞ -algebroid over a singular foliation \mathcal{F} . Every Lie ∞ -morphism $\Phi: (\mathfrak{g}[1], [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q)$ with \mathfrak{g} of finite dimension induces a Lie ∞ -algebroid $(E \oplus \mathfrak{g}[1], Q')$ with*

$$Q' := d^{CE} + Q + \sum_{k \geq 1, i_1, \dots, i_k = 1, \dots, \dim(\mathfrak{g})} \frac{1}{k!} \xi^{i_1} \odot \dots \odot \xi^{i_k} \Phi_{k-1}(\xi_{i_1}, \dots, \xi_{i_k}), \quad (26)$$

where d^{CE} is the Chevalley-Eilenberg complex of \mathfrak{g} , $\xi^1, \dots, \xi^{\dim(\mathfrak{g})} \in \mathfrak{g}^*$ is the dual basis of some basis $\xi_1, \dots, \xi_{\dim(\mathfrak{g})} \in \mathfrak{g}$ and for all $k \geq 0$, $\Phi_k: S^{k+1} \mathfrak{g}[1] \rightarrow \mathfrak{X}_{-k}(E)[1]$ is the k -th Taylor coefficients of Φ .

In the dual point of view, (26) corresponds to a Lie ∞ -algebroid over the complex

$$\cdots \xrightarrow{\ell_1} E_{-3} \xrightarrow{\ell_1} E_{-2} \xrightarrow{\ell_1} \mathfrak{g}[1] \oplus E_{-1} \xrightarrow{\rho'} TM \quad (27)$$

whose brackets satisfy

- (1) the anchor map ρ' sends an element $x \oplus e \in \mathfrak{g}[1] \oplus E_{-1}$ to $\varrho(x) + \rho(e) \in \varrho(\mathfrak{g}) + T\mathcal{F}$, where $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(W)$ is the weak symmetry action induced by Φ ,
- (2) the binary bracket satisfies

$$\ell_2(\Gamma(E_{-1}), \Gamma(E_{-1})) \subset \Gamma(E_{-1}) \quad \text{and} \quad \ell_2(\Gamma(E_{-1}), x) \subset \Gamma(E_{-1}), \quad \forall x \in \mathfrak{g}[1]$$

- (3) the $\mathfrak{g}[1]$ -component of the binary bracket on constant sections of $\mathfrak{g}[1] \times M$ is the Lie bracket of $\mathfrak{g}[1]$.

Conversely, if there exists a Lie ∞ -algebroid (E', Q') whose underlying complex of vector bundles is of the form (27) and that satisfies item 1, 2 and 3, then there is a Lie ∞ -morphism

$$\Phi: (\mathfrak{g}[1], [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_{Q'})$$

which is defined on a given basis ξ_1, \dots, ξ_d of \mathfrak{g} by:

$$\Phi_{k-1}(\xi_{i_1}, \dots, \xi_{i_k}) = \text{pr} \circ [\cdots [[Q', \iota_{\xi_{i_1}}], \iota_{\xi_{i_2}}], \dots, \iota_{\xi_{i_k}}] \in \mathfrak{X}(E)[1], \quad k \in \mathbb{N}, \quad (28)$$

where pr stands for the projection map $\mathfrak{X}(E')[1] \rightarrow \mathfrak{X}(E)[1]$.

Proof. We explain the idea of the proof. A direct computation gives the first implication. Conversely, let us denote by Q' the homological vector fields of Lie ∞ -algebroid whose underlying complex of vector bundles is of the form (27). The map defined in Equation (28) is indeed a lift into a Lie ∞ -morphism of the weak symmetry action ϱ :

- It is not difficult to check that for any $\xi \in \mathfrak{g}[1]$, one has $[Q, \Phi_0(\xi)] = 0$.
- The fact that Φ defines a Lie ∞ -morphism can be found using Voronov trick [28], i.e., doing Jacobi's identity inside the null derivation

$$0 = \text{pr} \circ [\cdots [[[Q', Q'], \iota_{\xi_{i_1}}], \iota_{\xi_{i_2}}], \dots, \iota_{\xi_{i_k}}]. \quad (29)$$

A direct computation of Equation (29) falls exactly on the requirements of Definition B.3. \square

Remark 3.2. Proposition 3.1 is stated in the finite dimensional context, i.e., it needs \mathfrak{g} to be finite dimensional and requires the existence of a geometric resolution for the singular foliation \mathcal{F} . The next theorem proves that: given a weak symmetry action of a Lie algebra \mathfrak{g} (maybe of infinite dimensional) on a Lie-Rinehart algebra $\mathcal{F} \subset \mathfrak{X}(M)$ (we do not require \mathcal{F} being locally finitely generated), such a Lie ∞ -algebroid (maybe of infinite dimension in the sense of Definition 1.14 in [20]) with the properties (1), (2) and (3) described at the sections level of the complex (27) in Proposition 3.1 exists.

Theorem 3.3. Let $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ be a weak symmetry action on a singular foliation \mathcal{F} . Let $((\mathcal{K}_{-i})_{i \geq 1}, d, \rho)$ be a free resolution² of the singular foliation \mathcal{F} over M . The complex of trivial vector bundles over M

$$\cdots \xrightarrow{d} E_{-3} \xrightarrow{d} E_{-2} \xrightarrow{d} \mathfrak{g}[1] \oplus E_{-1} \xrightarrow{\rho'} TM \quad (30)$$

where $\Gamma(E_{-1}) = \mathcal{K}_{-1}$, comes equipped with a Lie ∞ -algebroid structure

²Possibly of infinite dimension or infinite length.

- (1) whose unary bracket is d and whose anchor map ρ' , sends an element $x \oplus e \in \mathfrak{g}[1] \oplus E_{-1}$ to $\varrho(x) + \rho(e) \in \varrho(\mathfrak{g}) + T\mathcal{F}$,
- (2) the binary bracket satisfies

$$\ell_2(\Gamma(E_{-1}), \Gamma(E_{-1})) \subset \Gamma(E_{-1}) \quad \text{and} \quad \ell_2(\Gamma(E_{-1}), \Gamma(\mathfrak{g}[1])) \subset \Gamma(E_{-1}),$$

- (3) the $\mathfrak{g}[1]$ -component of the binary bracket on constant sections of $\mathfrak{g}[1] \times M$ is the Lie bracket of \mathfrak{g} .

For a proof, see Appendix C.

Remark 3.4. When we have $\varrho(\mathfrak{g}) \cap T_m\mathcal{F} = 0$ for all m in M , Equation (30) is a free resolution of the singular foliation $C^\infty(M)\varrho(\mathfrak{g}) + \mathcal{F}$ and we can apply directly the Theorem 2.1 in [20]. Otherwise, we need to show there is no obstruction in degree -1 while doing the construction of the brackets if the result still needs to hold.

4. ON WEAK AND STRICT SYMMETRIES: AN OBSTRUCTION THEORY

In this section, we apply theorems in Section 2 to define a class obstructing the existence of strict symmetry action equivalent to a given weak symmetry action. We apply these results to the problem of extending a strict Lie algebra action on an affine sub-variety to the ambient space.

Let us start with some generalities that we will use throughout of this section. Assume we are given

- a Lie algebra \mathfrak{g} with Lie bracket $[\cdot, \cdot]_{\mathfrak{g}}$,
- a universal Lie ∞ -algebroid (E, Q) of a singular foliation \mathcal{F} ,
- a weak symmetry action $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ of \mathfrak{g} on a singular foliation \mathcal{F} , together with $\eta: \wedge^2 \mathfrak{g} \rightarrow \Gamma(E_{-1})$ such that $x, y \in \mathfrak{g}$

$$\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] = \rho(\eta(x, y)). \quad (31)$$

Theorem 2.4 states that $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ admits a lift to a Lie ∞ -morphism

$$\Phi: (\mathfrak{g}[1], [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q). \quad (32)$$

Equivalently, if \mathfrak{g} is of finite dimension, the Lie ∞ -morphism (32) corresponds (by Proposition 3.1) to a Lie ∞ -algebroid (E', Q') over M such that

- (E, Q) is included as a sub-Lie ∞ -algebroid in a Lie algebroid (E', Q') over M ,
- its underlying complex is, $E'_{-1} := \mathfrak{g}[1] \oplus E_{-1}$, and for any $i \geq 2$, $E'_{-i} = E_{-i}$, namely

$$\dots \xrightarrow{d=\ell_1} E_{-3} \xrightarrow{d=\ell_1} E_{-2} \xrightarrow{d=\ell_1} \mathfrak{g}[1] \oplus E_{-1} \xrightarrow{\rho'} TM, \quad (33)$$

- we have,

$$\ell'_2(x \oplus 0, y \oplus 0) = [x, y]_{\mathfrak{g}} \oplus \eta(x, y)$$

and

$$\ell'_2(x, \Gamma(E_{-1})) \subset \Gamma(E_{-1})$$

for all $x \in \mathfrak{g}[1]$.

Remark 4.1. It is important to notice that the Lie ∞ -algebroid (E', Q') can be constructed even if \mathfrak{g} and (E, d) are of infinite dimensions (see Theorem 3.3).

Remark 4.2. In Equation (33), the complex (E, ℓ_1) can be chosen to be minimal at a point $m \in M$, i.e., $\ell_1|_m = 0$, provided that a geometric resolution of \mathcal{F} exists. By Proposition 4.14 in [19] the isotropy Lie algebra $\mathfrak{g}_m = \frac{\mathcal{F}(m)}{\mathcal{I}_m \mathcal{F}}$ of the singular foliation \mathcal{F} at the point $m \in M$ is isomorphic to $\ker(\rho_m)$. This allows to denote the latter space also by \mathfrak{g}_m .

Lemma 4.3. *Let $m \in M$ be a fixed point of the \mathfrak{g} -action ϱ . Assume that the underlying complex (E, ℓ_1) is minimal at a point m , i.e., $\ell_1|_m = 0$. The map*

$$\nu: \mathfrak{g} \longrightarrow \text{End}(\mathfrak{g}_m), \quad x \longmapsto \ell'_2(x, \cdot)|_m \quad (34)$$

satisfies

- (a) $\nu([x, y]_{\mathfrak{g}}) - [\nu(x), \nu(y)] + \ell_2(\cdot, \eta(x, y))|_m = 0$,
- (b) $\nu(z) (\eta(x, y)|_m) - \eta([x, y]_{\mathfrak{g}}, z)|_m + \odot(x, y, z) = 0$.

Proof. The map ν in (34) is well-defined since $\varrho|_m = 0$ and $\mathfrak{g}_m = \ker \rho_m$. The Jacobi identity on elements $x, y \in \mathfrak{g}[1]$, $e \in \Gamma(E_{-1})$, evaluated at the point m , implies that

$$\nu([x, y]_{\mathfrak{g}})(e|_m) - [\nu(x), \nu(y)](e|_m) + \ell_2(\eta(x, y), e)|_m = 0.$$

This proves item (a). Likewise, Jacobi identity on elements $x, y, z \in \mathfrak{g}[1]$ together with $\ell_1|_m = 0$ give:

$$\begin{aligned} \ell'_2(\ell'_2(x, y), z)|_m + \odot(x, y, z) = 0 &\implies \ell'_2([x, y]_{\mathfrak{g}}, z)|_m + \ell'_2(\eta(x, y), z)|_m + \odot(x, y, z) = 0, \\ &\implies \nu(z) (\eta(x, y)|_m) - \eta([x, y]_{\mathfrak{g}}, z)|_m + \odot(x, y, z) = 0. \end{aligned}$$

Here we have used the definition of ℓ'_2 on degree -1 elements and Jacobi identity for the bracket $[\cdot, \cdot]_{\mathfrak{g}}$. This proves item (b). \square

By Lemma 4.3, \mathfrak{g}_m is equipped with a \mathfrak{g} -module structure when $\eta(x, y)|_m$ is for all $x, y \in \mathfrak{g}$ valued in the center of the isotropy Lie algebra \mathfrak{g}_m . The following proposition is built on this last point. It defines an obstruction class mentioned earlier in the introduction of the paper as an obstruction of the possibility of turning a weak symmetry action into a strict symmetry action. Recall that η is defined by Equation (31). Notice that if $m \in M$ is a fixed point of the \mathfrak{g} -action ϱ , then this implies in particular that $\eta(x, y)|_m \in \ker \rho_m$.

Proposition 4.4. *Let $m \in M$ be a fixed point of the \mathfrak{g} -action ϱ . Assume that*

- the underlying complex (E, ℓ_1) of (E, Q) is minimal at m ,
- for all $x, y \in \mathfrak{g}$, $\eta(x, y)|_m$ is valued in the center³ $Z(\mathfrak{g}_m)$ of \mathfrak{g}_m .

Then,

- (1) For every $x \in \mathfrak{g}[1]$, $\ell'_2(x, \cdot)$ preserves $Z(\mathfrak{g}_m)$, and the restriction of the 2-ary bracket

$$\ell'_2: \mathfrak{g} \otimes Z(\mathfrak{g}_m) \longrightarrow Z(\mathfrak{g}_m)$$

endows $Z(\mathfrak{g}_m)$ with a \mathfrak{g} -module structure which does not depend neither on the choice of a weak symmetry action ϱ nor of a universal Lie ∞ -algebroid of \mathcal{F} , nor of the Lie ∞ -morphism $\Phi: \mathfrak{g}[1] \rightsquigarrow \mathfrak{X}(E)[1]$.

- (2) the restriction of the map $\eta: \wedge^2 \mathfrak{g} \longrightarrow E_{-1}$ at m

$$\eta|_m: \wedge^2 \mathfrak{g} \longrightarrow Z(\mathfrak{g}_m)$$

is a 2-cocycle for the Chevalley-Eilenberg complex of \mathfrak{g} valued in $Z(\mathfrak{g}_m)$,

³In particular, when the 2-ary bracket ℓ_2 is zero at m when applied to two elements of degree -1 , i.e., \mathfrak{g}_m is Abelian, we have $Z(\mathfrak{g}_m) = \mathfrak{g}_m$.

- (3) *the cohomology class of this cocycle does not depend on the representatives of the equivalence class of ϱ , nor on the choices made in the construction,*
- (4) *if ϱ is equivalent to a strict symmetry action, then $\eta|_m$ is exact.*

Proof. (E, ℓ_1) being minimal at m , $\ell_2|_m$ satisfies the Jacobi identity. In particular, for every $x \in \mathfrak{g}[1]$, $\ell_2'(x, \cdot)|_m$ preserves $Z(\mathfrak{g}_m)$. By item (a) of Lemma 4.3, the restriction of the 2-ary bracket

$$\ell_2': \mathfrak{g} \otimes Z(\mathfrak{g}_m) \longrightarrow Z(\mathfrak{g}_m)$$

endows $Z(\mathfrak{g}_m)$ with a \mathfrak{g} -module structure, since $\ell_2(\cdot, \eta(x, y))|_m = 0$ by assumption. It is easy to see that if we change the action ϱ to $\varrho + \rho \circ \beta$ for some vector bundle morphism $\beta: \mathfrak{g} \longrightarrow E_{-1}$ such that $\beta|_m: \mathfrak{g} \longrightarrow Z(\mathfrak{g}_m)$, the new 2-ary bracket between sections of $\mathfrak{g}[1]$ and E_{-1} constructed as in the proof of Theorem 3.3 is modified by $(x, e) \mapsto \ell_2'(x, e) + \ell_2(\beta(x), e)$. The second term of the latter vanishes at m , by definition of $\beta|_m$. As a result, the action of ℓ_2' on $Z(\mathfrak{g}_m)$ does not depend on the choices made in the construction. This proves item 1.

Item 2 follows from item (b) of Lemma 4.3 that tells that $\eta|_m: \wedge^2 \mathfrak{g} \longrightarrow Z(\mathfrak{g}_m)$ is a 2-cocycle for the Chevalley-Eilenberg complex of \mathfrak{g} valued in $Z(\mathfrak{g}_m)$.

Let ϱ' be a weak symmetry action of \mathfrak{g} on \mathcal{F} which is equivalent to ϱ , i.e., there exists a vector bundle morphism $\beta: \mathfrak{g} \longrightarrow E_{-1}$ (with $\beta|_m: \mathfrak{g} \longrightarrow Z(\mathfrak{g}_m)$) such that $\varrho'(x) = \varrho(x) + \rho(\beta(x))$ for all $x \in \mathfrak{g}$. Let $\eta': \wedge^2 \mathfrak{g} \longrightarrow E_{-1}$ be such that $\varrho'([x, y]_{\mathfrak{g}}) - [\varrho'(x), \varrho'(y)] = \rho(\eta'(x, y))$ for all $x, y \in \mathfrak{g}$. Following the constructions in the proof of Theorem 3.3, this implies that

$$\eta'(x, y) = \eta(x, y) + \beta([x, y]_{\mathfrak{g}}) - \ell_2'(x, \beta(y)) + \ell_2'(y, \beta(x)) - \ell_2(\beta(x), \beta(y)), \quad \text{for all } x, y \in \mathfrak{g}. \quad (35)$$

Equation (35) implies that $\eta'(x, y)|_m - \eta(x, y)|_m = d^{CE}(\beta|_m)(x, y)$, where d^{CE} stands for the Chevalley-Eilenberg differential. As a consequence, $\eta'|_m$ and $\eta|_m$ define the same class in the Chevalley-Eilenberg complex of \mathfrak{g} valued in $Z(\mathfrak{g}_m)$. This proves item 3 and 4. \square

Remark 4.5. Even when $\ell_2|_m \neq 0$, we can have some obstruction, but they are not given by cohomology classes because they are not given by linear equations. More precisely, it is obvious that the weak symmetry action ϱ is not equivalent to strict one if the Maurer-Cartan-like equation (35) has no solution β with $\eta'|_m = 0$.

The following is a direct consequence of Proposition 4.4.

Corollary 4.6. *Let $m \in M$ be a fixed point for the \mathfrak{g} -action ϱ . Assume that the isotropy Lie algebra \mathfrak{g}_m of \mathcal{F} at m is Abelian. Then*

- (1) \mathfrak{g}_m is a \mathfrak{g} -module.
- (2) *The bilinear map, $\eta|_m: \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}_m$, is a Chevalley-Eilenberg 2-cocycle of \mathfrak{g} valued in \mathfrak{g}_m .*
- (3) *Its class $\text{cl}(\eta|_m) \in H^2(\mathfrak{g}, \mathfrak{g}_m)$ does not depend on the choices made in the construction.*
- (4) *Furthermore, $\text{cl}(\eta|_m)$ is an obstruction of having a strict symmetry action equivalent to ϱ .*

Example 4.7. We return to Example 1.7 and consider the \mathfrak{g}_m^k -action on $\mathcal{I}_m^{k+1}\mathcal{F}$. Every point $m \in M$ is a fixed point for the \mathfrak{g}_m^k -action of item 2 of Example 1.7. Since the isotropy Lie algebra \mathfrak{g}_m^k is Abelian for every $k \geq 2$ the following assertions hold by Corollary 4.6:

- (1) For each $k \geq 1$, the vector space \mathfrak{g}_m^{k+1} is a \mathfrak{g}_m^k -module.
- (2) The obstruction of having a strict symmetry action equivalent to ϱ_k is a Chevalley-Eilenberg cocycle valued in \mathfrak{g}_m^{k+1} .

Here is a particular case of this example.

Example 4.8. Let $\mathcal{F} := \mathcal{I}_0^3 \mathfrak{X}(\mathbb{R}^n)$ be the singular foliation generated by vector fields vanishing to order 3 at the origin. The quotient $\mathfrak{g} := \frac{\mathcal{I}_0^2 \mathfrak{X}(\mathbb{R}^n)}{\mathcal{I}_0^3 \mathfrak{X}(\mathbb{R}^n)}$ is a trivial Lie algebra. There is a weak symmetry action of \mathfrak{g} on \mathcal{F} which assigns to an element in \mathfrak{g} a representative in $\mathcal{I}_0^2 \mathfrak{X}(\mathbb{R}^n)$. In this case, the isotropy Lie algebra of \mathcal{F} at zero is Abelian and $\ell'_2(\mathfrak{g}, \mathfrak{g}_0)|_0 = 0$. Thus, the action of \mathfrak{g} on \mathfrak{g}_0 is trivial. One can choose $\eta: \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}_0$ such that $\eta\left(\overline{x_i^2 \frac{\partial}{\partial x_i}}, \overline{x_i^2 \frac{\partial}{\partial x_j}}\right) = 2e_{ij}$, with e_{ij} a constant section in a set of generators of degree -1 whose image by the anchor is $x_i^3 \frac{\partial}{\partial x_j}$. Therefore, $\eta|_0\left(\overline{x_i^2 \frac{\partial}{\partial x_i}}, \overline{x_i^2 \frac{\partial}{\partial x_j}}\right) \neq 0$. This implies that the class of η is not zero at the origin. Therefore, by item 2 of Corollary 4.6 the weak symmetry action of \mathfrak{g} on \mathcal{F} is not equivalent to a strict one.

Also, we have the following consequence of Corollary 4.6 for Lie algebra actions on affine varieties, as in Example 1.11. Before going to Corollary 4.12 let us write definitions and some facts.

Settings: Let W be an affine variety realized as a subvariety of \mathbb{C}^d , and defined by some ideal $\mathcal{I}_W \subset \mathbb{C}[x_1, \dots, x_d]$. We denote by $\mathfrak{X}(W) := \text{Der}(\mathcal{O}_W)$ the Lie algebra of vector fields on W , where \mathcal{O}_W is coordinates ring of W .

Definition 4.9. A point $p \in W$ is said to be *strongly singular* if for all $f \in \mathcal{I}_W$, $d_p f \equiv 0$ or equivalently if for all $f \in \mathcal{I}_W$ and $X \in \mathfrak{X}(\mathbb{C}^d)$, one has $X[f](p) \in \mathcal{I}_p$.

Example 4.10. Any singular point of a hypersurface W defined by a polynomial $\varphi \in \mathbb{C}[x_1, \dots, x_d]$ is strongly singular.

The lemma below is immediate.

Lemma 4.11. *In a strongly singular point, the isotropy Lie algebra of the singular foliation $\mathcal{F} = \mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$ is Abelian.*

The following corollary answers the question of Example 1.11. Here, $cl(\eta|_p)$ is as in Corollary 4.6.

Corollary 4.12. *Let $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(W)$ be a Lie algebra morphism.*

- (1) *Any extension $\tilde{\varrho}$ as in Example 1.11 is a weak symmetry action for the singular foliation $\mathcal{F} = \mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$.*
- (2) *Let $p \in W$ be a fixed point for the \mathfrak{g} -action ϱ which is also a strongly singular point p in W . If class $cl(\eta|_p)$ does not vanish, then the Lie algebra morphism $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(W)$ can not be extended to a Lie algebra morphism $\tilde{\varrho}: \mathfrak{g} \rightarrow \mathfrak{X}(\mathbb{C}^d)$.*

Proof. This first item follows from Example 1.11. By Lemma 4.11, the isotropy Lie algebra of $\mathcal{F} = \mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$ is Abelian in every strongly singular points of an affine variety W . Thus, item 2 of Corollary 4.12 follows from Corollary 4.6. \square

Let us give examples of Lie algebra actions on an affine variety that do not extend to the ambient space.

Example 4.13. Let $W \subset \mathbb{C}^2$ be the affine variety generated by the polynomial $\varphi = FG$ with $F, G \in \mathbb{C}[x, y] := \mathcal{O}$. We consider the vector fields $U = F\mathcal{X}_G, V = G\mathcal{X}_F \in \mathfrak{X}(\mathbb{C}^2)$, where \mathcal{X}_F and \mathcal{X}_G are Hamiltonian vector fields w.r.t the Poisson structure $\{x, y\} := 1$. Note that U, V are tangent to W , i.e. $U[\varphi], V[\varphi] \in \langle \varphi \rangle$. It is easily checked that $[U, V] = \varphi \mathcal{X}_{\{F, G\}}$.

The action of the trivial Lie algebra $\mathfrak{g} = \mathbb{R}^2$ on W that sends its canonical basis (e_1, e_2) to U , and V respectively, is a weak symmetry action on the singular foliation $\mathcal{F}^\varphi := \langle \varphi \rangle \mathfrak{X}(\mathbb{C}^2)$ of vector fields vanishing on W , and induces a Lie algebra map,

$$\varrho: \mathfrak{g} \longrightarrow \mathfrak{X}(W). \quad (36)$$

A universal Lie ∞ -algebroid of \mathcal{F}^φ is a Lie algebroid (see Example 3.19 of [20]) because,

$$0 \longrightarrow \mathcal{O}\mu \otimes_{\mathcal{O}} \mathfrak{X}(\mathbb{C}^2) \xrightarrow{\varphi \frac{\partial}{\partial \mu} \otimes_{\mathcal{O}} \text{id}} \mathcal{F}^\varphi$$

is a \mathcal{O} -module isomorphism, (\mathcal{F}^φ is a projective module). Here, μ is a degree -1 variable, so that $\mu^2 = 0$. The universal algebroid structure over that resolution is given on the set of generators by:

$$\ell_2 \left(\mu \otimes_{\mathcal{O}} \frac{\partial}{\partial x}, \mu \otimes_{\mathcal{O}} \frac{\partial}{\partial y} \right) := \frac{\partial \varphi}{\partial x} \mu \otimes_{\mathcal{O}} \frac{\partial}{\partial y} - \frac{\partial \varphi}{\partial y} \mu \otimes_{\mathcal{O}} \frac{\partial}{\partial x} \quad (37)$$

and $\ell_k := 0$ for every $k \geq 3$. Since $\mathcal{X}_{\{F,G\}} = \frac{\partial\{F,G\}}{\partial y} \frac{\partial}{\partial x} - \frac{\partial\{F,G\}}{\partial x} \frac{\partial}{\partial y}$, we have

$$\eta(e_1, e_2) := \frac{\partial\{F,G\}}{\partial y} \mu \otimes_{\mathcal{O}} \frac{\partial}{\partial x} - \frac{\partial\{F,G\}}{\partial x} \mu \otimes_{\mathcal{O}} \frac{\partial}{\partial y}. \quad (38)$$

Take, for example, $F(x, y) = y - x^2$ and $G(x, y) = y + x^2$. The isotropy Lie algebra $\mathfrak{g}_{(0,0)}$ of \mathcal{F}^φ is Abelian by Equation (37), i.e. $\ell_2|_{(0,0)} = 0$. By Corollary 4.6 (1), $\mathfrak{g}_{(0,0)}$ is a \mathbb{R}^2 -module. A direct computation shows that the action on $\mathfrak{g}_{(0,0)}$ is not trivial, but takes values in $\mathcal{O}\mu \otimes_{\mathcal{O}} \frac{\partial}{\partial x}$. Besides, Equation (38) applied to $\{F, G\} = 4x$ gives

$$\eta(e_1, e_2) = -4\mu \otimes_{\mathcal{O}} \frac{\partial}{\partial y}. \quad (39)$$

If $\eta|_{(0,0)}$ were a coboundary of Chevalley-Eilenberg, we would have (in the notations of Proposition 4.4) that

$$\eta(x, y)|_{(0,0)} = \beta([x, y]_{\mathbb{R}^2}) - \ell'_2(x, \beta(y)) + \ell'_2(y, \beta(x)) \in \mathcal{O}\mu \otimes_{\mathcal{O}} \frac{\partial}{\partial x}, \quad \text{for all } x, y \in \mathfrak{g} \quad (40)$$

for some linear map⁴ $\beta: \mathfrak{g} \longrightarrow \mathfrak{g}_{(0,0)}$. Therefore, Equation (40) has no solution in view of Equation (39), since $\eta|_{(0,0)} \neq 0$. In other words, its class $\text{cl}(\eta)$ does not vanish at $(0, 0)$. By Corollary 4.12 (2), the action ϱ given in Equation (36) cannot be extended to the ambient space.

5. BI-SUBMERSION TOWERS AND SYMMETRIES

We end the paper by introducing the notion ‘‘bi-submersion towers’’. The work contained in this section is entirely original, except for the notion given in Definition 5.3 that arose in a discussion between C. Laurent-Gengoux, L. Ryvkin, and I, and will be the object of a separate study.

⁴Here, ℓ'_2 is as Proposition 4.4.

5.1. Definitions and existence. Let us firstly recall the definition of bi-submersion introduced in [1].

Definition 5.1. Let M be a manifold endowed with a singular foliation \mathcal{F} . A *bi-submersion*

$B \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{t} \end{array} M$ over \mathcal{F} is a triple (B, s, t) where:

- B is a manifold,
- $s, t: B \rightarrow M$ are submersions, respectively called *source* and *target*,

such that the pull-back singular foliations $s^{-1}\mathcal{F}$ and $t^{-1}\mathcal{F}$ are both equal to the space of vector fields of the form $\xi + \zeta$ with $\xi \in \Gamma(\ker(ds))$ and $\zeta \in \Gamma(\ker(dt))$. Namely,

$$s^{-1}\mathcal{F} = t^{-1}\mathcal{F} = \Gamma(\ker(ds)) + \Gamma(\ker(dt)). \quad (41)$$

In that case, we also say that (B, s, t) is a bi-submersion over (M, \mathcal{F}) .

Example 5.2. Let \mathcal{F} be a singular foliation on a manifold M . Let $x \in M$. Let X_1, \dots, X_n be vector fields in \mathcal{F} whose class in $\mathcal{F}_x := \mathcal{F}/\mathcal{I}_x\mathcal{F}$ generate the latter. We know from [1] that there is an open neighborhood \mathcal{W} of $(x, 0) \in M \times \mathbb{R}^n$ such that (\mathcal{W}, t, s) is a bi-submersion over \mathcal{F} , here

$$s(x, y) = x \quad \text{and} \quad t(x, y) = \exp_x \left(\sum_{i=1}^n y_i X_i \right) = \varphi_1^{\sum_{i=1}^n y_i X_i}(x) \quad (42)$$

where for $X \in \mathfrak{X}(M)$, φ_1^X denotes the time-1 flow of X . Such bi-submersions are called *path holonomy bi-submersions* [4].

Now we can introduce the following definition.

Definition 5.3. A *bi-submersion tower over a singular foliation \mathcal{F} on M* is a (finite or infinite) sequence of manifolds and maps as follows

$$\mathcal{T}_B : \cdots \begin{array}{c} \xrightarrow{s_{i+1}} \\ \xleftarrow{t_{i+1}} \end{array} B_{i+1} \begin{array}{c} \xrightarrow{s_i} \\ \xleftarrow{t_i} \end{array} B_i \begin{array}{c} \xrightarrow{s_{i-1}} \\ \xleftarrow{t_{i-1}} \end{array} \cdots \begin{array}{c} \xrightarrow{s_1} \\ \xleftarrow{t_1} \end{array} B_1 \begin{array}{c} \xrightarrow{s_0} \\ \xleftarrow{t_0} \end{array} B_0, \quad (43)$$

together with a sequence \mathcal{F}_i of singular foliations on B_i , with the convention that $B_0 = M$ and $\mathcal{F}_0 = \mathcal{F}$, such that

- for all $i \geq 1$, $\mathcal{F}_i \subset \Gamma(\ker ds_{i-1}) \cap \Gamma(\ker dt_{i-1})$,
- for each $i \geq 1$, $B_{i+1} \begin{array}{c} \xrightarrow{s_i} \\ \xleftarrow{t_i} \end{array} B_i$ is a bi-submersion over \mathcal{F}_i .

A bi-submersion tower over (M, \mathcal{F}) shall be denoted as $(B_{i+1}, s_i, t_i, \mathcal{F}_i)_{i \geq 0}$. The bi-submersion tower over \mathcal{F} in (43) is said to be of *length* $n \in \mathbb{N}$ if $B_j = B_n, s_j = t_j = \text{id}$ and $\mathcal{F}_j = \{0\}$ for all $j \geq n$.

Remark 5.4. Let us spell out some consequences of the axioms. For $i \geq 1$, two points $b, b' \in B_i$ of the same leaf of \mathcal{F}_i satisfy $s_{i-1}(b) = s_{i-1}(b')$ and $t_{i-1}(b) = t_{i-1}(b')$. Also, for all $b \in B_i$, $T_b \mathcal{F}_i \subset (\ker ds_{i-1})|_b \cap (\ker dt_{i-1})|_b$.

Let us explain how such towers can be constructed out of a singular foliation. Let \mathcal{F} be a singular foliation on M . Then,

- (1) By Proposition 2.10 in [1], there always exists a bi-submersion $B_1 \begin{array}{c} \xrightarrow{s_0} \\ \xleftarrow{t_0} \end{array} M$ over \mathcal{F} .

- (2) The $C^\infty(B_1)$ -module $\Gamma(\ker ds_0) \cap \Gamma(\ker dt_0)$ is closed under Lie bracket. When it is locally finitely generated, it is a singular foliation on B_1 . Then, it admits a bi-submersion

$$B_2 \begin{array}{c} \xrightarrow{s_1} \\ \xleftarrow{t_1} \end{array} B_1 .$$

- We now have obtained the two first terms of a bi-submersion tower.
- (3) We can then continue this construction provided that $\Gamma(\ker ds_1) \cap \Gamma(\ker dt_1)$ is locally finitely generated as a $C^\infty(B_2)$ -module, and that it is so at each step⁵.

Definition 5.5. A bi-submersion tower $\mathcal{T}_B = (B_{i+1}, s_i, t_i, \mathcal{F}_i)$ over (M, \mathcal{F}) is called *exact bi-submersion tower over (M, \mathcal{F})* when $\mathcal{F}_{i+1} = \Gamma(\ker(ds_i)) \cap \Gamma(\ker(dt_i))$ for all $i \geq 0$. It is called a *path holonomy bi-submersion tower* (resp. *path holonomy atlas bi-submersion tower*)

if $B_{i+1} \begin{array}{c} \xrightarrow{s_i} \\ \xleftarrow{t_i} \end{array} B_i$ is a path holonomy bi-submersion (resp. an Androulidakis-Skandalis' path

holonomy atlas⁶ over \mathcal{F}_i for each $i \geq 0$. When a path holonomy bi-submersion tower is exact, we speak of *exact path holonomy bi-submersion tower*.

The following theorem gives a condition which is equivalent to the existence of a bi-submersion tower over a singular foliation. The proof uses Proposition 5.16 and Lemma 5.17 which are stated in the next section.

Theorem 5.6. *Let \mathcal{F} be a singular foliation on M . The following items are equivalent:*

- (1) \mathcal{F} admits a geometric resolution.
- (2) There exists an exact path holonomy bi-submersion tower over (M, \mathcal{F}) .

Convention 5.7. For a submersion $\phi: M \rightarrow N$ and a smooth map $\psi: M \rightarrow N$, we denote by ${}^\psi\Gamma(\ker d\phi)$ the space of ψ -projectable vector fields in $\Gamma(\ker d\phi) \subset \mathfrak{X}(M)$.

Proof. (1) \Rightarrow (2): Assume that \mathcal{F} admits a geometric resolution (E, d, ρ) . In particular, (E_{-1}, ρ) is an anchored bundle over \mathcal{F} . We need to show by recursion on $i \geq 0$ that $\Gamma(\ker ds_i) \cap \Gamma(\ker dt_i)$ is locally finitely generated because $\ker d^{(i+1)}$ or $\ker \rho$ is locally finitely generated. We actually repeat at each step $i \geq 0$, the general fact that the pull-back complex of vector bundle by the submersion $\varphi = t_0 \circ t_1 \circ \dots \circ t_i: B_{i+1} \rightarrow M$

$$\dots \longrightarrow \varphi^* E_{-i-3} \xrightarrow{\varphi^* d^{(i+3)}} \varphi^* E_{-i-2} \xrightarrow{\varphi^* d^{(i+2)}} \varphi^* E_{-i-1} \longrightarrow TB_{i+1} \quad (44)$$

remains exact at the sections level (at degree⁷ ≤ -1), since $C^\infty(B_{i+1})$ is a flat $C^\infty(M)$ -module. In addition, for $i \geq 1$, the complex (44) defines a geometric resolution of \mathcal{F}_{i+1} .

Let (B_1, s_0, t_0) be a path holonomy bi-submersion over (M, \mathcal{F}) . Consider the map

$$R: \Gamma(t_0^* E_{-1}) \longrightarrow {}^{t_0}\Gamma(\ker ds_0) \subset \mathfrak{X}(B_1) \quad (45)$$

$$t_0^* e \longmapsto \vec{e}$$

defined as in Proposition 5.16. By Lemma 5.17 (1), the map R in (45) comes from a vector bundle morphism $t_0^* E_{-1} \rightarrow \ker ds_0$ and is surjective on an open subset $V_1 \subset B_1$, by item 2 of Lemma 5.17. In particular the map R in (45) restricts to a surjective map

$$\ker(t_0^* \rho) \longrightarrow \ker(dt_0|_{\Gamma(\ker ds_0)}) = \Gamma(\ker ds_0) \cap \Gamma(\ker dt_0) \subset \mathfrak{X}(B_1) \quad (46)$$

⁵In the real analytic case, the module $\Gamma(\ker ds_1) \cap \Gamma(\ker dt_1)$ is locally finitely generated because of the noetherianity of the ring of germs of real analytic functions [11, 26].

⁶See [1], Example 3.4 (3).

⁷We shall understand that the degree of elements of $\varphi^* E_{-i-j}$ is $-j$.

By exactness in degree -1 , $\ker \rho = d^{(2)}(\Gamma(E_{-2}))$. Therefore, $\ker(t_0^*\rho)$ is locally finitely generated. By surjectivity of the map (46), $\Gamma(\ker ds_0) \cap \Gamma(\ker dt_0) =: \mathcal{F}_1$ is also locally finitely generated on V_1 , in particular \mathcal{F}_1 is a singular foliation on $V_1 \subset B_1$ (we may assume that $B_1 = V_1$). Thus, one can take a path holonomy bi-submersion (B_2, s_1, t_1) over (B_1, \mathcal{F}_1) . The proof continues the same as the previous step.

Let us make a step further for clarity. The composition

$$\Gamma(E_{-2}) \xrightarrow{d^{(2)}} \text{im}(d^{(2)}) = \ker \rho \xrightarrow{\twoheadrightarrow} \mathcal{F}_1$$

together with E_{-2} is an anchored bundle over \mathcal{F}_1 . Let $\varphi = t_0 \circ t_1$. Just like in the first step, define the surjective $C^\infty(B_2)$ -linear map,

$$\begin{aligned} \Gamma(\varphi^* E_{-2}) &\longrightarrow {}^{t_1}\Gamma(\ker ds_1) \subset \mathfrak{X}(B_2) \\ \varphi^* e &\longmapsto \overleftarrow{e}. \end{aligned} \quad (47)$$

By Lemma 5.17 (2), the map 47 restricts (upon taking B_2 smaller) to a surjective map

$$\ker \left(\Gamma(\varphi^* E_{-2}) \xrightarrow{\varphi^* d^{(2)}} \Gamma(\varphi^* E_{-1}) \right) \longrightarrow \ker(dt_1|_{\Gamma(\ker ds_1)}) = \Gamma(\ker ds_1) \cap \Gamma(\ker dt_1) \subset \mathfrak{X}(B_2) \quad (48)$$

By exactness in degree -2 , the $C^\infty(M)$ -module

$$\ker \left(\Gamma(E_{-2}) \xrightarrow{d^{(2)}} \Gamma(E_{-1}) \right) = d^{(3)}(\Gamma(E_{-3}))$$

is (locally) finitely generated, hence $\mathcal{F}_2 := \Gamma(\ker ds_1) \cap \Gamma(\ker dt_1)$ is a singular foliation on B_2 . Thus, one can take a path holonomy bi-submersion (B_3, s_2, t_2) over (B_2, \mathcal{F}_2) . By recursion on $i \geq 1$, we use a path holonomy bi-submersion (B_{i+1}, s_i, t_i) over (B_i, \mathcal{F}_i) and construct an anchor bundle over \mathcal{F}_i by the composition

$$\Gamma(\varphi^* E_{-i-1}) \xrightarrow{\varphi^* d^{(i+1)}} \text{im}(\varphi^* d^{(i+1)}) = \ker(\varphi^* d^{(i)}) \xrightarrow{\twoheadrightarrow} \mathcal{F}_i$$

with $\varphi = t_0 \circ t_1 \circ \dots \circ t_{i-1}: B_i \longrightarrow M$ and show as for $i = 0, 1$ that $\mathcal{F}_{i+1} := \Gamma(\ker ds_i) \cap \Gamma(\ker dt_i)$ is a singular foliation on B_{i+1} . The proof follows.

(2) \Rightarrow (1) is proven by Lemma 5.8 and Remark 5.9 below. \square

In the following lemma we deduce out of any bi-submersion tower over a singular foliation, a complex of vector bundles over different base manifolds, and discuss exactness. In Remark 5.9, we give conditions to have a complex vector bundles over M .

Lemma 5.8. *Let \mathcal{F} be a singular foliation on M . Assume that there exists a bi-submersion tower $\mathcal{T}_B = (B_i, t_i, s_i, \mathcal{F}_i)_{i \geq 0}$ over \mathcal{F} . Then,*

$$\begin{array}{ccccccc} \dots & \longrightarrow & \ker ds_2 & \xrightarrow{dt_2} & \ker ds_1 & \xrightarrow{dt_1} & \ker ds_0 & \xrightarrow{dt_0} & TM \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & B_3 & \xrightarrow{t_2} & B_2 & \xrightarrow{t_1} & B_1 & \xrightarrow{t_0} & M. \end{array} \quad (49)$$

is a complex of vector bundles, which is exact at the sections level⁸ if \mathcal{T}_B is an exact bi-submersion tower, i.e., if $\mathcal{F}_i = \Gamma(\ker ds_{i-1}) \cap \Gamma(\ker dt_{i-1})$ for all $i \geq 1$.

⁸Let us explain the notion of exactness at the level of sections when the base manifolds are not the same: what we mean is that for all $n \geq 0$, $\Gamma(\ker dt_n) \cap \Gamma(\ker ds_n)$ is equal to the t_{n+1} -projectable vector fields in $\Gamma(\ker ds_{n+1})$.

Proof. For any element $b \in B_{i+1}$ and any vector $v \in \ker ds_i \subset T_b B_{i+1}$ one has,

$$\begin{aligned} dt_i(v) &\in T_{t_i(b)} \mathcal{F}_i, \quad (\text{since } \Gamma(\ker ds_i) \subset t_i^{-1}(\mathcal{F}_i)). \\ \implies dt_i(v) &\in (\ker ds_{i-1} \cap \ker dt_{i-1})|_{t_i(b)}, \quad \text{by Definition 5.3.} \\ \implies dt_i(v) &\in \ker ds_{i-1} \quad \text{and} \quad dt_{i-1} \circ dt_i(v) = 0, \quad \text{for all } i \geq 1. \end{aligned}$$

This shows the sequence (49) is a well-defined complex of vector bundles.

Let us prove that it is exact when $\mathcal{F}_i = \Gamma(\ker ds_{i-1}) \cap \Gamma(\ker dt_{i-1})$ for all $i \geq 1$. Let $\xi \in \Gamma(\ker ds_{i-1})$ be a t_{i-1} -projectable vector field that projects to zero, i.e. $dt_{i-1}(\xi) = 0$. This implies that $\xi \in \Gamma(\ker ds_{i-1}) \cap \Gamma(\ker dt_{i-1}) = \mathcal{F}_i$. Since t_i is a submersion, there exists a t_i -projectable vector field $\zeta \in t_i^{-1}(\mathcal{F}_i)$ that satisfies $dt_i(\zeta) = \xi$. The vector field ζ can be written as $\zeta = \zeta_1 + \zeta_2$ with $\zeta_1 \in \Gamma(\ker dt_i)$ and $\zeta_2 \in \Gamma(\ker ds_i)$, because $t_i^{-1}(\mathcal{F}_i) = \Gamma(\ker ds_i) + \Gamma(\ker dt_i)$. One has, $dt_i(\zeta_2) = \xi$. A similar argument shows that the map, $\Gamma(\ker ds_0) \xrightarrow{dt_0} t_0^* \mathcal{F}$, is surjective. This proves exactness in all degree. \square

Remark 5.9. One of the consequence of Lemma 5.8 is that:

- (1) If there exists a sequence of maps

$$M \xrightarrow{\varepsilon_0} B_1 \xrightarrow{\varepsilon_1} B_2 \xrightarrow{\varepsilon_2} \dots \quad (51)$$

where for all $i \geq 0$, ε_i is a section for both s_i and t_i then the pull-back of (49) on M through the sections $(\varepsilon_i)_{i \geq 0}$ i.e.,

$$\dots \xrightarrow{dt_3} \varepsilon_{2,0}^* \ker ds_2 \xrightarrow{dt_2} \varepsilon_{1,0}^* \ker ds_1 \xrightarrow{dt_1} \varepsilon_0^* \ker ds_0 \xrightarrow{dt_0} TM \quad (52)$$

is a complex of vector bundles, with the convention $\varepsilon_{n,0} = \varepsilon_n \circ \dots \circ \varepsilon_0$. If \mathcal{T}_B is an exact bi-submersion tower then, (52) is a geometric resolution of \mathcal{F} .

- (2) In case that \mathcal{T}_B is an exact path holonomy bi-submersion tower, such a sequence (51) always exists, since the bi-submersions (B_{i+1}, s_i, t_i) are as in Example 5.2. For such bi-submersions, the zero section $x \mapsto (x, 0)$ is a section for both s_i and t_i .

Theorem 5.6 is now proven. Here is a consequence.

Corollary 5.10. *Let \mathcal{F} be a singular foliation on M . Assume that there exists an exact bi-submersion tower $\mathcal{T}_B = (B_i, t_i, s_i, \mathcal{F}_i)_{i \geq 0}$ over \mathcal{F} of length $n + 1$. Then, the pull-back of the sequence of vector bundles*

$$\begin{array}{ccccccc} \ker ds_n & \xrightarrow{dt_n} & t_n^* \ker ds_{n-1} & \longrightarrow & \dots & \xrightarrow{dt_2} & t_{2,n}^* \ker ds_1 & \xrightarrow{dt_1} & TB_{n+1} \times_{TM} \ker ds_0 & \xrightarrow{\text{pr}_1} & TB_{n+1} \\ & & & & & & & & \downarrow & & \\ & & & & & & & & B_{n+1} & & \end{array} \quad (53)$$

is a geometric resolution of the pull-back foliation $t_{0,n}^{-1}(\mathcal{F}) \subset \mathfrak{X}(B_{n+1})$, where pr_1 is the projection on TB_{n+1} and for $i \geq 1$, $t_{i,j}$ is the composition $t_i \circ \dots \circ t_j: B_{j+1} \rightarrow B_i$.

Equivalently, it means that the pull-back of the vector bundles in (53) to any one of the manifold B_m with $m \geq n$ is exact at the level of sections, i.e.,

$$\Gamma(t_{n+1,m}^* \ker ds_{n+1}) \xrightarrow{dt_{n+1}} \Gamma(t_{n,m}^* \ker ds_n) \xrightarrow{dt_n} \Gamma(t_{n-1,m}^* \ker ds_{n-1}) \quad (50)$$

is a short exact sequence of $C^\infty(B_m)$ -modules, with $t_{n,m} = t_n \circ \dots \circ t_m$ for all $m \geq n$.

Proof. By Lemma 5.8, the complex in Equation (53) is exact. By construction, the projection of the fiber product $TB_{n+1} \times_{TM} \ker ds_0$ to TB_{n+1} induces the singular foliation $t_{0,n}^{-1}(\mathcal{F})$. \square

5.2. Lift of a symmetry to the bi-submersion tower. Let us investigate what an action $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ of a Lie algebra \mathfrak{g} on a singular foliation (M, \mathcal{F}) would induce on a bi-submersion tower \mathcal{T}_B over \mathcal{F} .

We start with some vocabulary and preliminary results.

Definition 5.11. Let (B, s, t) be a bi-submersion over a singular foliation \mathcal{F} on a manifold M . We call *lift* of a vector field $X \in \mathfrak{X}(M)$ to the bi-submersion (B, s, t) a vector field $\tilde{X} \in \mathfrak{X}(B)$ which is both s -projectable on X and t -projectable on X .

The coming proposition means that the notion of lift to a bi-submersion only makes sense for symmetries of the singular foliation.

Proposition 5.12. *If a vector field on M admits a lift to a bi-submersion (B, s, t) over a singular foliation \mathcal{F} , then it is a symmetry of \mathcal{F} .*

Proof. Let $\tilde{X} \in \mathfrak{X}(B)$ be a lift of $X \in \mathfrak{X}(M)$. Since \tilde{X} is s -projectable, $[\tilde{X}, \Gamma(\ker ds)] \subset \Gamma(\ker ds)$. Since \tilde{X} is t -projectable, $[\tilde{X}, \Gamma(\ker dt)] \subset \Gamma(\ker dt)$. Hence:

$$\begin{aligned} [\tilde{X}, s^{-1}(\mathcal{F})] &= [\tilde{X}, \Gamma(\ker ds) + \Gamma(\ker dt)] \\ &= [\tilde{X}, \Gamma(\ker ds)] + [\tilde{X}, \Gamma(\ker dt)] \\ &\subset \Gamma(\ker ds) + \Gamma(\ker dt) = s^{-1}(\mathcal{F}). \end{aligned}$$

In words, \tilde{X} is a symmetry of $s^{-1}\mathcal{F}$. Since \tilde{X} projects through s to X , X is a symmetry of \mathcal{F} . \square

We investigate the existence of lifts of symmetries of \mathcal{F} to bi-submersions over \mathcal{F} .

Remark 5.13. For $X, Y \in \mathfrak{s}(\mathcal{F})$,

- (1) the lift \tilde{X} to a given bi-submersion is not unique, even when it exists. However, two different lifts of a $X \in \mathfrak{s}(\mathcal{F})$ to a bi-submersion (B, s, t) differ by an element of the intersection $\Gamma(\ker(ds)) \cap \Gamma(\ker(dt))$.
- (2) the lift \tilde{X} is a symmetry of $\Gamma(\ker(ds)) \cap \Gamma(\ker(dt))$, i.e., $[\tilde{X}, \Gamma(\ker(ds)) \cap \Gamma(\ker(dt))] \subset \Gamma(\ker(ds)) \cap \Gamma(\ker(dt))$, since \tilde{X} is s -projectable and t -projectable.
- (3) If the lifts \tilde{X} and \tilde{Y} exist, then $[\tilde{X}, \tilde{Y}]$ exists and

$$[\tilde{X}, \tilde{Y}] - [\tilde{X}, \tilde{Y}] \in \Gamma(\ker ds) \cap \Gamma(\ker dt). \quad (54)$$

As the following example shows, the lift of a symmetry to a bi-submersion may not exist.

Example 5.14. Consider the trivial foliation $\mathcal{F} := \{0\}$ on M . For any diffeomorphism $\phi: M \rightarrow M$, (M, id, ϕ) is a bi-submersion over \mathcal{F} . Every vector field $X \in \mathfrak{X}(M)$ is a symmetry of \mathcal{F} . If it exists, its lift has to be given by $\tilde{X} = X$ since the source map is the identity. But $\tilde{X} = X$ is t -projectable if and only if X is ϕ -invariant. A non ϕ -invariant vector field X therefore admits no lift to (M, id, ϕ) .

However, internal symmetries, i.e., elements in \mathcal{F} admit lifts to any bi-submersion.

Proposition 5.15. *Let (B, s, t) be a bi-submersion over a singular foliation \mathcal{F} on a manifold M . Every internal symmetry, i.e., every vector field in $X \in \mathcal{F}$, admits a lift $\tilde{X} \in \mathfrak{X}(B)$ to (B, s, t) . Moreover, \tilde{X} can be chosen to be of the form*

$$\tilde{X} := X_s^t + X_t^s$$

with $X_s^t \in \Gamma(\ker(ds))$ and $X_t^s \in \Gamma(\ker(dt))$.

Proof. Let $X \in \mathcal{F}$. Since $s: B \rightarrow M$ is a submersion, there exists $X^s \in \mathfrak{X}(B)$ s -projectable on X . Since t is a submersion, there exists $X^t \in \mathfrak{X}(B)$ t -projectable on X . By construction $X^s \in s^{-1}(\mathcal{F})$ and $X^t \in t^{-1}(\mathcal{F})$. Using the property (41) of the bi-submersion (B, s, t) , the vector fields X^s and X^t decompose as

$$\begin{cases} X^s = X_s^s + X_t^s & \text{with } X_s^s \in \Gamma(\ker(ds)), X_t^s \in \Gamma(\ker(dt)), \\ X^t = X_s^t + X_t^t & \text{with } X_s^t \in \Gamma(\ker(ds)), X_t^t \in \Gamma(\ker(dt)). \end{cases}$$

By construction, X_t^s is s -projectable to X and t -projectable to 0 while X_s^t is s -projectable to 0 and t -projectable to X . It follows that, $\tilde{X} := X_s^t + X_t^s$, is a lift of X to the bi-submersion (B, s, t) . \square

The proof we give for Theorem 5.6 uses the notion of left-invariant, resp. right-invariant, vector fields on a bi-submersion over a singular foliation. We define the latter in the next proposition. It uses the notion of anchored bundle over a singular foliation and almost Lie algebroid, see [19, 25] for more details.

Proposition 5.16. *Let (B, s, t) be a bi-submersion over a singular foliation \mathcal{F} on a manifold M . Let (A, ρ) be an anchored bundle over \mathcal{F} , i.e., $A \rightarrow M$ is a vector bundle and $\rho: A \rightarrow TM$ is a vector bundle morphism such that $\rho(\Gamma(A)) = \mathcal{F}$. There exists two maps*

$$\begin{aligned} \Gamma(A) &\longrightarrow \mathfrak{X}(B) \\ a &\longmapsto \overleftarrow{a} \\ a &\longmapsto \overrightarrow{a} \end{aligned} \tag{55}$$

fulfilling the following conditions:

- (1) the vector field $\overrightarrow{a} \in \mathfrak{X}(B)$ (resp. $\overleftarrow{a} \in \mathfrak{X}(B)$) is t -related (resp. s -related) with $\rho(a) \in \mathcal{F}$,
- (2) the vector field \overrightarrow{a} (resp. \overleftarrow{a}) is tangent to the fibers of s (resp. t),
- (3) $\overrightarrow{fa} = t^*(f)\overrightarrow{a}$ and $\overleftarrow{fa} = s^*(f)\overleftarrow{a}$ for all $a \in \Gamma(A)$, $f \in C^\infty(M)$.

The vector fields \overleftarrow{a} (resp. \overrightarrow{a}) for $a \in \Gamma(A)$ are called left-invariant (resp. right-invariant) vector fields of (B, s, t) .

Proof. By Proposition 5.15, given a section $a \in \Gamma(A)$ the vector field $\rho(a) \in \mathcal{F}$ admits a lift $\widetilde{\rho(a)} \in \mathfrak{X}(B)$ on (B, s, t) of the form

$$\widetilde{\rho(a)} := \rho(a)_s^t + \rho(a)_t^s$$

with $\rho(a)_s^t \in \Gamma(\ker(ds))$ and $\rho(a)_t^s \in \Gamma(\ker(dt))$. Also, $dt(\rho(a)_s^t) = \rho(a)$ and $ds(\rho(a)_t^s) = \rho(a)$. Let $b \in B$ and \mathcal{U}_b an open neighborhood of b . Let (a_1, \dots, a_r) be a local trivialization of A on the open subset $\mathcal{U} = s(\mathcal{U}_b) \subset M$. We define a map $R_{\mathcal{U}}$ on local generators by

$$\begin{aligned} R_{\mathcal{U}}: \Gamma_{\mathcal{U}_b}(t^*A) &\longrightarrow \Gamma_{\mathcal{U}_b}(\ker(ds)) \\ t^*a_i &\longmapsto \rho(a_i)_s^t \end{aligned} \tag{56}$$

and extend by $C^\infty(\mathcal{U}_b)$ -linearity. These maps can be glued using partitions of unity. More precisely, let $(\chi_\lambda)_{\lambda \in \Lambda}$ be a partition of unity subordinate to an open cover $(\mathcal{U}_\lambda)_{\lambda \in \Lambda}$ by open sets that trivialize the vector bundle A . We define a map R on $\Gamma(t^*A)$ as

$$\sum_{\lambda \in \Lambda} \chi_\lambda R_{\mathcal{U}_\lambda}.$$

Now for $a \in \Gamma(a)$ we define $\overrightarrow{a} := R(s^*a)$. The map $\overleftarrow{\bullet}$ is defined similarly. Item 1, 2 and 3 hold by construction.

Assume that (A, ρ) is equipped with an almost Lie algebroid bracket $[\cdot, \cdot]_A$. For all $a, b \in \Gamma(A)$ one has

$$\begin{aligned} ds \left(\overleftarrow{[a, b]_A} - [\overleftarrow{a}, \overleftarrow{b}] \right) &= \rho([a, b]_A) - [\rho(a), \rho(b)] \\ &= 0, \end{aligned}$$

because \overleftarrow{a} is s -projectable to $\rho(a)$ and ρ is a morphism of brackets. Since left-invariant vector fields are tangent to the fibers of t , one has $dt \left(\overleftarrow{[a, b]_A} - [\overleftarrow{a}, \overleftarrow{b}] \right) = 0$. The proof is similar for $\overrightarrow{[a, b]_A} - [\overrightarrow{a}, \overrightarrow{b}]$. This ends the proof. \square

The following lemma is important in the proof of Theorem 5.6.

Lemma 5.17. *Let (B, s, t) be any bi-submersion over a singular foliation \mathcal{F} on a manifold M , and (A, ρ) an anchored bundle over \mathcal{F} .*

- (1) *There exists vector bundle morphisms $R: t^*A \rightarrow \ker ds$ and $L: s^*A \rightarrow \ker dt$ inducing (55).*
- (2) *Let $x \in M$. If (B, s, t) is a path holonomy bi-submersion over \mathcal{F} near $(x, 0)$ then, every $b \in B$ such that $t(b) = x$ admits a neighborhood V such that every t -projectable vector field of $\Gamma_V(\ker ds)$ is of the form $R(\xi)$ for some $\xi \in \Gamma_V(t^*A)$.*

Remark 5.18. In item 1 of Lemma 5.17, by ‘‘inducing (55)’’ we mean that for every $a \in \Gamma(A)$, $\overrightarrow{a} := R(t^*a)$ and $\overleftarrow{a} := L(s^*a)$.

Proof. Item 1 is obtained in the proof of Proposition 5.16. For instance, the $C^\infty(B)$ -linear map $R: \Gamma(t^*A) \rightarrow \Gamma(\ker ds)$ defined in (56) corresponds to a morphism of vector bundles $t^*A \rightarrow \ker ds$. Let us prove item 2. By assumption, B is a neighborhood of $(0, x)$ in $M \times \mathbb{R}^n$ with $n = \text{rk}_x(\mathcal{F}) = \dim(\mathcal{F}_x := \mathcal{F}/\mathcal{I}_x\mathcal{F})$ near $x \in M$ (see Example 5.2). Let $b \in B$ and \mathcal{U}_b an open neighborhood of b . Let (a_1, \dots, a_r) be a local trivialization of A on the open subset $\mathcal{U} = t(\mathcal{U}_b) \subset M$. One has by definition of right-invariant vector fields of (B, s, t) that $dt(\overrightarrow{a}_i) = t^*\rho(a_i)$ for $i = 1, \dots, \text{rk}(A)$. The vector fields $\rho(a_i)$ are generators of \mathcal{F} on \mathcal{U} . We necessarily have $n \leq \text{rk}(A)$. Since the $\rho(a_i)(x)$'s are generators of \mathcal{F}_x , without loss of generality we can assume that $\rho(a_1)(x), \dots, \rho(a_n)(x)$ is a basis of \mathcal{F}_x . Since $\text{rk}(\ker ds) = n$, $(\overrightarrow{a}_i(b))_{i=1, \dots, n}$ form a basis of $\ker ds|_x$. Therefore, the \overrightarrow{a}_i 's are independent at every point of some neighborhood $V \subset \mathcal{U}_b$ of b i.e., they form a local trivialization of the vector bundle $\ker ds \rightarrow B$. As a result, vector fields of $\Gamma_V(\ker ds)$ are of the form $\sum_i^n f_i \overrightarrow{a}_i$ with $f_i \in C^\infty(V)$ for $i = 1, \dots, n$. This ends the proof. \square

We can now state one of the important results of this section, which is the converse of Proposition 5.12. It uses several concepts introduced in [1], which are recalled in the proof.

Proposition 5.19. *Let \mathcal{F} be a singular foliation on a manifold M . Every symmetry $X \in \mathfrak{s}(\mathcal{F})$ admits a lift*

- (1) to any path holonomy bi-submersion (B, s, t) ,
- (2) to Androulidakis-Skandalis' path holonomy atlas,
- (3) to a neighborhood of any point in a bi-submersion through which there exists a local bisection that induces the identity.

Proof (of Proposition 5.19). Let $X \in \mathfrak{s}(\mathcal{F})$. Assume that $(B, s, t) = (\mathcal{W}, s_0, t_0)$ is a path holonomy bi-submersion associated to some generators $X_1, \dots, X_n \in \mathcal{F}$ as in Example 5.2. Fix $(u, y = (y_1, \dots, y_n)) \in \mathcal{W} \subset M \times \mathbb{R}^n$, set $Y := \sum_{i=1}^n y_i X_i$. By assumption, $[Y, X] \in \mathcal{F}$. This implies that $d\varphi_1^Y(X) = (\varphi_1^Y)_*(X) \in X + \mathcal{F}$. Indeed, for t such that the flow φ_t^Y of Y is defined, one has

$$\begin{aligned} d\varphi_1^Y(X) &= d\varphi_0^Y(X) + \int_0^1 \frac{d}{dt}(d\varphi_t^Y) dt \\ &= X + \int_0^1 \underbrace{d\varphi_t^Y([Y, X])}_{\in \mathcal{F}} dt, \quad \text{since } d\varphi_t^Y(\mathcal{F}) = \mathcal{F}. \end{aligned}$$

Let $Z_y = \int_0^1 d\varphi_t^Y([Y, X]) dt$. When \mathcal{F} is closed for Fréchet topology, it is clear that Z_y belongs to \mathcal{F} . We claim that it is in fact always true: Upon restricting to an open subset of M if necessary, Item 2 in Remark 2.9 implies that one can find local generators X_1, \dots, X_r of \mathcal{F} , such that

$$d\varphi_t^Y([Y, X]) = \sum_{i=1}^r F_t^i X_i$$

for some smooth functions F_t^i depending smoothly on the variable t . By integration, $Z_y = \sum_{i=1}^r \int_0^1 F_t^i dt X_i$ belongs to \mathcal{F} . In conclusion, there exists $Z_y \in \mathcal{F}$ depending smoothly on y such that $dt_0(X, 0) = X + Z_y$. Take $\tilde{Z}_y \in t_0^{-1}(\mathcal{F})$ such that $dt_0(\tilde{Z}_y) = Z_y$. One has,

$$dt_0\left((X, 0) - \tilde{Z}_y\right) = X.$$

Also, we can write $\tilde{Z}_y = \tilde{Z}_y^1 + \tilde{Z}_y^2$, with $\tilde{Z}_y^1 \in \Gamma(\ker ds_0)$, $\tilde{Z}_y^2 \in \Gamma(\ker dt_0)$, since $\tilde{Z}_y \in t_0^{-1}(\mathcal{F})$. By construction, the vector field $\tilde{X} := (X, -\tilde{Z}_y^1)$ is a lift of X to the bi-submersion (\mathcal{W}, s_0, t_0) . This proves item 1.

If $X_B \in \mathfrak{X}(B)$ and $X_{B'} \in \mathfrak{X}(B')$ are two lifts of the symmetry X on the path holonomy bi-submersions (B, s, t) and (B', s', t') respectively, then $(X_B, X_{B'})$ is a lift of X on the composition bi-submersion $B \circ B'$. This proves item 2, since the Androulidakis-Skandalis' path holonomy atlas is made of fibered products and inverse of holonomy path holonomy bi-submersions [1].

Item 2 in Proposition 2.10 of [1] states that if the identity of M is carried by (B, s, t) at some point $v \in B$, then there exists an open neighborhood $V \subset B$ of v that satisfies $s|_V = s_0 \circ g$ and $t|_V = t_0 \circ g$, for some submersion $g: V \rightarrow \mathcal{W}$, for \mathcal{W} of the form as in item 1. Thus, for all $X \in \mathfrak{s}(\mathcal{F})$ there exists a vector field $\tilde{X} \in \mathfrak{X}(V)$ fulfilling $ds|_V(\tilde{X}) = dt|_V(\tilde{X}) = X$. This proves item 3. \square

Definition 5.20. A symmetry of the tower of bi-submersion $\mathcal{T}_B = (B_{i+1}, s_i, t_i, \mathcal{F}_i)_{i \geq 0}$ over a singular foliation $\mathcal{F}_0 = \mathcal{F}$ is a family $(X^i)_{i \geq 0}$ of vector fields with the i -th component X^i in $\mathfrak{s}(\mathcal{F}_i)$ such that $ds_{i-1}(X^i) = dt_{i-1}(X^i) = X^{i-1}$ for all $i \geq 1$. We denote by $\mathfrak{s}(\mathcal{T}_B)$ the Lie algebra of symmetries of \mathcal{T}_B .

The next theorem gives a class of bi-submersion tower to which any symmetry of the base singular foliation \mathcal{F} lifts.

Theorem 5.21. *Let \mathcal{F} be a foliation. Let \mathcal{T}_B be an exact path holonomy bi-submersion tower (or an exact path holonomy atlas bi-submersion tower). A vector field $X \in \mathfrak{X}(M)$ is a symmetry of \mathcal{F} , i.e. $[X, \mathcal{F}] \subset \mathcal{F}$, if and only if it is the component on M of a symmetry of \mathcal{T}_B .*

Proof. For any symmetry $(X^i)_{i \geq 0}$ of the bi-submersion tower \mathcal{T}_B the first component $X^0 \in \mathfrak{X}(M)$ is a symmetry of \mathcal{F} , by Proposition 5.12. The other implication is a consequence of item 1. resp. item 2. in Proposition 5.19 and Remark 5.13. It is due to the fact that the tower \mathcal{T}_B is generated by path holonomy bi-submersions, and then we can lift symmetries at every stage of the tower \mathcal{T}_B . Indeed, assume for instance that \mathcal{T}_B is an exact path holonomy bi-submersion tower. By Proposition 5.19, any symmetry $X \in \mathfrak{s}(\mathcal{F})$ admits a lift $X^1 \in \mathfrak{X}(B_1)$. Moreover, X^1 is a symmetry of the singular foliation $\mathcal{F}_1 = \Gamma(\ker(ds_0)) \cap \Gamma(\ker(dt_0))$, by Remark 5.13(3). We continue by recursion to construct $X^i \in \mathfrak{s}(\mathcal{F}_i)$ for $i \geq 2$. \square

Remark 5.22. Let $(X^i)_{i \geq 0}$ be a lift of $X^0 := X \in \mathfrak{s}(\mathcal{F})$. For $i \geq 1$, $\nabla_X^i := \text{ad}_{X^i}$ preserves $\Gamma(\ker ds_{i-1})$, since X^i is s_{i-1} -projectable. Altogether, they define a chain map $(\nabla_X^i)_{i \geq 0}$ at the section level of the complex (49), on projectable vector fields in (50), since for every $i \geq 0$ and any t_i -projectable vector field $\xi \in \ker ds_i$,

$$\begin{aligned} dt_i([X^{i+1}, \xi]) &= [dt_i(X^{i+1}), dt_i(\xi)] \\ &= [X^i, dt_i(\xi)], \end{aligned}$$

that is $dt_i \circ \nabla_X^{i+1} = \nabla_X^i \circ dt_i$.

Remark 5.23. In [14], under some assumptions, it is shown that if a Lie group G acts on a foliated manifold (M, \mathcal{F}) , then it acts on its holonomy groupoid. It is likely that this result follows from Theorem 5.21, this will be addressed in another study.

5.3. Lifts of actions of a Lie algebra on a bi-submersion tower. We end the section with the following constructions and some natural questions.

Let $\mathcal{T}_B = (B_{i+1}, s_i, t_i, \mathcal{F}_i)_{i \geq 0}$ be an exact path holonomy bi-submersion tower over a singular foliation (M, \mathcal{F}) of length $n + 1$.

By Theorem 5.21, any vector field $X \in \mathfrak{s}(\mathcal{F})$ lifts to a symmetry $(X^i)_{i \geq 0}$ of \mathcal{T}_B . Once a lift is chosen, we can define a linear map,

$$X \in \mathfrak{s}(\mathcal{F}) \mapsto (X^i)_{i \geq 1} \in \mathfrak{s}(\mathcal{T}_B).$$

Let $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ be a strict symmetry action of a Lie algebra \mathfrak{g} on (M, \mathcal{F}) . For $x \in \mathfrak{g}$, there exists $(\varrho(x)^i)_{i \geq 0}$, with $\varrho(x)^i \in \mathfrak{s}(\mathcal{F}_i) \subset \mathfrak{X}(B_i)$ a symmetry of \mathcal{T}_B such that $X_0 = \varrho(x) \in \mathfrak{s}(\mathcal{F})$, by Theorem 5.21. Consider the composition,

$$x \in \mathfrak{g} \longmapsto \varrho(x) \in \mathfrak{s}(\mathcal{F}) \longmapsto (\varrho(x)^i)_{i \geq 0} \in \mathfrak{s}(\mathcal{T}_B) \mapsto \varrho(x)^1 \in \mathfrak{X}(B_1). \quad (57)$$

Lemma 5.24. *For all $x, y \in \mathfrak{g}$,*

$$[\varrho(x), \varrho(y)]^1 - [\varrho(x)^1, \varrho(y)^1] = dt_1(C_1(x, y))$$

with $C_1(x, y) \in \Gamma(\ker ds_1 \rightarrow B_2)$ a t_1 -projectable vector field, for some bilinear map

$$C_1: \wedge^2 \mathfrak{g} \longrightarrow \Gamma(\ker ds_1 \rightarrow B_2).$$

Proof. This follows from Lemma 5.8, because $[\varrho(x), \varrho(y)]^1 - [\varrho(x)^1, \varrho(y)^1] \in \Gamma(\ker ds_0) \cap \Gamma(\ker dt_0)$. \square

Theorem 5.25. *The map $C_1: \wedge^2 \mathfrak{g} \rightarrow \Gamma(\ker ds_1 \rightarrow B_2)$ of Lemma 5.24 satisfies for all $x, y, z \in \mathfrak{g}$,*

$$C_1([x, y]_{\mathfrak{g}}, z) + \nabla_{\varrho(x)}^2(C_1(y, z)) + \odot(x, y, z) = dt_2(C_2(x, y, z)) \quad (58)$$

for some tri-linear map $C_2: \wedge^3 \mathfrak{g} \rightarrow \Gamma(\ker ds_2 \rightarrow B_3)$. Here, ∇^2 is, as in Remark 5.22.

Proof. For $x, y, z \in \mathfrak{g}$,

$$\begin{aligned} dt_1(C_1([x, y]_{\mathfrak{g}}, z)) + \odot(x, y, z) &= [\varrho([x, y]_{\mathfrak{g}}), \varrho(z)]^1 - [\varrho([x, y]_{\mathfrak{g}})^1, \varrho(z)^1] + \odot(x, y, z) \\ &= \underline{[[\varrho(x), \varrho(y)], \varrho(z)]^1} - [[\varrho(x), \varrho(y)]^1, \varrho(z)^1] + \odot(x, y, z) \\ &= -\underline{[[\varrho(x)^1, \varrho(y)^1], \varrho(z)^1]} + [dt_1(C_1(x, y)), \varrho(z)^1] + \odot(x, y, z) \\ &= dt_1([C_1(x, y), \varrho(z)^2]) + \odot(x, y, z). \end{aligned}$$

We have used Jacobi identity and $dt_1(\varrho(z)^2) = \varrho(z)^1$. This implies that

$$dt_1(C_1([x, y]_{\mathfrak{g}}, z) - [C_1(x, y), \varrho(z)^2]) + \odot(x, y, z) = 0. \quad (59)$$

Again Lemma 5.8 implies the result. \square

Here is a natural question:

Question: Can we construct a Lie ∞ -algebra structure on $\oplus_{i=0}^{+\infty} {}^t \Gamma(\ker ds_i)$ such that the construction in Theorem 5.25 continues to a Lie ∞ -morphism from $\mathfrak{g}[1]$ to $\oplus_{i=0}^{+\infty} {}^t \Gamma(\ker ds_i)$? where ${}^t \Gamma(\ker ds_i)$ is defined as in Convention 5.7.

APPENDIX A. UNIVERSAL LIE ∞ -ALGEBROIDS

Let us now recall the definition of Lie ∞ -algebroids over a manifold and their morphisms and homotopies. Most definitions of this section can be found in [6, 19, 20] and our conventions and notations are those of [19, 20].

In the definition below, we restrict ourselves to the case of finite rank. Recall that finitely generated projective modules, by Serre-Swan theorem [27], are the module of sections of vector bundles. This is the setting in this article, except for Theorem 3.3 where infinite rank Lie algebroid are allowed see e.g., [20].

Definition A.1. A Lie ∞ -algebroid over M is the datum of a sequence $E = (E_{-i})$, $1 \leq i < \infty$ of vector bundles over M together with a structure of Lie ∞ -algebra $(\ell_k)_{k \geq 1}$ on the sheaf of sections of E and a vector bundle morphism, $\rho: E_{-1} \rightarrow TM$, called *anchor map* such that the k -ary brackets ℓ_k , $k \neq 2$ are \mathcal{O} -multilinear and such that

$$\ell_2(e_1, fe_2) = \rho(e_1)[f]e_2 + f\ell_2(e_1, e_2) \quad (60)$$

for all $e_1 \in \Gamma(E_{-1})$, $e_2 \in \Gamma(E_{\bullet})$ and $f \in \mathcal{O}$.

The sequence

$$\cdots \xrightarrow{\ell_1} E_{-2} \xrightarrow{\ell_1} E_{-1} \xrightarrow{\rho} TM, \quad (61)$$

is a complex called the *linear part* of the Lie ∞ -algebroid.

Remark A.2. Any Lie ∞ -algebroid on M has an induced singular foliation on M which is given by the image of the anchor map, that we call the *basic singular foliation*.

There is an alternative definition for Lie ∞ -algebroids in terms of Q -manifolds with purely non-negative degrees.

Definition A.3. A *splitted NQ -manifold* is a pair (E, Q) where $E \rightarrow M$ is a sequence of vector bundles over M indexed by negative integers and where Q is a homological vector field of degree $+1$, i.e., $Q \in \text{Der}_1(\Gamma(S^\bullet(E^*)))$ is such that $[Q, Q] = 0$.

We denote by \mathcal{E} and call *functions* on the splitted NQ -manifold $E \rightarrow M$ the sheaf of graded commutative \mathcal{O} -algebras made of sections of $S^\bullet(E^*)$.

There is a one-to-one correspondence between splitted NQ -manifolds and Lie ∞ -algebroids [6, 23, 28]. This formulation allows to write in a compact manner morphisms of Lie ∞ -algebroids i.e., simply as chain maps. Homotopy equivalence can also be defined, see Section 3.4.2 in [19] or [20] for more details. From now on, we write (E, Q) to denote a Lie ∞ -algebroid over M .

Let us recall from [19, 20] the following definition and theorem.

Definition A.4. Let $\mathcal{F} \subset \mathfrak{X}(M)$ be a singular foliation on a manifold M . A *geometric resolution* of the singular foliation \mathcal{F} is a projective resolution $((P_{-i})_{i \geq 1}, (d^{(i)})_{i \geq 2}, \rho)$ of \mathcal{F} as a \mathcal{O} -module that corresponds to a sequence of vector bundles $(E, \bar{d}, \bar{\rho})$ over M

$$\cdots \xrightarrow{\bar{d}^{(3)}} E_{-2} \xrightarrow{\bar{d}^{(2)}} E_{-1} \xrightarrow{\bar{\rho}} TM, \quad (62)$$

i.e.,

- for $i \geq 1$ the \mathcal{O} -module of sections of E_{-i} is $P_{-i} = \Gamma(E_{-i})$
- for $i \geq 2$, the induced maps at the sections level

$$\bar{d}^{(i)}: \Gamma(E_{-i}) \rightarrow \Gamma(E_{-i+1}) \quad \text{or} \quad \bar{\rho}: \Gamma(E_{-1}) \rightarrow \mathcal{F}$$

coincide with $d^{(i)}: P_{-i} \rightarrow P_{-i+1}$ or with $\rho: P_{-1} \rightarrow \mathcal{F}$ respectively.

For convenience, we denote by \bar{d} and $\bar{\rho}$ the same as d and ρ respectively. Also, we call $\rho: E_{-1} \rightarrow TM$ the *geometric resolution anchor*. A geometric resolution is said to be *minimal* at a point $m \in M$ if, for all $i \geq 2$, the linear maps $d_{|m}^{(i)}: E_{-i}|_m \rightarrow E_{-i+1}|_m$ vanish.

Theorem A.5. [19, 20, 22] *Let \mathcal{F} be a singular foliation on M . Any geometric resolution of \mathcal{F}*

$$\cdots \xrightarrow{d} E_{-3} \xrightarrow{d} E_{-2} \xrightarrow{d} E_{-1} \xrightarrow{\rho} TM \quad (63)$$

comes equipped with a Lie ∞ -algebroid structure whose unary bracket is d and whose anchor map is ρ . Such a Lie ∞ -algebroid structure is unique up to homotopy and is called a universal Lie ∞ -algebroid of \mathcal{F} .

Remark A.6. For a given Lie ∞ -algebroid (E, Q) , the triple $(\mathfrak{X}_\bullet(E), [\cdot, \cdot], \text{ad}_Q)$ is a differential graded Lie algebra, where $\mathfrak{X}_\bullet(E)$ stands for the module of graded vector fields (=graded derivations of \mathcal{E}) on E , the bracket $[\cdot, \cdot]$ is the graded commutator of derivations and $\text{ad}_Q := [Q, \cdot]$.

APPENDIX B. LIE ∞ -MORPHISMS OF DGLA AND HOMOTOPIES

Let us recall the definitions of Lie ∞ -morphisms and homotopies between differential graded Lie algebras in terms of coderivations. We restrict ourselves to a special case that we need for this paper.

Comorphisms and coderivations. Let \mathfrak{g} and \mathfrak{h} be graded vector spaces over \mathbb{K} .

Definition B.1. A linear map $\Phi: S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1]) \rightarrow S_{\mathbb{K}}^{\bullet}(\mathfrak{h}[1])$ is said to be of *arity* $r \in \mathbb{N}_0$, if it sends $S_{\mathbb{K}}^k(\mathfrak{g}[1])$ to $S_{\mathbb{K}}^{k-r}(\mathfrak{h}[1])$, for all $k \geq r$. Every linear map Φ can be decomposed as formal sum:

$$\Phi = \sum_{k \in \mathbb{Z}} \Phi^{(k)} \quad (64)$$

where, for all $k \in \mathbb{N}_0$, $\Phi^{(k)}: S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1]) \rightarrow S_{\mathbb{K}}^{\bullet-k}(\mathfrak{h}[1])$ is a linear map of arity k . Therefore, a linear map $\Phi: S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1]) \rightarrow S_{\mathbb{K}}^{\bullet}(\mathfrak{h}[1])$ is of arity $r \in \mathbb{N}_0$ if and only if $\Phi^{(r)}$ is the unique non-zero term, namely $\Phi^{(k)} = 0$, for $k \neq r$.

Let us denote by Δ the natural coalgebra structure of $S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1])$ and by Δ' the one on $S_{\mathbb{K}}^{\bullet}(\mathfrak{h}[1])$. Given any linear map $\Phi: S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1]) \rightarrow \mathfrak{h}[1]$, we denote by $\Phi_k: S_{\mathbb{K}}^{k+1}(\mathfrak{g}[1]) \rightarrow \mathfrak{h}[1]$ for $k \in \mathbb{N}_0$ the restriction of Φ to $S_{\mathbb{K}}^{k+1}(\mathfrak{g}[1])$. The linear map Φ can be extended to a unique comorphism $\bar{\Phi}: S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1]) \rightarrow S_{\mathbb{K}}^{\bullet}(\mathfrak{h}[1])$ by taking for $r \in \mathbb{N}$ the component on $S_{\mathbb{K}}^r \mathfrak{h}[1]$ to be for $x_1, \dots, x_k \in \mathfrak{g}[1]$

$$\sum_{i_1 + \dots + i_r = k} \sum_{\sigma \in \mathfrak{S}(i_1, \dots, i_r)} \epsilon(\sigma) \frac{1}{r!} \prod_{j=1}^r \Phi_{i_j-1}(x_{\sigma(i_1 + \dots + i_{j-1} + 1)}, \dots, x_{\sigma(i_1 + \dots + i_j)}). \quad (65)$$

where $\mathfrak{S}(i_1, \dots, i_r)$ is the set of (i_1, \dots, i_r) -shuffles, with $i_1, \dots, i_r \in \mathbb{N}$. Also, \prod stands for the product of $S_{\mathbb{K}}^{\bullet}(\mathfrak{h}[1])$.

Every comorphism from $S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1])$ to $S_{\mathbb{K}}^{\bullet}(\mathfrak{h}[1])$ is of this form [16]. That is, a comorphism $\Phi: S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1]) \rightarrow S_{\mathbb{K}}^{\bullet}(\mathfrak{h}[1])$ is entirely determined by the collection indexed by $k \in \mathbb{N}$ of maps called *k-th Taylor coefficient*:

$$\Phi_k: S_{\mathbb{K}}^{k+1}(\mathfrak{g}[1]) \xrightarrow{\Phi} S_{\mathbb{K}}^{\bullet}(\mathfrak{h}[1]) \xrightarrow{\text{pr}} \mathfrak{h}[1], \quad (66)$$

with pr being the projection onto the term of arity 1, i.e. $\text{pr}: S_{\mathbb{K}}^{\bullet}(\mathfrak{h}[1]) \rightarrow S_{\mathbb{K}}^1(\mathfrak{h}[1]) \simeq \mathfrak{h}[1]$. Notice that the component $\Phi^{(k)}$ of arity k of Φ coincides with k -th Taylor coefficient Φ_k on $S_{\mathbb{K}}^{k+1}(\mathfrak{g}[1])$. Hence, a comorphism $\Phi: S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1]) \rightarrow S_{\mathbb{K}}^{\bullet}(\mathfrak{h}[1])$ admits a decomposition of the form:

$$\Phi = \sum_{k \geq 0} \Phi^{(k)} \quad (67)$$

with a sum that runs on $k \geq 0$ and not on $k \in \mathbb{Z}$.

Definition B.2. Let $\Phi: S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1]) \mapsto S_{\mathbb{K}}^{\bullet}(\mathfrak{h}[1])$ be a graded comorphism. A Φ -*coderivation* of degree k on $S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1])$ is a degree $k \in \mathbb{N}_0$ linear map $\mathcal{H}: S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1]) \mapsto S_{\mathbb{K}}^{\bullet}(\mathfrak{h}[1])$ which satisfies the so-called (co)Leibniz identity:

$$\Delta' \circ \mathcal{H} = (\mathcal{H} \otimes \Phi) \circ \Delta + (\Phi \otimes \mathcal{H}) \circ \Delta. \quad (68)$$

When $\mathfrak{g} = \mathfrak{h}$ and $\Phi = \text{id}$, we say that \mathcal{H} is a *coderivation*.

The same results on comorphisms hold for coderivations [16].

Lie ∞ -morphisms of differential graded Lie algebras. Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a Lie algebra and (E, Q) a Lie ∞ -algebroid over M . The graded symmetric Lie bracket on $\mathfrak{X}(E)[1]$ is of degree +1 and given on homogeneous elements $u, v \in \mathfrak{X}(E)[1]$ as

$$\{u, v\} := (-1)^{|v|} [u, v].$$

In the sequel, we write $(\mathfrak{X}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ instead of $(\mathfrak{X}(E)[1], \{\cdot, \cdot\}, \text{ad}_Q)$.

Let $(S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1]), Q_{\mathfrak{g}})$ respectively $(S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1]), \bar{Q})$ be the corresponding formulations in terms of coderivations of the differential graded Lie algebras $(\mathfrak{g}[1], [\cdot, \cdot]_{\mathfrak{g}})$ and $(\mathfrak{X}(E)[1], [\cdot, \cdot], \text{ad}_Q)$.

Precisely, $Q_{\mathfrak{g}}$ is the coderivation defined for every homogeneous monomial $x_1 \wedge \cdots \wedge x_k \in S_{\mathbb{K}}^k(\mathfrak{g}[1])$ by

$$Q_{\mathfrak{g}}(x_1 \wedge \cdots \wedge x_k) := \sum_{1 \leq i < j \leq k} (-1)^{i+j-1} [x_i, x_j]_{\mathfrak{g}} \wedge x_1 \wedge \cdots \widehat{x}_i \cdots \widehat{x}_j \cdots \wedge x_k, \quad (69)$$

and $\bar{Q} = \bar{Q}^{(0)} + \bar{Q}^{(1)}$ is the coderivation of degree +1 where the only non-zero Taylor's coefficients are, $\bar{Q}^{(0)}: S_{\mathbb{K}}^1(\mathfrak{X}(E)[1]) \xrightarrow{\text{ad}_Q} \mathfrak{X}(E)[1]$ and $\bar{Q}^{(1)}: S_{\mathbb{K}}^2(\mathfrak{X}(E)[1]) \xrightarrow{\{\cdot, \cdot\}} \mathfrak{X}(E)[1]$.

Definition B.3. [18] A *Lie* ∞ -morphism $\Phi: (\mathfrak{g}[1], [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ is a graded coalgebra morphism $\bar{\Phi}: (S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1]), Q_{\mathfrak{g}}) \longrightarrow (S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1]), \bar{Q})$ of degree zero which satisfies,

$$\bar{\Phi} \circ Q_{\mathfrak{g}} = \bar{Q} \circ \bar{\Phi}. \quad (70)$$

In order words, it is the datum of degree zero linear maps $(\bar{\Phi}_k: S_{\mathbb{K}}^{k+1}\mathfrak{g}[1] \longrightarrow \mathfrak{X}_{-k}(E)[1])_{k \geq 0}$ that satisfies

$$\begin{aligned} \sum_{1 \leq i < j \leq n+2} (-1)^{i+j-1} \bar{\Phi}_n([x_i, x_j]_{\mathfrak{g}}, x_1, \dots, \widehat{x}_{ij}, \dots, x_{n+2}) &= [Q, \bar{\Phi}_{n+1}(x_1, \dots, x_{n+2})] + \\ &\sum_{\substack{i+j=n \\ i \leq j \\ \sigma \in \mathfrak{S}_{i+1, j+1}}} \epsilon(\sigma) [\bar{\Phi}_i(x_{\sigma(1)}, \dots, x_{\sigma(i+1)}), \bar{\Phi}_j(x_{\sigma(i+2)}, \dots, x_{\sigma(n+2)})] \end{aligned} \quad (71)$$

where \widehat{x}_{ij} means that we take x_i, x_j out of the list. When there is no risk of confusion, we write Φ for $\bar{\Phi}$.

Convention B.4. In the sequel, $Q_{\mathfrak{g}}$ and \bar{Q} will be in implicit.

Remark B.5. It is important to notice that:

- (1) Definition B.3 and Definition 3.45 in [19] are compatible when $M = \{\text{pt}\}$. Therefore, morphisms in both sense match.
- (2) In [24], Definition B.3 corresponds to the definition of actions of a Lie ∞ -algebras of finite dimension on Lie ∞ -algebroids of finite rank. Here, we only have a Lie algebra. In contrast to theirs, we do not assume that \mathfrak{g} is finite dimensional.

Remark B.6. It follows from the axioms (71) that for all $x, y \in \mathfrak{g}[1]$, $[Q, \Phi_0(x)] = 0$ and

$$\Phi_0([x, y]_{\mathfrak{g}}) - [\Phi_0(x), \Phi_0(y)] = [Q, \Phi_1(x, y)]. \quad (72)$$

The following lemma explains what the 0-Taylor coefficient of a Lie ∞ -morphism as in Definition B.3 induces on the linear part of (E, Q) . More details will be given in Proposition B.13.

Convention B.7. Let E, F be graded vector bundles over a manifold M . For a \mathbb{K} -linear map $P: \Gamma(S^{\bullet}(E^*)) \longrightarrow \Gamma(S^{\bullet}(E^*))$ we denote by $P^{(k)}: \Gamma(S^N(E^*)) \longrightarrow \Gamma(S^{N+k}(E^*))$, $N \geq 0$, the k -th polynomial degree component of P and is called the arity k component of P .

Lemma B.8. The 0-th Taylor coefficient $\Phi_0: \mathfrak{g}[1] \longrightarrow \mathfrak{X}_0(E)$ of a Lie ∞ -morphism Φ as in Definition B.3 induces

- (1) a linear map $\varrho: \mathfrak{g} \longrightarrow \mathfrak{X}(M)$, $x \longmapsto (\varrho(x)[f] := \Phi_0(x)[f], f \in \mathcal{O})$ and
- (2) a linear map $x \in \mathfrak{g} \longmapsto \nabla_x \in \text{Der}_0(E)$, i.e., for each $x \in \mathfrak{g}$, $\nabla_x: \Gamma(E) \longrightarrow \Gamma(E)$ is a degree zero map that satisfies

$$\nabla_x(fe) = f\nabla_x(e) + \varrho(x)[f]e, \text{ for } f \in \mathcal{O}, e \in \Gamma(E).$$

such that

$$\langle \Phi_0(x)^{(0)}(\alpha), e \rangle = \varrho(x)[\langle \alpha, e \rangle] - \langle \alpha, \nabla_x(e) \rangle, \text{ for all } \alpha \in \Gamma(E^*), e \in \Gamma(E). \quad (73)$$

$\Phi_0(x)^{(0)}$ stands for the arity zero component of $\Phi_0(x)$.

Proof. We have for every $x \in \mathfrak{g}[1]$, and $e \in \Gamma(E)$,

$$[\Phi_0(x), \iota_e]^{(-1)} = \iota_{\nabla_x e},$$

for some \mathbb{K} -bilinear map $\nabla_x: \Gamma(E_{-\bullet}) \rightarrow \Gamma(E_{-\bullet})$ that depends linearly on $x \in \mathfrak{g}[1]$ and that satisfies

$$\nabla_x(fe) = f\nabla_x(e) + \varrho(x)[f]e, \text{ for } f \in \mathcal{O}, e \in \Gamma(E). \quad (74)$$

To see (74), compute $[\Phi_0(x), \iota_{fe}]^{(-1)}$:

$$\begin{aligned} \iota_{\nabla_x(fe)} &= [\Phi_0(x), \iota_{(fe)}]^{(-1)} \\ &= \Phi_0(x)[f]\iota_e + f[\Phi_0(x), \iota_e]^{(-1)} \\ &= \iota_{\varrho(x)[f]e + \nabla_x e}. \end{aligned}$$

In particular, one has for all $\alpha \in \Gamma(E^*)$, $e \in \Gamma(E)$,

$$\begin{aligned} \langle \Phi_0(x)^{(0)}(\alpha), e \rangle &= \Phi_0(x)^{(0)}[\langle \alpha, e \rangle] - [\Phi_0(x)^{(0)}, \iota_e]^{(-1)}(\alpha) \\ &= \varrho(x)[\langle \alpha, e \rangle] - \langle \alpha, \nabla_x(e) \rangle. \end{aligned}$$

□

Homotopies. Now we are defining homotopy between Lie ∞ -morphisms.

Definition B.9. Let $\bar{\Phi}, \bar{\Psi}: (S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1]), Q_{\mathfrak{g}}) \rightsquigarrow (S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1]), \bar{Q})$ be Lie ∞ -morphisms. We say $\bar{\Phi}, \bar{\Psi}$ are *homotopic over the identity of M* if the following conditions hold:

- (1) there a piecewise rational continuous path $t \in [a, b] \mapsto \Xi_t: (S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1]), Q_{\mathfrak{g}}) \rightsquigarrow (S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1]), \bar{Q})$ made of Lie ∞ -morphisms that coincide with $\bar{\Phi}$ and $\bar{\Psi}$ at $t = a$ and b , respectively,
- (2) and a piecewise rational path $t \in [a, b] \mapsto H_t$ of Ξ_t -coderivations of degree -1 such that

$$\frac{d\Xi_t}{dt} = \bar{Q} \circ H_t + H_t \circ Q_{\mathfrak{g}}. \quad (75)$$

Remark B.10. Homotopy equivalence in the sense of the Definition B.9 is an equivalence relation, and it is compatible with composition of Lie ∞ -morphisms. Also, we can “glue” infinitely many equivalences, as in Lemma 1.39 in [20].

Remark B.11. Definition B.9 is slightly more general than the equivalence relation [24]. In [24], it is explained that Lie ∞ -oid morphisms are Maurer-Cartan elements in some Lie ∞ -algebroid $\mathfrak{g}[1] \oplus E$ of certain form, and they define equivalence as gauge-equivalence of the Maurer-Cartan elements. This gauge equivalence corresponds to homotopies as above, for which all functions are smooth. Also, we do not require nilpotence unlike in Definition 5.1 of [24]. Last, we do not assume \mathfrak{g} to be of finite dimension.

The following Proposition shows that the notion of homotopy given in Definition B.9 implies the usual notion of homotopy between chain maps.

Proposition B.12. Let $\Phi, \Psi: (\mathfrak{g}[1], [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ be Lie ∞ -morphisms which are homotopic. Then,

$$\Psi - \Phi = \bar{Q} \circ H + H \circ Q_{\mathfrak{g}} \quad (76)$$

for some \mathcal{O} -linear map $H: S_{\mathbb{K}}^{\bullet}(\mathfrak{g}[1]) \rightarrow S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1])$ of degree -1 .

Proof. The proof follows by applying the property that says the variation of a piecewise- C^∞ map is equal to the integral of its derivative on (75). \square

Proposition B.13. *Let \mathfrak{g} be a Lie algebra and (E, Q) a Lie ∞ -algebroid over M .*

- (1) *Any Lie ∞ -morphism $\Phi: (\mathfrak{g}[1], [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q)$ induces a weak symmetry action of \mathfrak{g} on the basic singular foliation $\mathcal{F} = \rho(\Gamma(E_{-1}))$ of (E, Q) .*
- (2) *Homotopic Lie ∞ -morphisms $\Phi, \Psi: (\mathfrak{g}[1], [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q)$ induce equivalent weak symmetry actions ϱ_a, ϱ_b of \mathfrak{g} on the basic singular foliation \mathcal{F} .*

Proof. Item 1. is a consequence of Remark B.6. Indeed, take $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ as in Lemma B.8(1). We claim that ϱ is a weak symmetry action of \mathfrak{g} on \mathcal{F} : Let $x, y \in \mathfrak{g}[1]$, and $e \in \Gamma(E_{-1})$ and $f \in \mathcal{O}$.

- $[\Phi_0(x), Q] = 0$ entails,

$$\begin{aligned} \langle \Phi_0(x)^{(0)} [Q^{(1)}(f)], e \rangle &= \langle Q^{(1)} (\Phi_0(x)^{(0)}[f]), e \rangle \\ \varrho(x)[\langle Q[f], e \rangle] - \langle Q[f], \nabla_x(e) \rangle &= \rho(e)[\varrho(x)], \quad (\text{by Lemma B.8 (2.)}) \\ \varrho(x)[\rho(e)][f] - \rho(\nabla_x(e))[f] &= \rho(e)[\varrho(x)] \end{aligned}$$

By consequence, $[\varrho(x), \rho(e)] = \rho(\nabla_x(e)) \in \mathcal{F}$. Therefore, $[\varrho(x), \mathcal{F}] \subseteq \mathcal{F}$.

- There exists a skew-symmetric linear map $\eta: \wedge^2 \mathfrak{g} \rightarrow \Gamma(E_{-1})$ such that $\Phi_1(x, y)^{(-1)} = \iota_{\eta(x, y)}$. Therefore, the arity zero of Equation (72) evaluated at an arbitrary function $f \in \mathcal{O}$ yields:

$$\begin{aligned} \Phi_0([x, y]_{\mathfrak{g}})^{(0)}(f) - [\Phi_0(x), \Phi_0(y)]^{(0)}(f) &= [Q, \Phi_1(x, y)]^{(0)}(f) \\ \implies \Phi_0([x, y]_{\mathfrak{g}})(f) - [\Phi_0(x)^{(0)}, \Phi_0(y)^{(0)}](f) &= [Q^{(1)}, \Phi_1(x, y)^{(-1)}](f) \\ \implies \varrho([x, y]_{\mathfrak{g}})[f] - [\varrho(x), \varrho(y)][f] &= [Q^{(1)}, \iota_{\eta(x, y)}](f) \\ &= \rho(\eta(x, y))[f]. \end{aligned}$$

Since f is arbitrary, this proves item 1. Using Proposition B.12, $\Phi \sim \Psi$ implies for $x \in \mathfrak{g}[1]$ that

$$\begin{aligned} \Psi(x) - \Phi(x) &= \bar{Q} \circ H(x) + \underline{H \circ Q_{\mathfrak{g}}(x)} \\ &= [Q, H(x)] \end{aligned} \tag{77}$$

with $H: \mathfrak{g}[1] \rightarrow \mathfrak{X}_{-1}(E)$ a linear map. Let $\beta: \mathfrak{g}[1] \rightarrow \Gamma(E_{-1})$ be a linear map such that $H(x)^{(-1)} = \iota_{\beta(x)}$. Taking the arity zero of both sides in Equation (77) and evaluating at $f \in \mathcal{O}$ we obtain that

$$(\varrho_a(x) - \varrho_b(x))[f] = [Q^{(1)}, H(x)^{(-1)}] = [Q^{(1)}, \iota_{\beta(x)}][f] = \rho(\beta(x))[f].$$

Since f is arbitrary, this proves item 2. \square

Proposition B.13 tells us that Lie ∞ -morphism $\Phi: (\mathfrak{g}[1], [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q)$ induces weak symmetry action on the base manifold M . In Section 2, we investigate the opposite direction. We respond to the following question: Do any weak symmetry action of a Lie algebra on a singular foliation comes from a Lie ∞ -morphism? If so, can we extend in a unique manner?

APPENDIX C. PROOF OF THEOREM 3.3

In this section, we prove Theorem 3.3. For \mathfrak{g} a Lie algebra, let $\Gamma(\mathfrak{g})$ stand for sections of $\mathfrak{g} \times M \rightarrow M$.

Proof. (of Theorem 3.3). Let $\varphi: \wedge^2 \mathfrak{g} \rightarrow \mathcal{F}$ be such that $\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] = \varphi(x, y) \in \mathcal{F}$ for all $x, y \in \mathfrak{g}$. Notice that $\mathcal{A} := (\Gamma(\mathfrak{g}) \oplus \mathcal{F}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ is a Lie-Rinehart algebra over \mathcal{O} , whose Lie bracket and anchor map are given respectively on a set of generators $(x_i)_{i \in I}$ of $\Gamma(\mathfrak{g}[1])$ and $(X_{\lambda})_{\lambda \in \Lambda}$ of \mathcal{F} by:

(1) for $i, j \in I$ and $\lambda, \beta \in \Lambda$,

$$[(x_i, X_{\lambda}), (x_j, X_{\beta})]_{\mathcal{A}} := ([x_i, x_j]_{\mathfrak{g}}, [X_{\lambda}, \varrho(x_j)] - [X_{\beta}, \varrho(x_i)] - \varphi(x_i, x_j) + [X_{\lambda}, X_{\beta}]) \quad (78)$$

(2) for $i \in I$ and $\lambda \in \Lambda$

$$\rho_{\mathcal{A}}(x_i, X_{\lambda}) = \varrho(x_i) + X_{\lambda} \quad (79)$$

We extend the bracket (78) by Leibniz identity. Also, $\rho_{\mathcal{A}}$ in (79) is extended by \mathcal{O} -linearity (it is a morphism by construction).

For any free resolution $(\mathcal{K}_{\bullet}, \ell_1, \rho)$ of \mathcal{F} , the sequence

$$\cdots \xrightarrow{\ell_1} \mathcal{K}_{-2} \xrightarrow{\ell_1} \Gamma(\mathfrak{g}[1]) \oplus \mathcal{K}_{-1} \xrightarrow{\pi = \text{id} \oplus \rho} \Gamma(\mathfrak{g}) \oplus \mathcal{F} \quad (80)$$

is a free resolution of $\mathcal{A} = \Gamma(\mathfrak{g}) \oplus \mathcal{F}$. By Theorem 2.1 of [20], the complex (80) can be equipped with a Lie ∞ -algebroid whose unary bracket is ℓ_1 and whose anchor is $\rho' := \rho_{\mathcal{A}} \circ \pi$. But we claim that we can add some constraint on the k -ary brackets that appear in Theorem 2.1. This requires to adapt its proof to this particular setting.

Construction of a 2-ary bracket on $\mathfrak{g}[1] \oplus E_{-1}$: Let us denote by $(e_{\lambda}^{(-1)})_{\lambda \in \Lambda}$ a free basis of \mathcal{K}_{-1} . The set $\{X_{\lambda} = \rho(e_{\lambda}^{(-1)}) \in \mathcal{F} \mid \lambda \in \Lambda\}$ is a set of generators of \mathcal{F} . Let $(x_i)_{i \in I}$ be a basis for $\Gamma(\mathfrak{g}[1])$. There exists elements $c_{\lambda\beta}^k \in \mathcal{O}$ and satisfying the skew-symmetry condition $c_{\lambda\beta}^{\alpha} = -c_{\beta\lambda}^{\alpha}$ together with

$$[X_{\lambda}, X_{\beta}] = \sum_{\alpha \in \Lambda} c_{\lambda\beta}^{\alpha} X_{\alpha} \quad \forall \lambda, \beta \in \Lambda. \quad (81)$$

By definition of the weak symmetry action ϱ , one has

$$[\varrho(x_i), \rho(e_{\lambda}^{(-1)})] \in \mathcal{F} \quad \text{and} \quad \varrho([x_i, x_j]_{\mathfrak{g}}) - [\varrho(x_i), \varrho(x_j)] \in \mathcal{F} \quad \text{for all } (i, j) \in I^2 \text{ and } \lambda \in \Lambda. \quad (82)$$

Since $(\mathcal{K}_{\bullet}, \ell_1, \rho)$ is a free resolution of \mathcal{F} , there exists two \mathcal{O} -bilinear maps $\chi: \Gamma(\mathfrak{g}[1]) \times \mathcal{K}_{-1} \rightarrow \mathcal{K}_{-1}$ and $\eta: \Gamma(\mathfrak{g}[1]) \times \Gamma(\mathfrak{g}[1]) \rightarrow \mathcal{K}_{-1}$ defined on generators $x_i, e_{\lambda}^{(-1)}$ by the relations

$$[\varrho(x_i), \rho(e_{\lambda}^{(-1)})] = \rho(\chi(x_i, e_{\lambda}^{(-1)})) \quad \text{and} \quad \varrho([x_i, x_j]_{\mathfrak{g}}) - [\varrho(x_i), \varrho(x_j)] = \rho(\eta(x_i, x_j)).$$

We define a 2-ary bracket on $\Gamma(\mathfrak{g}[1]) \oplus \mathcal{K}_{-1}$ as follows:

- (1) an anchor map by $\rho'(0 \oplus e_{\lambda}^{(-1)}) = X_{\lambda}$, and $\rho'(x_i \oplus 0) = \varrho(x_i)$, for all $i \in I$ and $\lambda \in \Lambda$,
- (2) a degree +1 graded symmetric operation ℓ'_2 on $\Gamma(\mathfrak{g}[1]) \oplus \mathcal{K}_{-1}$ as follows: for all i, j and $\lambda, \beta \in \Lambda$

- (a) $\ell'_2(0 \oplus e_{\lambda}^{(-1)}, 0 \oplus e_{\beta}^{(-1)}) = 0 \oplus \sum_{\alpha \in \Lambda} c_{\lambda\beta}^{\alpha} e_{\alpha}^{(-1)}$,
- (b) $\ell'_2(x_i \oplus 0, 0 \oplus e_{\lambda}^{(-1)}) = 0 \oplus \chi(x_i, e_{\lambda}^{(-1)})$,
- (c) $\ell'_2(x_i \oplus 0, x_j \oplus 0) = [x_i, x_j]_{\mathfrak{g}} \oplus \eta(x_i, x_j)$.

We extend ℓ'_2 to $\Gamma(\mathfrak{g}[1]) \oplus \mathcal{K}_{-1}$ using Leibniz identity with respect to the anchor map ρ' .

By construction ℓ'_2 satisfies the Leibniz identity with respect to the anchor ρ' . Moreover, ρ' is a bracket morphism. We continue exactly as in the proof of Lemma 2.23 in [20] and construct a 2-ary bracket ℓ_2 of degree +1 whose restriction to $\Gamma(\mathfrak{g}[1]) \oplus \mathcal{K}_{-1}$ is ℓ'_2 . It equips the complex (80) with an almost Lie algebroid such that $\rho'(\Gamma(\mathfrak{g}[1]) \oplus \mathcal{K}_{-1}) = \mathcal{A}$ and whose restriction to $\mathfrak{g}[1]$ is the Lie bracket of $\mathfrak{g}[1]$. A close look at the construction in Theorem 2.1 of [20] implies that this almost differential graded Lie algebroid can be extended to a Lie ∞ -algebroid on the complex (80). \square

REFERENCES

- [1] Iakovos Androulidakis and Georges Skandalis. The holonomy groupoid of a singular foliation. *Journal für die reine und angewandte Mathematik*, 2009(626):1–37, 2009.
- [2] Iakovos Androulidakis and Georges Skandalis. Pseudodifferential calculus on a singular foliation. *Journal of noncommutative geometry*, 5(1):125–152, 2011.
- [3] Iakovos Androulidakis and Georges Skandalis. A Baum–Connes conjecture for singular foliations. *Annals of K-Theory*, 4(4):561 – 620, 2019.
- [4] Iakovos Androulidakis and Marco Zambon. Smoothness of holonomy covers for singular foliations and essential isotropy. *Mathematische Zeitschrift*, 275(3-4):921–951, 2013.
- [5] Iakovos Androulidakis and Marco Zambon. Holonomy transformations for singular foliations. *Advances in mathematics (New York. 1965)*, 256:348–397, 2014.
- [6] Giuseppe Bonavolontà and Norbert Poncin. On the category of Lie n -algebroids. *Journal of geometry and physics*, 73:70–90, 2013.
- [7] Ricardo Campos. Homotopy equivalence of shifted cotangent bundles. *Journal of Lie Theory*, 29, 2019.
- [8] Dominique Cerveau. Distributions involutives singulières. *Ann. Inst. Fourier (Grenoble)*, 29(3):xii, 261–294, 1979.
- [9] Claire Debord. Holonomy Groupoids of Singular Foliations. *J. Differential Geom.*, 58(3):467–500, 07 2001.
- [10] Klas Diederich, Gilbert Hector, and Ulrich Hirsch. *Introduction to the Geometry of Foliations, Part A: Foliations on Compact Surfaces, Fundamentals for Arbitrary Codimension, and Holonomy*, volume 1 of *Aspects of Mathematics*. Springer Vieweg. in Springer Fachmedien Wiesbaden GmbH, Wiesbaden, second edition. edition, 1986.
- [11] Jacques Frisch. Points de platitude d’un morphisme d’espaces analytiques complexes. *Inventiones mathematicae*, 4(2):118–138, 1967.
- [12] Alfonso Garmendia and Ori Yudilevich. On the inner automorphisms of a singular foliation. *Mathematische Zeitschrift*, 2018.
- [13] Alfonso Garmendia and Marco Zambon. Hausdorff Morita equivalence of singular foliations. *Ann. Global Anal. Geom.*, 55(1):99–132, 2019.
- [14] Alfonso Garmendia and Marco Zambon. Quotients of singular foliations and lie 2-group actions. *Journal of noncommutative geometry*, 15(4):1251–1283, 2021.
- [15] Robert Hermann. The differential geometry of foliations, II. *Journal of Mathematics and Mechanics*, 11(2):303–315, 1962.
- [16] Christian Kassel. *Quantum Groups*, volume 155 of *Graduate Texts in Mathematics*. Springer, 2012.
- [17] Alexei Kotov, Camille Laurent-Gengoux, and Vladimir Salnikov. Normal forms of \mathbb{Z} -graded Q -manifolds. arxiv:2212.05579. 2022.
- [18] Tom Lada and Martin Markl. Strongly homotopy Lie algebras. *Commun. Algebra*, 23(6):2147–2161, 1995.
- [19] Camille Laurent-Gengoux, Sylvain Lavau, and Thomas Strobl. The universal Lie ∞ -algebroid of a singular foliation. *Doc. Math.*, 25:1571–1652, 2020.
- [20] Camille Laurent-Gengoux and Ruben Louis. Lie-Rinehart algebras \simeq acyclic Lie ∞ -algebroids. *Journal of algebra*, 594:1–53, 2022.
- [21] Camille Laurent-Gengoux and Leonid Ryvkin. The holonomy of a singular leaf. *Selecta Mathematica*, 28, 2022.
- [22] Sylvain Lavau. Lie ∞ -algebroids and singular foliations. arxiv:1703.07404. 2017.

- [23] Madeleine Jotz Lean, Rajan Amit Mehta, and Theodoris Papantonis. Modules and representations up to homotopy of lie n -algebroids, <https://arxiv.org/abs/2001.01101v2>. 2020.
- [24] Rajan Amit Mehta and Marco Zambon. L_∞ -algebra actions. *Differential Geometry and its Applications*, 30(6):576–587, 2012.
- [25] Camille Laurent-Gengoux Ruben Louis and Leonid Ryvkin. *Geometry of singular foliations: a draft of an introduction*. CRM Barcelona, Poisson geometry summer school, 2022.
- [26] Yum-Tong Siu. Noetherianness of rings of holomorphic functions on stein compact subsets. *Proceedings of the American Mathematical Society*, 21(2):483–489, 1969.
- [27] Richard G. Swan. Vector Bundles and Projective modules. *Transactions of the American Mathematical Society*, 105(2):264–277, 1962.
- [28] Theodore Voronov. Q -manifolds and higher analogs of Lie algebroids. In *XXIX Workshop on Geometric Methods in Physics*, volume 1307 of *AIP Conf. Proc.*, pages 191–202. Amer. Inst. Phys., Melville, NY, 2010.

UNIVERSITÉ DE LORRAINE, CNRS, IECL, F-57000 METZ, FRANCE.