Does it exist a general form for *adaptation/learning* algorithms?

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In collaboration with :
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gipsa-lab

Adaptation /Learning Algorithms = *The Fiddler's Paradise!*

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- Conjugate gradients
- Nesterov algorithm
- Back momentum
- Average gradients
- Integral + Proportional
- Integral +Proportional+ Derivative
- ……
- ……

Without memory

- Leakage algorithm
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- *Number of algorithms have been re-discovered (often without reference to the original).*
- *Many algorithms differs just by the way that the equations are written.*
- *Absence of comparisons.*
- *Atentive reading allows to reduce significantly the number of truly « original » algorithms.*
- *Absence of serious theoretical analysis (most of the cases).*
- *The stability issues seldom discussed (most of the cases)*

What should you remember from this talk ?

$$\hat{\theta}(t+1) = \hat{\theta}(t) + correcting\ term$$

Vector of parameters

$$\hat{\theta}(t+1) = \frac{1}{1-q^{-1}}\, correcting\ term$$

Integrator

- New estimated parameter vector = the correcting term filtered by an « integrator »
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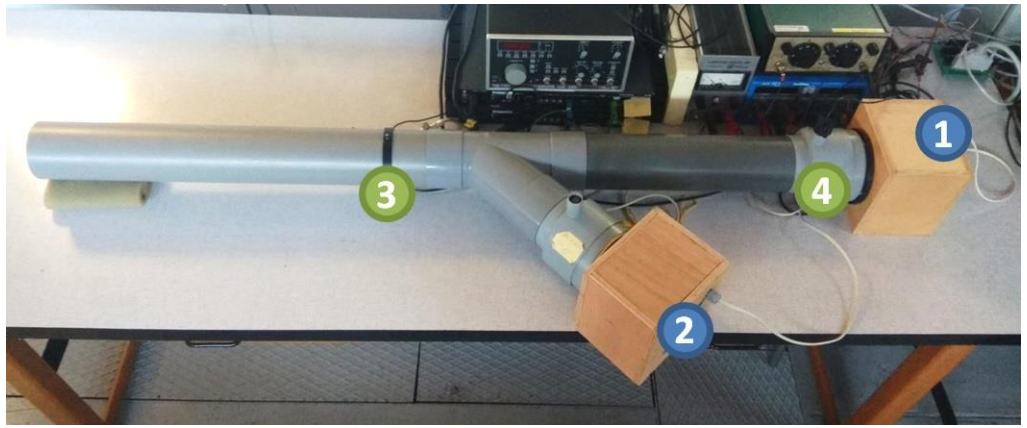
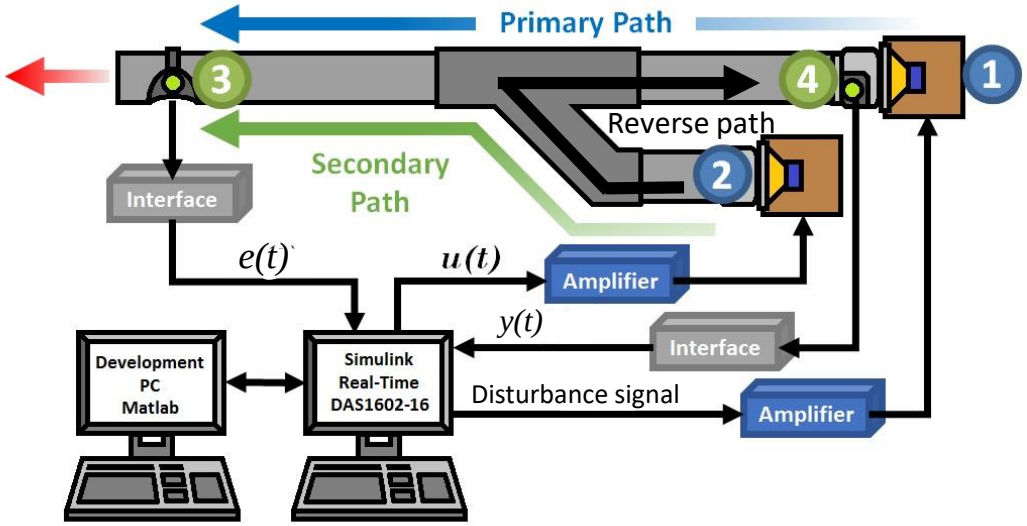
  (one can generate an infinite number of adaptation/learning algorithms!)
- *Stability of adaptation/learning algorithms is a very important issue.*
- ***Stability*** *of the adaptation/learning alg. for a large range of adaptation gains/learning rates requires that the filter acting on the gradient be characterized by a* ***positive real transfer function*** *(* $phase\ lag \leq |90|°\ for\ all\ freq$)
- *Some of the algorithms* ***can improve*** *the performance of the gradient algorithm.*

An example in Adaptive Feedorward Noise Attenuation





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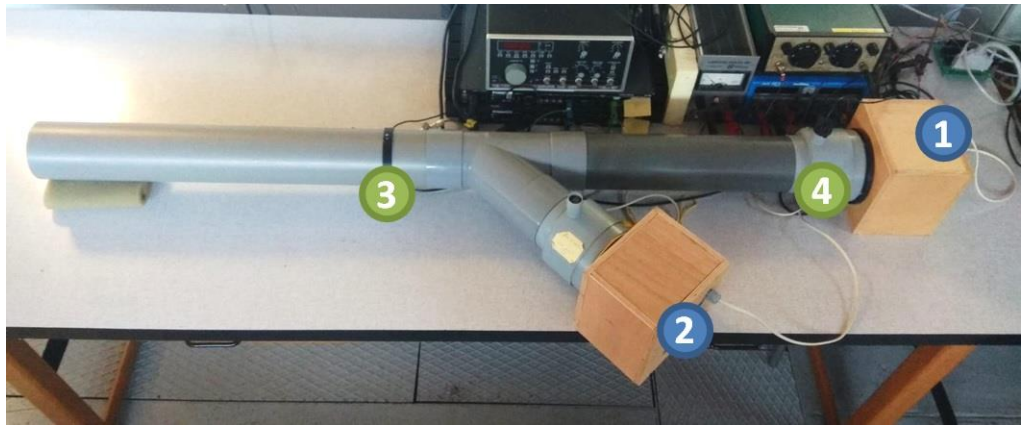

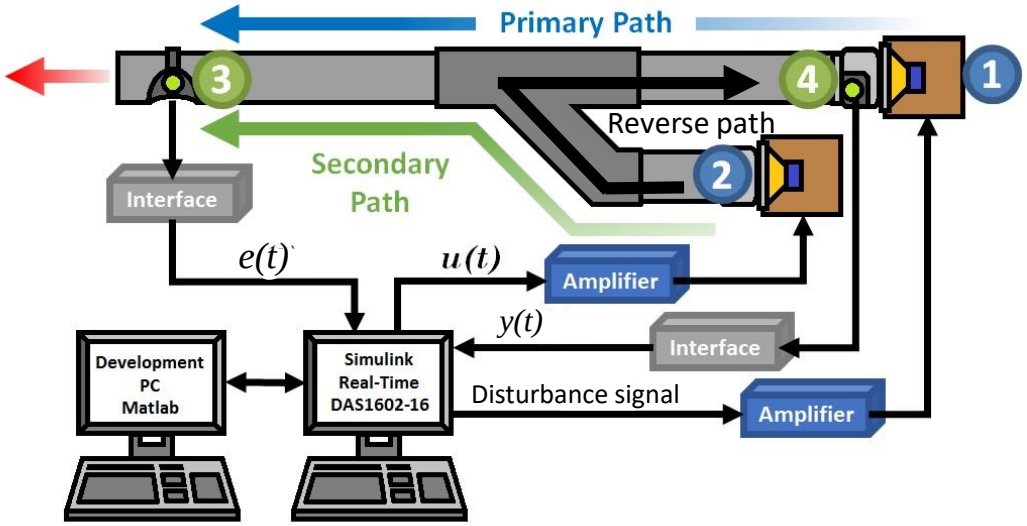

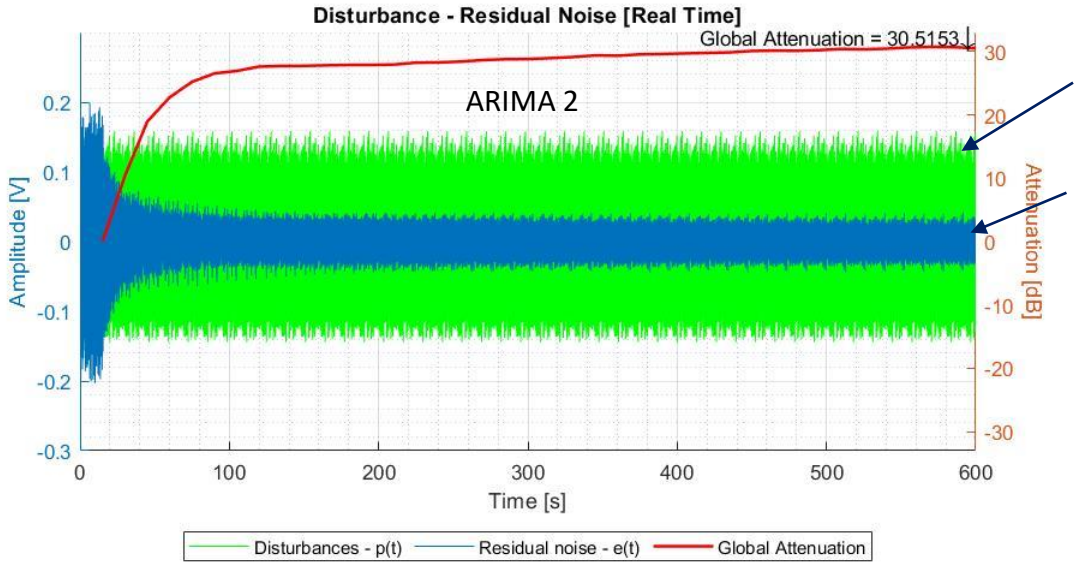

Broad band Noise dist.

Residual Noise

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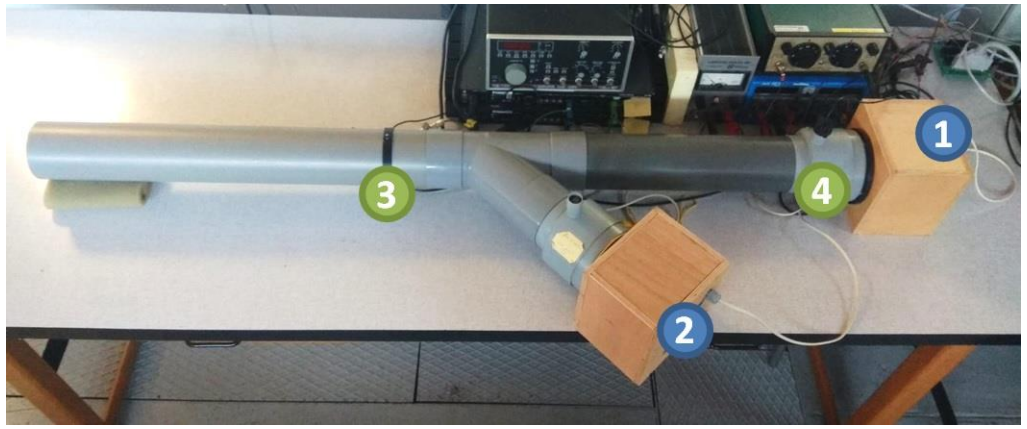

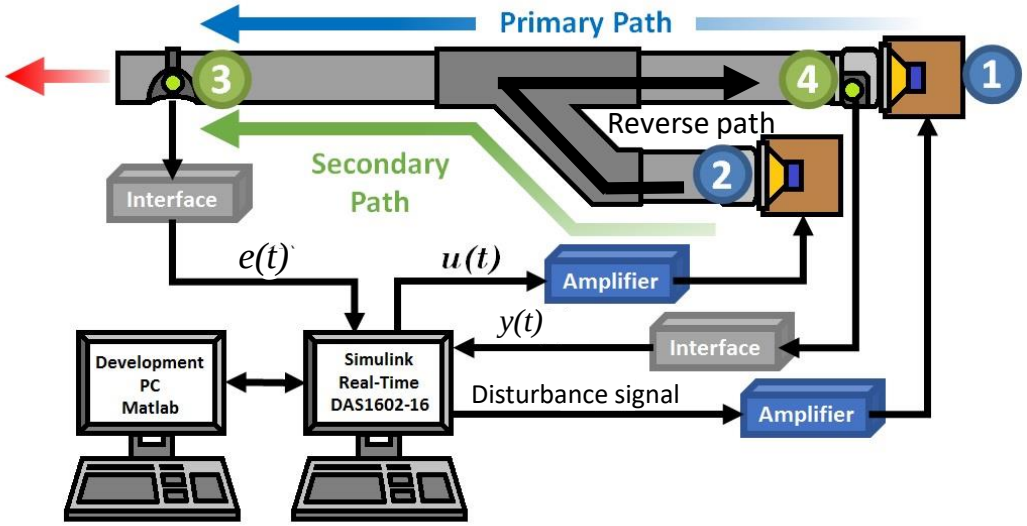

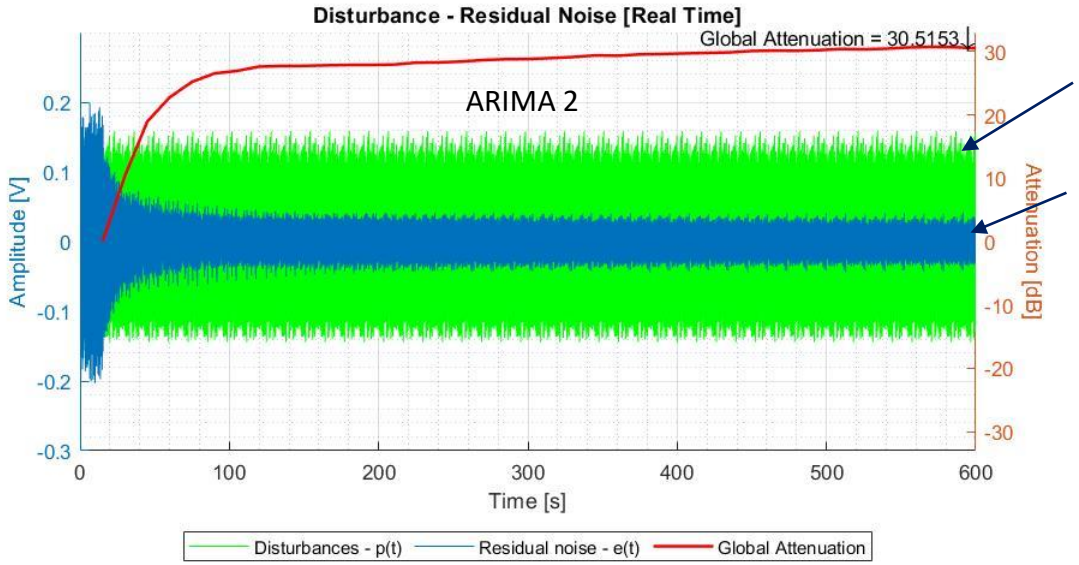

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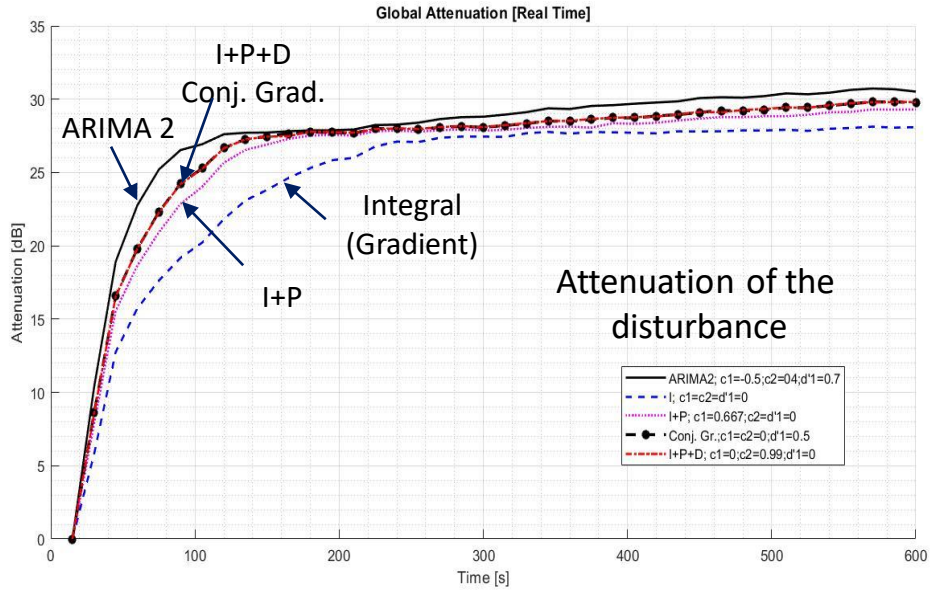

Outline

- Revisiting the gradient algorithm – Feedback interpretation and stability issues
- A general form for adaptation/learning algorithms
- Stability issues for high and low adaptation gains/learning rates (slow and fast adaptation/learning)
- The ARIMA 2 (I+P+DF) algorithm (new)
- A review of existing adaptation/learning algorithms
- Comparisons – Simulation results
- Comparisons – Real time results (adaptive active noise control)
- Concluding remarks

Algorithms for parameter estimation

Discrete time plant model (unknown parameters))

$$y(t+1) = -a_1 y(t) + b_1 u(t) = \theta^T \phi(t)$$

$\theta^T = [a_1, b_1]$ ← Parameter vector ; $\phi(t)^T = [-y(t), u(t)]$

Measurement vector

Vector of attributes

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Adjustable prediction model (*a priori*)

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$\theta(t)^T = [\hat{a}_1(t), \hat{b}_1(t)]$ ← Vector of adjustable parameters

Prediction error (a priori) $\varepsilon^o(t+1) = y(t+1) - \hat{y}^o(t+1) = \varepsilon^o(t+1, \hat{\theta}(t))$

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Parameter adaptation algorithm (learning algorithm) ?

$$\hat{\theta}(t+1) = \hat{\theta}(t) + \Delta\hat{\theta}(t+1) = \hat{\theta}(t) + f\left(\hat{\theta}(t), \phi(t), \varepsilon^o(t+1)\right)$$

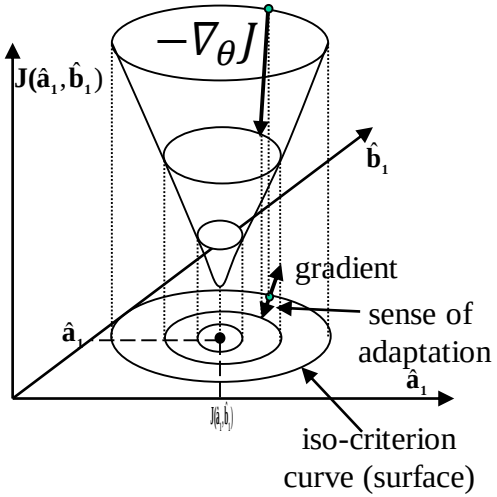
PAA – Gradient algorithm (steepest descent)

Criterion to be minimized (objective): $\min_{\hat{\theta}(t+1)} J(t+1) = [\varepsilon(t+1)]^2$

Other criteria can be considered

Gradient strategy: (*) $\hat{\theta}(t+1) = \hat{\theta}(t) - F\frac{\delta J(t+1)}{\delta \hat{\theta}(t+1)}$ $F = \alpha I \quad (\alpha > 0)$ **(*I* = unit matrix)**

(α – adaptation gain/learning rate)



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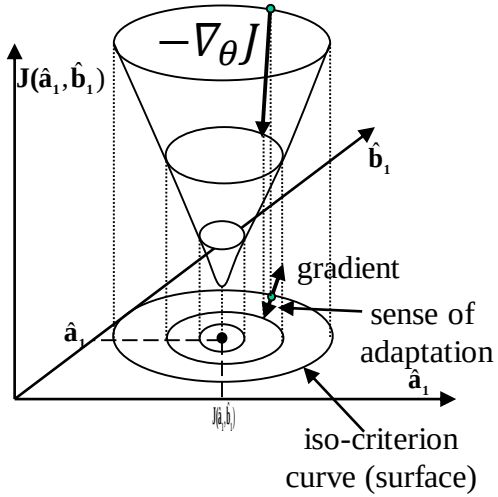

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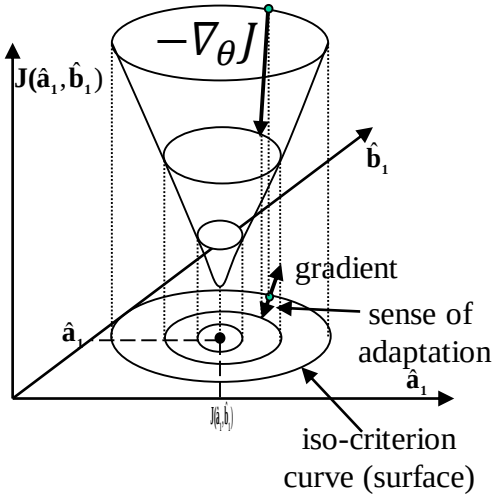

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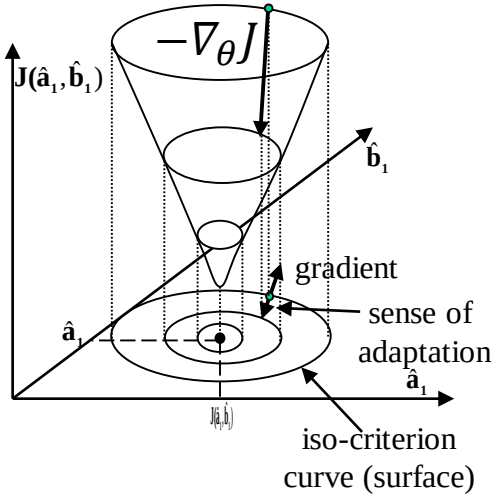

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measurable

(*) $\hat{\theta}(t+1) = \hat{\theta}(t) + F\phi(t)\varepsilon(t+1)$ ← $\varepsilon(t+1) = \dfrac{\varepsilon^o(t+1)}{1+\phi(t)^T F\phi(t)}$

$$\hat{\theta}(t+1) = \hat{\theta}(t) + \frac{F\phi(t)\varepsilon^o(t+1)}{1+\phi(t)^T F\phi(t)}$$

Stable for any $F > 0$ ($\alpha > 0$)

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Parameter adaptation/learning algorithms: a feedback interpretation

Even for the estimation of the parameters of a static model, the adaptation/learning algorithm introduces a dynamic feedback

$y(t+1) = bu(t)$ Unknown model

$\hat{y}(t+1) = \hat{b}(t+1)u(t)$ Adjustable predictor model (a poseriori)

$\varepsilon(t+1) = y(t+1) - \hat{y}(t+1) = \left[b - \hat{b}(t+1)\right]u(t)$ (*) *a posteriori* Prediction error

$\hat{b}(t+1) = \hat{b}(t) + \alpha u(t)\varepsilon(t+1);\ \alpha > 0$ (**) Adaptation/learning alg.

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$\tilde{b}(t+1) = \hat{b}(t+1) - b$ Parameter error

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Feedback system

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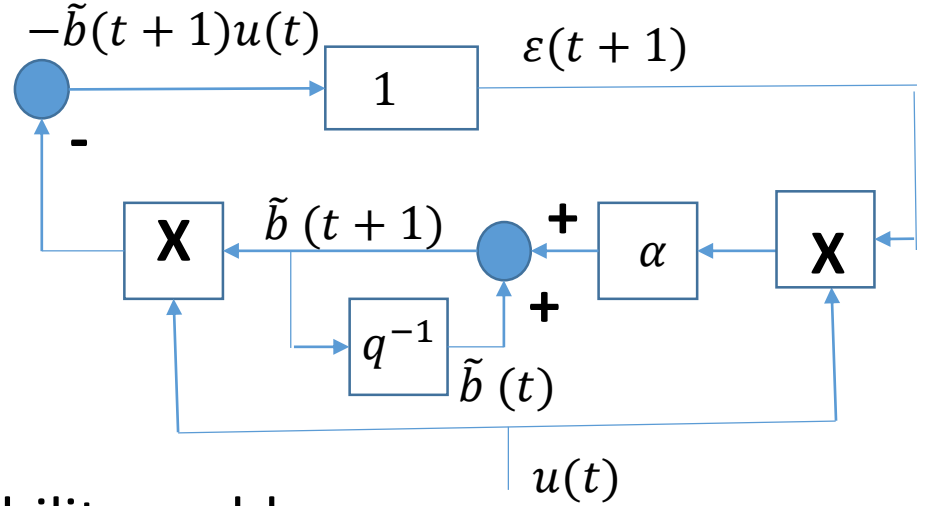

There is an associated asymptotic stability problem

(will the system be **as. stable** for all values of the adaptation gain/learning rate $\alpha > 0$?)

Parameter adaptation/learning algorithms: a feedback interpretation

$$\hat{\theta}(t+1) = \hat{\theta}(t) + F\phi(t)\varepsilon(t+1)$$

$$\tilde{\theta}(t) = \hat{\theta}(t) - \theta$$

$$\phi(t)^T\tilde{\theta}(t+1) = \phi(t)^T\tilde{\theta}(t) + \phi(t)^T F\phi(t)\varepsilon(t+1)$$

Feedback path

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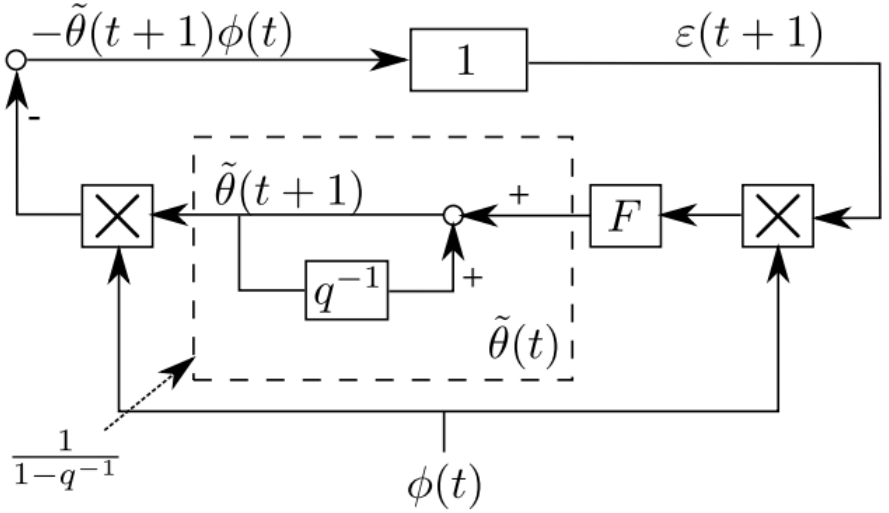
Feedforward path



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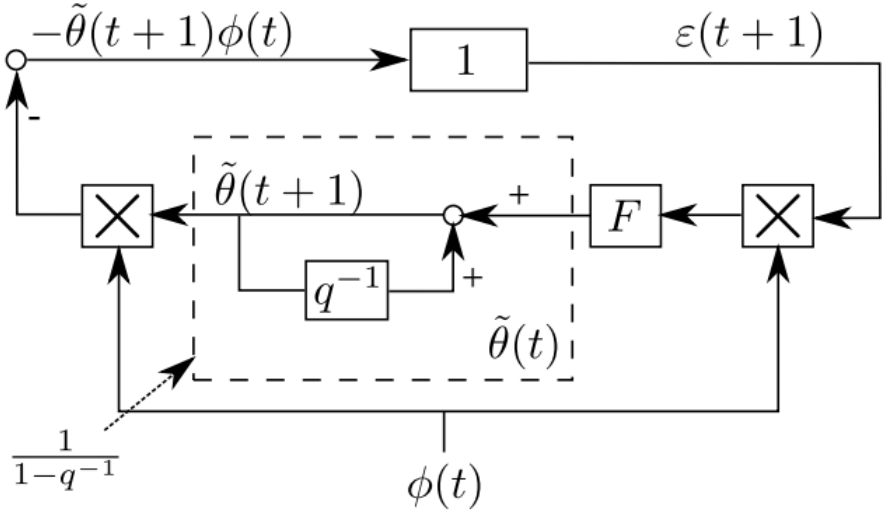
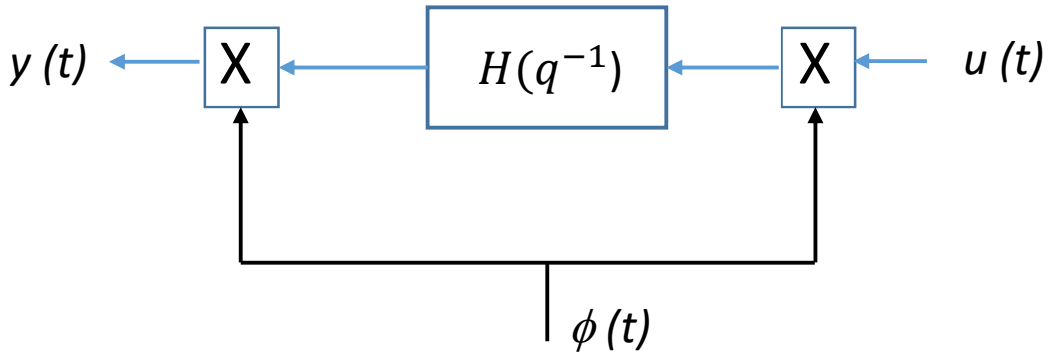
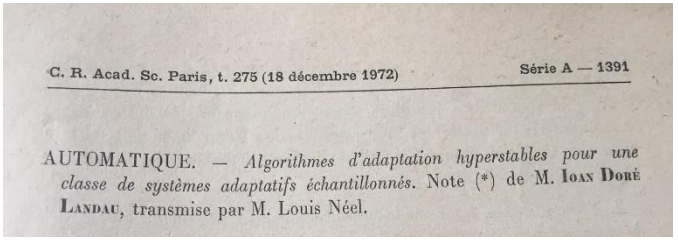


Since the equivalent feedforward path is characterized by a stricly positive real transfer function (it is stricly passive) the feedback system is Globally Asymptotically Stable for **all the equivalent feedback paths which are passive.**
(based on V. M. Popov – Hyperstability, 1962)

An old result (1972)





C. R. Acad. Sc. Paris, t. 275 (18 décembre 1972) Série A — 1391

AUTOMATIQUE. — *Algorithmes d'adaptation hyperstables pour une classe de systèmes adaptatifs échantillonnés.* Note (*) de M. IOAN DORÉ LANDAU, transmise par M. Louis Néel.

Passivity: $\sum_0^{t_1} y(t,\tau)u(t) \geq -\gamma^2 \; ; \gamma^2 < \infty; \; \forall \; t_1 \geq 0 \; (\tau \leq t)$

The system (input u –output y) is **passive** provided that $H(z^{-1})$ is a **positive real transfer function**

Positive real transfer function : stable, $-90° \leq$ input-output phase lag/advance $\leq 90°$)

A Transfer Operator interpretation of the Gradient Algorithm

$$\hat{\theta}(t+1) = \hat{\theta}(t) + F\phi(t)\varepsilon(t+1)$$

$\Rightarrow$ $\hat{\theta}(t+1)(1-q^{-1}) = \alpha\,\phi(t)\epsilon(t+1)$ $\Rightarrow$ $\hat{\theta}(t+1) = \frac{1}{1-q^{-1}}\alpha\,\phi(t)\epsilon(t+1)$

$$F = \alpha I \quad (\alpha > 0)$$

$$\hat{\theta}(t+1) = \hat{\theta}(t) - F\nabla_{\theta} J(t+1)$$

$\Rightarrow$ $\hat{\theta}(t+1) = \frac{1}{1-q^{-1}}\alpha[-\nabla_{\theta} J(t+1)]$

Gradient of J with respect to θ

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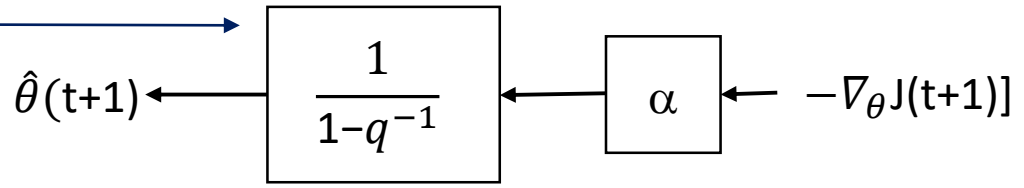
Gradient of J with respect to θ

- It is an « integrator » filter
- It is a *passive* system
- Characterized by a *positive real* transfer function
- $-90° \leq$ input-output phase lag/advance $\leq 90°$



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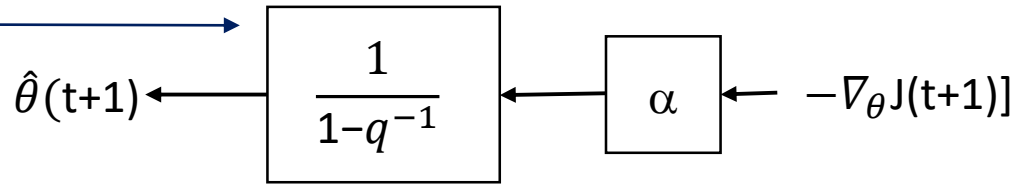
- Characterized by a *positive real* transfer function
- $-90° \leq$ input-output phase lag/advance $\leq 90°$



Adaptation gain/learning rate

$$\hat{\theta}(t+1) = H_{PAA}(q^{-1})\alpha[-\bigtriangledown_\theta J(t+1)]$$

MIMO diagonal transfer operator with identical terms ($H^{ii}(q^{-1}) = \frac{1}{1-q^{-1}}$ for gradient alg.)

Main Result

$$\hat{\theta}(t+1) = H_{PAA}(q^{-1})\alpha[-\nabla_{\theta} J(t+1)]$$

$$H_{PAA}=\begin{bmatrix} H_{11} & & \\ & H_{ii} & \\ & & H_{nn} \end{bmatrix}$$

Objective : $\lim_{t\to\infty} \varepsilon(t+1) = 0$ *for any initial conditions* $\theta(0), \varepsilon(0)$

The global asymptotic stability is assured for any positive adaptation gain/learning rate provided that the H_{ii} operators are characterized by a **positive real transfer function** with a pole at z=1 (for memory)

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ARIMA Structure

$$H^{ii}(q^{-1}) = \frac{1 + c_1 q^{-1} + c_2 q^{-2} + .. + c_{nC} q^{-nC}}{(1-q^{-1})(1 - d'_1 q^{-1} - d'_2 q^{-2} - ..d'_{n_{D'}} q^{-n'_D})}$$

$$= \frac{C(q^{-1})}{(1-q^{-1})D'(q^{-1})} = \frac{C(q^{-1})}{D(q^{-1})} = \frac{1 + c_1 q^{-1} + c_2 q^{-2} + .. + c_{nC} q^{-nC}}{1 - d_1 q^{-1} - d_2 q^{-2} .. - d_{n_D} q^{-n_D}}$$

$$d_i = (d'_i - d'_{i-1}) \;\; ; i = 1, ...n_D; d'_0 = -1, \; d'_{n_D} = 0$$

ARIMA2 (I+P+DF) Algorithm (new)

$$H^{ii}(q^{-1}) = \frac{1 + c_1 q^{-1} + c_2 q^{-2}}{1 - d_1 q^{-1} - d_2 q^{-2}} = \frac{1 + c_1 q^{-1} + c_2 q^{-2}}{(1 - q^{-1})(1 - d_1' q^{-1})}$$

$$\hat{\theta}(t+1) = d_1 \hat{\theta}(t) + d_2 \hat{\theta}(t-1) + F[\phi(t)\varepsilon(t+1) + c_1 \phi(t-1)\varepsilon(t) + c_2 \phi(t-2)\varepsilon(t-1)]$$

$$d_1 = (1 + d_1'); \quad d_2 = -d_1'$$

(to assure the presence of the integrator)

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*If we want to guarantee stability for **any value** of the adaptation gain/learning rate:*

The weights: d_1', c_1, c_2 should be chosen such that:

H^{ii} be characterized by a **positive real** transfer function

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*If we want to guarantee stability for **any value** of the adaptation gain/learning rate:*

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H^{ii} be characterized by a **positive real** transfer function

Define: $\delta = \frac{1 + c_1 + c_2}{1 - d_1'}; \quad \gamma = \frac{d_1' c_1 + d_1'^2 + c_2}{d_1' - 1}$

PR conditions (B. Vau) :

$$-1 < d_1' < 1 \; ; \quad 0 \le \delta \le 2 \quad ; \quad -1 \le d_1' - \frac{\gamma}{1 - \delta/2} \le 1$$

ARIMA2 (I+P+DF) Algorithm (new)

This algorithm can be interpreted as an *Integral + Proportional + Filtered Derivative* adaptation algorithm

$$H^{ii}(q^{-1}) = \boxed{\frac{\alpha_I}{1-q^{-1}} + \alpha_P + \alpha_D \frac{(1-q^{-1})}{(1-d_1' q^{-1})}} = \frac{1 + c_1 q^{-1} + c_2 q^{-2}}{(1-q^{-1})(1-d_1' q^{-1})} \qquad (*)$$

Bringing (*) to a common denominator, one gets:

$$c_1 = \frac{-\alpha_I d_1' - \alpha_P(1+d_1') - 2\alpha_D}{\alpha_T} \qquad c_2 = \frac{d_1' \alpha_P + \alpha_D}{\alpha_T} \qquad \alpha_T = \alpha_I + \alpha_P + \alpha_D$$

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Conversely : given c_1, c_2, d_1' one gets:

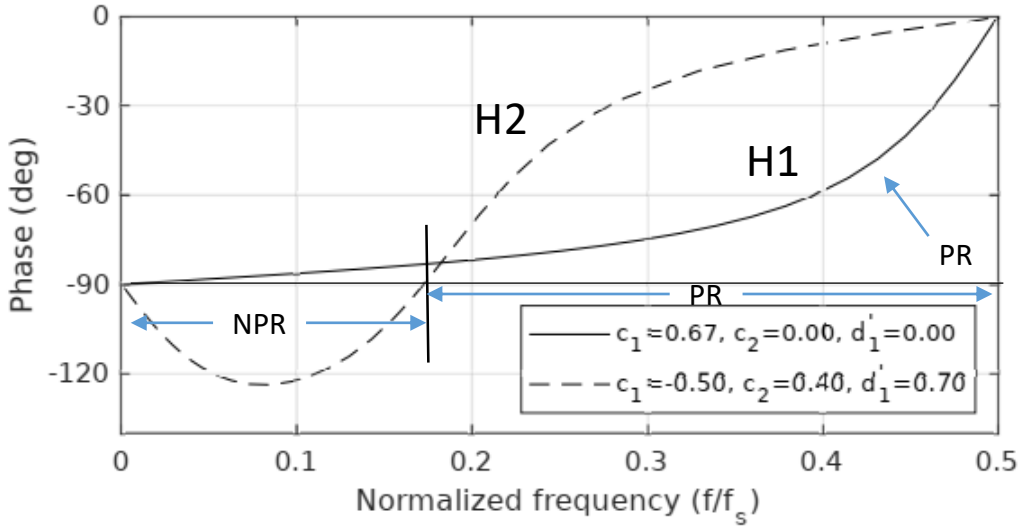
$$\alpha_I = \frac{1 + c_1 + c_2}{1 - d_1'} \qquad \alpha_P = -\frac{c_1 + c_2(2 - d_1') + d_1'}{(1-d_1')^2} \qquad \alpha_D = c_2 - \alpha_P d_1' \qquad \alpha_T = \alpha_I + \alpha_P + \alpha_D = 1$$

The new value of the estimated parameters is a weighted sum (with minus sign) of: the integral of the gradient, the gradient and its filtered derivative

$$\hat{\theta}(t+1) = H_{PAA}(q^{-1}) \alpha [- \nabla_\theta J(t+1)]$$

Is the PR condition necessary when operating withsmall adaptation gain/learning rate?

Answer: No. There are however signal dependent relaxed conditions to be fullfilled.



H_{ii} can be « non positive real » in some frequency regions provided that the regions where it is « positive real » exceed those where it is not PR.
(the averaged input/output energy should be positive)

An example :

Assume that the excitation signal cover the region from 0.1 f/fs to 0.4 f/fs with an almost constant energy.
The system H2 is non PR from 0.1f/fs to 0.17f/fs and PR from 0.17 to 0.4.f/fs. It will works for small adaptation gains
One can say that is « PR in the average »

Review of PALA algorithms

$$H^{ii}(q^{-1}) = \frac{1 + c_1 q^{-1} + c_2 q^{-2}}{1 - d_1 q^{-1} - d_2 q^{-2}} = \frac{1 + c_1 q^{-1} + c_2 q^{-2}}{(1 - q^{-1})(1 - d'_1 q^{-1})}$$

IMA

- Integral + Proportional: $c_1 \neq 0$; c_2=0; d'_1=0
- Int. + Prop. + Derivative: $c_1 \neq 0$; $c_2 \neq 0$; d'_1=0
- Averaged gradient : $c_1 \neq 0$; $c_2 \neq 0$; d'_1=0 ($c_i \neq 0$)

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ARI

- Conjugate gradients: $c_1 = 0$; c_2=0; $d'_1 \neq 0$
- Nesterov Algorithm: $c_1 = 0$; c_2=0; $d'_1 \neq 0$
- Momentum back propagation: $c_1 = 0$; c_2=0; $d'_1 \neq 0$ $\alpha' = \alpha(1 - d'_1)$

Leakage algorithm: $H^{ii}(q^{-1}) = \frac{1}{1 - \sigma q^{-1}}$; $0 < \sigma < 1$

Review of PALA algorithms

$$H^{ii}(q^{-1}) = \frac{1 + c_1 q^{-1} + c_2 q^{-2}}{1 - d_1 q^{-1} - d_2 q^{-2}} = \frac{1 + c_1 q^{-1} + c_2 q^{-2}}{(1 - q^{-1})(1 - d'_1 q^{-1})}$$

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Leakage algorithm: $H^{ii}(q^{-1}) = \frac{1}{1 - \sigma q^{-1}}$; $0 < \sigma < 1$

ARIMA2 can be viewed as a combination of I+P+D and Conjugate gradients

Challenge: Find a parameter adaptation/learning algorithm which does not have an ARMA structure.

Simulation Results

Estimation of the parameters of: $S = \dfrac{q^{-2} + 0.5q^{-3}}{1 - 1.5q^{-1} + 0.7q^{-2}}$ (Input: PRBS)

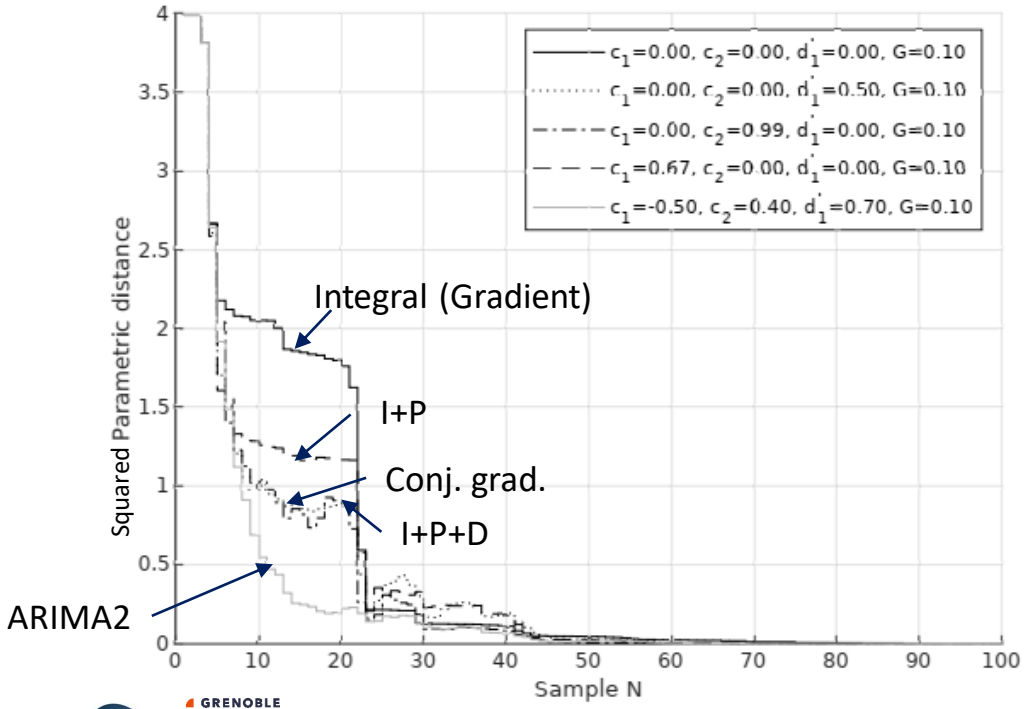
Performance indices:

$$J_\varepsilon(N) = \sum_{t=0}^{N} \varepsilon^2(t+1)$$

$$D^2(t) = \{[\theta - \hat{\theta}(t)]^T [\theta - \hat{\theta}(t)]\}$$

Squared Parametric Distance

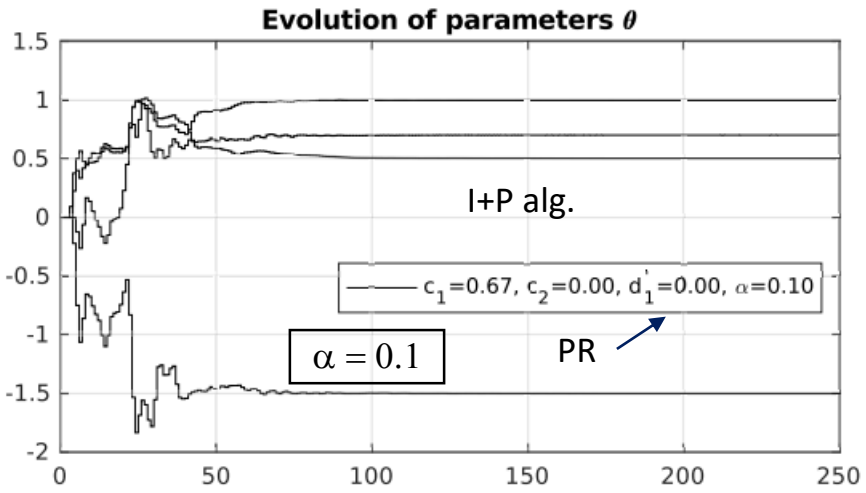
$$J_D(N) = \sum_{t=0}^{N} D^2(t)$$

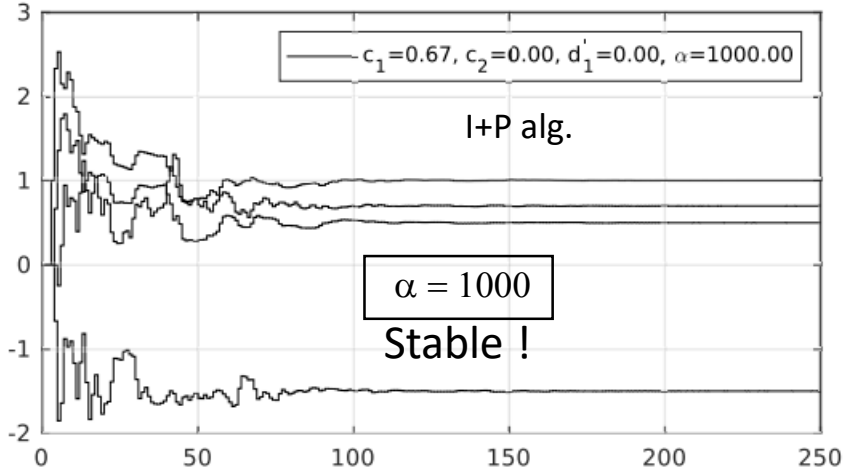


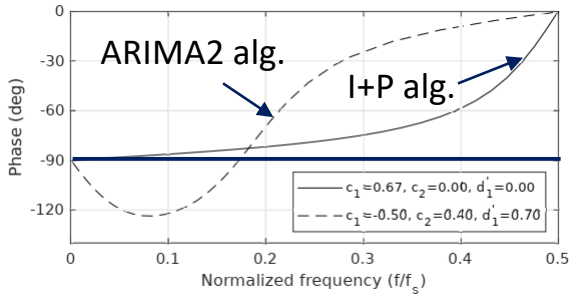
$\alpha = 0.1$ (adaptation gain)

Algorithm	PR	c_1	c_2	d_1'	$J_D(N)$	$J_\epsilon(N)$
Integral (gradient)	Y	0	0	0	**51.65**	**13.32**
Conj.Gr/Nest..	N	0	0	0.5	37.15	12.09
I+P+D ($\alpha_P = -2\alpha_D$)	N	0	0.99	0	34.58	11.95
I+P	Y	0.667	0	0	41.41	12.45
ARIMA 2	N	−0.5	0.4	0.7	**26.62**	**9.67**

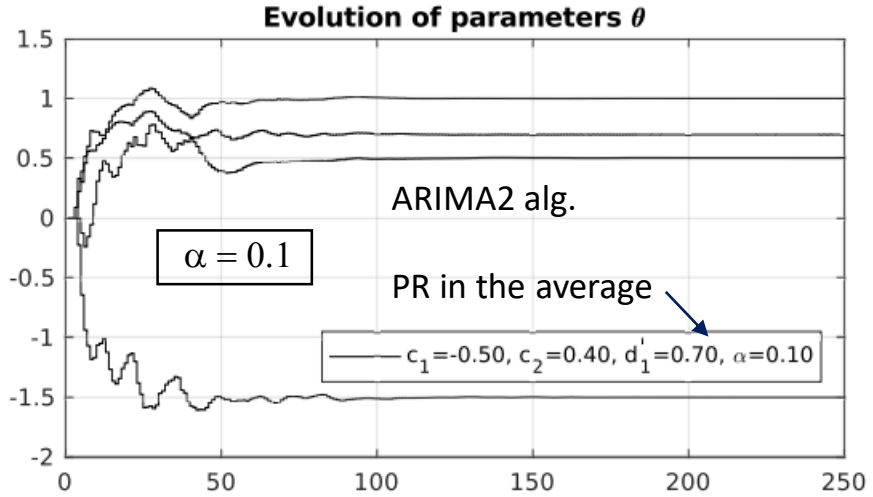
Simulation results – Stability issues

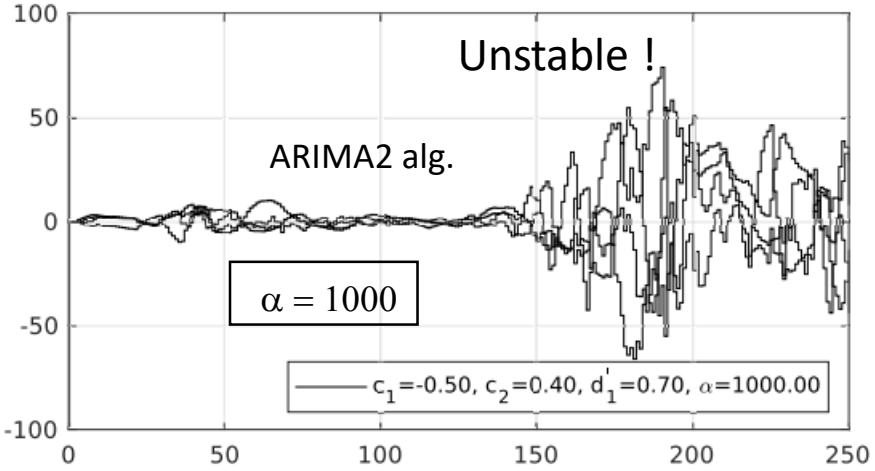




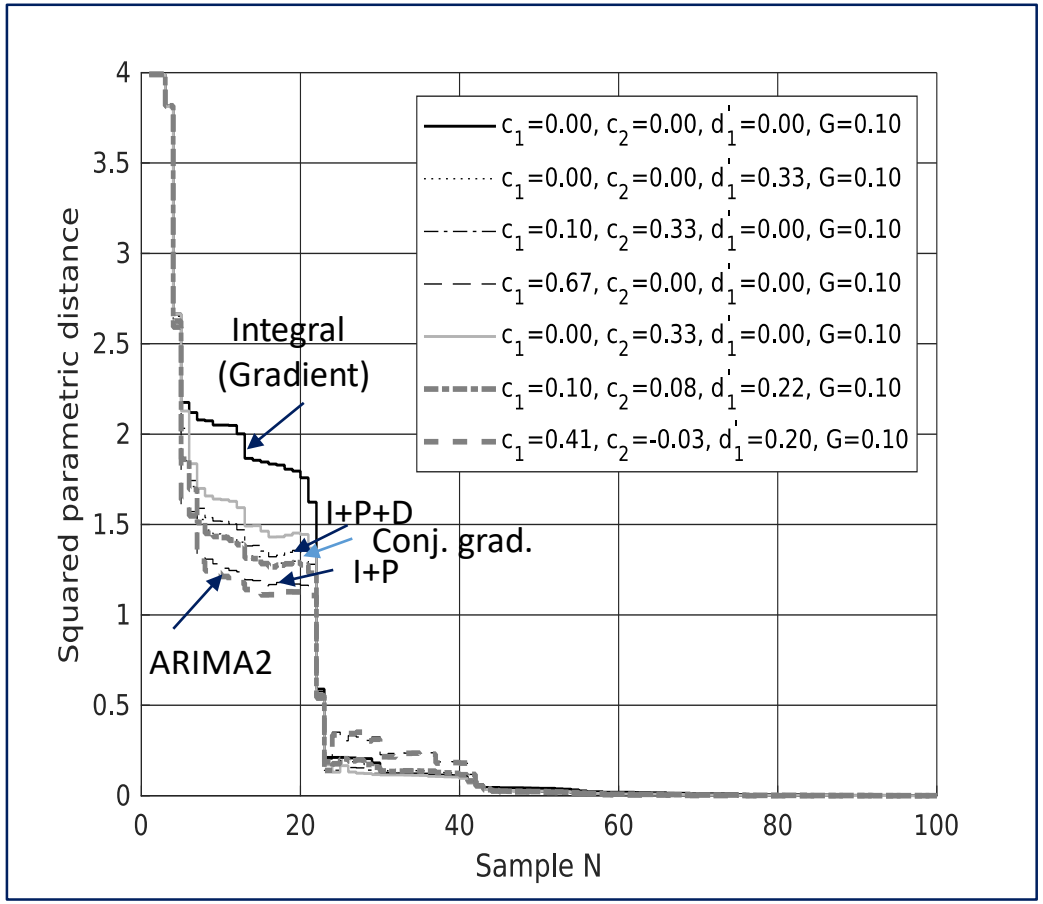


α= adaptation gain





Simulation results under the « positive real » constraint



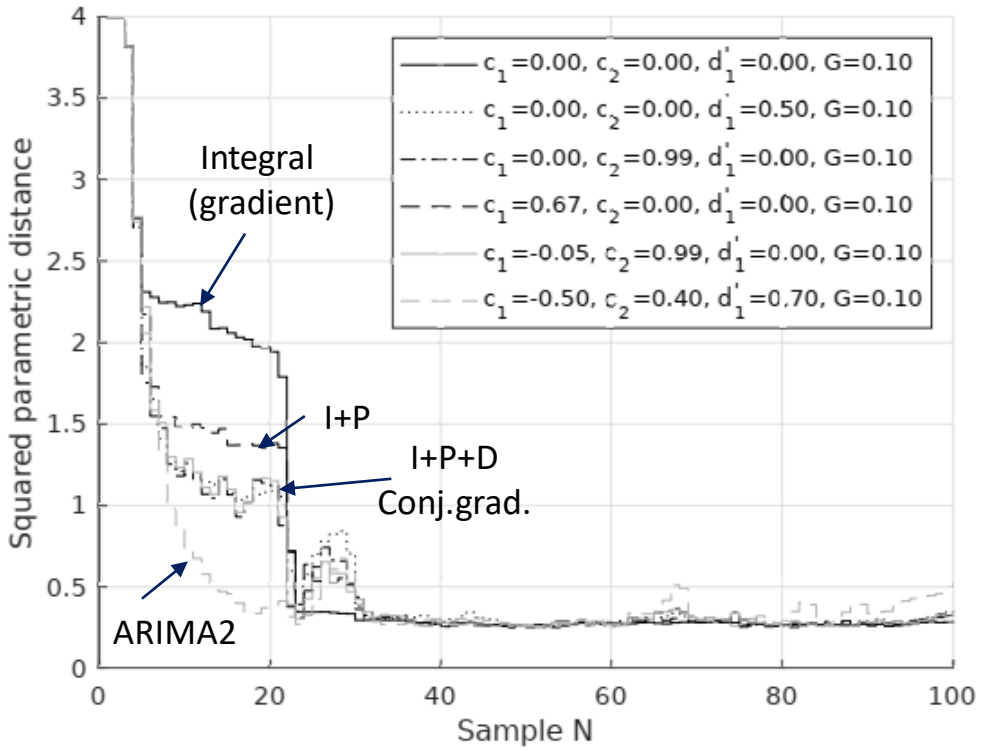
Algorithm	PR	c_1	c_2	d_1'	$J_D(N)$	$J_\epsilon(N)$
Integral	Y	0	0	0	**51.65**	**13.32**
Conj.Gr/Nest..	Y	0	0	0.333	42.16	**11.99**
I+P+D	Y	0.1	0.333	0	42.91	12.04
I+P	Y	0.667	0	0	41.41	12.45
I+P+D/Av.Gr	Y	0	0.33	0	44.655	12.21
ARIMA 2	Y	0.0989	0.0789	0.22	41.96	**11.99**
ARIMA 2	Y	0.408	−0.032	0.2	**40.59**	12.39

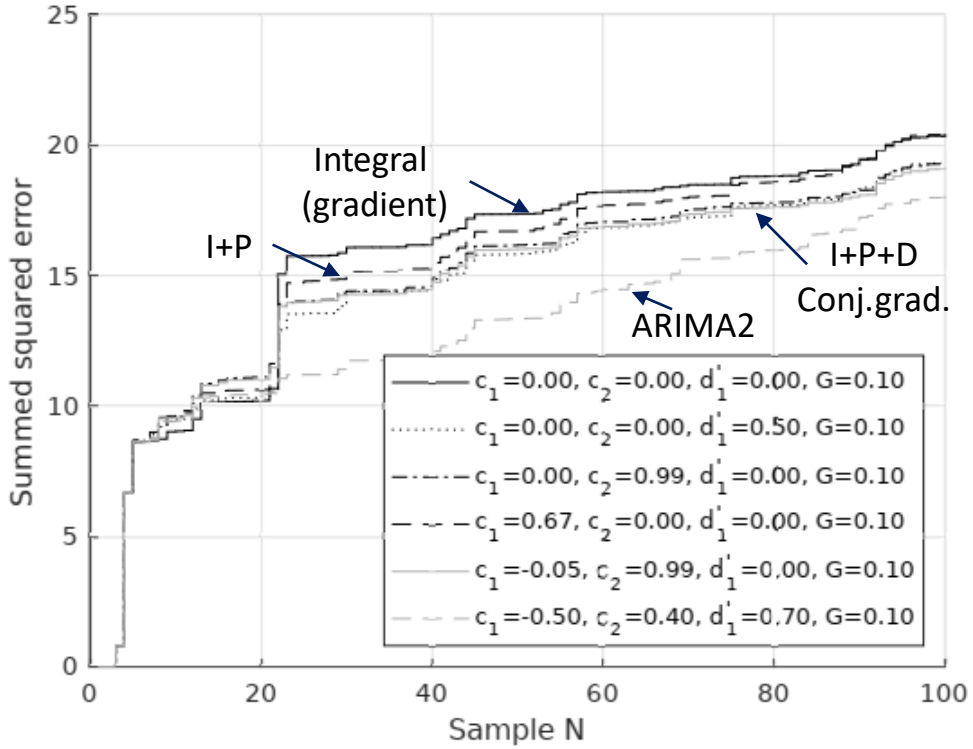
- The improvement in performance is less significant
- Small differences in performance between various algorithms
- Are these weights values the best ?

Simulation results – Imperfect matching

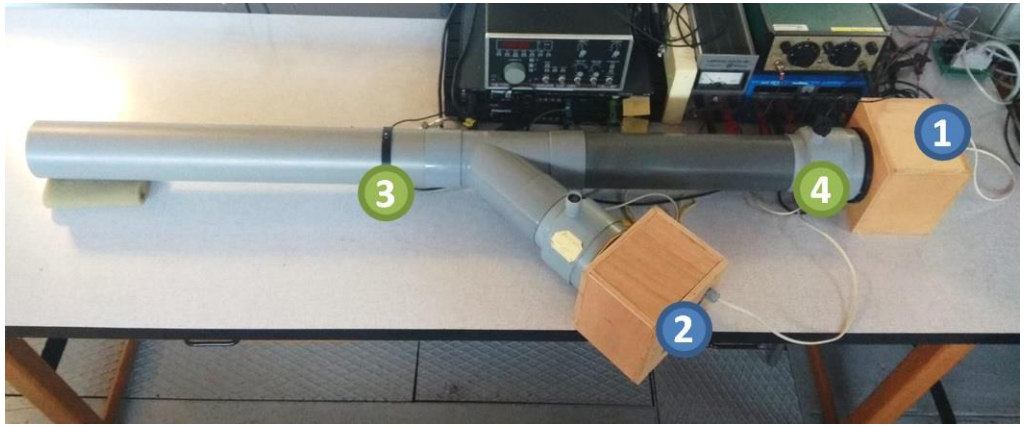
$$S = \frac{q^{-2} + 0.5q^{-3}}{1 - 1.5q^{-1} + 0.7q^{-2}} \qquad \hat{S} = \frac{\hat{b}_2 q^{-2}}{1 + \hat{a}_1 q^{-1} + \hat{a}_2 q^{-2}}$$

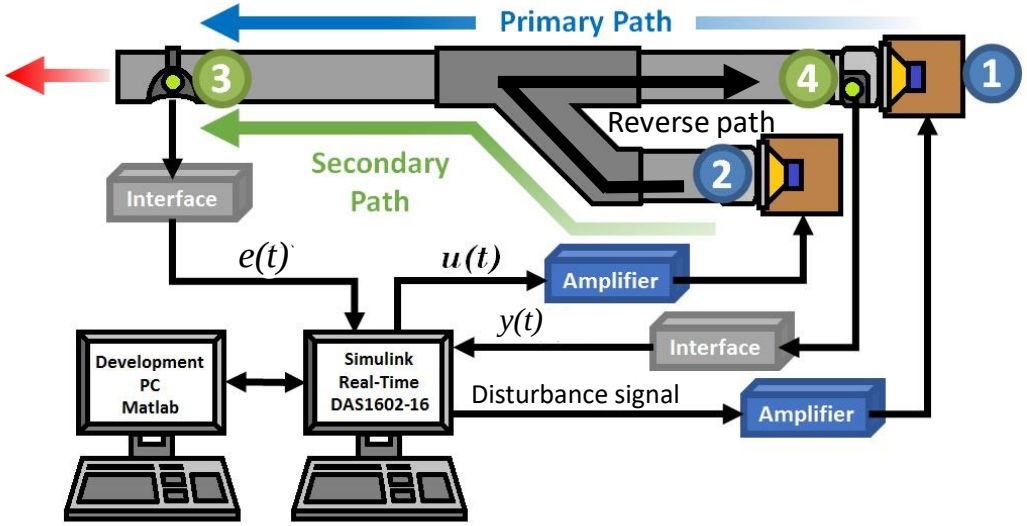
The prediction (adaptation) error and the parametric distance will no go to zero !

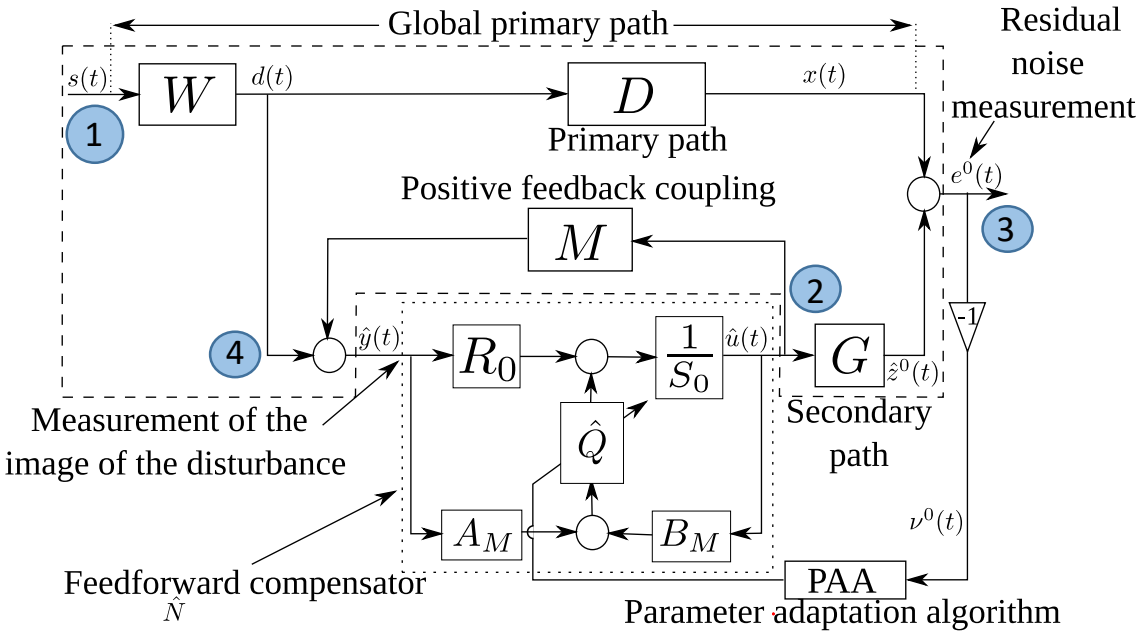




Experimental Results: Adaptive Feedorward Noise Attenuation



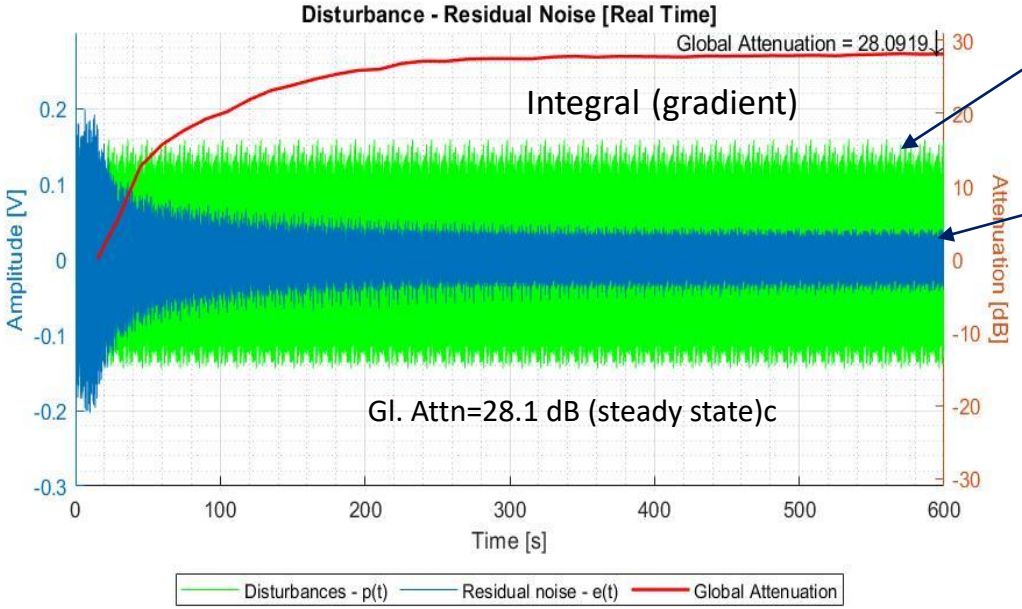




- Broad band noise disturbance (70 – 170 Hz)
- Youla-Kucera parametrized feedforward compensator
- Q- adjustable FIR filter (60 parameters)
- Adaptation gain/learning rate: $\alpha = 0.2$

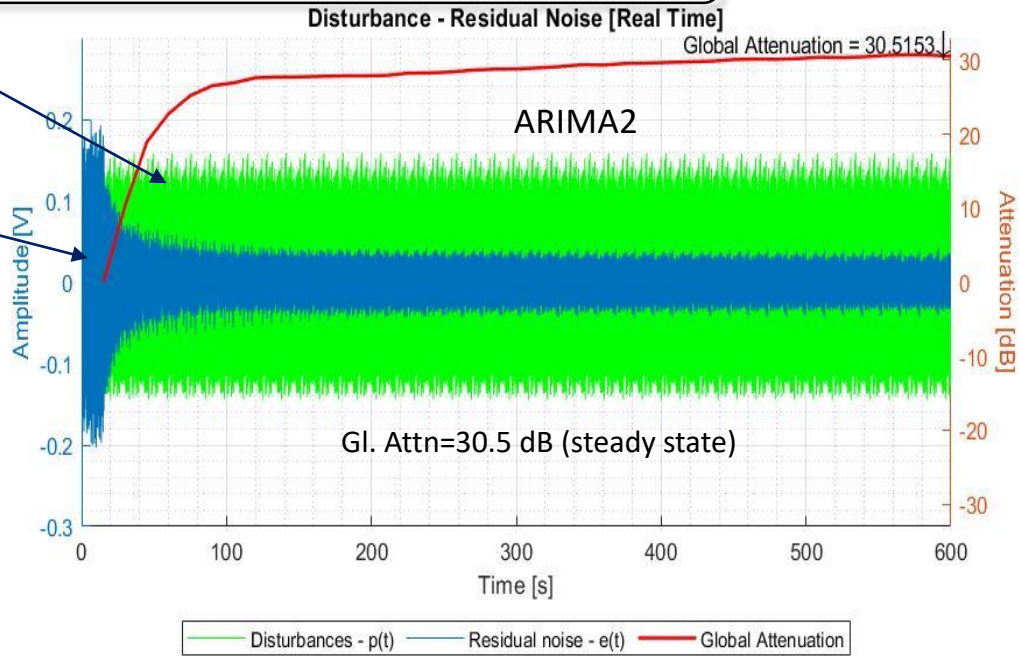
I.D. Landau & al. "Does it exists a general form for adaptation/learning algorithms"

Experimental Results: Adaptive Feedorward Noise Attenuation



Broad band Noise dist. (70 -170 Hz)

Residual Noise



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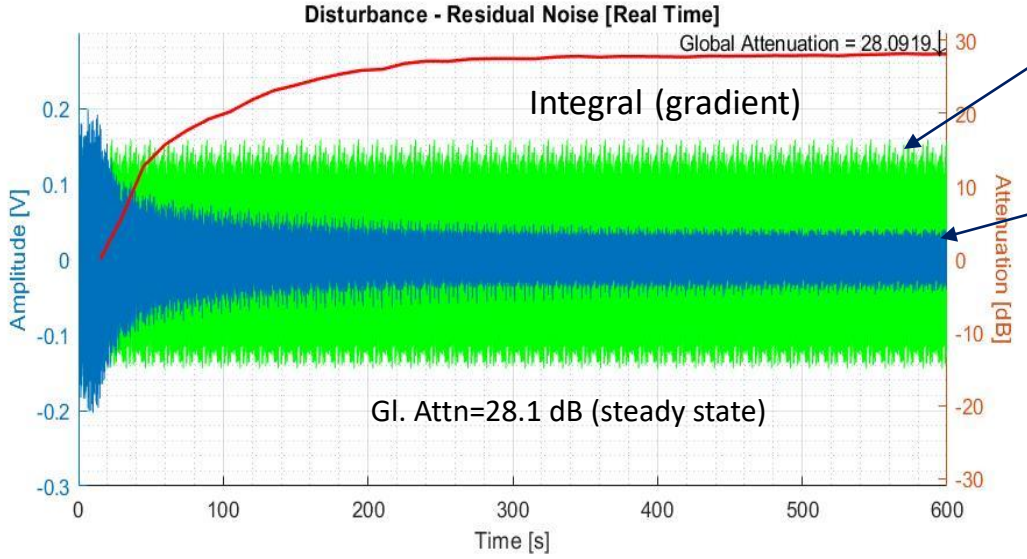

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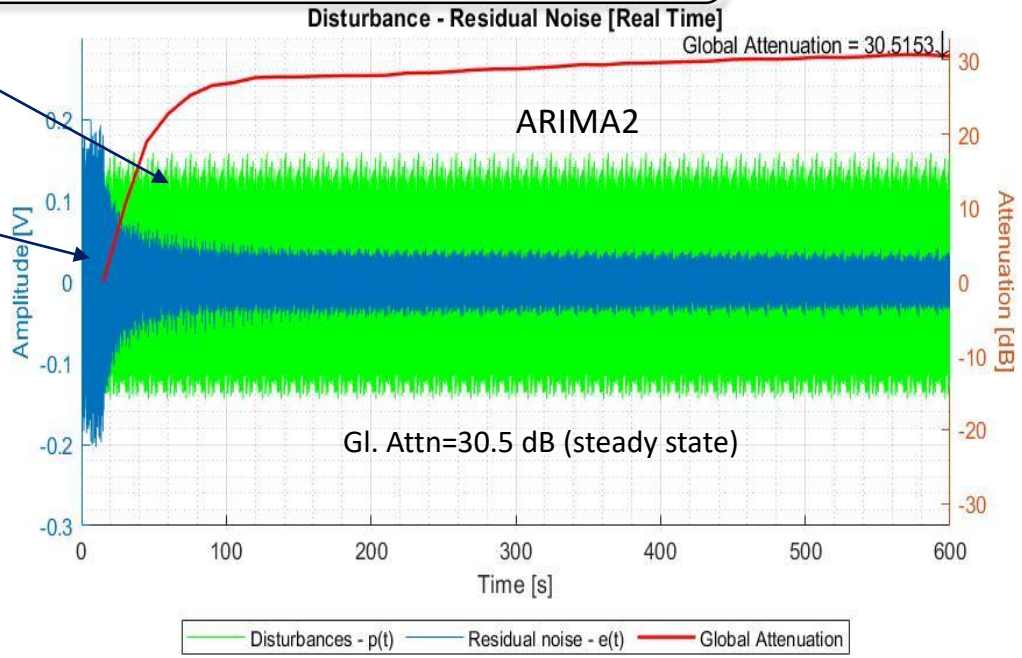

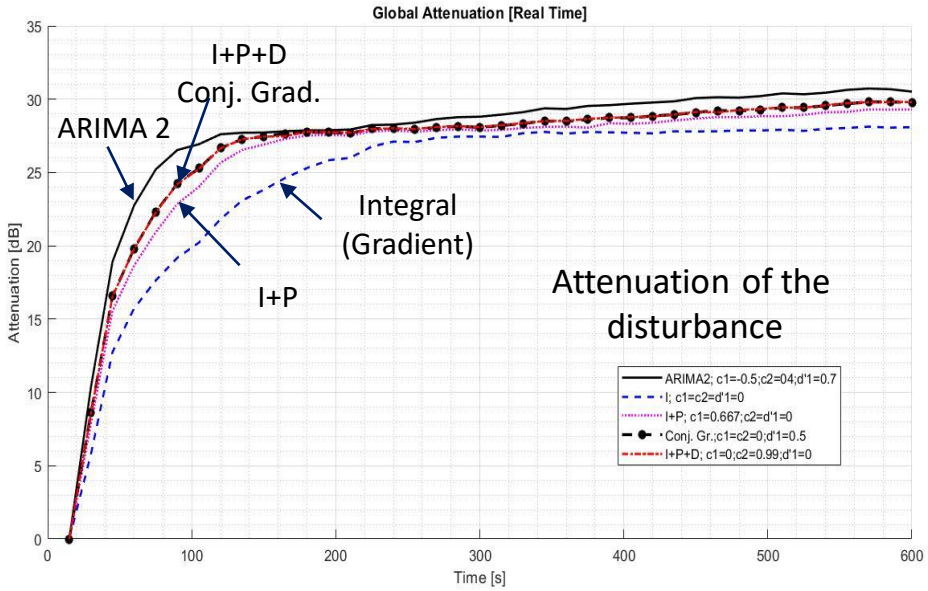

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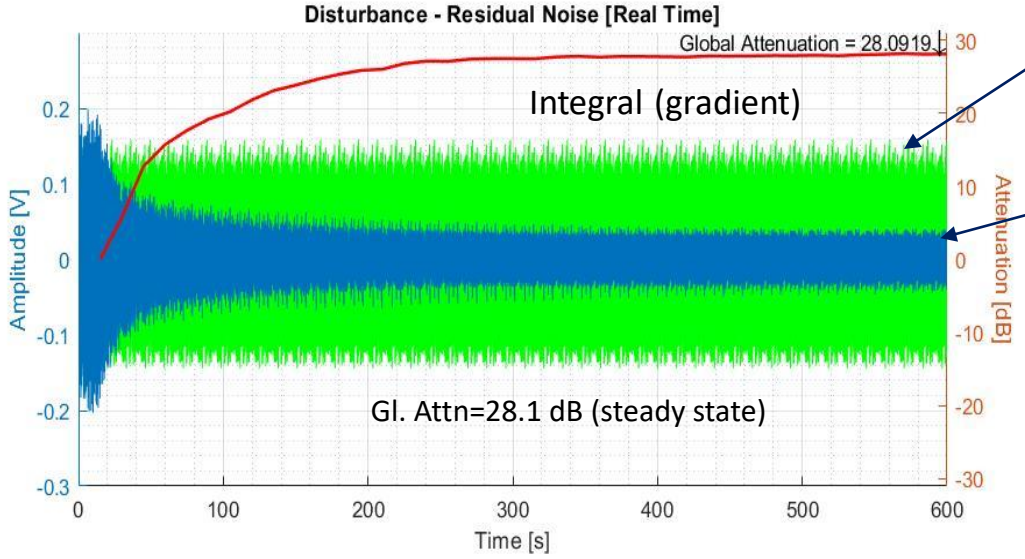

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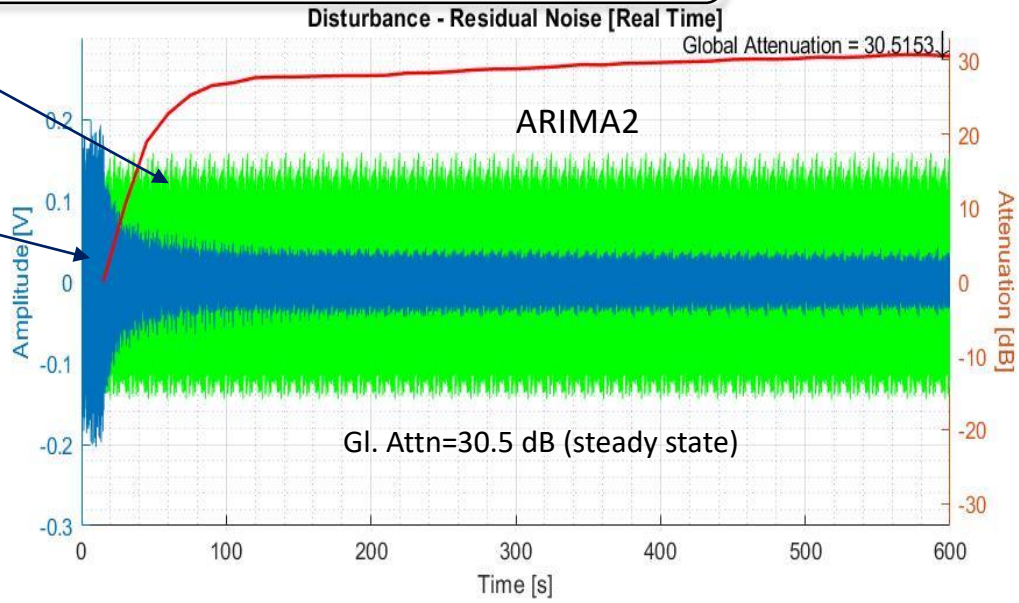
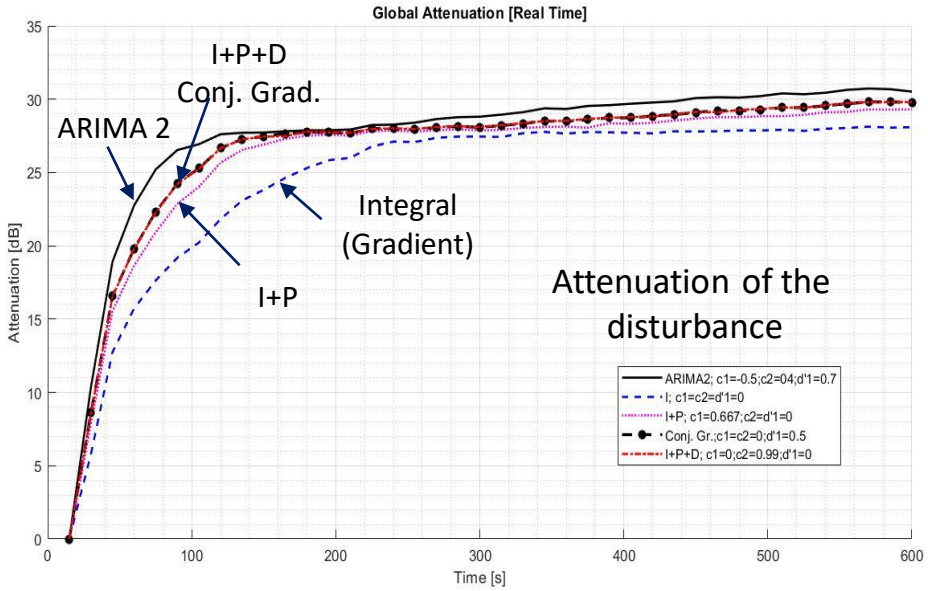
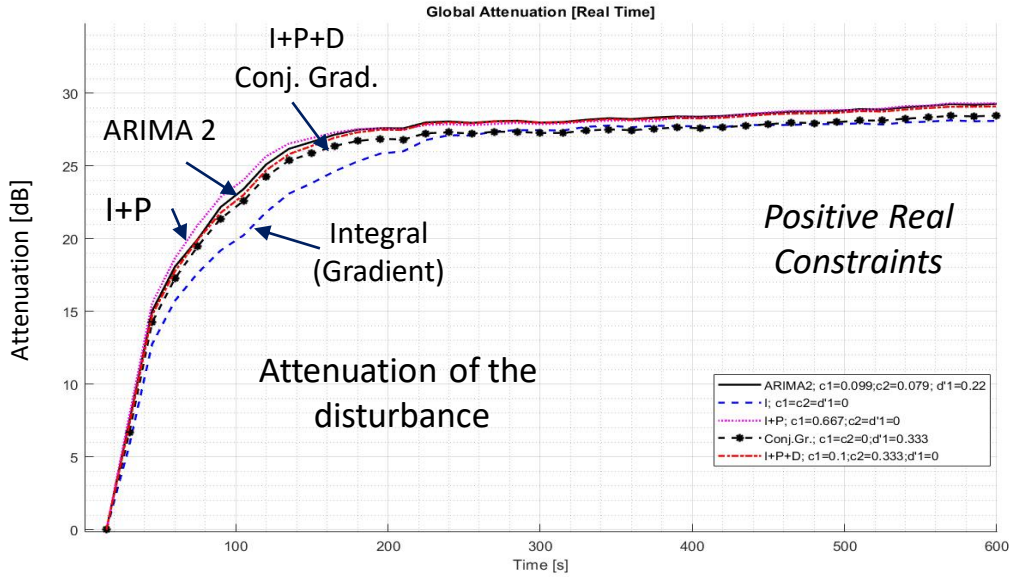






A new concept: *Dynamic adaptation gain/learning rate* (frequency dependend adaptation gain/learning rate)

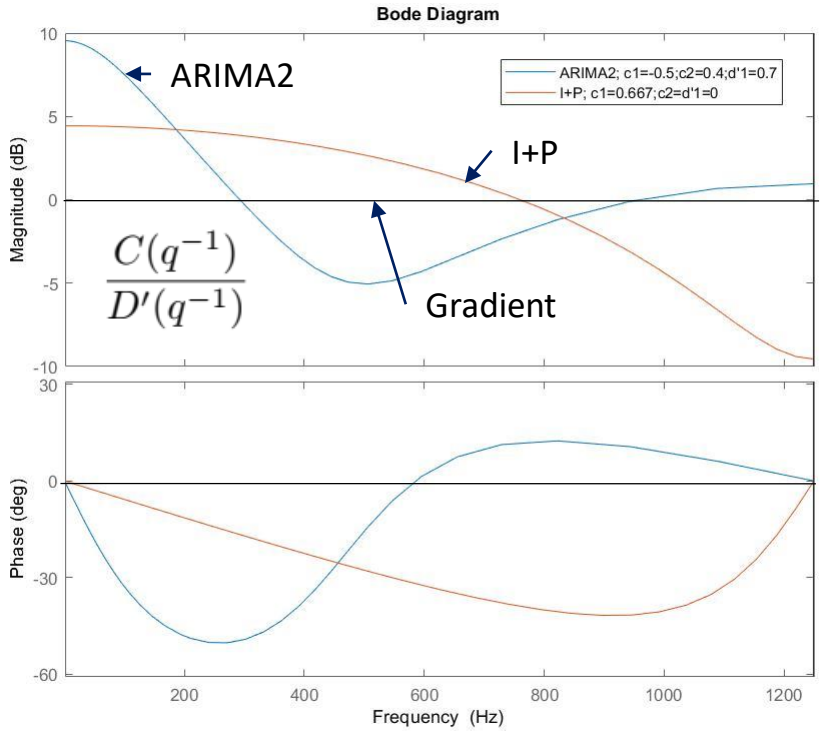
$$\hat{\theta}(t+1) = H_{PAA}(q^{-1})\alpha[-\nabla_\theta J(t+1)]$$

$$H_{PAA} = \begin{bmatrix} H_{11} & & \\ & H_{ii} & \\ & & H_{nn} \end{bmatrix}$$

$$H^{ii}(q^{-1}) = \frac{C(q^{-1})}{(1-q^{-1})D'(q^{-1})}$$

$$\hat{\theta}(t+1) = \hat{\theta}(t) + \alpha\frac{C(q^{-1})}{D'(q^{-1})}[-\nabla_\theta J(t+1)]$$

Dynamic adaptation gain/learning rate



If C and D' have all the zeros inside the unit circle ➞ average gain=0!

Concluding Remarks

- The « gradient » classical strategy can be generalized (the integrator is replaced by an ARIMA filter)
- Stability of the adaptive/learning scheme for any magnitude of the adaptation gain/learning rate requires that this embedded filter be characterized by a *positive real* transfer function
- For slow adaptation/learning the above condition can be relaxed
- The algorithms using an ARIMA filter can improve the performance of the « gradient » algorithm
- The optimal choice of the coefficients of this embedded filter is a partial open problem

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- For slow adaptation/learning the above condition can be relaxed
- The algorithms using an ARIMA filter can improve the performance of the « gradient » algorithm
- The optimal choice of the coefficients of this embedded filter is a partial open problem
- The results hold also for « continous time » formulation of adaptive/learning systems
- When using « approximations » of the gradient as « correcting » term, an additional SPR condition has to be satisfied for stability of the adaptive/learning system

Thank you for your attention!

Implementation

$$H^{ii}(q^{-1}) = \frac{1 + c_1 q^{-1} + c_2 q^{-2} + .. + c_{n_C} q^{-n_C}}{1 - d_1 q^{-1} - d_2 q^{-2} .. - d_{n_D} q^{-n_D}} = \frac{C(q^{-1})}{D(q^{-1})}$$

$$\hat{\theta}(t+1) = d_1 \hat{\theta}(t) + d_2 \hat{\theta}(t-1) + \ldots + d_{n_D} \hat{\theta}(t-n_D)$$
$$+ F[\phi(t)\varepsilon(t+1) + c_1 \phi(t-1)\varepsilon(t) + c_2 \phi(t-2)\varepsilon(t-1)$$
$$+ \ldots + c_{n_C} \phi(t-n_C)\varepsilon(t-n_C+1)]$$

Gradient algorithm ($d_1=1$)

Simulation Results

Estimation of the parameters of: $S = \dfrac{q^{-2} + 0.5q^{-3}}{1 - 1.5q^{-1} + 0.7q^{-2}}$ (Input: PRBS)

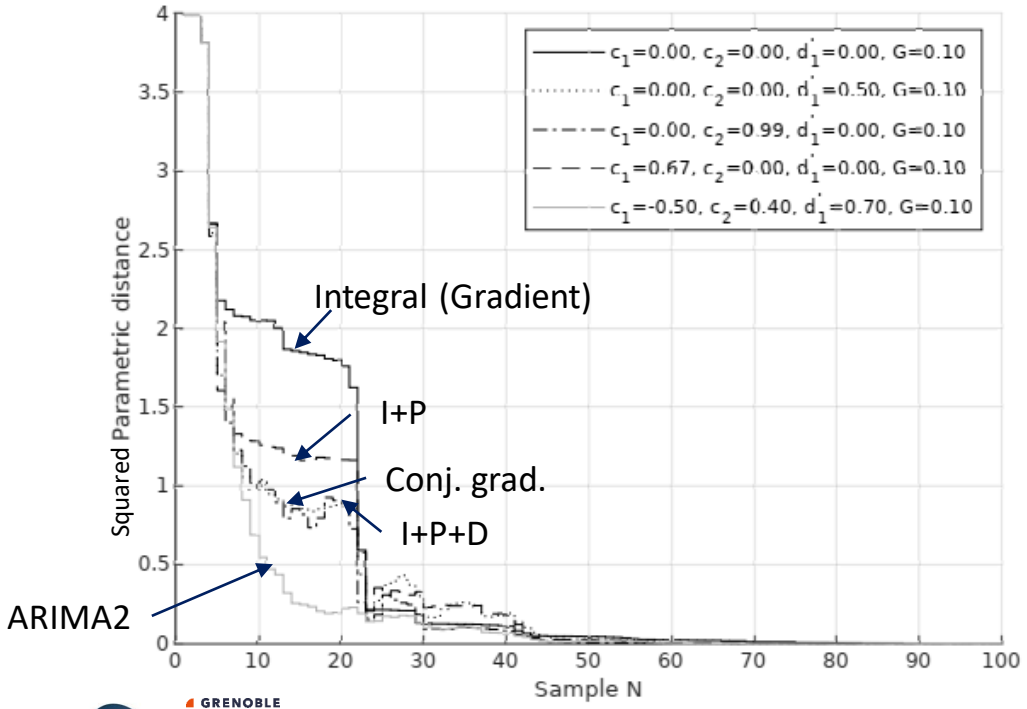
Performance indices:

$$J_\varepsilon(N) = \sum_{t=0}^{N} \varepsilon^2(t+1)$$

$$D^2(t) = \{[\theta - \hat{\theta}(t)]^T [\theta - \hat{\theta}(t)]\}$$

Squared Parametric Distance

$$J_D(N) = \sum_{t=0}^{N} D^2(t)$$

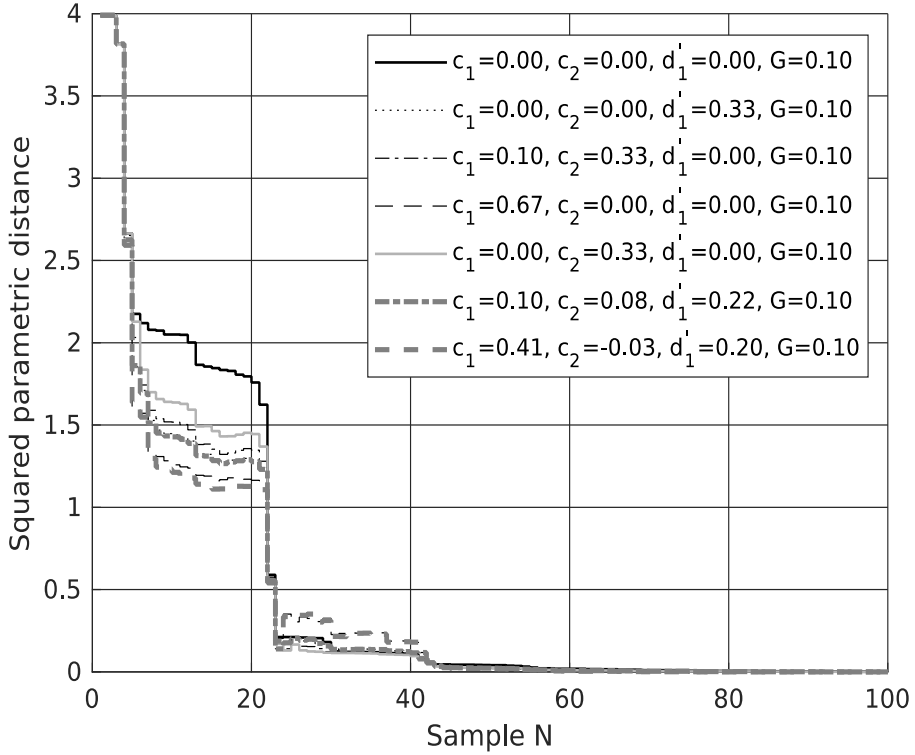


$\alpha = 0.1$ (adaptation gain)

Algorithm	PR	c_1	c_2	d_1'	$J_D(N)$	$J_\epsilon(N)$
Integral (gradient)	Y	0	0	0	**51.65**	**13.32**
Conj.Gr/Nest..	N	0	0	0.5	37.15	12.09
I+P+D ($\alpha_P = -2\alpha_D$)	N	0	0.99	0	34.58	11.95
I+P	Y	0.667	0	0	41.41	12.45
ARIMA 2	N	−0.5	0.4	0.7	**26.62**	**9.67**

(α_I=1; α_P=-2,66 α_D=2)

Simulation results under the « positive real » constraint



Algorithm	PR	c_1	c_2	d'_1	$J_D(N)$	$J_\epsilon(N)$
Integral	Y	0	0	0	**51.65**	**13.32**
Conj.Gr/Nest..	Y	0	0	0.333	42.16	**11.99**
I+P+D	Y	0.1	0.333	0	42.91	12.04
I+P	Y	0.667	0	0	41.41	12.45
I+P+D/Av.Gr	Y	0	0.33	0	44.655	12.21
ARIMA 2	Y	0.0989	0.0789	0.22	41.96	**11.99**
ARIMA 2	Y	0.408	−0.032	0.2	**40.59**	12.39

(α_I=1; α_P=-0.5; α_D=0.081)

- The improvement in performance is less significant
- Small differences in performance between various algorithms
- Are these weights values the best ?