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ON SYMMETRIES OF SINGULAR FOLIATIONS

RUBEN LOUIS

ABSTRACT. This paper shows that a weak symmetry action of a Lie algebra \mathfrak{g} on a singular foliation \mathcal{F} induces a unique up to homotopy Lie ∞ -morphism from \mathfrak{g} to the DGLA of vector fields on a universal Lie ∞ -algebroid of \mathcal{F} . We deduce from this general result several geometrical consequences. For instance, we give an example of a Lie algebra action on a sub-affine variety which cannot be extended on the ambient space. Last, we introduce the notion of tower of bi-submersions over a singular foliations and lift symmetries to those.

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INTRODUCTION

Singular foliations arise frequently in differential or algebraic geometry. Here following [18, 1, 5, 8, 9] we define a singular foliation on a smooth, complex, algebraic, real analytic manifold M with sheaf of functions \mathcal{O} to be a subsheaf $\mathcal{F}: U \rightarrow \mathcal{F}(U)$ of the sheaf of vector fields \mathfrak{X} , which is closed under the Lie bracket and locally finitely generated as an \mathcal{O} -module. By Hermann's theorem [15], this is enough to induce a partition of the manifold M into embedded submanifolds of possibly different dimensions, called *leaves* of the singular foliation. Singular

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foliations appear for instance as orbits of Lie group actions with possibly different dimensions or as symplectic leaves of a Poisson structure. When all the leaves have the same dimension, we recover the usual “regular foliations” [10].

The purpose of this paper is to look at symmetries of singular foliations. Let (M, \mathcal{F}) be a foliated manifold. A *global symmetry* of a singular foliation \mathcal{F} on M is a diffeomorphism $\phi: M \rightarrow M$ which preserves \mathcal{F} , that is, $\phi_*(\mathcal{F}) = \mathcal{F}$. The image of a leaf through a global symmetry is again a leaf (not necessarily the same leaf). For G a Lie group, a *strict symmetry action* of G on a foliated manifold (M, \mathcal{F}) is a smooth action $G \times M \rightarrow M$ that acts by global symmetries [14]. Infinitesimally, it corresponds to a Lie algebra morphism $\mathfrak{g} \rightarrow \mathfrak{X}(M)$ between the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ of G and the Lie algebra of symmetries of \mathcal{F} .

A strict symmetry action of G on M goes down to the leaf space M/\mathcal{F} , even though the latter space does not exist as a manifold. The opposite direction is more sophisticated since a strict symmetry action of G on M/\mathcal{F} does not induce a strict action over M in general. However, it makes sense to consider linear maps $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ that satisfy $[\varrho(x), \mathcal{F}] \subset \mathcal{F}$ for all $x \in \mathfrak{g}$, and which are Lie algebra morphisms up \mathcal{F} , namely, $\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] \in \mathcal{F}$ for all $x, y \in \mathfrak{g}$. The latter actions are called “*weak symmetry actions*”. These actions induce a “strict symmetry action” on the leaf space i.e. a Lie algebra morphism $\mathfrak{g} \rightarrow \mathfrak{X}(M/\mathcal{F})$, whenever M/\mathcal{F} is a manifold.

In view of [21, 18] it is shown that behind every singular foliation or more generally any Lie-Rinehart algebras [19] there exists a Lie ∞ -algebroid structure which is unique up to homotopy called the *universal Lie ∞ -algebroid*. Here is a natural question: what does a symmetry of a singular foliation \mathcal{F} induce on an universal Lie ∞ -algebroid of \mathcal{F} ? Theorem 2.2 of this paper gives an answer to that question. It states that any weak symmetry action of a Lie algebra on a singular foliation \mathcal{F} can be lifted to a Lie ∞ -morphism valued in the DGLA of vector fields on an universal Lie ∞ -algebroid of \mathcal{F} . Such Lie ∞ -morphism will be called a *lift of the symmetry action*. This goes in the same direction as [14] who already underlined Lie-2-group structures associated to strict symmetry action of Lie groups. Furthermore, Theorem 2.2 says this lift is unique modulo homotopy equivalence.

This result gives several geometric consequences. Here is an elementary question: can a Lie algebra action $\mathfrak{g} \rightarrow \mathfrak{X}(W)$ on an affine variety $W \subset \mathbb{C}^d$ be extended to a Lie algebra action $\mathfrak{g} \rightarrow \mathfrak{X}(\mathbb{C}^d)$ on \mathbb{C}^d ? Said differently: it is trivial that any vector field on W extends to \mathbb{C}^d , but can this extension be done in such a manner that it preserves the Lie bracket? Although no “ ∞ -oids” appears in the question, which seems to be a pure algebraic geometry question, we claim that the answer goes through Lie ∞ -algebroids and singular foliations. More precisely, by Theorem 2.2, we know that it is possible to lift any symmetry action of singular foliation into a Lie ∞ -morphism. Is it possible to build such a Lie ∞ -morphism where the arity -1 of the second order Taylor coefficient is zero? There are cohomological obstructions. The idea is then to say that any \mathfrak{g} -action on W induces a weak symmetry action on the singular foliation $\mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$ of all vector fields vanishing on W (here \mathcal{I}_W is the ideal that defines W). In some specific cases, obstruction classes of extending this action to the ambient space appear on some cohomology, although in general the obstruction is rather a Maurer-Cartan-like equations that may or may not have solutions.

The outline of this paper is made as follows: In Section 1 and Appendix A we present some definitions and facts on symmetry action of a Lie algebra on singular foliation and give some

examples. Also, we review Lie ∞ -algebroid structures and their morphisms in order to fix notations. In Section 2 we state the main results of this paper and present their proofs. In Section 3 we describe the relation between weak symmetry actions and Lie ∞ -algebroids that have some special properties. In Section 4 we define an obstruction class of extending a Lie algebra action on an affine variety to ambient space. Finally, in Section 5 we look at symmetries of bi-submersions. Afterwards, we introduce the notion of tower of bi-submersions over a singular foliation and point out some observations related to their symmetries.

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1. DEFINITIONS AND EXAMPLES OF WEAK AND STRICT SYMMETRY ACTIONS

Convention 1.1. *Throughout this paper, M stands for a smooth or complex manifold, or an affine variety over \mathbb{C} . We will denote the sheaf of smooth or complex, or regular functions on M by \mathcal{O} and the sheaf of vector fields on M by $\mathfrak{X}(M)$, and $X[f]$ stands for a vector field $X \in \mathfrak{X}(M)$ applied to $f \in \mathcal{O}$. Also, \mathbb{K} stands for \mathbb{R} or \mathbb{C} .*

Definition 1.2. Let $\mathcal{F} \subset \mathfrak{X}(M)$ be a singular foliation over M .

- A diffeomorphism $\phi: M \rightarrow M$ is said to be a *symmetry* of \mathcal{F} , if $\phi_*(\mathcal{F}) = \mathcal{F}$.
- A vector field $X \in \mathfrak{X}(M)$ is said to be a *infinitesimal symmetry* of \mathcal{F} , if $[X, \mathcal{F}] \subset \mathcal{F}$. The Lie algebra of infinitesimal symmetries of \mathcal{F} is denoted by $\text{sym}(\mathcal{F})$.

In particular, $\mathcal{F} \subset \text{sym}(\mathcal{F})$, since $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$.

Proposition 1.3. [1, 12] *Let M be a smooth or complex manifold. The flow of an infinitesimal symmetry of \mathcal{F} , if it exists, is a symmetry of \mathcal{F} .*

As we will see later, one of the consequences of our future results is that any symmetry $X \in \text{sym}(\mathcal{F})$ of a singular foliation \mathcal{F} admits a lift to a degree zero vector field on any universal NQ -manifold over \mathcal{F} that commutes with the homological vector field Q . This will allow us to have an alternative proof and interpretation of Proposition 1.3 (see Section 2).

Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a Lie algebra over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , depending on the context. From now on and in the sequel \mathfrak{g} is concentrated in degree -1 .

Definition 1.4. A *weak symmetry action* of the Lie algebra \mathfrak{g} on a singular foliation \mathcal{F} on M is an \mathbb{K} -linear map $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ that satisfies:

- $\forall x \in \mathfrak{g}, [\varrho(x), \mathcal{F}] \subseteq \mathcal{F}$,
- $\forall x, y \in \mathfrak{g}, \varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] \in \mathcal{F}$.

When $x \mapsto \varrho(x)$ is a Lie algebra morphism we speak of *strict symmetry action* of \mathfrak{g} on \mathcal{F} . There is an equivalence relation on the set of weak symmetry actions which is defined as follows: two weak symmetry actions, $\varrho, \varrho': \mathfrak{g} \rightarrow \mathfrak{X}(M)$ are said to be *equivalent* if there exists a linear map $\varphi: \mathfrak{g} \rightarrow \mathcal{F}$ such that $\varrho - \varrho' = \varphi$.

Remark 1.5. It is important to notice that when \mathcal{F} is a regular foliation and M/\mathcal{F} is a manifold, any weak symmetry action of a Lie algebra \mathfrak{g} on \mathcal{F} induces a strict action of \mathfrak{g} over M/\mathcal{F} . Definition 1.4 is a way of extending this idea to all singular foliations.

Here is a list of some examples.

Example 1.6. Let $\pi: M \rightarrow N$ be a submersion. Since any vector field on N comes from a π -projectable vector field on M , therefore any Lie algebra morphism $\mathfrak{g} \rightarrow \mathfrak{X}(N)$ can be lifted to a weak symmetry action $\mathfrak{g} \rightarrow \mathfrak{X}(M)$ on the regular foliation $\Gamma(\ker d\pi)$, and any two such lifts are equivalent.

Furthermore, any weak action of a Lie algebra \mathfrak{g} on a singular foliation \mathcal{F} on N can be lifted up to a class of weak symmetry actions on the pull-back foliation $\pi^{-1}(\mathcal{F})$, (see Definition 1.9 in [1]).

Example 1.7. Let \mathcal{F} be a singular foliation on M . For any point $m \in M$, the set $\mathcal{F}(m) = \{X \in \mathcal{F} \mid X(m) = 0\}$ is a Lie subalgebra of \mathcal{F} . Put $\mathcal{I}_m = \{f \in C^\infty(M) \mid f(m) = 0\}$. The quotient space $\mathfrak{g}_m = \frac{\mathcal{F}(m)}{\mathcal{I}_m\mathcal{F}}$ is a Lie algebra, since $\mathcal{I}_m\mathcal{F} \subseteq \mathcal{F}(m)$ is a Lie ideal. The Lie algebra \mathfrak{g}_m is called the *isotropy Lie algebra of \mathcal{F} at m* (see [4]). Let us denote by $[\cdot, \cdot]_{\mathfrak{g}_m}$, its Lie bracket.

- (1) Consider $\varrho: \mathfrak{g}_m \rightarrow \mathcal{F}(m) \subset \mathfrak{X}(M)$ a section of the projection map

$$\mathcal{I}_m\mathcal{F} \hookrightarrow \mathcal{F}(m) \begin{array}{c} \xleftarrow{\varrho} \\ \xrightarrow{\varrho} \end{array} \mathfrak{g}_m \quad (1)$$

Then $[\varrho(x), \mathcal{I}_m\mathcal{F}] \subset \mathcal{I}_m\mathcal{F}$ and $\varrho([x, y]_{\mathfrak{g}_m}) - [\varrho(x), \varrho(y)] \in \mathcal{I}_m\mathcal{F}$. Hence, the map $\varrho: \mathfrak{g}_m \rightarrow \mathfrak{X}(M)$ is a weak symmetry action of the singular foliation $\mathcal{I}_m\mathcal{F}$. A different section ϱ' of the projection map yields an equivalent weak symmetry action of \mathfrak{g}_m on $\mathcal{I}_m\mathcal{F}$. An obstruction class for having a strict symmetry action equivalent to ϱ will be given later in Section 4.

- (2) In particular, for $k \geq 1$, let us denote by \mathfrak{g}_m^k the isotropy Lie algebra of the singular foliation $\mathcal{I}_m^k\mathcal{F}$ at m . Any section $\varrho_k: \mathfrak{g}_m^k \rightarrow \mathfrak{X}(M)$ of the projection map

$$\mathcal{I}_m^{k+1}\mathcal{F} \hookrightarrow \mathcal{I}_m^k\mathcal{F} \begin{array}{c} \xleftarrow{\varrho_k} \\ \xrightarrow{\varrho_k} \end{array} \mathfrak{g}_m^k \quad (2)$$

is a weak symmetry action of the Lie algebra \mathfrak{g}_m^k on the singular foliation $\mathcal{I}_m^{k+1}\mathcal{F}$.

Example 1.8. The following example is taken from [20], and follows the same patterns as in Examples 1.6 and 1.7. Let (M, \mathcal{F}) be a singular foliation on a manifold M and $L \subset M$ a leaf. Let $[L, M]$ be a neighbourhood of L in M equipped with some projection $\pi: M \rightarrow L$. According to [20], upon replacing $[L, M]$ by a smaller neighborhood of L if necessary, there exists an Ehresmann connection (that is a vector sub-bundle $H \subset T[L, M]$ with $H \oplus \ker(d\pi) = T[L, M]$) which satisfies that $\Gamma(H) \subset \mathcal{F}$. Such an Ehresmann connection is called an *Ehresmann \mathcal{F} -connection* and induces a $C^\infty(L)$ -linear section $\varrho_H: \mathfrak{X}(L) \rightarrow \mathcal{F}^{\text{proj}}$ of the surjection $\mathcal{F}^{\text{proj}} \rightarrow \mathfrak{X}(L)$, where $\mathcal{F}^{\text{proj}}$ stands for vector fields of \mathcal{F} π -projectable on elements of $\mathfrak{X}(L)$.

The section ϱ_H is a weak symmetry action of $\mathfrak{X}(L)$ on the *transverse* foliation $\mathcal{T} := \Gamma(\ker d\pi) \cap \mathcal{F}$. When the Ehresmann connection H is flat, ϱ_H is bracket-preserving, and defines a strict symmetry of $\mathfrak{X}(L)$ on the transverse foliation \mathcal{T} .

Example 1.9. Consider for a fixed k , the singular foliation $\mathcal{F}_k := \mathcal{I}_0^k \mathfrak{X}(\mathbb{R}^d)$ generated by all vector fields vanishing to order k at the origin. The action of the Lie algebra $\mathfrak{gl}(\mathbb{R})$ on \mathbb{R}^d which is given by,

$$\mathfrak{gl}(\mathbb{R}) \longrightarrow \mathfrak{X}(\mathbb{R}^d), (a_{ij})_{1 \leq i, j \leq d} \longmapsto \sum_{1 \leq i, j \leq d} a_{ij} x_i \frac{\partial}{\partial x_j}$$

is a strict symmetry of \mathcal{F}_k .

Example 1.10. Let $\varphi := (\varphi_1, \dots, \varphi_r)$ be a r -tuple of homogeneous polynomial functions in d variables over \mathbb{K} . Consider the singular foliation \mathcal{F}_φ (see [19] Section 3.2.1) which is generated by all polynomial vector fields $X \in \mathfrak{X}(\mathbb{K}^d)$ that satisfy $X[\varphi_i] = 0$ for all $i \in \{1, \dots, r\}$. The action $\mathbb{K} \rightarrow \mathfrak{X}(\mathbb{K}^d)$, $\lambda \mapsto \lambda \vec{E}$, is a strict symmetry of \mathcal{F}_φ . Here \vec{E} stands for the Euler vector field.

Example 1.11. Let W be an affine variety realized as a subvariety of \mathbb{C}^d and $\mathcal{I}_W \subset \mathbb{C}[x_1, \dots, x_d]$ its corresponding ideal. Let us denote by $\mathfrak{X}(W) := \text{Der}(\mathbb{C}[x_1, \dots, x_d]/\mathcal{I}_W)$ the Lie algebra of vector field of W . Let $\mathcal{F}_W := \mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$ the singular foliation made of vector fields vanishing on W . Since every vector field on W can be extended to a vector field on \mathbb{C}^d tangent to W . Any Lie algebra morphism $\varrho: \mathfrak{g} \longrightarrow \mathfrak{X}(W)$ extends to a linear map $\tilde{\varrho}: \mathfrak{g} \longrightarrow \mathfrak{X}(\mathbb{C}^d)$ that makes this diagram commutes

$$\begin{array}{ccc} & & \mathfrak{X}(\mathbb{C}^d) \\ & \nearrow \tilde{\varrho} & \downarrow \\ \mathfrak{g} & \xrightarrow{\varrho} & \mathfrak{X}(W) \end{array}$$

This extension $\tilde{\varrho}$ is a weak symmetry action of \mathfrak{g} on \mathcal{F}_W over the ambient space \mathbb{C}^d . Two different extensions yield equivalent symmetry actions.

2. A LIE ∞ -MORPHISM LIFTING A WEAK SYMMETRY OF A FOLIATION

We refer to Appendix A for the notion of (universal) Lie ∞ -algebroid of a singular foliation. We denote them by (E, Q) and their functions by \mathcal{E} . Also, see Appendix B for the notion of Lie ∞ -morphism of differential graded Lie algebras and notations.

Definition 2.1. Let \mathcal{F} be a singular foliation over M and (E, Q) a Lie ∞ -algebroid over \mathcal{F} . Consider a weak symmetry action $\varrho: \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ of \mathfrak{g} on \mathcal{F} .

- We say that a Lie ∞ -morphism of differential graded Lie algebras

$$\Phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \rightarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$$

lifts the weak symmetry action ϱ to (E, Q) if for all $x \in \mathfrak{g}$, $f \in \mathcal{O}$, $\Phi_0(x)(f) = \varrho(x)[f]$.

- When Φ exists we say then Φ is a lift of ϱ on (E, Q) .

We now state the main theorem of this paper.

Theorem 2.2. Let \mathcal{F} a be singular foliation over a smooth manifold (or an affine variety) M and \mathfrak{g} a Lie algebra. Let $\varrho: \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ be a weak symmetry action of \mathfrak{g} on \mathcal{F} . The following assertions hold:

- (1) for any universal Lie ∞ -algebroid (E, Q) of the singular foliation \mathcal{F} , there exists a Lie ∞ -morphism $\Phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \longrightarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ that lifts ϱ to (E, Q) ,
- (2) any two such Lie ∞ -morphism are homotopy equivalent over the identity of M ,

- (3) any two such lifts of any two equivalent weak symmetry actions of \mathfrak{g} on \mathcal{F} are homotopy equivalent.

Remark 2.3. Item (1) in Theorem 2.2 means that

- (1) there exists a linear map $\Phi_0: \mathfrak{g} \rightarrow \mathfrak{X}_0(E)$ such that

$$\Phi_0(x)[f] = \varrho(x)[f], \text{ and } [Q, \Phi_0(x)] = 0, \quad \forall x \in \mathfrak{g}, f \in \mathcal{O}. \quad (3)$$

This morphism is not a graded Lie algebra morphism, but there exist a linear map $\Phi_1: \wedge^2 \mathfrak{g} \rightarrow \mathfrak{X}_{-1}(E)$ such that for all $x, y, z \in \mathfrak{g}$,

$$\Phi_0([x, y]_{\mathfrak{g}}) - [\Phi_0(x), \Phi_0(y)] = [Q, \Phi_1(x, y)].$$

Also,

$$\Phi_1([x, y]_{\mathfrak{g}}, z) - [\Phi_0(x), \Phi_1(y, z)] + \circlearrowleft(x, y, z) = [Q, \Phi_2(x, y, z)]$$

for some linear map $\wedge^3 \mathfrak{g} \rightarrow \mathfrak{X}_{-2}(E)$. These sets of compatibility conditions continue to higher multilinear maps.

- (2) For every element $x \in \mathfrak{g}$ and $i \geq 1$, there is a degree zero map $\nabla_x \in \text{Der}(E)$ (i.e. $\nabla_x(fe) = f\nabla_x(e) + \varrho(x)[f]e$, for $f \in \mathcal{O}, e \in \Gamma(E)$) depending linearly on x , such that

$$\langle \Phi_0(x)^{(0)}(\alpha), e \rangle = \varrho(x)[\langle \alpha, e \rangle] - \langle \alpha, \nabla_x(e) \rangle, \text{ for all } \alpha \in \Gamma(E^*), e \in \Gamma(E). \quad (4)$$

Therefore, by using Equation (3),(4) and the dual correspondence between Lie ∞ -algebroids and NQ -manifolds [25, 24, 22], we obtain theses compatibility conditions:

$$\ell_1 \circ \nabla_x = \nabla_x \circ \ell_1 \quad \text{and} \quad \rho \circ \nabla_x = \text{ad}_{\varrho(x)} \circ \rho.$$

Where $\Phi_0(x)^{(0)}$ stands for the arity zero of $\Phi_0(x)$, and ℓ_1 stands for the corresponding unary bracket of (E, Q) . Also, for $X \in \mathfrak{X}(M)$, $\text{ad}_X := [X, \cdot]$.

In general, the map $\mathfrak{g} \rightarrow \text{Der}(E)$, $x \mapsto \nabla_x$ is not a Lie algebra morphism even when the action ϱ is strict. In fact, there exists a bilinear map $\gamma: \wedge^2 \mathfrak{g} \rightarrow \text{End}(E)[1]$ of degree 0 that satisfies

$$\nabla_{[x, y]_{\mathfrak{g}}} - [\nabla_x, \nabla_y] = \gamma(x, y) \circ \ell_1 - \ell_1 \circ \gamma(x, y) + \ell_2(\eta(x, y), \cdot),$$

here ℓ_2 is the corresponding 2-ary bracket of (E, Q) , and $\eta: \wedge^2 \mathfrak{g} \rightarrow \Gamma(E_{-1})$ is such that $\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] = \rho(\eta(x, y))$.

Corollary 2.4. Any symmetry $X \in \mathfrak{X}(M)$ of the singular foliation \mathcal{F} can be lifted to a degree zero vector field $Z \in \mathfrak{X}_0(E)$ that commutes with Q , i.e. such that $[Z, Q] = 0$.

Proof. To construct Z , it suffices to apply Theorem 2.2 for $\mathfrak{g} = \mathbb{R}$ and take Z to be the image of 1 through $\Phi_0: \mathbb{R} \rightarrow \mathfrak{X}_0(E)$. \square

Remark 2.5. In particular Corollary 2.4 has the following consequences:

- (1) for any admissible t , the flow $\Phi_t^Z: \mathcal{E} \rightarrow \mathcal{E}$ of Z induces an isomorphism of vector bundles $E_{-1} \rightarrow E_{-1}$. Since $[Q, Z] = 0$, the following diagram commutes,

$$\begin{array}{ccc} \Gamma(E_{-1}) & \xrightarrow{(\Phi_t^Z)^{(0)}} & \Gamma(E_{-1}) \\ \downarrow \rho & & \downarrow \rho \\ \mathfrak{X}(M) & \xrightarrow{(\varphi_t^X)_*} & \mathfrak{X}(M) \end{array}$$

where ϕ_t^X is the flow of X at t .

- (2) Consequently, for any open set $U \subset M$ which is stable under φ_t^X , there exists an invertible matrix \mathfrak{M}_X^t with coefficients in $\mathcal{O}(U)$ that satisfies

$$(\phi_t^X)_* \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \mathfrak{M}_X^t \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix},$$

for some generators X_1, \dots, X_n of \mathcal{F} over U . As announced earlier, we recover Proposition 1.3, that is, $(\phi_t^X)_*(\mathcal{F}) = \mathcal{F}$.

Let (E, Q) and (E', Q') be two universal Lie ∞ -algebroids of \mathcal{F} . A direct consequence of Ricardo Campos's Theorem 4.1 in [7] is that the differential graded Lie algebras $(\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q)$ and $(\mathfrak{X}_\bullet(E')[1], [\cdot, \cdot], \text{ad}_{Q'})$ are homotopy equivalent over the identity of M . This leads to the following statement.

Corollary 2.6. *Let $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ be a weak symmetry action of a Lie algebra \mathfrak{g} on \mathcal{F} . Then, there exist Lie ∞ -morphisms, $\Phi: \mathfrak{g} \rightarrow (\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q)$ and $\Psi: \mathfrak{g} \rightarrow (\mathfrak{X}_\bullet(E')[1], [\cdot, \cdot], \text{ad}_{Q'})$ that lift ϱ , and Φ, Ψ make the following diagram commute up to homotopy*

$$\begin{array}{ccc} & \mathfrak{g} & \\ \Phi \swarrow & & \searrow \Psi \\ (\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q) & \xleftrightarrow{\sim} & (\mathfrak{X}_\bullet(E')[1], [\cdot, \cdot], \text{ad}_{Q'}) \end{array} \quad (5)$$

Proof. The composition of Φ with the horizontal map in the diagram (5) is a lift of the action ϱ . It is necessarily homotopy equivalent to Ψ by item (2) in Theorem 2.2. \square

2.1. Cohomology of longitudinal graded vector fields. In this Section we study the cohomology of longitudinal vector fields, which will help in proving the main results stated in the beginning of Section 2.

Let \mathcal{F} be a singular foliation over M .

Definition 2.7. Let E be a splitted graded manifold over M with sheaf of function \mathcal{E} . A vector field $L \in \mathfrak{X}(E)$ is said to be a *longitudinal vector field for \mathcal{F}* if there exists vector fields $X_1, \dots, X_k \in \mathcal{F}$ and functions $\Theta_1, \dots, \Theta_k \in \mathcal{E}$ such that

$$L(f) = \sum_{i=1}^k X_i[f]\Theta_i, \quad \forall f \in \mathcal{O}. \quad (6)$$

Example 2.8. Here are some examples.

- (1) Vertical vector fields are longitudinal.
- (2) For any Q -manifold (E, Q) over a manifold M . The homological vector field $Q \in \mathfrak{X}(E)$ is a longitudinal vector field for $\mathcal{F} := \rho(\Gamma(E_{-1}))$.

- (3) For (E, Q) a Q -manifold and $\mathcal{F} := \rho(\Gamma(E_{-1}))$ its basic singular foliation. For any extension of a symmetry $X \in \text{sym}(\mathcal{F})$ of \mathcal{F} to a degree zero vector field $\widehat{X} \in \mathfrak{X}(E)$, the degree +1 vector field $[Q, \widehat{X}]$ is longitudinal for \mathcal{F} .

Let us show this last point using local coordinates (x_1, \dots, x_n) on M and a local trivialisation ξ^1, ξ^2, \dots of graded sections in $\Gamma(E^*)$. The vector fields Q and \widehat{X} take the form:

$$\begin{aligned} Q &= \sum_j \sum_{k, |\xi^k|=1} Q_k^j(x) \xi^k \frac{\partial}{\partial x_j} + \sum_j \sum_{k, \iota_1, \dots, \iota_k} \frac{1}{k!} Q_{\iota_1, \dots, \iota_k}^j(x) \xi^1 \odot \dots \odot \xi^k \frac{\partial}{\partial \xi^j} \\ \widehat{X} &= X + \sum_j \sum_{k, \iota_1, \dots, \iota_k} \frac{1}{k!} X_{\iota_1, \dots, \iota_k}^j(x) \xi^1 \odot \dots \odot \xi^k \frac{\partial}{\partial \xi^j} \end{aligned} \quad (7)$$

where $X = \sum_{i=1}^n X_i(x) \frac{\partial}{\partial x_i}$. By using Equation (7) we note that all the terms of $[Q, \widehat{X}]$ are vertical except maybe for the ones where the vector field X appears. For $k \geq 1$, the vector field $[Q_{\iota_1, \dots, \iota_k}^j \xi^1 \odot \dots \odot \xi^k \frac{\partial}{\partial \xi^j}, X]$ is vertical; and for every fix k , one has

$$\left[\sum_{j=1}^n Q_k^j \xi^k \frac{\partial}{\partial x_j}, X \right] = \xi^k \left[\sum_{j=1}^n Q_k^j \frac{\partial}{\partial x_j}, X \right].$$

Thus, $\left[\sum_{j=1}^n Q_k^j \frac{\partial}{\partial x_j}, X \right] \in \mathcal{F}$, since X is a symmetry for \mathcal{F} and $\sum_{j=1}^n Q_k^j \frac{\partial}{\partial x_j} \in \mathcal{F}$.

Remark 2.9. Longitudinal vector fields are stable under the graded Lie bracket.

Remark 2.10. Let us study vector fields on E .

- (1) Sections of E are identified with derivations under the isomorphism mapping $e \in \Gamma(E) \mapsto \iota_e \in \mathfrak{X}(E)$. This allows us to identify a vertical vector field with (maybe infinite) sums of tensor products of the form $\Theta \otimes e$ with $\Theta \in \mathcal{E}, e \in \Gamma(E)$.
- (2) Any connection on $\Gamma(E^*)$ induces vector field of degree zero $\widetilde{\nabla}_X \in \mathfrak{X}(E)$ by setting for $f \in \mathcal{O}$, $\widetilde{\nabla}_X(f) := X[f]$. Once a connection is chosen, we have for all $k \in \mathbb{Z}$

$$\mathfrak{X}_k(E) \simeq \bigoplus_{j \geq 1} \mathcal{E}_{k+j} \otimes_{\mathcal{O}} \Gamma(E_{-j}) \oplus \mathcal{E}_k \otimes_{\mathcal{O}} \mathfrak{X}(M) \simeq \bigoplus_{j \geq 1} \Gamma(S(E^*)_{k+j} \otimes E_{-j}) \oplus \Gamma(S(E^*)_k \otimes TM).$$

Thus, one can realize a vector field $P \in \mathfrak{X}_k(E)$ as a sequence $P = (p_0, p_1, \dots)$, where $p_0 \in \Gamma(S(E^*)_k \otimes TM)$ and $p_i \in \Gamma(S(E^*)_{k+i} \otimes E_{-i})$ for $i \geq 1$ are called *components* of P . In the diagram (9), $P = (p_0, p_1, \dots)$ is represented as an element of the anti-diagonal and p_i is on column i . We say that P is of *depth* $n \in \mathbb{N}$ if $p_i = 0$ for all $i < n$. In particular, vector fields of depth greater or equal to 1 are vertical. Under the decomposition (2), the differential map ad_Q takes the form

$$D = D^h + \sum_{s \geq 0} D^{v_s} \quad (8)$$

with $D^2 = 0$. Here $D^h = \text{id} \otimes d$ or $\text{id} \otimes \rho$, and $D^{v_s}: \Gamma(S(E^*)_k \otimes E_{-i}) \rightarrow \Gamma(S(E^*)_{k+s+1} \otimes E_{-i-s})$ for $i \geq 1, s \geq 0$. We denote the latter complex by (\mathcal{L}, D) . They can be represented as anti-diagonal lines in the following commutative diagram whose lines are complexes of \mathcal{O} -modules

$$\begin{array}{ccccc}
 & \vdots & & \vdots & & \vdots \\
 \cdots & \longrightarrow & \Gamma(S(E^*)_{k+2} \otimes E_{-2}) & \xrightarrow{\text{id} \otimes d} & \Gamma(S(E^*)_{k+2} \otimes E_{-1}) & \xrightarrow{\text{id} \otimes \rho} & \Gamma(S(E^*)_{k+2} \otimes TM) \\
 & & \uparrow Q \otimes \text{id} + \cdots & & \uparrow Q \otimes \text{id} + \cdots & & \uparrow Q \otimes \text{id} + \cdots \\
 \cdots & \longrightarrow & \Gamma(S(E^*)_{k+1} \otimes E_{-2}) & \xrightarrow{\text{id} \otimes d} & \Gamma(S(E^*)_{k+1} \otimes E_{-1}) & \xrightarrow{\text{id} \otimes \rho} & \Gamma(S(E^*)_{k+1} \otimes TM) \\
 & & \uparrow Q \otimes \text{id} + \cdots & & \uparrow Q \otimes \text{id} + \cdots & & \uparrow Q \otimes \text{id} + \cdots \\
 \cdots & \longrightarrow & \Gamma(S(E^*)_k \otimes E_{-2}) & \xrightarrow{\text{id} \otimes d} & \Gamma(S(E^*)_k \otimes E_{-1}) & \xrightarrow{\text{id} \otimes \rho} & \Gamma(S(E^*)_k \otimes TM) \\
 & & \uparrow \vdots & & \uparrow \vdots & & \uparrow \vdots \\
 & & \text{column 2} & & \text{column 1} & & \text{column 0}
 \end{array} \tag{9}$$

Under this correspondence we understand longitudinal vector fields as the following.

Lemma 2.11. *A graded vector field $P = (p_0, p_1, \dots) \in \mathfrak{L}$ is longitudinal if $p_0 \in \mathcal{E} \otimes \mathcal{O} \mathcal{F}$.*

The following theorem is crucial for the rest of this paper.

Theorem 2.12. *Let (E, Q) be a universal Q -manifold of \mathcal{F} .*

(1) *Longitudinal vector fields form an acyclic complex.*

More precisely, any longitudinal vector field on E which is a ad_Q -cocycle is the image through ad_Q of some vertical vector field on E .

(2) *More generally, if a vector field on E of depth n is a ad_Q -cocycle, then it is the image through ad_Q of some vector field on E of depth $n + 1$.*

Proof. Since (E, Q) is an universal Q -manifold of \mathcal{F} , lines in (9) are exact. It is now a diagram chasing phenomena. Let $P = (p_0, p_1, \dots) \in \mathfrak{L}$ be a longitudinal element which is a D -cocycle. By longitudinality there exists an element $b_1 \in \Gamma(S(E^*) \otimes E_{-1})$ such that $(\text{id} \otimes \rho)(b_1) = p_0$. Set $P_1 = (0, b_1, 0, \dots)$, that is we extend b_1 by zero on $\Gamma(S(E^*) \otimes E_{\leq -2})$ and $\Gamma(S(E^*) \otimes TM)$. It is clear that $P - D(P_1) = (0, p'_1, p'_2, \dots)$ is also a D -cocycle. In particular we have $D^h(p'_1) = 0$ by exactness there exists $b_2 \in \Gamma(S(E^*) \otimes E_{-2})$ such that $D^h(b_2) = p'_1$. As before put $P_2 = (0, 0, b_2, 0, \dots)$. Similarly, $P - D(P_1) - D(P_2) = (0, 0, p''_2, p''_3, \dots)$ is a D -cocycle. By recursion we end up to construct P_1, P_2, \dots that satisfy $P - D(P_1) - D(P_2) + \dots = 0$, that is, there exists an element $B = (0, b_1, b_2, \dots) \in \mathfrak{L}$ such that $D(B) = P$. This proves item 1.

To prove item 2 it suffices to cross out in the diagram (9) the columns number $0, \dots, n - 1$, which does not break exactness. The proof now follows as for item 1. \square

In particular we deduce from Theorem 2.12 the following exact subcomplex.

Corollary 2.13. *Let (E, Q) be a universal Q -manifold of \mathcal{F} . The subcomplex \mathfrak{V}_Q of $(\mathfrak{X}(E), \text{ad}_Q)$ made of vertical vector fields $P \in \mathfrak{X}(E)$ that satisfy $P \circ Q(f) = 0$ for all $f \in \mathcal{O}$ is exact.*

Proof. Let $P \in \mathfrak{X}(E)$ be a vertical vector field which is a ad_Q -cocycle (note that we have automatically $P \circ Q(f) = 0$ for all $f \in \mathcal{O}$). By Theorem 2.12 there exists a vertical vector field $\tilde{P} \in \mathfrak{X}(E)$ such that $[Q, \tilde{P}] = P$. Moreover, $\tilde{P} \in \mathfrak{V}_Q$, since for all $f \in \mathcal{O}$,

$$0 = [Q, \tilde{P}](f) = (-1)^{|\tilde{P}|} \tilde{P} \circ Q(f).$$

□

2.2. Proof of the main results. This section is devoted to the proof of the main results stated in Section 2.

Let \mathcal{F} be a singular foliation, and (E, Q) a universal Lie ∞ -algebroid of \mathcal{F} . We start with the following lemma.

Lemma 2.14. *For every weak symmetry Lie algebra action of \mathfrak{g} on \mathcal{F} there exists a linear map, $\Phi_0: \mathfrak{g} \rightarrow \mathfrak{X}_0(E)$, such that $[Q, \Phi_0(x)] = 0$ and $\Phi_0(x)[f] = \varrho(x)[f]$ for all $x \in \mathfrak{g}$, $f \in \mathcal{O}$.*

Proof. For $x \in \mathfrak{g}$, let $\widehat{\varrho(x)} \in \mathfrak{X}_0(E)$ be any arbitrary extension of $\varrho(x) \in \text{sym}(\mathcal{F})$ to a degree zero vector field on E . Since $\varrho(x)$ is a symmetry of \mathcal{F} , the degree +1 vector field $[\widehat{\varrho(x)}, Q]$ is also a longitudinal vector field on E , see Example 2.8 item 3. In addition, $[\widehat{\varrho(x)}, Q]$ is a ad_Q -cocycle. By item 1 of Theorem 2.12, there exists a vertical vector field $Y(x) \in \mathfrak{X}_0(E)$ of degree zero such that

$$[Q, Y(x) + \widehat{\varrho(x)}] = 0. \quad (10)$$

Let us set for $x \in \mathfrak{g}$, $\Phi_0(x) := Y(x) + \widehat{\varrho(x)}$. By construction we have, $[Q, \Phi_0(x)] = 0$ and $\Phi_0(x)[f] = \varrho(x)[f]$ for all $x \in \mathfrak{g}$, $f \in \mathcal{O}$. □

We will need the following lemma.

Lemma 2.15. *Assume (E, Q) is a universal Lie ∞ -algebroid over M . Let $\bar{\Phi}: (S_{\mathbb{K}}^{\bullet} \mathfrak{g}, Q_{\mathfrak{g}}) \rightarrow (S_{\mathbb{K}}^{\bullet} \mathfrak{X}(E)[1], \bar{Q})$ be a coalgebra morphism which is a Lie ∞ -morphism up to arity $n \geq 0$, i.e. $(\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi})^{(i)} = 0$ for all integer $i \in \{0, \dots, n\}$. Then, $\bar{\Phi}$ can be lengthened to an ∞ -morphism up to arity $n + 1$.*

Proof. For convenience, we omit the variables. The identity,

$$\bar{Q} \circ (\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi}) + (\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi}) \circ Q_{\mathfrak{g}} = 0$$

taken in arity $n + 1$ yields,

$$0 = (\bar{Q} \circ (\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi}))^{(n+1)} = [Q, (\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi})^{(n+1)}],$$

since $Q_{\mathfrak{g}}^{(0)} = 0$ and $(\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi})^{(i)} = 0$ for $i \in \{0, \dots, n\}$. It is clear that for all $n \geq 0$ the map $(\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi})^{(n+1)}: S_{\mathbb{K}}^{n+2} \mathfrak{g} \rightarrow \mathfrak{X}_{-n}(E)[1]$ take value in vertical vector fields on E . By virtue of Lemma 2.13 there exists a vector field $\zeta \in \mathfrak{X}_{-n-1}(E)[1]$ of degree $-n - 1$ such that

$$[Q, \bar{\Phi}^{(n+1)} + \zeta] = \bar{\Phi}^{(n)} \circ Q_{\mathfrak{g}}^{(1)} - \bar{Q}^{(1)} \circ \bar{\Phi}^{(n)}. \quad (11)$$

By redefining the arity $n + 1$ of $\bar{\Phi}$ as $\bar{\Phi}^{(n+1)} := \bar{\Phi}^{(n+1)} + \zeta$. One obtains a Lie ∞ -morphism up to arity $n + 1$. The proof continues by recursion. □

Proof of Theorem 2.2. Let us show Item 1. Note that Lemma 2.14 gives the existence of a linear map $\Phi_0: \mathfrak{g} \rightarrow (\mathfrak{X}_0(E)[1], [\cdot, \cdot], \text{ad}_Q)$ such that, $[Q, \Phi_0(x)] = 0$ for all $x \in \mathfrak{g}$. For $x, y \in \mathfrak{g}$, consider

$$\Lambda(x, y) = \Phi_0([x, y]_{\mathfrak{g}}) - [\Phi_0(x), \Phi_0(y)]. \quad (12)$$

Since $\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] \in \mathcal{F}$ for all $x, y \in \mathfrak{g}$, and since $\rho: \Gamma(E_{-1}) \rightarrow \mathcal{F}$ surjective, we have $\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] = \rho(\eta(x, y))$ for some element $\eta(x, y) \in \Gamma(E_{-1})$ depending linearly on x and y . Now we consider the vertical vector field of degree -1 , $\iota_{\eta(x, y)} \in \mathfrak{X}_{-1}(\mathbb{U}^{\mathcal{F}})$ which is defined on $\Gamma(E^*)$ as:

$$\iota_{\eta(x, y)}(\alpha) := \langle \alpha, \eta(x, y) \rangle \text{ for all } \alpha \in \Gamma(E^*),$$

and extended it by derivation on the whole space. For every $f \in \mathcal{O}$,

$$\begin{aligned} (\Lambda(x, y) - [Q, \iota_{\eta(x, y)}])(f) &= (\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] - \rho(\eta(x, y)))(f) && \text{(by definition of } \Phi_0) \\ &= 0 && \text{(by definition of } \eta) \end{aligned}$$

It is clear that $\Lambda(x, y) + [Q, \iota_{\eta(x, y)}]$ is a ad_Q -cocycle, hence by Corollary 2.13 it is of the form $[Q, \Upsilon(x, y)]$ for some vertical vector field $\Upsilon(x, y) \in \mathfrak{X}_{-1}(E)$ of degree -1 . For all $x, y \in \mathfrak{g}$, we define the Taylor coefficient $\Phi_1: \wedge^2 \mathfrak{g} \rightarrow \mathfrak{X}(E)$ as $\Phi_1(x, y) := \Upsilon(x, y) + \iota_{\eta(x, y)}$. By construction, we have the following relation,

$$\Phi_0([x, y]_{\mathfrak{g}}) - [\Phi_0(x), \Phi_0(y)] = [Q, \Phi_1(x, y)], \quad \forall x, y \in \mathfrak{g}. \quad (13)$$

Consider for $x, y, z \in \mathfrak{g}$,

$$\vartheta(x, y, z) = \Phi_1([x, y]_{\mathfrak{g}}, z) - [\Phi_0(x), \Phi_1(y, z)] + \circlearrowleft(x, y, z). \quad (14)$$

Here $\circlearrowleft(x, y, z)$ stands for circular permutation of x, y and z . For degree reason $\vartheta(x, y, z)$ is \mathcal{O} -linear. Moreover, $\vartheta(x, y, z)$ is a ad_Q -cocycle:

$$\begin{aligned} [Q, \Phi_1([x, y]_{\mathfrak{g}}, z)] + \circlearrowleft(x, y, z) &= -[\Phi_0([x, y]_{\mathfrak{g}}), \Phi_0(z)] + \circlearrowleft(x, y, z) \\ &= [[\Phi_0(z), Q], \Phi_1(x, y)] - [[\Phi_1(x, y), \Phi_0(z)], Q] + \circlearrowleft(x, y, z) \\ &= [Q, [\Phi_0(x), \Phi_1(y, z)]] + \circlearrowleft(x, y, z). \end{aligned}$$

Where we have used the fact that $[Q, \Phi_0(x)] = 0$ for all $x \in \mathfrak{g}$, and the Jacobi identity for the Lie brackets $[\cdot, \cdot]_{\mathfrak{g}}$ and $[\cdot, \cdot]$. By Corollary 2.13, there exists a derivation of degree -2 denoted by $\Phi_2(x, y, z) \in \mathfrak{X}_{-2}(E)[1]$ that satisfies,

$$\vartheta(x, y, z) = [Q, \Phi_2(x, y, z)]. \quad (15)$$

So far, in the construction of the Lie ∞ -morphism, we have shown the existence of a Lie ∞ -morphism $\bar{\Phi}: S_{\mathbb{K}}^{\bullet} \mathfrak{g} \rightarrow S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1])$ up to arity 2 that is $(\bar{\Phi} \circ Q_{\mathfrak{g}})^{(i)} = (\bar{Q} \circ \bar{\Phi})^{(i)}$ with $i = 0, 1, 2$. The proof continues by recursion or by applying directly Lemma 2.15. This proves the part 1. of the theorem. \square

Before proving item 3 of Theorem 2.2 we will need the following lemma. For convenience, we sometimes omit the variables in \mathfrak{g} .

Lemma 2.16. *For any two Lie ∞ -morphisms $\Gamma, \Omega: (S_{\mathbb{K}}^{\bullet} \mathfrak{g}, Q_{\mathfrak{g}}) \rightarrow (S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1]), \bar{Q})$ which coincide up to arity $n \geq 1$, i.e. $\Gamma^{(i)} = \Omega^{(i)}$, for $0 \leq i \leq n$, their difference in arity $n + 1$, namely,*

$$\Gamma^{(n+1)} - \Omega^{(n+1)}: S_{\mathbb{K}}^{n+2} \mathfrak{g} \rightarrow \mathfrak{X}_{-n-1}(E)[1]$$

is a ad_Q -coboundary.

Proof. Indeed, a direct computation yields

$$\begin{aligned} \bar{Q} \circ (\Gamma - \Omega) &= (\Gamma - \Omega) \circ Q_{\mathfrak{g}} \implies \bar{Q}^{(0)} \circ (\Gamma - \Omega)^{(n+1)} - \underbrace{((\Gamma - \Omega) \circ Q_{\mathfrak{g}})^{(n+1)}}_{=0} = 0 \\ &\implies [Q, \Gamma^{(n+1)} - \Omega^{(n+1)}] = 0 \\ &\implies \Gamma^{(n+1)} - \Omega^{(n+1)} = [Q, H^{(n+1)}] \quad (\text{by item 1 of Theorem 2.12}) \end{aligned}$$

for some linear map $H^{(n+1)}: S_{\mathbb{K}}^{n+2}\mathfrak{g} \longrightarrow \mathfrak{X}_{-n-2}(E)[1]$. \square

Let us show item 2 of Theorem 2.2. Let $\Phi, \Psi: \mathfrak{g} \longrightarrow \mathfrak{X}(E)[1]$ be two different lifts of the action $\mathfrak{g} \longrightarrow \mathfrak{X}(M)$. We denote by $\bar{\Phi}, \bar{\Psi}: S_{\mathbb{K}}^{\bullet}\mathfrak{g} \longrightarrow S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1])$ the unique comorphisms given respectively by the Taylor's coefficients

$$\begin{cases} \bar{\Phi}^{(r)}: S_{\mathbb{K}}^{r+1}\mathfrak{g} \xrightarrow{\Phi_r} \mathfrak{X}_{-r}(E)[1] \\ \bar{\Psi}^{(r)}: S_{\mathbb{K}}^{r+1}\mathfrak{g} \xrightarrow{\Psi_r} \mathfrak{X}_{-r}(E)[1] \end{cases}, \text{ for } 0 \leq r \leq \dim \mathfrak{g}. \quad (16)$$

For any $x \in \mathfrak{g}$, the degree zero vector field $\Phi_0(x) - \Psi_0(x) \in \mathfrak{X}_0(E)$ is vertical. Moreover we have, $[Q, \Phi_0(x) - \Psi_0(x)] = 0$. By Corollary 2.13 there exists a vector field $H_0 \in \mathfrak{X}_{-1}(E)$ of degree -1 , such that $\Psi_0(x) - \Phi_0(x) = [Q, H_0(x)]$

$$\begin{array}{ccc} & & \mathfrak{g} \\ & & \downarrow \Psi_0 - \Phi_0 \\ & H_0 \swarrow & \\ \mathfrak{X}_{-1}(E)[1] & \xrightarrow{\text{ad}_Q} & \mathfrak{X}_0(E)[1] \end{array} \quad (17)$$

Consider the following differential equation

$$\begin{cases} \frac{d\Xi_t}{dt} = \bar{Q} \circ H_t + H_t \circ Q_{\mathfrak{g}}, & t \in [0, 1] \\ \Xi_0 = \bar{\Phi} \end{cases} \quad (18)$$

where $(\Xi_t)_{t \in [0, 1]}$ is as in Definition B.7, and for $t \in [0, 1]$, H_t is the unique Ξ_t -coderivation where the only non-zero arity is $H^{(0)} = H_0$. Equation (18) gives a homotopy between $\bar{\Phi}$ and Ξ_1 . When we consider the arity zero component in Equation (18), one obtains

$$\begin{aligned} \frac{d\Xi_t^{(0)}}{dt} &= \bar{Q}^{(0)} \circ H_t^{(0)} + H_t^{(0)} \circ Q_{\mathfrak{g}}^{(0)} \\ &= [Q, H_0] \\ &= \Psi_0 - \Phi_0 = \bar{\Psi}^{(0)} - \bar{\Phi}^{(0)}. \end{aligned}$$

Therefore, $\Xi_t^{(0)} = \bar{\Phi}^{(0)} + t(\bar{\Psi}^{(0)} - \bar{\Phi}^{(0)})$, and $\bar{\Phi} \sim \Xi_1$ with $\bar{\Psi}^{(0)} = \Xi_1^{(0)}$. Using Lemma 2.16, the image of any element through the map $\bar{\Psi}^{(1)} - \Xi_1^{(1)}: S_{\mathbb{K}}^2\mathfrak{g} \longrightarrow \mathfrak{X}_{-1}(E)[1]$ is a ad_Q -coboundary. Thus, $\bar{\Psi}^{(1)} - \Xi_1^{(1)}$ can be written as

$$\bar{\Psi}^{(1)} - \Xi_1^{(1)} = [Q, H^{(1)}], \quad \text{with } H^{(1)}: S_{\mathbb{K}}^2\mathfrak{g} \longrightarrow \mathfrak{X}_{-2}(E)[1]. \quad (19)$$

Let us go one step further by considering the differential equation on $[0, 1]$ given by

$$\begin{cases} \frac{d\Theta_t}{dt} = \bar{Q} \circ H_t + H_t \circ Q_{\mathfrak{g}} \\ \Theta_0 = \bar{\Xi}_1 \end{cases} \quad (20)$$

Here H_t is the extension of $H^{(1)}$ as the unique Θ_t -coderivation where all its arities vanish except the arity 1 which is given by $H^{(1)}$. In arity zero, $(\Theta_t^{(0)})_{t \in [0,1]}$ is constant and has value $\Theta_1^{(0)} = \bar{\Psi}^{(0)}$. In arity one we have,

$$\begin{aligned} \frac{d\Theta_t^{(1)}}{dt} &= \bar{Q}^{(0)} \circ H_t^{(1)} \\ &= [Q, H^{(1)}] = \bar{\Psi}^{(1)} - \Xi_1^{(1)}. \end{aligned}$$

Hence, $\Theta_t^{(1)} = \bar{\Phi}^{(1)} + t(\bar{\Psi}^{(1)} - \Xi_1^{(1)})$ with $\bar{\Psi}^{(i)} = \Theta_1^{(i)}$ for $i = 0, 1$. We then continue this procedure by gluing all these homotopies until we reach the dimension of the Lie algebra \mathfrak{g} . We will obtain at last a Lie ∞ -morphism Ω such that $\bar{\Phi} \sim \Omega$ and $\Omega^{(i)} = \bar{\Psi}^{(i)}$ for $i = 1, \dots, \dim \mathfrak{g}$. That means $\Omega = \bar{\Psi}$, therefore $\bar{\Phi} \sim \bar{\Psi}$. This proves item 2. of Theorem 2.2.

Let us prove item 3 of Theorem 2.2. Given two equivalent weak symmetry actions ϱ, ϱ' of \mathfrak{g} on a singular foliation \mathcal{F} , i.e. ϱ, ϱ' differ by a linear map $\mathfrak{g} \rightarrow \mathfrak{X}(M)$ of the form $x \mapsto \rho(\beta(x))$ for some linear map $\beta: \mathfrak{g} \rightarrow \Gamma(E_{-1})$. Let $\Phi, \Phi': \mathfrak{g} \rightarrow (\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q)$ be a lift into a Lie ∞ -morphism of the action ϱ and ϱ' respectively. One has for all $x \in \mathfrak{g}$ and $f \in \mathcal{O}$,

$$\begin{aligned} (\Phi_0(x) - \Psi_0(x) - [Q, \iota_{\varphi(x)}])(f) &= \rho(\varphi(x))[f] - \langle Q(f), \varphi(x) \rangle \\ &= 0. \end{aligned}$$

Since $[Q, \Phi_0(x) - \Psi_0(x) - [Q, \iota_{\varphi(x)}]] = 0$, by Corollary 2.13 there exists a vertical derivation $\hat{H}(x) \in \mathfrak{X}_{-1}(E)$ of degree -1 depending linearly on $x \in \mathfrak{g}$ such that

$$\Phi_0(x) - \Psi_0(x) = [Q, \hat{H}(x) + \iota_{\varphi(x)}].$$

Let $H(x) := \hat{H}(x) + \iota_{\varphi(x)}$, for $x \in \mathfrak{g}$. The proof continues the same as for item 2 of Theorem 2.2

2.3. Particular examples. We recall that for a regular foliation \mathcal{F} on a manifold M , the Lie algebroid $TF \subset TM$, whose sections form \mathcal{F} , is a universal Lie ∞ -algebroid of \mathcal{F} . Its corresponding Q -manifold is given by the leafwise De Rham differential on $\Gamma(\wedge^\bullet T^*F)$.

Example 2.17. Let \mathcal{F} be a regular foliation on a manifold M . Any weak symmetry action $\mathfrak{g} \rightarrow \mathfrak{X}(M)$, $x \mapsto \varrho(x)$, of \mathcal{F} , can be lifted up to Lie ∞ -morphism $\Phi: \mathfrak{g} \rightarrow (\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q)$ that satisfies $[\text{d}_{\text{dR}}, \Phi] = 0$, given explicitly as follows:

$$x \in \mathfrak{g} \mapsto \Phi_0(x) = \mathcal{L}_{\varrho(x)} \in \mathfrak{X}_0(\wedge^\bullet T^*F) \quad (21)$$

$$x \wedge y \in \wedge^2 \mathfrak{g} \mapsto \Phi_1(x, y) = \iota_{\chi(x, y)} \in \mathfrak{X}_{-1}(\wedge^\bullet T^*F) \quad (22)$$

and $(\Phi_i: \wedge^{i+1} \mathfrak{g} \rightarrow \mathfrak{X}_{-i}(\wedge^\bullet T^*F)) \equiv 0$, for all $i \geq 2$, where $\chi(x, y) := \varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)]$ for $x, y \in \mathfrak{g}$. Also, \mathcal{L}_X stands for the Lie derivative on multi-forms w.r.t $X \in \mathfrak{X}(M)$, and ι_X is the internal product.

Example 2.18. Let \mathcal{F} be a singular foliation on a manifold M together with a strict symmetry action $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ such that $\underline{\mathfrak{g}} \subset \mathcal{F}$. Hence, $C^\infty(M)\underline{\mathfrak{g}}$ is a singular foliation which is the image of the transformation Lie algebroid $\mathfrak{g} \times M$. The universality theorem (see [18, 19]) provides the existence of a Lie ∞ -morphism $\nu: \mathfrak{g} \rightarrow \mathbb{U}^{\mathcal{F}}$. Let us call its Taylor coefficients $\nu_n: \wedge^{n+1} \mathfrak{g} \rightarrow E_{-n-1}$, $n \geq 0$. We may take for example the 0-th and 1-th Taylor coefficients

of a Lie ∞ -morphism that lifts ϱ as:

$$\begin{aligned}\Phi_0(x) &:= [Q, \iota_{\nu_0(x)}] \in \mathfrak{X}_0(\mathbb{U}^{\mathcal{F}}), \text{ for } x \in \mathfrak{g}. \\ \Phi_1(x, y) &:= [Q, \iota_{\nu_1(x, y)}]^{(-1)} - \sum_{k \geq 0} [[Q, \iota_{\nu_0(x)}], \iota_{\nu_0(y)}]^{(k)} \in \mathfrak{X}_{-1}(\mathbb{U}^{\mathcal{F}}), \text{ for } x, y \in \mathfrak{g}.\end{aligned}$$

Note that in this case the action ϱ is equivalent to zero, therefore by item 3 of Theorem 2.2 the Lie ∞ -morphism Φ is homotopic to zero.

3. LIFTS OF STRICT SYMMETRY ACTIONS AND LIE ∞ -ALGEBROIDS

In this Section, we consider the finite dimensional Lie algebra \mathfrak{g} as the trivial vector bundle over M with fiber \mathfrak{g} .

The following theorem says that any lift of strict symmetry action of \mathfrak{g} on a singular foliation \mathcal{F} induces a Lie ∞ -algebroids with some special properties and vice versa.

Proposition 3.1. *Let (E, Q) be a Lie ∞ -algebroid over a singular foliation \mathcal{F} . Any Lie ∞ -morphism $\Phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \rightarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ induces a Lie ∞ -algebroid $(E \oplus \mathfrak{g}, Q')$ with*

$$Q' := d^{CE} + Q + \sum_{k \geq 1, i_1, \dots, i_k = 1, \dots, \dim(\mathfrak{g})} \frac{1}{k!} \xi^{i_1} \odot \dots \odot \xi^{i_k} \Phi_{k-1}(\xi_{i_1}, \dots, \xi_{i_k}), \quad (23)$$

where d^{CE} is the Chevalley-Eilenberg complex of \mathfrak{g} , $\xi^1, \dots, \xi^{\dim(\mathfrak{g})} \in \mathfrak{g}^*$ is the dual basis of some basis $\xi_1, \dots, \xi_{\dim(\mathfrak{g})} \in \mathfrak{g}$ and for all $k \geq 0$, $\Phi_k: S^{k+1}\mathfrak{g} \rightarrow \mathfrak{X}_{-k}(E)[1]$ is the k -th Taylor coefficients of Φ .

In the dual point of view, (23) corresponds to a Lie ∞ -algebroid over the complex

$$\dots \xrightarrow{\ell_1} E_{-3} \xrightarrow{\ell_1} E_{-2} \xrightarrow{\ell_1} \mathfrak{g} \oplus E_{-1} \xrightarrow{\rho'} TM \quad (24)$$

whose brackets satisfy

- (1) the anchor map ρ' sends an element $x \oplus e \in \mathfrak{g} \oplus E_{-1}$ to $\varrho(x) + \rho(e) \in \varrho(\mathfrak{g}) + T\mathcal{F}$,
- (2) the binary bracket satisfies

$$\ell_2(\Gamma(E_{-1}), \Gamma(E_{-1})) \subset \Gamma(E_{-1}) \quad \text{and} \quad \ell_2(\Gamma(E_{-1}), x) \subset \Gamma(E_{-1}), \quad \forall x \in \mathfrak{g}$$

- (3) the \mathfrak{g} -component of the binary bracket on constant sections of $\mathfrak{g} \times M$ is the Lie bracket of \mathfrak{g} .

Conversely, if there exists a Lie ∞ -algebroid (E', Q') whose underlying complex of vector bundles is of the form (24) and that satisfies item (1), (2) and (3), then there is Lie ∞ -morphism $\Phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \rightarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ which is defined on a given basis ξ_1, \dots, ξ_d of \mathfrak{g} by:

$$\Phi_{k-1}(\xi_{i_1}, \dots, \xi_{i_k}) = \text{pr} \circ [\dots [[Q, \iota_{\xi_{i_1}}], \iota_{\xi_{i_2}}], \dots, \iota_{\xi_{i_k}}] \in \mathfrak{X}(E)[1], \quad k \in \mathbb{N}, \quad (25)$$

where pr stands for the projection map $\mathfrak{X}(E')[1] \rightarrow \mathfrak{X}(E)[1]$.

Remark 3.2. Note that every Lie ∞ -morphism $\Phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \rightarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ induces a weak symmetry action of \mathfrak{g} on \mathcal{F} that maps $x \in \mathfrak{g}$ to the base vector field of $\Phi_0(x) \in \mathfrak{X}_0(E)$ on M .

Proof. A direct computation gives the first implication. Conversely, let us denote by Q' the homological vector fields of Lie ∞ -algebroid whose underlying complex of vector bundles is of the form (24). The map defined in Equation (25) is indeed a lift into a Lie ∞ -morphism of the weak symmetry action ϱ :

- It is not difficult to check that of any $\xi \in \mathfrak{g}$, one has $[Q, \Phi_0(\xi)] = 0$.
- The fact that Φ defines a Lie ∞ -morphism can be found using Voronov trick [24], i.e, doing Jacobi's identity inside the null derivation

$$0 = \text{pr} \circ [\cdots [[Q', Q'], \iota_{\xi_{i_1}}, \iota_{\xi_{i_2}}, \dots, \iota_{\xi_{i_k}}]. \quad (26)$$

A direct computation of Equation (26) falls exactly on the requirements of Definition B.3.

Let us compute Equation (26) for a small number of generators (e.g $k = 2, 3$) in order to show how it works: from the identity

$$\left[\left[[Q', Q'], \iota_{\xi_{i_1}} \right], \iota_{\xi_{i_2}} \right] = 0,$$

one obtains by using twice the Jacobi identity the following relation,

$$[Q', [[Q', \xi_{i_1}], \xi_{i_2}]] - [[Q', \xi_{i_1}], [Q', \xi_{i_2}]] = 0. \quad (27)$$

One should notice that $[[Q', \xi_{i_1}], \xi_{i_2}]$ splits into two parts. One part where the Chevalley Eilenberg acts to give $[[d^{\text{CE}}, \xi_{i_1}], \xi_{i_2}] = \iota_{[\xi_{i_1}, \xi_{i_2}]_{\mathfrak{g}}}$, while the other part is $[[Q' - d^{\text{CE}}, \xi_{i_1}], \xi_{i_2}]$. Hence, by putting them in Equation (27), afterwards projecting on $\mathfrak{X}(S^\bullet(E^*))$, we get

$$\text{pr} \circ [Q', \iota_{[\xi_{i_1}, \xi_{i_2}]_{\mathfrak{g}}}] + \text{pr} \circ [Q', [[Q' - d^{\text{CE}}, \xi_{i_1}], \xi_{i_2}]] - \text{pr} \circ [[Q', \xi_{i_1}], [Q', \xi_{i_2}]] = 0.$$

From there we deduce that

$$\Phi_0([\xi_{i_1}, \xi_{i_2}]_{\mathfrak{g}}) = [Q, \Phi_1(\xi_{i_1}, \xi_{i_2})] + [\Phi_0(\xi_{i_1}), \Phi_0(\xi_{i_2})].$$

□

Notice that Proposition 3.1 assumes the Lie ∞ -algebroid over \mathcal{F} exists, not necessarily a universal one (i.e. Lie ∞ -algebroid that is built on a geometric resolution) which always exists by [18, 19]. In fact we can be more general, we do not need a geometric resolution. The following Theorem states that given a weak symmetry action of a Lie algebra \mathfrak{g} on singular foliation \mathcal{F} and a universal Lie ∞ -algebroid of \mathcal{F} seen as a Lie-Rinehart algebra (i.e. Lie ∞ -algebroid that is built on a free resolution of \mathcal{F} , that is, resolutions that do not need to be geometric) there always exists a Lie ∞ -algebroid whose underlying complex of vector bundles is of the form (24) and that satisfies item (1), (2) and (3) of Proposition 3.1.

Theorem 3.3. *Let $\varrho: \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ be a weak symmetry action on a singular foliation \mathcal{F} . Let $((\mathcal{K}_{-i})_{i \geq 1}, d, \rho)$ be a free resolution of the singular foliation \mathcal{F} over M . The complex of trivial vector bundles over M*

$$\cdots \xrightarrow{d} E_{-3} \xrightarrow{d} E_{-2} \xrightarrow{d} \mathfrak{g} \oplus E_{-1} \xrightarrow{\rho'} TM \quad (28)$$

where $\Gamma(E_{-1}) = \mathcal{K}_{-i}$, comes equipped with a Lie ∞ -algebroid structure

- (1) whose unary bracket is d and whose anchor map ρ' , sends an element $x \oplus e \in \mathfrak{g} \oplus E_{-1}$ to $\varrho(x) + \rho(e) \in \varrho(\mathfrak{g}) + T\mathcal{F}$,
- (2) the binary bracket satisfies

$$\ell_2(\Gamma(E_{-1}), \Gamma(E_{-1})) \subset \Gamma(E_{-1}) \quad \text{and} \quad \ell_2(\Gamma(E_{-1}), \Gamma(\mathfrak{g})) \subset \Gamma(E_{-1}),$$

- (3) the \mathfrak{g} -component of the binary bracket on constant sections of $\mathfrak{g} \times M$ is the Lie bracket of \mathfrak{g} .

For a proof see Appendix C.

Remark 3.4. When we have $\varrho(\mathfrak{g}) \cap T_m \mathcal{F} = 0$ for all m in M , the Equation (28) is free resolution of the singular foliation $C^\infty(M)\varrho(\mathfrak{g}) + \mathcal{F}$ and we can apply directly the Theorem 2.1 in [19]. Otherwise, we need to show there is no obstruction in degree -1 while doing the construction of the brackets if the result still needs to hold.

4. ON WEAK AND STRICT SYMMETRIES: AN OBSTRUCTION THEORY

In this section we apply theorems in Section 2 to define a class obstructing the existence of strict symmetry action equivalent to a given weak symmetry action.

Let us start with some generalities. Assume we are given

- a Lie algebra \mathfrak{g} .
- a Lie ∞ -algebroid (E, Q_E) over M , with anchor map $\rho: E_{-1} \rightarrow TM$,
- included as a sub-Lie ∞ -algebroid in a Lie algebroid (E', Q) over M , whose underlying complex is, $E'_{-1} := \mathfrak{g} \oplus E_{-1}$, and for any $i \geq 2$, $E'_{-i} = E_{-i}$, namely

$$\cdots \xrightarrow{d} E_{-3} \xrightarrow{d} E_{-2} \xrightarrow{d} \mathfrak{g} \oplus E_{-1} \xrightarrow{\rho'} TM. \quad (29)$$

We also assume that

$$\ell'_2(x \oplus 0, y \oplus 0) = [x, y]_{\mathfrak{g}} \oplus \eta(x, y)$$

for all $x, y \in \mathfrak{g}$, for some $\eta: \wedge^2 \mathfrak{g} \rightarrow E_{-1}$, and

$$\ell'_2(x, \Gamma(E_{-1})) \subset \Gamma(E_{-1})$$

for all $x \in \mathfrak{g}$.

In particular, this induces a weak symmetry action ϱ of the Lie algebra \mathfrak{g} on the singular foliation $\mathcal{F} := \rho(\Gamma(E_{-1}))$ which is given by $x \in \mathfrak{g} \mapsto \varrho(x) := \rho'(x)$ and satisfies for all $x, y \in \mathfrak{g}$

$$\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] = \rho(\eta(x, y)). \quad (30)$$

Lemma 4.1. *Let $m \in M$. Assume that the underlying complex (E, ℓ_1) is minimal at a point m , i.e. $\ell_1|_m = 0$. The map*

$$\nu: \mathfrak{g} \rightarrow \text{End}\left(E_{-1}|_m\right), \quad x \mapsto \ell'_2(x, \cdot)|_m$$

satisfies

- (a) $\nu[x, y]_{\mathfrak{g}} - [\nu(x), \nu(y)] + \ell_2(\cdot, \eta(x, y))|_m = 0$,
- (b) $\nu(z) (\eta(x, y)|_m) - \eta([x, y]_{\mathfrak{g}}, z)|_m + \odot(x, y, z) = 0$.

Proof. Since $\ell_1|_m = 0$, $E'_{-1}|_m$ is a Lie algebra. The Jacobi identity on elements $x, y \in \mathfrak{g}$, $e \in \Gamma(E_{-1})$, evaluated at the point m , implies that

$$\nu([x, y]_{\mathfrak{g}})(e|_m) - [\nu(x), \nu(y)](e|_m) + \ell_2(\eta(x, y), e)|_m = 0.$$

This proves item (a). Likewise, Jacobi identity on elements $x, y, z \in \mathfrak{g}$ and since $\ell_1|_m = 0$ give:

$$\begin{aligned} \ell'_2(\ell'_2(x, y), z)|_m + \odot(x, y, z) = 0 &\implies \ell'_2([x, y]_{\mathfrak{g}}, z)|_m + \ell'_2(\eta(x, y), z)|_m + \odot(x, y, z) = 0, \\ &\implies \nu(z) (\eta(x, y)|_m) - \eta([x, y]_{\mathfrak{g}}, z)|_m + \odot(x, y, z) = 0. \end{aligned}$$

Here we have used the definition of ℓ'_2 on degree -1 elements and Jacobi identity for the bracket $[\cdot, \cdot]_{\mathfrak{g}}$. This proves item (b). \square

By Lemma 4.1, E_{-1} is equipped with a \mathfrak{g} -module structure when $\eta(x, y)|_m$ is for all $x, y \in \mathfrak{g}$ valued in the center the Lie algebra $E_{-1}|_m$. The following proposition generalizes this remark.

Proposition 4.2. *Let $m \in M$ and assume that*

- *the underlying complex (E, ℓ_1) of (E, Q) is minimal at m ,*
- *for all $x, y \in \mathfrak{g}$, $\eta(x, y)|_m$ is valued in the center¹ $Z(E_{-1}|_m)$ of $E_{-1}|_m$.*

Then,

- (1) *the restriction of the 2-ary bracket*

$$\ell'_2: \mathfrak{g} \otimes Z(E_{-1}|_m) \longrightarrow Z(E_{-1}|_m)$$

endows $Z(E_{-1}|_m)$ with a \mathfrak{g} -module structure which does not depend neither on the choice of weak symmetry action ϱ nor a universal Lie ∞ -algebroid of \mathcal{F} , nor of the Lie ∞ -morphism $\Phi: \mathfrak{g} \longrightarrow \mathfrak{X}(E)$.

- (2) *the restriction of the map $\eta: \wedge^2 \mathfrak{g} \longrightarrow E_{-1}$ at m*

$$\eta|_m: \wedge^2 \mathfrak{g} \longrightarrow Z(E_{-1}|_m)$$

is a 2-cocycle for the Chevalley-Eilenberg complex of \mathfrak{g} valued in $Z(E_{-1}|_m)$,

- (3) *the cohomology class of this cocycle does not depend on the representatives of the equivalence class of ϱ ,*
- (4) *If ϱ is equivalent to a strict symmetry action, then $\eta|_m$ is exact.*

Proof. We may assume that $Z(E_{-1}|_m) = E_{-1}|_m$, i.e. $\ell_{2|m} = 0$ on $E_{-1}|_m$. The first clause of item (1) follows from item (a) of Lemma 4.1 when $\ell_{2|m} = 0$. It is easy to see that if we change the action ϱ to $\varrho + \rho \circ \beta$ for some vector bundle morphism $\beta: \mathfrak{g} \longrightarrow E_{-1}$, the new 2-ary bracket between sections of \mathfrak{g} and E_{-1} made in the proof of Theorem 3.3 is modified by $(x, e) \mapsto \ell'_2(x, e) + \ell_2(\beta(x), e)$. Therefore, under the assumption, $\ell_{2|m} = 0$, we obtain the last clause of item (1). Item (2) follows from Item (b) of Lemma 4.1 that tells that $\eta|_m: \wedge^2 \mathfrak{g} \longrightarrow E_{-1}|_m$ is a 2-cocycle for the Chevalley-Eilenberg complex of \mathfrak{g} valued in $E_{-1}|_m$.

Let ϱ' be a weak action of \mathfrak{g} on \mathcal{F} which is equivalent to ϱ , i.e. there exists a vector bundle morphism $\beta: \mathfrak{g} \longrightarrow E_{-1}$ such that $\varrho'(x) = \varrho(x) + \rho(\beta(x))$ for all $x \in \mathfrak{g}$. Let $\eta': \wedge^2 \mathfrak{g} \longrightarrow E_{-1}$ be such that $\varrho'([x, y]_{\mathfrak{g}}) - [\varrho'(x), \varrho'(y)] = \rho(\eta'(x, y))$ for all $x, y \in \mathfrak{g}$. Following the constructions in the proof of Theorem 3.3, this implies that

$$\eta'(x, y) = \eta(x, y) + \beta([x, y]_{\mathfrak{g}}) - \ell'_2(x, \beta(y)) + \ell'_2(y, \beta(x)) - \ell_2(\beta(x), \beta(y)), \quad \text{for all } x, y \in \mathfrak{g}. \quad (31)$$

Hence, if $\eta'|_m \in H^2(\mathfrak{g}, E_{-1}|_m)$ is exact, i.e. there exists a linear map $\lambda: \mathfrak{g} \longrightarrow E_{-1}|_m$ such that $d^{CE}(\lambda) = \eta'|_m$. Using Equation (31) and $\ell_{2|m} = 0$, one gets $d^{CE}(\beta|_m + \lambda) = \eta|_m$. This proves items (3) and (4). \square

Remark 4.3. When $\ell_{2|m} \neq 0$. The weak symmetry action ϱ is equivalent to strict one if the Maurer-Cartan-like equation (31) has no solution with $\eta'|_m = 0$.

Let \mathcal{F} be a singular foliation. Let us choose a universal algebroid (E, Q) such that (E, ℓ_1) is minimal at a point $m \in M$. Such a structure always exists. By Proposition 4.14 in [18] the isotropy Lie algebra \mathfrak{g}_m of the singular foliation \mathcal{F} at the point $m \in M$ is isomorphic to $\ker(\rho_m)$. The following is a direct consequence of Proposition 4.2.

Corollary 4.4. *Let $m \in M$ be a point of M . Assume that the isotropy Lie algebra \mathfrak{g}_m of \mathcal{F} at m is Abelian. Then, for any weak symmetry action ϱ of a Lie algebra action \mathfrak{g} on \mathcal{F} such that $\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] \in \mathcal{F}(m)$ for all $x, y \in \mathfrak{g}$*

- (1) *\mathfrak{g}_m is a \mathfrak{g} -module.*

¹In particular, when the 2-ary bracket ℓ_2 is zero at m , on elements of degree -1 we have, $Z(E_{-1}|_m) = E_{-1}|_m$.

- (2) The bilinear map, $\eta|_m : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}_m$, is 2-cocycle for the Chevalley-Eilenberg complex of \mathfrak{g} valued in \mathfrak{g}_m .
- (3) Its class $\text{cl}(\eta) \in H^2(\mathfrak{g}, \mathfrak{g}_m)$ does not depend on the choices made in the construction.
- (4) Furthermore, $\text{cl}(\eta)$ is an obstruction of having a strict symmetry action equivalent to ϱ .

Example 4.5. We return to Example 1.7 with $m \in M$ a leaf of \mathcal{F} . Since the isotropy Lie algebra \mathfrak{g}_m^k is Abelian for every $k \geq 2$ the following assertions hold by Corollary 4.4:

- (1) For each $k \geq 1$, the vector space \mathfrak{g}_m^{k+1} is a \mathfrak{g}_m^k -module.
- (2) The obstruction of having a strict symmetry action equivalent to ϱ_k is a Chevalley-Eilenberg cocycle valued in \mathfrak{g}_m^{k+1} .

Example 4.6. Let $\mathcal{F} := \mathcal{I}_0^3 \mathfrak{X}(\mathbb{R}^n)$ be the singular foliation generated by vector fields vanishing to order 3 at the origin. The quotient $\mathfrak{g} := \frac{\mathcal{I}_0^2 \mathfrak{X}(\mathbb{R}^n)}{\mathcal{I}_0^3 \mathfrak{X}(\mathbb{R}^n)}$ is a trivial Lie algebra. There is a weak symmetry action of \mathfrak{g} on \mathcal{F} which assigns to an element in \mathfrak{g} a representative in $\mathcal{I}_0^2 \mathfrak{X}(\mathbb{R}^n)$. In this case, the isotropy Lie algebra of \mathcal{F} at zero is Abelian and $\ell'_2(\mathfrak{g}, \mathfrak{g}_0)|_0 = 0$. Thus, the action of \mathfrak{g} on \mathfrak{g}_0 is trivial. One can choose $\eta: \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}_0$ such that $\eta\left(\overline{x_i^2 \frac{\partial}{\partial x_i}}, \overline{x_i^2 \frac{\partial}{\partial x_j}}\right) = 2e_{ij}$, with e_{ij} a constant section in a set of generators of degree -1 whose image by the anchor is $x_i^3 \frac{\partial}{\partial x_j}$. Therefore, $\eta|_0\left(\overline{x_i^2 \frac{\partial}{\partial x_i}}, \overline{x_i^2 \frac{\partial}{\partial x_j}}\right) \neq 0$. This implies that the class of η is not zero at the origin. Therefore, by item 2 of Corollary 4.4 the weak symmetry action of \mathfrak{g} on \mathcal{F} is not equivalent to a strict one.

Also, we have the following consequence of Corollary 4.4 for Lie algebra actions on affine varieties, as in Example 1.11. Before going to Corollary 4.10 let us write definitions and some facts.

Settings: Let W be an affine variety realized as a subvariety of \mathbb{C}^d , and defined by some ideal $\mathcal{I}_W \subset \mathbb{C}[x_1, \dots, x_d]$. We denote by $\mathfrak{X}(W) := \text{Der}(\mathcal{O}_W)$ the Lie algebra of vector fields on W , where \mathcal{O}_W is coordinates ring of W .

Definition 4.7. A point $p \in W$ is said to be *strongly singular* if for all $f \in \mathcal{I}_W$, $d_p f \equiv 0$ or equivalently if for all $f \in \mathcal{I}_W$ and $X \in \mathfrak{X}(\mathbb{C}^d)$, one has $X[f](p) \in \mathcal{I}_p$.

Example 4.8. Any singular point of a hypersurface W defined by a polynomial $\varphi \in \mathbb{C}[x_1, \dots, x_d]$ is strongly singular.

The lemma below is immediate.

Lemma 4.9. *In a strongly singular point, the isotropy Lie algebra of the singular foliation $\mathcal{F} = \mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$ is Abelian.*

Corollary 4.10. *Let $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(W)$ be a Lie algebra morphism.*

- (1) Any extension $\tilde{\varrho}$ as in Example 1.11 is a weak symmetry action for the singular foliation $\mathcal{F} = \mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$.
- (2) For any strongly singular point p in W if the class $\text{cl}(\eta)$ does not vanish the strict action $\varrho: \mathfrak{g} \rightarrow \text{Der}(\mathcal{O}_W)$ can not be extended to the ambient space.

Let us give an examples of Lie algebra actions on an affine variety that do not extend to the ambient space.

Example 4.11. Let $W \subset \mathbb{C}^2$ be the affine variety generated by the polynomial $\varphi = FG$ with $F, G \in \mathbb{C}[x, y] =: \mathcal{O}$. We consider the vector fields $U = F\mathcal{X}_G, V = G\mathcal{X}_F \in \mathfrak{X}(\mathbb{C}^2)$, where \mathcal{X}_F

and \mathcal{X}_G are Hamiltonian vector fields w.r.t the Poisson structure $\{x, y\} := 1$. Note that U, V are tangent to W , i.e. $U[\varphi], V[\varphi] \in \langle \varphi \rangle$. It is easily checked that $[U, V] = \varphi \mathcal{X}_{\{F, G\}}$.

The action of the trivial Lie algebra $\mathfrak{g} = \mathbb{R}^2$ on W that sends its canonical basis (e_1, e_2) to U , and V respectively, is a weak symmetry action on the singular foliation $\mathcal{F}^\varphi := \langle \varphi \rangle \mathfrak{X}(\mathbb{C}^2)$, and induces a Lie algebra map,

$$\varrho: \mathfrak{g} \longrightarrow \mathfrak{X}(W). \quad (32)$$

A universal Lie algebroid of \mathcal{F}^φ is a Lie algebroid (see Example 3.19 of [19]) because,

$$0 \longrightarrow \mathcal{O}\mu \otimes_{\mathcal{O}} \mathfrak{X}(\mathbb{C}^2) \xrightarrow{\varphi \frac{\partial}{\partial \mu} \otimes_{\mathcal{O}} \text{id}} \mathcal{F}^\varphi$$

is a \mathcal{O} -module isomorphism. Here μ is a degree -1 variable, so that $\mu^2 = 0$. The universal algebroid structure over that resolution is given on the set of generators by:

$$\ell_2 \left(\mu \otimes_{\mathcal{O}} \frac{\partial}{\partial x}, \mu \otimes_{\mathcal{O}} \frac{\partial}{\partial y} \right) := \frac{\partial \varphi}{\partial x} \mu \otimes_{\mathcal{O}} \frac{\partial}{\partial y} - \frac{\partial \varphi}{\partial y} \mu \otimes_{\mathcal{O}} \frac{\partial}{\partial x} \quad (33)$$

and $\ell_k := 0$ for every $k \geq 3$. Write $\mathcal{X}_{\{F, G\}} = \frac{\partial \{F, G\}}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \{F, G\}}{\partial x} \frac{\partial}{\partial y}$. Therefore, we can put

$$\eta(e_1, e_2) := \frac{\partial \{F, G\}}{\partial y} \mu \otimes_{\mathcal{O}} \frac{\partial}{\partial x} - \frac{\partial \{F, G\}}{\partial x} \mu \otimes_{\mathcal{O}} \frac{\partial}{\partial y}. \quad (34)$$

Take for example, $F(x, y) = y - x^2$ and $G(x, y) = y + x^2$. The isotropy Lie algebra $\mathfrak{g}_{(0,0)}$ of \mathcal{F}^φ is abelian, since zero is a strong singular point of W . By Corollary 4.4 (1), $\mathfrak{g}_{(0,0)}$ is a \mathbb{R}^2 -module. A direct computation shows that the action on $\mathfrak{g}_{(0,0)}$ is not trivial but takes value in $\mathcal{O}\mu \otimes_{\mathcal{O}} \frac{\partial}{\partial x}$. Besides, Equation (34) applied to $\{F, G\} = 4x$ gives

$$\eta(e_1, e_2) = -4\mu \otimes_{\mathcal{O}} \frac{\partial}{\partial y}. \quad (35)$$

If $\eta|_{(0,0)}$ were a coboundary of Chevalley Eilenberg, we would have (in the notations of Proposition 4.2) that

$$\eta(x, y)|_{(0,0)} = \beta([x, y]_{\mathbb{R}^2}) - \ell'_2(x, \beta(y)) + \ell'_2(y, \beta(x)) \in \mathcal{O}\mu \otimes_{\mathcal{O}} \frac{\partial}{\partial x}, \quad \text{for all } x, y \in \mathfrak{g} \quad (36)$$

for some linear map $\beta: \mathfrak{g} \longrightarrow \mathfrak{g}_{(0,0)}$. Therefore, Equation (36) is impossible by Equation (35) and since $\eta|_{(0,0)} \neq 0$. In other words, its class $\text{cl}(\eta)$ does not vanish. By Corollary 4.10 (2), the action ϱ given in Equation (32) cannot be extended to ambient space.

5. SYMMETRIES OF BI-SUBMERSIONS

The concept of bi-submersion over singular foliations has been introduced in [1] and it is used in K -theory [3] or differential geometry [5, 13, 2]. Let us recall some definitions.

Definition 5.1. Let M be a manifold endowed with a singular foliation \mathcal{F} . A *bi-submersion* over \mathcal{F} is a triple (B, s, t) where:

- B is a manifold,
- $s, t: B \rightarrow M$ are surjective submersions, respectively called *source* and *target*,

such that the pull-back singular foliations $s^{-1}\mathcal{F}$ and $t^{-1}\mathcal{F}$ are both equal to the space of vector fields of the form $\xi + \zeta$ with $\xi \in \Gamma(\ker(ds))$ and $\zeta \in \Gamma(\ker(dt))$. Namely,

$$s^{-1}\mathcal{F} = t^{-1}\mathcal{F} = \Gamma(\ker(ds)) + \Gamma(\ker(dt)). \quad (37)$$

Example 5.2. Let \mathcal{F} be a singular foliation over a manifold M . For $x \in M$ and $X_1, \dots, X_n \in \mathcal{F}$ inducing a basis of $\mathcal{F}_x := \mathcal{F}/\mathcal{I}_x\mathcal{F}$. We know from [1] that there is an open neighborhood \mathcal{W} of $(x, 0) \in M \times \mathbb{R}^n$ such that (\mathcal{W}, t, s) is a bi-submersion over \mathcal{F} , where²

$$s(x, y) = x \quad \text{and} \quad t(x, y) = \exp_x \left(\sum_{i=1}^n y_i X_i \right). \quad (38)$$

Such bi-submersions are called *path holonomy bi-submersions* [4].

5.1. Lifts of symmetries. Let (B, s, t) be a bi-submersion of a singular foliation \mathcal{F} on a manifold M .

Definition 5.3. We call *lift* of a vector field $X \in \mathfrak{X}(M)$ to the bisubmerssion (B, s, t) a vector field $\widehat{X} \in \mathfrak{X}(B)$ which is both s -projectable on X and t -projectable on X .

The coming proposition means that the notion of lift to a bisubmersion only makes sense for symmetries of the singular foliation.

Proposition 5.4. *If a vector field on M admits a lift to (B, s, t) , then it is a symmetry of \mathcal{F} .*

Proof. Let $\widehat{X} \in \mathfrak{X}(B)$ be a lift of $X \in \mathfrak{X}(M)$. Since \widehat{X} is s -projectable, $[\widehat{X}, \Gamma(\ker ds)] \subset \Gamma(\ker ds)$. Since \widehat{X} is t -projectable, $[\widehat{X}, \Gamma(\ker dt)] \subset \Gamma(\ker dt)$. Hence:

$$\begin{aligned} [\widehat{X}, s^{-1}(\mathcal{F})] &= [\widehat{X}, \Gamma(\ker ds) + \Gamma(\ker dt)] \\ &= [\widehat{X}, \Gamma(\ker ds)] + [\widehat{X}, \Gamma(\ker dt)] \\ &\subset \Gamma(\ker ds) + \Gamma(\ker dt) = s^{-1}(\mathcal{F}). \end{aligned}$$

In words, \widehat{X} is a symmetry of $s^{-1}\mathcal{F}$. Since \widehat{X} projects through s to X , X is a symmetry for \mathcal{F} . \square

In the rest of the section, we investigate on existence of lifts of symmetries of \mathcal{F} to bisubmersions over \mathcal{F} .

Remark 5.5. For a given $X \in \text{sym}(\mathcal{F})$, the lift \widehat{X} to a given bisubmersion is not canonical even when it exists. However, two different lifts of a $X \in \mathfrak{X}(M)$ to a bisubmerssion (B, s, t) differ by an element of the intersection $\Gamma(\ker ds) \cap \Gamma(\ker dt)$.

As the following example shows, the lift of a symmetry to a bi-submersion may not exist.

Example 5.6. Consider the trivial foliation $\mathcal{F} := \{0\}$ on M . For any diffeomorphism $\phi: M \rightarrow M$, (M, id, ϕ) is a bi-submersion over \mathcal{F} . Every vector field $X \in \mathfrak{X}(M)$ is a symmetry of \mathcal{F} . If it exists, its lift has to be given by $\widehat{X} = X$ since the source map is the identity. But $\widehat{X} = X$ is t -projectable if and only if X is ϕ -invariant. A non-invariant vector field X therefore admits no lift to (M, id, ϕ) .

However, internal symmetries, i.e. elements in \mathcal{F} admit lifts to any bisubmersion.

Proposition 5.7. *Let (B, s, t) be any bi-submersion of a singular foliation \mathcal{F} on a manifold M . Every internal symmetry, i.e. every vector field in \mathcal{F} , admits a lift to (B, s, t) .*

²For a vector field $Y \in \mathfrak{X}(M)$ and $x \in M$, $\exp_x(Y)$ stands for the image of x by the time-1 flow of Y .

Proof. Let $X \in \mathcal{F}$. Since $s: B \rightarrow M$ is a submersions, there exists $X^s \in \mathfrak{X}(B)$ s -projectable on X . Since t is a submersion, there exists $X^t \in \mathfrak{X}(B)$ t -projectable on X . By construction $X^s \in s^{-1}(\mathcal{F})$ and $X^t \in t^{-1}(\mathcal{F})$. Using the property (37) of the bi-submersion (B, s, t) , the vector fields X^s and X^t decompose as

$$\begin{cases} X^s = X_s^s + X_t^s & \text{with } X_s^s \in \Gamma(\ker(ds)), X_t^s \in \Gamma(\ker(dt)), \\ X^t = X_s^t + X_t^t & \text{with } X_s^t \in \Gamma(\ker(ds)), X_t^t \in \Gamma(\ker(dt)). \end{cases}$$

By construction, X_t^s is s -projectable to X and t -projectable to 0 while X_s^t is s -projectable to 0 and t -projectable to X . It follows that, $\widehat{X} := X_s^t + X_t^s$, is a lift of X to the bi-submersion (B, s, t) . □

As the following lemma shows that the existence of a lift to a bi-submersion of a symmetry of \mathcal{F} is a local property.

Lemma 5.8. *Let $X \in \text{sym}(\mathcal{F})$. If X admits local lifts to the bi-submersion B , i.e. if every $b \in B$ admits an open neighborhood $U \subset B$ on which there exists $\widehat{X}_U \in \mathfrak{X}(U)$ such that $dt(\widehat{X}) = X|_{t(U)}$ and $ds(\widehat{X}) = X|_{s(U)}$, then X admits a lift $\widehat{X} \in \mathfrak{X}(B)$ to the bisubmerssion (B, s, t) .*

Proof. By assumption, there exists partition of unity $(U_i, \chi_i)_{i \in I}$ of B such that there exists \widehat{X}_{U_i} as in the statement. The vector field $\widehat{X} = \sum_{i \in I} \chi_i \widehat{X}_{U_i}$ is both s -projectable on X and t -projectable on X . □

We can now state the main result of this section. It uses several concepts introduced in [1], which are recalled in the proof.

Proposition 5.9. *Let \mathcal{F} be a singular foliation on a manifold M . Any symmetry $X \in \text{sym}(\mathcal{F})$ admits a lift*

- (1) *to any path holonomy bi-submersion (B, s, t) ,*
- (2) *to Androulidakis-Skandalis' path holonomy atlas,*
- (3) *to a neighborhood of any point in a bisubmersion through which there exists a local bisection that induces the identity.*

Remark 5.10. In the cases 1) or 2) in Proposition 5.9, a linear lift

$$X \rightarrow \widehat{X}$$

can be defined on the whole space $\text{sym}(\mathcal{F})$ of symmetries of \mathcal{F} . As an immediate consequence of Remark 5.5, we obtain that for all $X, Y \in \text{sym}(\mathcal{F})$,

$$[\widehat{X}, \widehat{Y}] - [\widehat{X}, \widehat{Y}] \in \Gamma(\ker ds) \cap \Gamma(\ker dt). \quad (39)$$

Proof of Proposition 5.9. Let $X \in \text{sym}(\mathcal{F})$. Asume that $(B, s, t) = (\mathcal{W}, s_0, t_0)$ is a path holonomy bi-submersion associated to some generators $X_1, \dots, X_n \in \mathcal{F}$. Fix $(y = (y_1, \dots, y_n), u) \in \mathcal{W} \subset \mathbb{R}^n \times M$, set $Y := \sum_{i=1}^n y_i X_i$. Since $d\varphi_Y^1(X) = \exp(\text{ad}_Y)(X) \in X + \mathcal{F}$, there exists $Z_y \in \mathcal{F}$ depending smoothly on y such that $dt_0(0, X) = X + Z_y$. Take $\widetilde{Z}_y \in t_0^{-1}(\mathcal{F})$ such that $dt_0(\widetilde{Z}_y) = Z_y$. One has,

$$dt_0 \left((0, X) - \widetilde{Z}_y \right) = ds_0(0, X) = X.$$

We can write $\widetilde{Z}_y = \widetilde{Z}_y^1 + \widetilde{Z}_y^2$, with $\widetilde{Z}_y^1 \in \Gamma(\ker ds_0)$, $\widetilde{Z}_y^2 \in \Gamma(\ker dt_0)$. Hence, $\widehat{X} := (-\widetilde{Z}_y^1, X)$ is a lift to the bi-submersion (\mathcal{W}, s_0, t_0) . This proves item 1.

If $X_a \in \mathfrak{X}(B)$ and $X_b \in \mathfrak{X}(B')$ are two lifts of the symmetry X on the path holonomy bi-submersions (B, s, t) and (B', s', t') respectively, then (X_a, X_b) is a lift of X on the composition bi-submersion $B \circ B'$. This proves item 2, since the path holonomy atlas is made of fibered products of holonomy path holonomy bi-submersions.

Item 2 in Proposition 2.10 of [1] states that if the identity of M is carried by (B, s, t) at some point $v \in B$ then there exists an open neighbourhood $V \subset B$ of v that satisfies $s|_V = s_0 \circ g$ and $t|_V = t_0 \circ g$, for some submersion $g: V \rightarrow \mathcal{W}$. Thus, for all $X \in \text{sym}(\mathcal{F})$ there exists a vector field $\widehat{X} \in \mathfrak{X}(V)$ fulfilling $ds|_V(\widehat{X}) = dt|_V(\widehat{X}) = X$. This proves item 3. \square

5.2. Tower of bi-submersions and symmetries. We end this paper by considering towers of bi-submersions. The work contained in this section is entirely original, except for the notion below that arose in a discussion between C. Laurent-Gengoux, L. Ryvkin, and myself, and will be the object of a separate study.

Definition 5.11. We call *tower of bi-submersion over a singular foliation \mathcal{F} on M* , a (finite or infinite) sequence of manifolds and maps as follows

$$\mathcal{T}_B : \cdots \begin{array}{c} \xrightarrow{s_{i+1}} \\ \xleftarrow{t_{i+1}} \end{array} B_{i+1} \begin{array}{c} \xrightarrow{s_i} \\ \xleftarrow{t_i} \end{array} B_i \begin{array}{c} \xrightarrow{s_{i-1}} \\ \xleftarrow{t_{i-1}} \end{array} \cdots \begin{array}{c} \xrightarrow{s_1} \\ \xleftarrow{t_1} \end{array} B_1 \begin{array}{c} \xrightarrow{s_0} \\ \xleftarrow{t_0} \end{array} B_0, \quad (40)$$

together with a sequence \mathcal{F}_i of singular foliations on B_i , with the convention that $B_0 = M$ and $\mathcal{F}_0 = \mathcal{F}$, such that

- for all $i \geq 1$, $\mathcal{F}_i \subset \Gamma(\ker ds_{i-1}) \cap \Gamma(\ker dt_{i-1})$,
- for each $i \geq 1$, $B_{i+1} \begin{array}{c} \xrightarrow{s_i} \\ \xleftarrow{t_i} \end{array} B_i$ is a bi-submersion over \mathcal{F}_i .

Tower of bi-submersions over (M, \mathcal{F}) shall be denoted as $(B_{i+1}, s_i, t_i, \mathcal{F}_i)_{i \geq 0}$. The tower of bi-submersions over \mathcal{F} in (40) is said to be of *length* $n \in \mathbb{N}$ if $B_j = B_n$ and $\mathcal{F}_j = \{0\}$ for all $j \geq n$.

Remark 5.12. Let us spell out some consequences of the axioms. For $i \geq 1$, two points $b, b' \in B_i$ of the same leaf of \mathcal{F}_i satisfy $s_{i-1}(b) = s_{i-1}(b')$ and $t_{i-1}(b) = t_{i-1}(b')$. In fact, for all $b \in B_i$, $T_b \mathcal{F}_i \subset (\ker ds_{i-1})|_b \cap (\ker dt_{i-1})|_b$.

Let us explain how such towers can be constructed out of a singular foliation. Let \mathcal{F} be a singular foliation on M . Then,

- (1) By Proposition 2.10 in [1], there always exists a bi-submersion $B_1 \begin{array}{c} \xrightarrow{s_0} \\ \xleftarrow{t_0} \end{array} M$ over \mathcal{F} .
- (2) The $C^\infty(B_1)$ -module $\Gamma(\ker ds_0) \cap \Gamma(\ker dt_0)$ is closed under Lie bracket. When it is locally finitely generated, it is a singular foliation on B_1 . Then, it admits a bi-submersion $B_2 \begin{array}{c} \xrightarrow{s_1} \\ \xleftarrow{t_1} \end{array} B_1$. Therefore, we have obtained the two first terms of tower of bisubmersions.
- (3) We can then continue this construction provided that $\Gamma(\ker ds_1) \cap \Gamma(\ker dt_1)$ is locally finitely generated as a $C^\infty(B_2)$ -module, and that it will be so at each step³.

³In real analytic case, the module $\Gamma(\ker ds_1) \cap \Gamma(\ker dt_1)$ is locally finitely generated because of the noetherianity of the ring of germs of real analytic functions [11, 23].

Definition 5.13. A tower of bisubmersions $(B_{i+1}, s_i, t_i, \mathcal{F}_i)$ over (M, \mathcal{F}) is called *exact tower of bisubmersions over (M, \mathcal{F})* when $\mathcal{F}_i = \Gamma(\ker(ds_i)) \cap \Gamma(\ker(dt_i))$ for all $i \geq 0$. It is called a *path holonomy tower of bisubmersions* (resp. *path holonomy atlas tower of bisubmersions*) if (B_{i+1}, s_i, t_i) is a path holonomy bisubmersion (resp. a path holonomy atlas) for \mathcal{F}_i for each $i \geq 0$. When a path holonomy tower is exact we speak of *exact path holonomy tower*.

Definition 5.14. A *symmetry of the tower of bi-submersion $(B_{i+1}, s_i, t_i, \mathcal{F}_i)_{i \geq 0}$* is a family $X = (X_i)_{i \geq 0}$, with the i -th component X_i in $\text{sym}(\mathcal{F}_i)$, such that $ds_{i-1}(X_i) = dt_{i-1}(X_i) = X_{i-1}$ for all $i \geq 1$. We denote by $\text{sym}(\mathcal{T}_B)$ the Lie algebra of symmetries of \mathcal{T}_B .

The next theorem that gives a class of tower of bisubmersions to which any symmetry of the base singular foliation \mathcal{F} lifts.

Theorem 5.15. *Let \mathcal{F} be a foliation. Let \mathcal{T}_B be an exact path holonomy tower of bi-submersions (or an exact path holonomy atlas tower of bisubmersions). A vector field $X \in \mathfrak{X}(M)$ is a symmetry of \mathcal{F} , i.e. $[X, \mathcal{F}] \subset \mathcal{F}$, if and only if it is the component on M of a symmetry of \mathcal{T}_B .*

Proof. It is a direct consequence of item 2 in Proposition 5.9. \square

Lemma 5.16. *Let \mathcal{F} be a singular foliation on M . Assume that there exists a tower of bi-submersion $\mathcal{T}_B = (B_i, t_i, s_i, \mathcal{F}_i)_{i \geq 0}$ over \mathcal{F} . Then,*

$$\begin{array}{ccccccc} \dots & \longrightarrow & \ker ds_2 & \xrightarrow{dt_2} & \ker ds_1 & \xrightarrow{dt_1} & \ker ds_0 & \xrightarrow{dt_0} & TM \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & B_3 & \xrightarrow{t_2} & B_2 & \xrightarrow{t_1} & B_1 & \xrightarrow{t_0} & M. \end{array} \quad (41)$$

is a complex of vector bundles, which is exact on the sections level⁴ if \mathcal{T}_B is an exact tower of bisubmersions, i.e. if $\mathcal{F}_i = \Gamma(\ker ds_{i-1}) \cap \Gamma(\ker dt_{i-1})$ for all $i \geq 1$.

Proof. For any element $b \in B_{i+1}$ and any vector $v \in \ker ds_i \subset T_b B_{i+1}$ one has

$$\begin{aligned} dt_i(v) &\in T_{t_i(b)} \mathcal{F}_i, \quad (\text{since } \Gamma(\ker ds_i) \subset t_i^{-1}(\mathcal{F}_i)). \\ \implies dt_i(v) &\in (\ker ds_{i-1} \cap \ker dt_{i-1})|_{t_i(b)} \quad \text{by Remark 5.12.} \\ \implies dt_i(v) &\in \ker ds_{i-1} \quad \text{and} \quad dt_{i-1} \circ dt_i(v) = 0, \quad \text{for all } i \geq 1. \end{aligned}$$

This shows the the sequence (41) is a well-defined complex of vector bundles.

Let us prove that it is exact when $\mathcal{F}_i = \Gamma(\ker ds_{i-1}) \cap \Gamma(\ker dt_{i-1})$ for all $i \geq 1$. Let $\xi \in \Gamma(\ker ds_{i-1})$ be a t_{i-1} -projectable vector field that projects to zero, i.e. $dt_{i-1}(\xi) = 0$. This implies that $\xi \in \Gamma(\ker ds_{i-1}) \cap \Gamma(\ker dt_{i-1}) = \mathcal{F}_i$. Since t_i is a submersion there exists a t_i -projectable vector field $\zeta \in t_i^{-1}(\mathcal{F}_i)$ that satisfies $dt_i(\zeta) = \xi$. The vector field ζ can be written as $\zeta = \zeta_1 + \zeta_2$ with $\zeta_1 \in \Gamma(\ker dt_i)$ and $\zeta_2 \in \Gamma(\ker ds_i)$, because $t_i^{-1}(\mathcal{F}_i) = \Gamma(\ker ds_i) + \Gamma(\ker dt_i)$. One has, $dt_i(\zeta_2) = \xi$. A similar argument shows that the map, $\Gamma(\ker ds_0) \xrightarrow{dt_0} t_0^* \mathcal{F}$, is surjective. This proves exactness in all degree. \square

⁴Let us explain the notion of exactness of level of sections when the base manifolds are not the same: what we mean is that for all $n \geq 0$, $\Gamma(\ker dt_n) \cap \Gamma(\ker ds_n) = (t_{n+1})_*(\Gamma(\ker ds_{n+1}))$.

Equivalently, it means that the pull-back of the vector bundles to any one of the manifold B_m with $m \geq n$ is exact at the level of sections, i.e

$$\Gamma(t_{m,n+1}^* \ker ds_{n+1}) \xrightarrow{dt_{n+1}} \Gamma(t_{m,n}^* \ker ds_n) \xrightarrow{dt_n} \Gamma(t_{m,n-1}^* \ker ds_{n-1})$$

is a short exact sequence of $C^\infty(B_m)$ -modules, with $t_{m,n} = t_n \circ \dots \circ t_m$ for all $m \geq n$.

Corollary 5.17. *Under the assumptions of Lemma 5.16. Assume the tower of bi-submersion \mathcal{T}_B is of length $n + 1$. Then, the pull-back of the sequence of vector bundles*

$$\begin{array}{ccccccc} \ker ds_n & \xrightarrow{dt_n} & t_n^* \ker ds_{n-1} & \longrightarrow & \cdots & \xrightarrow{dt_2} & t_{2,n}^* \ker ds_1 & \xrightarrow{dt_1} & TB_{n+1} \times_{TM} \ker ds_0 & \xrightarrow{\text{pr}_1} & TB_{n+1} \\ & & & & & & & & \downarrow & & \swarrow \\ & & & & & & & & & & B_{n+1} \end{array} \quad (7)$$

is a geometric resolution of the pull-back foliation $t_{0,n}^{-1}(\mathcal{F}) \subset \mathfrak{X}(B_{n+1})$, where pr_1 is the projection on TB_{n+1} and for $i \geq 1$, $t_{i,j}$ is the composition $t_i \circ \cdots \circ t_j: B_{j+1} \rightarrow B_i$.

Proof. By Lemma 5.16, the complex in Equation (5.17) is exact. By construction, the projection of the fiber product $TB_{n+1} \times_{TM} \ker ds_0$ to TB_{n+1} induces the singular foliation $t_{0,n}^{-1}(\mathcal{F})$. \square

Remark 5.18. If there exists a sequence of maps

$$M \xrightarrow{\varepsilon_0} B_1 \xrightarrow{\varepsilon_1} B_2 \xrightarrow{\varepsilon_2} \cdots \quad (42)$$

where for all $i \geq 0$, ε_i is a section for both s_i and t_i then by Corollary 5.16, the pull-back of (41) on M through the sections $(\varepsilon_i)_{i \geq 0}$ is a geometric resolution of \mathcal{F} .

Let $(X_i)_{i \geq 0}$ be a lift of $X \in \text{sym}(\mathcal{F})$. For $i \geq 1$, ad_{X_i} preserves $\Gamma(\ker ds_{i-1})$. Altogether, they define a chain map between the complex (41) in itself. Let us give a precise meaning to this statement.

Proposition 5.19. *Let $\mathcal{T}_B = (B_{i+1}, s_i, t_i, \mathcal{F}_i)_{i \geq 0}$ be an exact path-holonomy tower of bisubmersions over a singular foliation (M, \mathcal{F}) of length $n + 1$.*

- (1) *Any symmetry $X \in \text{sym}(\mathcal{F})$ of \mathcal{F} is the zero component of a symmetry $(X_i)_{i \geq 0}$ of \mathcal{T}_B .*
- (2) *This, in turn, induces a \mathbb{K} -linear chain map*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \Gamma(t_{3,n}^* \ker ds_2) & \xrightarrow{dt_2} & \Gamma(t_{2,n}^* \ker ds_1) & \xrightarrow{dt_1} & \Gamma(TB_{n+1} \times_{TM} \ker ds_0) & \xrightarrow{\text{pr}_1} & t_{0,n}^{-1}(\mathcal{F}) \\ & & \downarrow \nabla_X^2 & & \downarrow \nabla_X^1 & & \downarrow \nabla_X^0 & & \downarrow \text{ad}_{X_{n+1}} \\ \cdots & \longrightarrow & \Gamma(t_{3,n}^* \ker ds_2) & \xrightarrow{dt_2} & \Gamma(t_{2,n}^* \ker ds_1) & \xrightarrow{dt_1} & \Gamma(TB_{n+1} \times_{TM} \ker ds_0) & \xrightarrow{\text{pr}_1} & t_{0,n}^{-1}(\mathcal{F}) \end{array} \quad (43)$$

that satisfies for all $i \geq 1$,

$$(a) \quad \nabla_X^{i-1}(f\xi) = X_{n+1}[f] + f\nabla_X^{i-1}(\xi), \text{ for all } f \in C^\infty(B_{n+1}), \xi \in \Gamma(t_{i,n}^* \ker ds_{i-1}).$$

$$(b) \quad \nabla_X^i \circ dt_i = dt_i \circ \nabla_X^{i+1} \quad \text{and} \quad \text{pr}_1 \circ \nabla_X^0 = \text{ad}_{X_{n+1}} \circ \text{pr}_1.$$

Proof. Let $U \subset B_{n+1}$ an open subset and $\xi_1, \dots, \xi_r \in t_{0,n}^{-1}(\mathcal{F})$ some local generators. Since $X_{n+1} \in \text{sym}(t_{0,n}^{-1}(\mathcal{F}))$, this implies $[X_{n+1}, \xi_i] = \sum_{p=1}^r c_i^p(x) \xi_p$ for some smooth functions $c_i^p(x) \in C^\infty(U)$ with $i, p = 1, \dots, r$. Pick a local trivialisation $\zeta_1, \dots, \zeta_d \in \Gamma(TB_{n+1} \times_{TM} \ker ds_0)|_U$. We define a map from $\Gamma(TB_{n+1} \times_{TM} \ker ds_0)$ to $\Gamma(TB_{n+1} \times_{TM} \ker ds_0)$ as follows,

$$\nabla_X^0 \zeta := \sum_{p=1}^r \left(\sum_{i=1}^r f_i c_i^p(x) + X_{n+1}[f_p] \right) \zeta_p. \quad (44)$$

Clearly, it satisfies the following Leibniz identity,

$$\nabla_X^0(f\xi) = X_{n+1}[f] + f\nabla_X^0(\xi), \text{ for all } f \in C^\infty(B_{n+1}), \xi \in \Gamma(t_{i,n}^* \ker ds_{i-1}). \quad (45)$$

Of course, the map ∇_X^0 is constructed to agree with ad_X and the first projection map pr_1 , namely

$$\text{pr}_1 \circ \nabla_X^0 = \text{ad}_{X_{n+1}} \circ \text{pr}_1. \quad (46)$$

Equation (46) and Corollary 5.17 imply that ∇_X restricts to a map over the kernel of pr_1 , that is,

$$\nabla_X: \ker \text{pr}_1 = \text{im}(dt_1) \rightarrow \ker \text{pr}_1 = \text{im}(dt_1).$$

A similar construction allows to construct $\nabla_X^1: \Gamma(t_{2,n}^* \ker ds_1) \rightarrow \Gamma(t_{2,n}^* \ker ds_1)$ on degree -2 elements that commutes with the differential map dt_1 . Again, by exactness, we are allowed to continue on every degree to obtain a map that makes the diagram (43) commutes and satisfies the Leibniz identity in (45) on every degree. \square

Remark 5.20. In [14], under some assumptions, it is shown that if a Lie group G acts on a foliated manifold (M, \mathcal{F}) it acts on its holonomy groupoid. It is likely that this result follows from Theorem 5.15, this will be addressed in an other study.

APPENDIX A. LIE ∞ -ALGEBROIDS AND THEIR MORPHISMS

Let us now recall the definition of Lie ∞ -algebroids over a manifold their morphisms and homotopies. Most definitions of this section can be found in [6, 18, 19] and our convention are those of [18, 19].

Definition A.1. A *Lie ∞ -algebroid over M* is the datum of a sequence $E = (E_{-i})$, $1 \leq i < \infty$ of vector bundles over M together with a structure of Lie ∞ -algebra $(\ell_k)_{k \geq 1}$ on the sheaf of sections of E and a vector bundle morphism, $\rho: E_{-1} \rightarrow TM$, called *anchor map* such that the k -ary brackets ℓ_k , $k \neq 2$ are \mathcal{O} -multi-linear and such that

$$\ell_2(e_1, fe_2) = \rho(e_1)[f]e_2 + f\ell_2(e_1, e_2) \quad (47)$$

for all $e_1 \in \Gamma(E_{-1})$, $e_2 \in \Gamma(E_\bullet)$ and $f \in \mathcal{O}$.

The sequence

$$\cdots \xrightarrow{\ell_1} E_{-2} \xrightarrow{\ell_1} E_{-1} \xrightarrow{\rho} TM, \quad (48)$$

is a complex called the *linear part* of the Lie ∞ -algebroid.

Remark A.2. Any Lie ∞ -algebroid on M has an induced singular foliation on M which is given by the image of the anchor map, that we call the *basic singular foliation*.

There is an alternative definition for Lie ∞ -algebroids in term of Q -manifolds with purely non-negative degrees.

Definition A.3. A *splitted NQ -manifold* is a pair (E, Q) where $E \rightarrow M$ is a sequence of vector bundles over M indexed by negative integers and where Q is a homological vector field of degree $+1$, i.e. $Q \in \text{Der}_1(\Gamma(S_{\mathbb{K}}^\bullet(E^*)))$ is such that $[Q, Q] = 0$.

We denote by \mathcal{E} and call *functions* on the splitted NQ -manifold $E \rightarrow M$ the sheaf of graded commutative \mathcal{O} -algebras made of sections of $S_{\mathbb{K}}(E^*)$.

There is a one-to-one correspondence between splitted NQ -manifolds and Lie ∞ -algebroids [25, 24, 6, 22]. This formulation allows to write in a compact manner morphisms of Lie ∞ -algebroids. From now on, we write (E, Q) to denote a Lie ∞ -algebroid over M .

Definition A.4. Let (E, Q) and (E', Q') be Lie ∞ -algebroids on a manifold M resp. M' , with sheaves of functions \mathcal{E}' and \mathcal{E} , respectively. A *Lie ∞ -algebroid morphism* from (E, Q) to (E', Q') , is a graded algebra morphism $\Phi: \mathcal{E} \rightarrow \mathcal{E}'$ (of degree 0) which interchanges Q with Q' :

$$\Phi \circ Q = Q' \circ \Phi. \quad (49)$$

Homotopy equivalence can also be defined, see Section 3.4.3 in [18] or Section 1.2.5 in [19].

Remark A.5. A Lie ∞ -algebroid morphism Φ induces a smooth map $\phi: M' \rightarrow M$ called the *base morphism*. It also induces a graded vector bundle morphism $\phi_0: E'_\bullet \rightarrow E_\bullet$ over ϕ which is called the *linear part of Φ* .

Let us recall from [18, 19] the following definition and theorem.

Definition A.6. Let $\mathcal{F} \subset \mathfrak{X}(M)$ be a singular foliation on a manifold M . A *geometric resolution* of the singular foliation \mathcal{F} is a projective resolution $((P_{-i})_{i \geq 1}, (d^{(i)})_{i \geq 2}, \rho)$ of \mathcal{F} as a \mathcal{O} -module that corresponds to a sequence of vector bundles $(E, \bar{d}, \bar{\rho})$ over M

$$\cdots \xrightarrow{\bar{d}^{(3)}} E_{-2} \xrightarrow{\bar{d}^{(2)}} E_{-1} \xrightarrow{\bar{\rho}} TM, \quad (50)$$

i.e.

- for $i \geq 1$ the \mathcal{O} -module of sections of E_{-i} is $P_{-i} = \Gamma(E_{-i})$
- for $i \geq 2$, the induced maps on the sections level

$$\bar{d}^{(i)}: \Gamma(E_{-i}) \longrightarrow \Gamma(E_{-i+1}) \quad \text{or} \quad \bar{\rho}: \Gamma(E_{-1}) \longrightarrow \mathcal{F}$$

coincide with $d^{(i)}: P_{-i} \longrightarrow P_{-i+1}$ or with $\rho: P_{-1} \longrightarrow \mathcal{F}$ respectively.

For convenience we denote by \bar{d} and $\bar{\rho}$ the same as d and ρ respectively. Also, we call $\rho: E_{-1} \longrightarrow TM$ the *geometric resolution anchor*. A geometric resolution is said to be *minimal* at a point $m \in M$ if, for all $i \geq 2$, the linear maps $d_{|m}^{(i)}: E_{-i}|_m \longrightarrow E_{-i+1}|_m$ vanish.

Theorem A.7. [18, 21, 19] *Let \mathcal{F} be a singular foliation over M . Any geometric resolution of \mathcal{F}*

$$\cdots \xrightarrow{d} E_{-3} \xrightarrow{d} E_{-2} \xrightarrow{d} E_{-1} \xrightarrow{\rho} TM \quad (51)$$

comes equipped with a Lie ∞ -algebroid structure whose unary bracket is d and whose anchor map is ρ . Such a Lie ∞ -algebroid structure is unique up to homotopy and is called a universal Lie ∞ -algebroid of \mathcal{F} .

Remark A.8. For a given Lie ∞ -algebroid (E, Q) , the triple $(\mathfrak{X}_\bullet(E), [\cdot, \cdot], \text{ad}_Q)$ is a differential graded Lie algebra, where $\mathfrak{X}_\bullet(E)$ stands of the module of graded vector fields (=graded derivations of \mathcal{E}) on E , the bracket $[\cdot, \cdot]$ is the graded commutator of derivations and $\text{ad}_Q := [Q, \cdot]$. We say a vector field on E is *vertical* if it is \mathcal{O} -linear.

APPENDIX B. LIE ∞ -MORPHISMS OF DIFFERENTIAL GRADED LIE ALGEBRAS AND HOMOTOPIES

Let us recall the definitions of Lie ∞ -morphisms and homotopies between differential graded Lie algebras in terms of coderivations. We restrict ourself to a special case that we need for this paper.

B.1. Comorphisms and coderivations. Let \mathfrak{g} and \mathfrak{h} be graded Lie algebras over \mathbb{K} .

Definition B.1. A linear map $\Phi: S_{\mathbb{K}}^{\bullet}\mathfrak{g} \rightarrow S_{\mathbb{K}}^{\bullet}\mathfrak{h}$ is said to be of *arity* $r \in \mathbb{N}_0$, if it sends polynomials of $S_{\mathbb{K}}^k\mathfrak{g}$ to those of $S_{\mathbb{K}}^{k-r}\mathfrak{h}$. Any linear map Φ can be decomposed as formal sum:

$$\Phi = \sum_{k \in \mathbb{Z}} \Phi^{(k)} \quad (52)$$

where for all $k \in \mathbb{N}_0$, $\Phi^{(k)}: S_{\mathbb{K}}^{\bullet}\mathfrak{g} \rightarrow S_{\mathbb{K}}^{\bullet-k}\mathfrak{h}$ is a linear map of arity k . Therefore, a linear map $\Phi: S_{\mathbb{K}}^{\bullet}\mathfrak{g} \rightarrow S_{\mathbb{K}}^{\bullet}\mathfrak{h}$ is of arity $r \in \mathbb{N}_0$ if and only if $\Phi^{(k)}$ is the unique non-zero term, namely $\Phi^{(k)} = 0$, for $k \neq r$.

Let us denote by Δ the coalgebra structure $S_{\mathbb{K}}^{\bullet}\mathfrak{g}$ and by Δ' the one on $S_{\mathbb{K}}^{\bullet}\mathfrak{h}$. Given any linear map $\Phi: S_{\mathbb{K}}^{\bullet}\mathfrak{g} \rightarrow \mathfrak{h}$. Denoting by $\Phi_k: S_{\mathbb{K}}^{k+1}\mathfrak{g} \rightarrow \mathfrak{h}$ for $k \in \mathbb{N}_0$ the restriction of Φ to $S_{\mathbb{K}}^{k+1}\mathfrak{g}$. The linear map Φ can be extended to a unique comorphism $\widehat{\Phi}: S_{\mathbb{K}}^{\bullet}\mathfrak{g} \rightarrow S_{\mathbb{K}}^{\bullet}\mathfrak{h}$ by taking for $r \in \mathbb{N}$ the component on $S_{\mathbb{K}}^r\mathfrak{h}$ to be for $x_1, \dots, x_k \in \mathfrak{g}$

$$\sum_{i_1 + \dots + i_r = k} \sum_{\sigma \in \mathfrak{S}(i_1, \dots, i_r)} \epsilon(\sigma) \frac{1}{r!} \prod_{j=1}^r \Phi_{i_j-1}(x_{\sigma(i_1 + \dots + i_{j-1} + 1)}, \dots, x_{\sigma(i_1 + \dots + i_j)}). \quad (53)$$

where $\mathfrak{S}(i_1, \dots, i_r)$ is the set of (i_1, \dots, i_r) -shuffles, with $i_1, \dots, i_r \in \mathbb{N}$.

Every comorphism from $S_{\mathbb{K}}^{\bullet}\mathfrak{g}$ to $S_{\mathbb{K}}^{\bullet}\mathfrak{h}$ is of this form [16]. That is, a comorphism $\Phi: S_{\mathbb{K}}^{\bullet}\mathfrak{g} \rightarrow S_{\mathbb{K}}^{\bullet}\mathfrak{h}$ is entirely determined by the collection indexed by $k \in \mathbb{N}$ of maps called its *k-th Taylor coefficients*:

$$\Phi_k: S_{\mathbb{K}}^{k+1}\mathfrak{g} \xrightarrow{\Phi} S_{\mathbb{K}}^{\bullet}\mathfrak{h} \xrightarrow{\text{pr}} \mathfrak{h}, \quad (54)$$

with pr being the projection onto the term of arity 1, i.e. $\text{pr}: S_{\mathbb{K}}^{\bullet}\mathfrak{g} \rightarrow S_{\mathbb{K}}^1\mathfrak{h} \simeq \mathfrak{h}$. Notice that the component $\Phi^{(k)}$ of arity k of Φ coincides with k -th Taylor coefficient Φ_k on $S_{\mathbb{K}}^{k+1}\mathfrak{g}$. Hence, a comorphism $\Phi: S_{\mathbb{K}}^{\bullet}\mathfrak{g} \rightarrow S_{\mathbb{K}}^{\bullet}\mathfrak{h}$ admits a decomposition of the form:

$$\Phi = \sum_{k \geq 0} \Phi^{(k)}. \quad (55)$$

Definition B.2. Let $\Phi: S_{\mathbb{K}}^{\bullet}\mathfrak{g} \rightarrow S_{\mathbb{K}}^{\bullet}\mathfrak{h}$ be a graded comorphism. A Φ -*coderivation* of degree N on $S_{\mathbb{K}}^{\bullet}\mathfrak{g}$ is a degree $k \in \mathbb{N}_0$ linear map $\mathcal{H}: S_{\mathbb{K}}^{\bullet}\mathfrak{g} \rightarrow S_{\mathbb{K}}^{\bullet}\mathfrak{h}$ which satisfies the so-called (co)Leibniz identity:

$$\Delta' \circ \mathcal{H} = (\mathcal{H} \otimes \Phi) \circ \Delta + (\Phi \otimes \mathcal{H}) \circ \Delta. \quad (56)$$

When $\mathfrak{g} = \mathfrak{h}$ and $\Phi = \text{id}$, we say that \mathcal{H} is a *coderivation*.

The same results on comorphisms hold for coderivations [16].

B.2. Lie ∞ -morphisms of differential graded Lie algebras. Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ a Lie algebra and (E, Q) a Lie ∞ -algebroid over M . In the sequel, the Lie algebra \mathfrak{g} is concentrated in degree -1 . The differential graded Lie algebra $(\mathfrak{X}(E), [\cdot, \cdot], \text{ad}_Q)$ of vector fields on E is shifted by 1, i.e. a derivation of degree k in $\mathfrak{X}_k(E)$ is of degree $k - 1$ as an element of the shifted space $\mathfrak{X}_k(E)[1]$. The graded symmetric Lie bracket on $\mathfrak{X}(E)[1]$ is of degree $+1$ and given on homogeneous elements $u, v \in \mathfrak{X}(E)[1]$ as

$$\{u, v\} := (-1)^{|v|} [u, v].$$

In the sequel we write $(\mathfrak{X}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ instead of $(\mathfrak{X}(E)[1], \{\cdot, \cdot\}, \text{ad}_Q)$.

Let $(S_{\mathbb{K}}^{\bullet} \mathfrak{g}, Q_{\mathfrak{g}})$ respectively $(S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1]), \bar{Q})$ be the corresponding formulations in term of coderivations of the differential graded Lie algebras $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ and $(\mathfrak{X}(E)[1], [\cdot, \cdot], \text{ad}_Q)$. Precisely, $Q_{\mathfrak{g}}$ is the coderivation defined by putting for every homogeneous monomial $x_1 \wedge \cdots \wedge x_k \in S_{\mathbb{K}}^k \mathfrak{g}$,

$$Q_{\mathfrak{g}}(x_1 \wedge \cdots \wedge x_k) := \sum_{1 \leq i < j \leq k} (-1)^{i+j-1} [x_i, x_j]_{\mathfrak{g}} \wedge x_1 \wedge \cdots \wedge \widehat{x}_i \cdots \wedge \widehat{x}_j \cdots \wedge x_k, \quad (57)$$

and $\bar{Q} = \bar{Q}^{(0)} + \bar{Q}^{(1)}$ is the coderivation of degree +1 where the only non zero Taylors coefficients are, $\bar{Q}^{(0)}: S_{\mathbb{K}}^1(\mathfrak{X}(E)[1]) \xrightarrow{\text{ad}_Q} \mathfrak{X}(E)[1]$ and $\bar{Q}^{(1)}: S_{\mathbb{K}}^2(\mathfrak{X}(E)[1]) \xrightarrow{\{\cdot, \cdot\}} \mathfrak{X}(E)[1]$.

Definition B.3. [17] A Lie ∞ -morphism $\Phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \longrightarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ is a graded coalgebra morphism $\bar{\Phi}: (S_{\mathbb{K}}^{\bullet} \mathfrak{g}, Q_{\mathfrak{g}}) \longrightarrow (S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1]), \bar{Q})$ of degree zero which satisfies,

$$\bar{\Phi} \circ Q_{\mathfrak{g}} = \bar{Q} \circ \bar{\Phi}. \quad (58)$$

In order words, it is the datum of degree zero linear maps $(\bar{\Phi}_k: S_{\mathbb{K}}^{k+1} \mathfrak{g} \longrightarrow \mathfrak{X}_{-k}(E)[1])_{k \geq 0}$ that satisfies

$$\begin{aligned} \sum_{1 \leq i < j \leq n+2} (-1)^{i+j-1} \bar{\Phi}_n([x_i, x_j]_{\mathfrak{g}}, x_1, \dots, \widehat{x}_{ij}, \dots, x_{n+2}) &= [Q, \bar{\Phi}_{n+1}(x_1, \dots, x_{n+2})] + \\ &\sum_{\substack{i+j=n \\ i \leq j \\ \sigma \in \mathfrak{S}_{i+1, j+1}}} \epsilon(\sigma) [\bar{\Phi}_i(x_{\sigma(1)}, \dots, x_{\sigma(i+1)}), \bar{\Phi}_j(x_{\sigma(i+2)}, \dots, x_{\sigma(n+2)})] \end{aligned}$$

where \widehat{x}_{ij} means that we take x_i, x_j out of the list. When there is no risk of confusion we write Φ for $\bar{\Phi}$.

Remark B.4. Definition B.3 and Definition A.4 are compatible when $M = \{\text{pt}\}$. Therefore, morphisms in both sense match.

Remark B.5. Its follows from these axioms that if the homological vector field Q vanishes at some point $m \in M$, then the map $x \longmapsto (P \in \mathfrak{X}(E), P|_m \mapsto [\Phi_0(x), P]|_m)$ endows the vector space $\mathfrak{X}(E)|_m \simeq (S(E^*) \otimes E)|_m$ with a \mathfrak{g} -module structure. Moreover, the restriction of the map $\Phi_1: \wedge^2 \mathfrak{g} \longrightarrow \mathfrak{X}_{-1}(E)|_m$ at m is a 2-cocycle of Chevalley-Eilenberg.

Remark B.6. Let (E, Q) be a Lie ∞ -algebroid and \mathcal{F} its basic singular foliation. Any Lie ∞ -morphism $\Phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \longrightarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ gives a weak symmetry action of \mathfrak{g} on \mathcal{F} . If $Q|_m = 0$ for some point $m \in M$, the \mathfrak{g} -action defined in Remark B.5, is independent of the equivalence class of the weak symmetry action.

B.3. Homotopies. Now we are defining homotopy between Lie ∞ -morphisms. A *homotopy* that joins two Lie ∞ -morphisms $\bar{\Phi}, \bar{\Psi}: (S_{\mathbb{K}}^{\bullet} \mathfrak{g}, Q_{\mathfrak{g}}) \longrightarrow (S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1]), \bar{Q})$ is the datum of an interval $[a, b] \subset \mathbb{R}$ and a chain map

$$\begin{aligned} (S_{\mathbb{K}}^{\bullet} \mathfrak{g}, \bar{Q}) &\xrightarrow{\mathcal{H}} (S_{\mathbb{K}}^{\bullet} \mathfrak{X}(E)[1] \otimes_{C^{\infty}([a, b])} \Omega^{\bullet}([a, b]), Q_{\mathfrak{g}} \otimes \text{id} + \text{id} \otimes \text{d}_{\text{dR}}) \\ v &\longmapsto \Xi(v) \otimes \alpha(t) + H(v) \otimes \beta(t) dt, \quad \text{for } t \in [a, b]. \end{aligned}$$

which is a coalgebra morphism as well and coincides with $\bar{\Phi}$ and $\bar{\Psi}$ at $t = a$ and b respectively. For further use, we will write $\Xi_t \otimes 1$ for $\Xi(v) \otimes \alpha(t)$ and $H_t \otimes dt$ for $H(v) \otimes \beta(t) dt$. The map \mathcal{H} induces for every $t \in [a, b]$ two different maps. One of them is of degree zero and the other one of degree -1 respectively:

$$\begin{cases} \Xi_t & : S_{\mathbb{K}}^{\bullet} \mathfrak{g} \longrightarrow S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1]) \\ H_t & : S_{\mathbb{K}}^{\bullet} \mathfrak{g} \longrightarrow S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1]). \end{cases}$$

Using the property of coalgebra-morphisms and chain map property respectively, one can check easily that for every $t \in [a, b]$, Ξ_t must be a Lie ∞ -morphism of differential graded Lie algebras and H_t a Ξ_t -coderivation satisfying the following additional condition over $[a, b]$

$$\frac{d\Xi_t}{dt} = \bar{Q} \circ H_t + H_t \circ Q_{\mathfrak{g}}. \quad (59)$$

We can formulate the definition of homotopies between Lie ∞ -morphisms the following manner.

Definition B.7. Let $\bar{\Phi}, \bar{\Psi}: (S_{\mathbb{K}}^{\bullet} \mathfrak{g}, Q_{\mathfrak{g}}) \rightarrow (S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1]), \bar{Q})$ be Lie ∞ -morphisms. We say $\bar{\Phi}, \bar{\Psi}$ are *homotopic over the identity of M* if the following conditions hold:

- (1) there is a family $(\Xi_t: (S_{\mathbb{K}}^{\bullet} \mathfrak{g}, Q_{\mathfrak{g}}) \rightarrow (S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1]), \bar{Q}))_{t \in [0,1]}$ made of Lie ∞ -morphisms which coincide with $\bar{\Phi}$ and $\bar{\Psi}$ at $t = 0$ and 1 , respectively,
- (2) and a family of Ξ_t -coderivations $(H_t)_{t \in [0,1]}$ of degree -1 such that

$$\frac{d\Xi_t}{dt} = \bar{Q} \circ H_t + H_t \circ Q_{\mathfrak{g}}. \quad (60)$$

Remark B.8. Homotopy equivalence in the sense of the Definition B.7 is an equivalence relation and it is compatible with composition of Lie ∞ -morphisms, see [19] Proposition 1.38.

APPENDIX C. PROOF OF THEOREM 3.3

Proof. (of Theorem 3.3) The complex of Equation (28) being exact everywhere except in degree -1 we cannot apply directly Theorem 2.1 in [19] but we can mimic the proof given for Theorem 2.1 in [19] to construct the higher brackets when there is no obstruction in degree -1 . For convenience, let us denote $\mathcal{R}_{-1} := \Gamma(\mathfrak{g}) \oplus \Gamma(E_{-1})$ and $\mathcal{R}_{-i} := \Gamma(E_{-i})$ for $i \geq 2$. Given a natural number $k \geq 0$, we consider the total complex $\left(\widehat{\mathfrak{Page}}_{\bullet}^{(k)}(\mathcal{R}), D = [d, \cdot]_{\mathbb{R}\mathbb{N}} \right)$ of the following bicomplex

$$\begin{array}{ccccc}
 & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & \\
 \cdots & \rightarrow & \text{Hom}_{\mathcal{O}} \left(\odot^{k+1} \mathcal{R}_{|-k-3}, \mathcal{R}_{-3} \right) & \xrightarrow{d} & \text{Hom}_{\mathcal{O}} \left(\odot^{k+1} \mathcal{R}_{|-k-3}, \mathcal{R}_{-2} \right) & \xrightarrow{d} & \text{Hom}_{\mathcal{O}} \left(\odot^{k+1} \mathcal{R}_{|-k-3}, d\mathcal{R}_{-2} \right) & \rightarrow & 0 \\
 & & \delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \\
 \cdots & \rightarrow & \text{Hom}_{\mathcal{O}} \left(\odot^{k+1} \mathcal{R}_{|-k-2}, \mathcal{R}_{-3} \right) & \xrightarrow{d} & \text{Hom}_{\mathcal{O}} \left(\odot^{k+1} \mathcal{R}_{|-k-2}, \mathcal{R}_{-2} \right) & \xrightarrow{d} & \text{Hom}_{\mathcal{O}} \left(\odot^{k+1} \mathcal{R}_{|-k-2}, d\mathcal{R}_{-2} \right) & \rightarrow & 0 \\
 & & \delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \\
 \cdots & \rightarrow & \text{Hom}_{\mathcal{O}} \left(\odot^{k+1} \mathcal{R}_{|-k-1}, \mathcal{R}_{-3} \right) & \xrightarrow{d} & \text{Hom}_{\mathcal{O}} \left(\odot^{k+1} \mathcal{R}_{|-k-1}, \mathcal{R}_{-2} \right) & \xrightarrow{d} & \text{Hom}_{\mathcal{O}} \left(\odot^{k+1} \mathcal{R}_{|-k-1}, d\mathcal{R}_{-2} \right) & \rightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & \\
 & & \text{"-3 column"} & & \text{"-2 column"} & & \text{"-1 column"} & & \\
 & & & & & & & & (61)
 \end{array}$$

The map δ stands for the vertical differential which is defined for all $\Phi \in \text{Hom}_{\mathcal{O}} \left(\odot^{k+1} \mathcal{R}, \mathcal{R} \right)$ by

$$\delta(\Phi)(r_1, \dots, r_{k+1}) := \Phi \circ d(r_1 \odot \dots \odot r_{k+1}), \quad \forall r_1, \dots, r_{k+1} \in \mathcal{R},$$

where here d acts as an \mathcal{O} -derivation on $r_1 \odot \dots \odot r_{k+1} \in \odot^k \mathcal{R}$ and the horizontal differential given by

$$\Phi \mapsto d \circ \Phi.$$

Since the line the bicomplex is exact the total complex $\left(\widehat{\mathfrak{Page}}_{\bullet}^{(k)}(\mathcal{R}), D = [d, \cdot]_{\mathbb{R}\mathbb{N}} \right)$ is also exact.

Construction of the 2-ary bracket: its construction is almost the same as in [19] we will adapt what has been done to our case. We first construct a 2-ary bracket on \mathcal{R}_{-1} to extend on every degree. For all $k \geq 1$, let us denote by $(e_i^{(-k)})_{i \in I_k}$ a basis of $\Gamma(E_{-k})$. The set $\{X_i = \rho(e_i^{(-1)}) \in \mathcal{F} \mid i \in I_1\}$ is a set of generators of \mathcal{F} . In particular, there exists elements $c_{ij}^k \in \mathcal{O}$ and satisfying the skew-symmetry condition $c_{ij}^k = -c_{ji}^k$ together with

$$[X_i, X_j] = \sum_{k \in I} c_{ij}^k X_k \quad \forall i, j \in I_1. \quad (62)$$

By definition of weak symmetry one has

$$[\varrho(\xi_i), \rho(e_j^{(-1)})] \in \mathcal{F} \quad \text{and} \quad \varrho([\xi_i, \xi_j]_{\mathfrak{g}}) - [\varrho(\xi_i), \varrho(\xi_j)] \in \mathcal{F} \quad \text{for all } (i, j) \in I_{\mathfrak{g}} \times I_{-1}. \quad (63)$$

Where $(\xi_i)_{i \in I_{\mathfrak{g}}}$ is a basis for \mathfrak{g} . Since $((E_{-i})_{i \geq 1}, d, \rho)$ is a geometric resolution of \mathcal{F} , there exists two \mathcal{O} -bilinear maps $\chi: \Gamma(\mathfrak{g}) \times \Gamma(E_{-1}) \rightarrow \Gamma(E_{-1})$, $\eta: \Gamma(\mathfrak{g}) \times \Gamma(\mathfrak{g}) \rightarrow \Gamma(E_{-1})$ defined on generators $\xi_i, e_j^{(-1)}$ by the relations

$$[\varrho(\xi_i), \rho(e_j^{(-1)})] = \rho(\chi(\xi_i, e_j^{(-1)})) \quad \text{and} \quad \varrho([\xi_i, \xi_j]_{\mathfrak{g}}) - [\varrho(\xi_i), \varrho(\xi_j)] = \rho(\eta(\xi_i, \xi_j)).$$

We first define a naive 2-ary bracket on $\Gamma(E_{-1})$ as follows:

- (1) an anchor map by $\rho'(e_i^{(-1)}) = X_i$, and $\rho'(\xi_i) = \varrho(\xi_i)$, for all $i \in I, I_{\mathfrak{g}}$,
- (2) a degree +1 graded symmetric operation $\tilde{\ell}_2$ on \mathcal{R}_{\bullet} as follows:
 - (a) $\tilde{\ell}_2(e_i^{(-1)}, e_j^{(-1)}) = \sum_{k \in I} c_{ij}^k e_k^{(-1)}$ for all $i, j \in I_{-1}$,
 - (b) $\tilde{\ell}_2(\xi_i, e_j^{(-1)}) = \chi(\xi_i, e_j^{(-1)})$,
 - (c) $\tilde{\ell}_2(\xi_i, \xi_j) = [\xi_i, \xi_j]_{\mathfrak{g}} + \eta(\xi_i, \xi_j)$,
 - (d) $\tilde{\ell}_2$ is zero on the other generators,
 - (e) we extend $\tilde{\ell}_2$ to \mathcal{R} using \mathcal{O} -bilinearity and Leibniz identity with respect to the anchor ρ' .

By (a), (b), (c), (d), (e), $\tilde{\ell}_2$ satisfies the Leibniz identity with respect to the anchor $\tilde{\rho}$ and (a), (b), (c) makes the latter a bracket morphism. The map defined for all homogeneous $r_1, r_2 \in \mathcal{R}_{\bullet}$ by

$$[d, \tilde{\ell}_2]_{\text{RN}}(r_1, r_2) = d \circ \tilde{\ell}_2(r_1, r_2) + \tilde{\ell}_2(dr_1, r_2) + (-1)^{|r_1|} \tilde{\ell}_2(r_1, dr_2), \quad (64)$$

is a graded symmetric degree +2 operation $(\mathcal{R} \otimes \mathcal{R})_{\bullet} \rightarrow \mathcal{R}_{\bullet+2}$, and $[d, \tilde{\ell}_2]_{\text{RN}}|_{\mathcal{R}_{-1}} = 0$. It is \mathcal{O} -bilinear, i.e. for all $f \in \mathcal{O}, r_1, r_2 \in \mathcal{R}$

$$[d, \tilde{\ell}_2]_{\text{RN}}(r_1, fr_2) - f[d, \tilde{\ell}_2]_{\text{RN}}(r_1, r_2) = 0.$$

We also have that $\rho([d, \tilde{\ell}_2]_{\text{RN}}(r_1, fr_2)) = \rho(\tilde{\ell}_2(dr_1, r_2)) = 0$, for all $r_1 \in \mathcal{R}_{-2}, r_2 \in \mathcal{R}_{-1}$, since $\rho \circ d = 0$. Thus, $[d, \tilde{\ell}_2]_{\text{RN}}|_{\mathcal{R}_{-2} \times \mathcal{R}_{-1}} \in d\mathcal{R}_{-2}$, because $((E_{-i})_{i \geq 1}, d, \rho)$ is a geometric resolution.

Therefore, $[d, \tilde{\ell}_2]_{\text{RN}}$ is a degree +2 element in the total complex $\widehat{\mathfrak{Page}}^{(1)}(\mathcal{R})$. The \mathcal{O} -bilinear operator $[d, \tilde{\ell}_2]_{\text{RN}}$ is D -closed in $\widehat{\mathfrak{Page}}^{(1)}(\mathcal{R})$, since $[d, [d, \tilde{\ell}_2]_{\text{RN}}]_{\text{RN}}|_{\mathcal{R}_{\leq -2}} = 0$. So there exists $\tau_2 \in \bigoplus_{j \geq 2} \text{Hom}_{\mathcal{O}}(\bigodot^2 \mathcal{R}_{-j-1}, \mathcal{R}_{-j})$ such as $D(\tau_2) = -[d, \tilde{\ell}_2]_{\text{RN}}$. By replacing $\tilde{\ell}_2$ by $\tilde{\ell}_2 + \tau_2$ we get a 2-ary bracket ℓ_2 of degree +1 which is compatible with the differential map d and the anchor map $\tilde{\rho}$.

Construction of higher brackets: notice by construction of the 2-ary bracket ℓ_2 one has, $\text{Jac}(r_1, r_2, r_3) \in d\mathcal{R}_{-2}$ for all $r_1, r_2, r_3 \in \mathcal{R}_{-1}$. In other words, $\text{Jac} \in \text{Hom}_{\mathcal{O}}(\bigodot^3 \mathcal{R}_{-1}, d\mathcal{R}_{-2})$. A

direct computation shows

$$d\text{Jac}(r_1, r_2, r_3) = \text{Jac}(dr_1, r_2, r_3) + (-1)^{|r_1|} \text{Jac}(r_1, dr_2, r_3) + (-1)^{|r_1|+|r_2|} \text{Jac}(r_1, r_2, dr_3)$$

for all $r_1, r_2, r_3 \in \mathcal{R}$. Which means, $[\text{Jac}, d]_{\text{RN}}(r_1, r_2, r_3) = 0$ for all $r_1, r_2, r_3 \in \mathcal{R}$.

Thus, $D(\text{Jac}) = 0$. It follows that, Jac is a D -coboundary, there exists an element $\ell_3 = \sum_{j \geq 2} \ell_3^j \in \widehat{\mathfrak{Page}}_1^{(2)}(\mathcal{R})$ with $\ell_3^j \in \text{Hom}(\odot^3 \mathcal{R}_{|-j-1}, \mathcal{R}_{-j})$ such that

$$D(\ell_3) = -\text{Jac}. \quad (65)$$

We choose the 3-ary bracket to be ℓ_3 . For degree reason the remaining terms of the k -ary brackets for $k \geq 3$ have trivial components on column -1 of the bicomplex (61). From this point, the proof continues exactly as in the proof of Theorem 2.1 in [19]. \square

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