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LOCAL EXACT CONTROLLABILITY TO THE STEADY STATES OF A PARABOLIC SYSTEM WITH COUPLED NONLINEAR BOUNDARY CONDITIONS

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ABSTRACT. In this article, we study the boundary local exact controllability to any steady state of a one-dimensional parabolic system with coupled nonlinear boundary conditions by means of only one control. The significant point is that the state components are interacting only at the boundary points with the assistance of some nonlinear terms. We consider two cases : either the control function is acting through a *mixed nonlinear boundary condition on the first component* or through a *Neumann condition on the second component*. The results are slightly different in the two cases.

To study this problem, we first consider the associated linearized systems around the given steady state. The method of moments let us to prove its controllability and to obtain a suitable estimate of the control cost of the form $Me^{M(T+\frac{1}{T})}$. To this end, we need to develop a precise spectral analysis of a non self-adjoint operator.

Thanks to those preliminary results, we can use the source term method developed in [29], followed by the Banach fixed point argument, to obtain the small-time boundary local exact controllability to the steady state for the original system.

1. Introduction.

1.1. The system under study. This paper is concerned with the local exact controllability to the equilibrium states of some 2×2 parabolic system by means of only one boundary control where the state components are only coupled through nonlinear boundary conditions. More precisely, we consider the following parabolic system (without any control for the moment), given by

$$\begin{cases} \partial_t z_1 - \partial_x^2 z_1 + \alpha_1 z_1 = 0 & \text{in } (0, T) \times (0, 1), \\ \partial_t z_2 - \partial_x^2 z_2 + \alpha_2 z_2 = 0 & \text{in } (0, T) \times (0, 1), \\ z_1(0, \cdot) = z_{0,1} & \text{in } (0, 1), \\ z_2(0, \cdot) = z_{0,2} & \text{in } (0, 1), \end{cases} \quad (1)$$

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with the boundary conditions given by

$$\begin{cases} \partial_x z_1(\cdot, 1) = 0 & \text{in } (0, T), \\ \partial_x z_2(\cdot, 1) - f_r(z_1(\cdot, 1)) = 0 & \text{in } (0, T), \end{cases} \quad (2)$$

at the right boundary point and

$$\begin{cases} \partial_x z_1(\cdot, 0) + f_\ell(z_2(\cdot, 0)) = 0 & \text{in } (0, T), \\ \partial_x z_2(\cdot, 0) = 0 & \text{in } (0, T), \end{cases} \quad (3)$$

at the left boundary point, where f_ℓ, f_r are some nonlinear functions, $\alpha_1, \alpha_2 \in \mathbb{R}$ are some parameters and $z_0 := (z_{0,1}, z_{0,2}) \in (L^2(0, 1))^2$ is the given initial data.

Our main assumption concerning f_ℓ and f_r is

$$f_\ell, f_r \in W^{2,\infty}(\mathbb{R}). \quad (4)$$

Such systems with boundary interactions appear in several biological or chemical models. A prototype of system (1)–(2)–(3) has been considered for instance in [33] where the authors studied the equilibrium states and oscillations of localized two-enzyme kinetics. In their work, the states z_1 and z_2 represent the concentrations of the effectors of two enzymes localized at $x = 0$ and $x = 1$ affecting each other only through the end points, and in that work, the authors consider some particular type of nonlinear functions f_ℓ, f_r according to the role of the enzymes (activation/inhibition) such as

$$s \mapsto \frac{s^2}{\gamma^2 + s^2}, \text{ (activation)} \quad \text{or} \quad s \mapsto \frac{\gamma^2}{\gamma^2 + s^2} \text{ (inhibition)}, \quad (5)$$

where $\gamma > 0$ is a fixed parameter. For more details, we refer to [33, Sections 1 & 2].

We consider now a steady-state $z^* := (z_1^*, z_2^*) \in (L^2(0, 1))^2$ for the system (1)–(2)–(3) that will be fixed all along this work. We will assume that

$$\beta_\ell := f'_\ell(z_2^*(0)) \neq 0, \quad \text{and} \quad \beta_r := f'_r(z_1^*(1)) \neq 0. \quad (6)$$

Remark 1.1. A simple computation helps us to find the steady state of the system (1)–(2)–(3); namely

$$\begin{aligned} z_1^*(x) &= a_1 \cosh(\sqrt{\alpha_1}(1-x)), \quad \forall x \in [0, 1], \\ z_2^*(x) &= a_2 \cosh(\sqrt{\alpha_2}x), \quad \forall x \in [0, 1], \end{aligned}$$

where a_1 and a_2 satisfy

$$\begin{aligned} a_2 \sqrt{\alpha_2} \sinh(\sqrt{\alpha_2}) &= f_r(a_1), \\ a_1 \sqrt{\alpha_1} \sinh(\sqrt{\alpha_1}) &= f_\ell(a_2). \end{aligned}$$

Depending on the particular forms of f_ℓ and f_r and the values of α_1, α_2 , there may exist one or many solutions (a_1, a_2) to this nonlinear system, each of them leading to a steady-state of our parabolic problem, as discussed for instance in [33].

In system (1)–(2)–(3), we observe that the coupling between the two components only occurs at the boundary points since there is no internal coupling in the system. The main mathematical challenge that we will deal with, is to analyze the controllability of such system by only one boundary control.

Main goal. In this paper, we are going to study the controllability properties of the system (1)–(2) by a single boundary control $v : (0, T) \rightarrow \mathbb{R}$ acting at the left

boundary point either in the case

$$\begin{cases} \partial_x z_1(\cdot, 0) + f_\ell(z_2(\cdot, 0)) = v & \text{in } (0, T), \\ \partial_x z_2(\cdot, 0) = 0 & \text{in } (0, T), \end{cases} \quad (8)$$

or in the case

$$\begin{cases} \partial_x z_1(\cdot, 0) + f_\ell(z_2(\cdot, 0)) = 0 & \text{in } (0, T), \\ \partial_x z_2(\cdot, 0) = v & \text{in } (0, T). \end{cases} \quad (9)$$

To be more precise, we look for a control $v \in L^2(0, T)$ acting through the boundary condition (8) or (9), such that both components can be driven to the steady state $z^* := (z_1^*, z_2^*)$ (at least locally, that is if the initial data is close enough from z^*) at any given time $T > 0$, that is to say,

$$z_1(T, \cdot) = z_1^*, \quad z_2(T, \cdot) = z_2^*.$$

1.2. Bibliographic comments and motivations. The controllability of a system of partial differential equations with less number of control(s) than equations is gaining genuine interests to the control community. In light of this, we first refer some pioneer works. The authors in [5, 3] established some generalized Kalman rank conditions which are necessary and sufficient for the distributed null-controllability of a class of linear parabolic systems. The boundary controllability of such system is much more intricate and most of the results are restricted to the 1-D case since the very powerful Carleman technique is often inefficient in this context. Indeed, the boundary controllability of a system of PDEs when there is less controls than equations, is no more equivalent to the distributed one unlike the scalar case, see for instance [19]. The authors in [19] also proved a necessary and sufficient condition for the boundary null-controllability of a 2×2 coupled parabolic system with a scalar Dirichlet control. A generalization of this result for the system of n parabolic equations with $m < n$ controls has been achieved in [4]. In the multi-dimensional case, we quote [2, 8], where controllability results are obtained in particular cylindrical geometries by exploiting on the one hand a sharp estimate of the control for the associated 1D problem and on the other hand spectral Lebeau-Robbiano inequalities ([27]) for any space dimensions, see also the discussion in [9, Section 2.4.2]. We also mention [1] where symmetric parabolic coupled systems are analyzed in any dimension, provided that the control region satisfies the Geometric Control Condition (GCC).

The above cases mainly dealt with internal coupling. Concerning the controllability with boundary coupling, we mention the book [14] and the survey paper [6] where the authors studied the controllability of some wave, heat and Schrödinger systems on metric graphs. We also refer to [10] where the boundary null-controllability of some coupled parabolic systems has been addressed where the boundary coupling is chosen by means of a Kirchhoff-type condition. In most of the known cases, the boundary conditions are linearly posed. Thus, dealing with the systems (1)–(2)–(8)/(9) is naturally more interesting since the concerned boundary couplings are nonlinear.

In the context of controllability of nonlinear systems, we mention first [21, Chapter I, Sec. 4] by Fursikov and Imanuvilov where a small-time local null-controllability of semilinear heat equations has been proved using a perturbation argument. In 2000, Barbu [7], and independently Fernández-Cara and Zuazua [20], proved the small-time global null-controllability of semilinear heat equations where the growth

of nonlinearities is slower than $|s| \ln^{3/2}(1 + |s|)$. In fact, the large-time global null-controllability of semilinear heat equations has recently been obtained in [26] for nonlinearities F such that $s \mapsto F(s)/(|s| \ln^\alpha(1 + |s|))$ tends to zero at $\pm\infty$ for some $\alpha > 0$, with either $F(s) > 0$ for $s > 0$ or $F(s) < 0$ for $s < 0$ and $1/F \in L^1([0, +\infty))$. Furthermore, in the context of controllability for evolutionary systems with nonlinear (interior) couplings, we mention, for instance, the works [12, 24, 25].

In the present work, we deal with the controllability of the concerned models (1)–(2)–(8)/(9) where the couplings arise on the boundary points in terms of the nonlinear functions (4) verifying (6).

1.3. Auxiliary control problems. We shall first make a change of unknowns to transform our control problem into a null-control problem, and then write its linearization around zero.

To this end, we introduce the following functions, that depend on the considered steady state z^* :

$$\begin{aligned}\tilde{f}_r(s) &:= f_r(s + z_1^*(1)) - f_r(z_1^*(1)), \quad \forall s \in \mathbb{R}, \\ \tilde{f}_\ell(s) &:= f_\ell(s + z_2^*(0)) - f_\ell(z_2^*(0)), \quad \forall s \in \mathbb{R}.\end{aligned}$$

Since z^* is a steady state solution of our uncontrolled system, we see that the study of the aforementioned local exact controllability problem is equivalent to the following local null-controllability problem for the new unknown $y = z - z^*$,

$$\begin{cases} \partial_t y_1 - \partial_x^2 y_1 + \alpha_1 y_1 = 0 & \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x^2 y_2 + \alpha_2 y_2 = 0 & \text{in } (0, T) \times (0, 1), \\ y_1(0, \cdot) = y_{0,1} & \text{in } (0, 1), \\ y_2(0, \cdot) = y_{0,2} & \text{in } (0, 1), \end{cases} \quad (10)$$

$$\begin{cases} \partial_x y_1(\cdot, 1) = 0 & \text{in } (0, T), \\ \partial_x y_2(\cdot, 1) - \tilde{f}_r(y_1(\cdot, 1)) = 0 & \text{in } (0, T), \end{cases} \quad (11)$$

with a control acting either through the mixed condition

$$\begin{cases} \partial_x y_1(\cdot, 0) + \tilde{f}_\ell(y_2(\cdot, 0)) = v & \text{in } (0, T), \\ \partial_x y_2(\cdot, 0) = 0 & \text{in } (0, T), \end{cases} \quad (12)$$

or, through the Neumann condition of y_2 ,

$$\begin{cases} \partial_x y_1(\cdot, 0) + \tilde{f}_\ell(y_2(\cdot, 0)) = 0 & \text{in } (0, T), \\ \partial_x y_2(\cdot, 0) = v & \text{in } (0, T), \end{cases} \quad (13)$$

where the initial data $y_0 := (y_{0,1}, y_{0,2})$ is simply defined by $y_{0,1} = z_{0,1} - z_1^*$ and $y_{0,2} = z_{0,2} - z_2^*$.

If we linearize the nonlinear systems (10)–(11)–(12)/(13) around 0, we obtain the same set of linear PDEs (10) along with the linearized boundary conditions

$$\begin{cases} \partial_x y_1(\cdot, 1) = 0 & \text{in } (0, T), \\ \partial_x y_2(\cdot, 1) - \beta_r y_1(\cdot, 1) = 0 & \text{in } (0, T), \end{cases} \quad (14)$$

at the right boundary and one of the following two situations on the left boundary: either

$$\begin{cases} \partial_x y_1(\cdot, 0) + \beta_\ell y_2(\cdot, 0) = v & \text{in } (0, T), \\ \partial_x y_2(\cdot, 0) = 0 & \text{in } (0, T), \end{cases} \quad (15)$$

or,

$$\begin{cases} \partial_x y_1(\cdot, 0) + \beta_\ell y_2(\cdot, 0) = 0 & \text{in } (0, T), \\ \partial_x y_2(\cdot, 0) = v & \text{in } (0, T). \end{cases} \quad (16)$$

We recall that the parameters β_ℓ and β_r have been introduced in (6). They only depends on f_ℓ, f_r and z^* and are assumed to be non zero as mentioned in (6).

The study of this linear system will be central in this paper.

1.4. Main results. Let us state the main result concerning our nonlinear systems. We will set $X = (L^2(0, 1))^2$.

Theorem 1.2. *Let f_ℓ, f_r satisfying (4) and $z^* \in X$ be a steady state of the system (1)–(2)–(3), that satisfies (6).*

Then, we have the following local controllability results.

1. *The system (1)–(2)–(8) is small time locally exact controllable to the equilibrium, that is to say, for any given time $T > 0$, there is a $\delta > 0$ such that for any initial data $z_0 \in X$ with $\|z_0 - z^*\|_X \leq \delta$, there exists a control $v \in L^2(0, T)$ such that the associated solution z satisfies $z(T) = z^*$.*
2. *There exists a non-empty discrete set $\mathcal{R} \subset \mathbb{R}$ such that if $\beta_\ell \beta_r \notin \mathcal{R}$, then the system (1)–(2)–(9) is also small time locally exact controllable to the equilibrium.*

The set \mathcal{R} will be explicitly specified later, namely in Lemma 4.1.

As we discussed earlier, the proof of Theorem 1.2 is equivalent to prove a local null-controllability result for the systems (10)–(11)–(12)/(13), and the first step in the analysis of the nonlinear systems is the study of the linearized systems (10)–(14)–(15)/(16). More precisely, we shall first prove the following theorem.

Theorem 1.3. *Let $\alpha_1, \alpha_2, \beta_\ell, \beta_r \in \mathbb{R}$ with $\beta_\ell \beta_r \neq 0$.*

1. *For any initial data $y_0 \in X$, and any time $T > 0$, there exists a control $v \in L^2(0, T)$ such that the solution of (10)–(14)–(15) satisfies $y(T) = 0$.*
2. *There exists a non-empty discrete set $\mathcal{R} \subset \mathbb{R}$ such that if $\beta_\ell \beta_r \notin \mathcal{R}$, then for any initial data $y_0 \in X$, and any time $T > 0$, there is a control $v \in L^2(0, T)$ such that the solution of (10)–(14)–(16) satisfies $y(T) = 0$.*

In both cases, the control can be chosen to satisfy the following estimate

$$\|v\|_{L^2(0, T)} \leq M e^{M(T + \frac{1}{T})} \|y_0\|_X, \quad (17)$$

where the constant $M > 0$ neither depend on T nor on y_0 but may depend on $\alpha_1, \alpha_2, \beta_\ell$ and β_r .

Outline of the paper. The paper is organized as follows.

- Section 2 is devoted to study the well-posedness results of the linearized control systems (10)–(14)–(15)/(16) by virtue of transposition method. Moreover, we formulate the null-control problems for the concerned models.
- The global boundary null-controllability results of the associated linear models are established in Section 4 using the so-called method of moments. To this end, we develop a concrete spectral analysis of the adjoint operator (which is non-self-adjoint) to the corresponding linear control systems, and this is described in Section 3. This approach let us in particular obtain the estimate (17) which is crucial to deduce the controllability results for the non-linear models.

- In Section 5, we prove the local null-controllability results for the systems (10)–(11)–(12)/(13).

To do so, we apply the source term method introduced in [29]; more precisely, we prove the null-controllability of our linearized models with additional boundary terms in $L^2(0, T)$ (exponentially decreasing while $t \rightarrow T^-$) on the boundary points where the non-linearities appear. Then, we use the Banach fixed-point argument to obtain the local (boundary) null-controllability for the concerned non-linear models.

- We conclude our paper by mentioning several remarks in Section 6 and finally in Appendix A, we present the proofs of some technical lemmas which are useful for our spectral study.

Notations. Throughout the paper, $C > 0$ denotes a generic constant that may vary line to line and may depend on $\alpha_1, \alpha_2, \beta_\ell, \beta_r$ but neither depend on T nor on the initial data y_0 . By the notation

$$\gamma_1(\cdot) = O(\gamma_2(\cdot)),$$

for some functions γ_1, γ_2 , we mean

$$|\gamma_1(\cdot)| \leq C|\gamma_2(\cdot)|, \quad \text{for some constant } C := C(\alpha_1, \alpha_2, \beta_\ell, \beta_r).$$

The set of all non-zero real numbers is denoted by \mathbb{R}^* . For any $z \in \mathbb{C}$, the real and complex parts are denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$ respectively. Moreover, in the whole paper we denote by $\sqrt{\cdot}$ the principal determination of the square root in the complex plane defined by

$$\sqrt{\rho e^{i\phi}} := \sqrt{\rho} e^{i\phi/2}, \quad \forall \rho \geq 0, \quad \forall \phi \in (-\pi, \pi],$$

and which is holomorphic in $\mathbb{C} \setminus \mathbb{R}_-$.

2. Well-posedness and formulation of the control problems. In this section, we shall discuss about the existence of the weak solutions to our linearized systems and formulate the associated null-control problems.

2.1. Existence of the semigroup. As mentioned before, we will work in the Hilbert space $X = (L^2(0, 1))^2$. We introduce the following elliptic operator \mathcal{A} associated to the system (10)–(14) with (15) or (16),

$$\mathcal{A} = \begin{pmatrix} -\partial_x^2 + \alpha_1 & 0 \\ 0 & -\partial_x^2 + \alpha_2 \end{pmatrix}, \quad (18)$$

with its domain

$$D(\mathcal{A}) := \left\{ \phi := (\phi_1, \phi_2) \in (H^2(0, 1))^2 \mid \begin{aligned} &\phi_1'(0) + \beta_\ell \phi_2(0) = 0, \quad \phi_2'(0) = 0, \\ &\phi_1'(1) = 0, \quad \phi_2'(1) - \beta_r \phi_1(1) = 0 \end{aligned} \right\}. \quad (19)$$

It is clear that $(\mathcal{A}, D(\mathcal{A}))$ is a non-self-adjoint operator due to the presence of such coupled boundary conditions. More precisely, we find that, the adjoint operator \mathcal{A}^* of \mathcal{A} has the same formal expression as (18), yet with a different domain given by

$$D(\mathcal{A}^*) = \left\{ u := (u_1, u_2) \in (H^2(0, 1))^2 \mid \begin{aligned} &u_1'(0) = 0, \quad u_2'(0) + \beta_\ell u_1(0) = 0, \\ &u_1'(1) - \beta_r u_2(1) = 0, \quad u_2'(1) = 0 \end{aligned} \right\}. \quad (20)$$

Proposition 2.1. *The operator $(-\mathcal{A}^*, D(\mathcal{A}^*))$ defined by (18)–(20), generates an analytic semigroup in X , denoted by $(e^{-t\mathcal{A}^*})_{t \geq 0}$.*

A similar proof shows that the same property holds for the operator $(-\mathcal{A}, D(\mathcal{A}))$ defined in (18)–(19).

Proof. We consider the following densely defined sesquilinear form h ; for all $u := (u_1, u_2)$, $\psi := (\psi_1, \psi_2) \in (H^1(0, 1))^2$,

$$\begin{aligned} h(u, \psi) &:= \int_0^1 \left(u_1'(x) \overline{\psi_1'(x)} + u_2'(x) \overline{\psi_2'(x)} \right) dx \\ &+ \int_0^1 \left(\alpha_1 u_1(x) \overline{\psi_1(x)} + \alpha_2 u_2(x) \overline{\psi_2(x)} \right) dx - \beta_\ell u_1(0) \overline{\psi_2(0)} - \beta_r u_2(1) \overline{\psi_1(1)}. \end{aligned}$$

It is clear that h is continuous in $(H^1(0, 1))^2$ and moreover, we have

$$|h(u, \psi)| \leq \kappa \|u\|_{(H^1(0,1))^2} \|\psi\|_{(H^1(0,1))^2},$$

where $\kappa > 0$ depends on the parameters $\alpha_1, \alpha_2, \beta_\ell$ and β_r . Denote by $(\tilde{\mathcal{A}}, D(\tilde{\mathcal{A}}))$ the operator associated with the form h , which is by definition given by

$$\begin{cases} D(\tilde{\mathcal{A}}) := \{u \in (H^1(0, 1))^2 \mid \exists w \in X \text{ s.t. } h(u, \psi) = (w, \psi)_X, \quad \forall \psi \in (H^1(0, 1))^2\}, \\ \tilde{\mathcal{A}}u := w \in X. \end{cases}$$

Since any $u \in (H^1(0, 1))^2$ satisfies

$$\operatorname{Re} (h(u, u)) \geq \kappa_1 \|u\|_{(H^1(0,1))^2}^2 - \kappa_2 \|u\|_X^2,$$

for some constants $\kappa_1, \kappa_2 > 0$, we know by [31, Proposition 1.51 and Theorem 1.52] that $-\tilde{\mathcal{A}}$ generates an analytic semigroup in X of angle $\pi/2 - \arctan \kappa$ for some $\kappa > 0$.

It remains to prove that $(\tilde{\mathcal{A}}, D(\tilde{\mathcal{A}}))$ is indeed equal to $(\mathcal{A}^*, D(\mathcal{A}^*))$.

- First, we fix some $u := (u_1, u_2) \in D(\mathcal{A}^*)$. Then for all $\psi := (\psi_1, \psi_2) \in (H^1(0, 1))^2$, we have

$$\begin{aligned} h(u, \psi) &= \int_0^1 \left(-u_1''(x) + \alpha_1 u_1(x) \right) \overline{\psi_1(x)} dx + \int_0^1 \left(-u_2''(x) + \alpha_2 u_2(x) \right) \overline{\psi_2(x)} dx \\ &= (\mathcal{A}^* u, \psi), \end{aligned}$$

where we performed an integration by parts using the boundary conditions satisfied by $u \in D(\mathcal{A}^*)$. Thus, for the chosen $u \in D(\mathcal{A}^*)$, there is a $w = \mathcal{A}^* u \in X$ such that $h(u, \psi) = (w, \psi)_X$, for all $\psi \in (H^1(0, 1))^2$, which concludes the inclusion $D(\mathcal{A}^*) \subseteq D(\tilde{\mathcal{A}})$ and the fact that $\tilde{\mathcal{A}} = \mathcal{A}^*$ on $D(\mathcal{A}^*)$.

- Conversely, let $u \in D(\tilde{\mathcal{A}})$. By definition, there exists some $w \in X$ such that $h(u, \psi) = (w, \psi)_X$ with $\tilde{\mathcal{A}}u = w$, for all $\psi \in (H^1(0, 1))^2$, and accordingly

$$\begin{aligned} &\int_0^1 \left(u_1'(x) \overline{\psi_1'(x)} + u_2'(x) \overline{\psi_2'(x)} \right) dx + \int_0^1 \left(\alpha_1 u_1(x) \overline{\psi_1(x)} + \alpha_2 u_2(x) \overline{\psi_2(x)} \right) dx \\ &- \beta_\ell u_1(0) \overline{\psi_2(0)} - \beta_r u_2(1) \overline{\psi_1(1)} = \int_0^1 \left(w_1(x) \overline{\psi_1(x)} + w_2(x) \overline{\psi_2(x)} \right) dx. \end{aligned}$$

In particular, by considering any $\psi = (\psi_1, 0) \in (H_0^1(0, 1))^2$, then $\psi = (0, \psi_2) \in (H_0^1(0, 1))^2$ we conclude that

$$w_1 = -u_1'' + \alpha_1 u_1,$$

$$w_2 = -u_2'' + \alpha_2 u_2.$$

Since $w_i \in L^2(0, 1)$ ($i = 1, 2$), we have $u_1, u_2 \in H^2(0, 1)$ and thus an integration by parts yields

$$\begin{aligned} & \int_0^1 \left(-u_1''(x) + \alpha_1 u_1(x) \right) \overline{\psi_1(x)} dx + \int_0^1 \left(-u_2''(x) + \alpha_2 u_2(x) \right) \overline{\psi_2(x)} dx \\ & - \beta_\ell u_1(0) \overline{\psi_2(0)} - \beta_r u_2(1) \overline{\psi_1(1)} + u_1'(1) \overline{\psi_1(1)} - u_1'(0) \overline{\psi_1(0)} \\ & + u_2'(1) \overline{\psi_2(1)} - u_2'(0) \overline{\psi_2(0)} = \int_0^1 \left(w_1(x) \overline{\psi_1(x)} + w_2(x) \overline{\psi_2(x)} \right) dx, \end{aligned}$$

for all $\psi \in (H^1(0, 1))^2$. We eventually obtain

$$\begin{aligned} u_1'(0) &= 0, & u_2'(0) + \beta_\ell u_1(0) &= 0, \\ u_2'(1) &= 0, & u_1'(1) - \beta_r u_2(1) &= 0. \end{aligned}$$

which are exactly the boundary conditions of an element of $D(\mathcal{A}^*)$ and thus $D(\tilde{\mathcal{A}}) \subseteq D(\mathcal{A}^*)$.

The proof is complete. \square

2.2. The backward in time adjoint system. The adjoint problem (backward in time) of our control systems (10)–(14) with (15) or (16) is

$$\left\{ \begin{array}{ll} -\partial_t q_1 - \partial_x^2 q_1 + \alpha_1 q_1 = \tilde{w}_1 & \text{in } (0, T) \times (0, 1), \\ -\partial_t q_2 - \partial_x^2 q_2 + \alpha_2 q_2 = \tilde{w}_2 & \text{in } (0, T) \times (0, 1), \\ \partial_x q_1(\cdot, 0) = 0 & \text{in } (0, T), \\ \partial_x q_2(\cdot, 0) + \beta_\ell q_1(\cdot, 0) = 0 & \text{in } (0, T), \\ \partial_x q_1(\cdot, 1) - \beta_r q_2(\cdot, 1) = 0 & \text{in } (0, T), \\ \partial_x q_2(\cdot, 1) = 0 & \text{in } (0, T), \\ q_1(T, \cdot) = \zeta_1 & \text{in } (0, 1), \\ q_2(T, \cdot) = \zeta_2 & \text{in } (0, 1), \end{array} \right. \quad (21)$$

where $\tilde{w} := (\tilde{w}_1, \tilde{w}_2) \in L^2(0, T; X)$ is some given right hand side and $\zeta = (\zeta_1, \zeta_2)$ is given final data.

Proposition 2.2. *For any given $\zeta \in X$ and $\tilde{w} \in L^2(0, T; X)$, there exists a unique weak solution $q := (q_1, q_2) \in \mathcal{C}^0([0, T]; X) \cap L^2(0, T; (H^1(0, 1))^2)$ to (21) that satisfies the following energy estimate*

$$\begin{aligned} \|q\|_{\mathcal{C}^0([0, T]; X)} + \|q\|_{L^2(0, T; (H^1(0, 1))^2)} + \|\partial_t q\|_{L^2(0, T; (H^{-1}(0, 1))^2)} \\ \leq C e^{CT} (\|\zeta\|_X + \|\tilde{w}\|_{L^2(0, T; X)}), \end{aligned}$$

where the constant $C > 0$ does not depend on $T > 0$.

Proof. We just give a short sketch of the proof. For $\zeta \in D(\mathcal{A}^*)$ and $\tilde{w} \in \mathcal{C}^1([0, T]; X)$ we indeed have the existence of a strong solution $q \in \mathcal{C}^1([0, T]; X) \cap \mathcal{C}^0([0, T]; D(\mathcal{A}^*))$ to (21). It remains to prove claimed the estimate to obtain the result by the usual density argument.

Let us test the first and second equation of (21) by q_1 and q_2 respectively, which provides

$$-\frac{1}{2} \frac{d}{dt} \|q(t)\|_X^2 + \|\partial_x q(t)\|_X^2 + \alpha_1 \|q_1(t)\|_{L^2(0, 1)}^2 + \alpha_2 \|q_2(t)\|_{L^2(0, 1)}^2$$

$$+ \beta_\ell q_1(t, 0)q_2(t, 0) - \beta_r q_1(t, 1)q_2(t, 1) \leq \frac{1}{2} (\|\tilde{w}(t)\|_X^2 + \|q(t)\|_X^2). \quad (22)$$

Using the following trace inequality

$$|\phi(0)| + |\phi(1)| \leq 2\|\phi\|_{L^2(0,1)}^{\frac{1}{2}}\|\phi\|_{H^1(0,1)}^{\frac{1}{2}}, \forall \phi \in H^1(0, 1),$$

we get from (22) that

$$-\frac{1}{2} \frac{d}{dt} \|q(t)\|_X^2 + \|q(t)\|_{(H^1(0,1))^2}^2 \leq C\epsilon \|q(t)\|_{(H^1(0,1))^2}^2 + C_\epsilon \|q(t)\|_X^2 + \|\tilde{w}(t)\|_X^2,$$

for some constant $C > 0$ that does not depend on ϵ .

Choosing a fixed and small enough $\epsilon > 0$ and by using Grönwall's lemma we deduce that

$$\|q\|_{C^0([0,T];X)}^2 \leq C e^{CT} (\|\zeta\|_X^2 + \|\tilde{w}\|_{L^2(0,T;X)}^2).$$

The other estimates can be proved in a standard fashion, more details can be found in [28, Chapter 4] (see also [16, Chapter 7]). \square

2.3. A non homogeneous forward system. Let us consider the following non homogeneous system

$$\left\{ \begin{array}{ll} \partial_t y_1 - \partial_x^2 y_1 + \alpha_1 y_1 = F_1 & \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x^2 y_2 + \alpha_2 y_2 = F_2 & \text{in } (0, T) \times (0, 1), \\ \partial_x y_1(\cdot, 0) + \beta_\ell y_2(\cdot, 0) = G_1 & \text{in } (0, T), \\ \partial_x y_2(\cdot, 0) = G_2 & \text{in } (0, T), \\ \partial_x y_1(\cdot, 1) = G_3 & \text{in } (0, T), \\ \partial_x y_2(\cdot, 1) - \beta_r y_1(\cdot, 1) = G_4 & \text{in } (0, T), \\ y_1(0, \cdot) = y_{0,1} & \text{in } (0, 1), \\ y_2(0, \cdot) = y_{0,2} & \text{in } (0, 1), \end{array} \right. \quad (23)$$

where $F_1, F_2 \in L^2(0, T; L^2(0, 1))$ and $G_j \in L^2(0, T; \mathbb{R})$ for $j = 1, 2, 3, 4$.

Recall that the adjoint operator $(-\mathcal{A}^*, D(\mathcal{A}^*))$ given by (18)–(20), defines an analytic semigroup in X , thanks to Proposition 2.1. Therefore, the existence of a unique weak solution to (23) can be shown using the method of transposition; see [13, 34]. More precisely, we test the PDEs in (23) against the solution q to the adjoint system (21) with $\tilde{w}_1 = \tilde{w}_2 = 0$. We express this below.

Theorem 2.3. *For any given $y_0 := (y_{0,1}, y_{0,2}) \in X$, $F := (F_1, F_2) \in L^2(0, T; X)$ and $G := (G_1, G_2, G_3, G_4) \in L^2(0, T; \mathbb{R}^4)$, there exists a unique weak solution y to (23) belonging to the space $C^0([0, T]; X) \cap L^2(0, T; (H^1(0, 1))^2)$ in the following sense: for any $t \in [0, T]$ and $\zeta := (\zeta_1, \zeta_2) \in X$, we have*

$$\begin{aligned} (y(t), \zeta)_X &= (y_0, e^{-t\mathcal{A}^*} \zeta)_X + \int_0^t (F(s), e^{-(t-s)\mathcal{A}^*} \zeta)_X ds \\ &\quad - \int_0^t \left\langle \begin{pmatrix} G_1(s) \\ G_2(s) \end{pmatrix}, \left(e^{-(t-s)\mathcal{A}^*} \zeta \right) \Big|_{\{x=0\}} \right\rangle_{\mathbb{R}^2} ds \\ &\quad + \int_0^t \left\langle \begin{pmatrix} G_3(s) \\ G_4(s) \end{pmatrix}, \left(e^{-(t-s)\mathcal{A}^*} \zeta \right) \Big|_{\{x=1\}} \right\rangle_{\mathbb{R}^2} ds. \end{aligned}$$

Moreover, we have

$$\|y\|_{C^0([0,T];X)} + \|y\|_{L^2(0,T;(H^1(0,1))^2)} + \|\partial_t y\|_{L^2(0,T;(H^{-1}(0,1))^2)}$$

$$\leq C e^{CT} (\|y_0\|_X + \|F\|_{L^2(0,T;X)} + \|G\|_{L^2(0,T;\mathbb{R}^4)}).$$

2.4. The control problems. First, observe that the existence of a unique weak solutions to our control systems (10)–(14) with (15) or (16) is clear from Theorem 2.3. Now, let us prescribe the formulation of the control problems in both cases. We hereby introduce the observation operators \mathcal{B}_1^* and \mathcal{B}_2^* associated with the control problems (25) and (26) respectively as follows,

$$\mathcal{B}_1^* = \mathbf{1}_{\{x=0\}} \begin{pmatrix} 1 & 0 \end{pmatrix} : (H^1(0,1))^2 \rightarrow \mathbb{R}, \quad (24a)$$

$$\mathcal{B}_2^* = \mathbf{1}_{\{x=0\}} \begin{pmatrix} 0 & 1 \end{pmatrix} : (H^1(0,1))^2 \rightarrow \mathbb{R}. \quad (24b)$$

Proposition 2.4. *Let any $y_0 \in X$, $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ and $T > 0$ be given, and assume that $\beta_\ell \beta_r \neq 0$.*

1. *A function $v \in L^2(0, T)$ is a null-control for the system (10) with the boundary conditions (14)–(15) if and only if it satisfies: for any $\zeta \in X$,*

$$(y_0, e^{-T\mathcal{A}^*} \zeta)_X = \int_0^T v(t) \mathcal{B}_1^* e^{-(T-t)\mathcal{A}^*} \zeta dt. \quad (25)$$

2. *A function $v \in L^2(0, T)$ is a null-control for the system (10) with the boundary conditions (14)–(16) if and only if it satisfies: for any $\zeta \in X$,*

$$(y_0, e^{-T\mathcal{A}^*} \zeta)_X = \int_0^T v(t) \mathcal{B}_2^* e^{-(T-t)\mathcal{A}^*} \zeta dt. \quad (26)$$

3. Spectral analysis. Let us write the eigenvalue problem $\mathcal{A}^* u = \lambda u$, for $\lambda \in \mathbb{C}$ as follows

$$\begin{cases} -u_1'' + \alpha_1 u_1 = \lambda u_1 & \text{in } (0, 1), \\ -u_2'' + \alpha_2 u_2 = \lambda u_2 & \text{in } (0, 1), \\ u_1'(0) = 0, \quad u_2'(0) + \beta_\ell u_1(0) = 0, \\ u_1'(1) - \beta_r u_2(1) = 0, \quad u_2'(1) = 0. \end{cases} \quad (27)$$

with the parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ and $\beta_\ell \beta_r \neq 0$.

3.1. \mathcal{A}^* has compact resolvent. Let us denote the spectrum (resp. the resolvent set) of \mathcal{A}^* by $\sigma(\mathcal{A}^*)$ (resp. by $\rho(\mathcal{A}^*)$). The sesquilinear form associated with the operator $\mathcal{A}^* - \xi \text{Id}$ for any $\xi \in \rho(\mathcal{A}^*)$ is

$$\begin{aligned} h(u, \phi) := & \int_0^1 \left(u_1'(x) \overline{\phi_1'(x)} + u_2'(x) \overline{\phi_2'(x)} \right) dx + (\alpha_1 - \xi) \int_0^1 u_1(x) \overline{\phi_1(x)} dx \\ & + (\alpha_2 - \xi) \int_0^1 u_2(x) \overline{\phi_2(x)} dx - \beta_\ell u_1(0) \overline{\phi_2(0)} - \beta_r u_2(1) \overline{\phi_1(1)}, \end{aligned}$$

for all $\phi := (\phi_1, \phi_2) \in (H^1(0,1))^2$.

One can show that h is continuous and that there exists constants $\gamma_1, \gamma_2 > 0$ such that we have

$$\text{Re}(h(u, u)) \geq \gamma_1 \|u'\|_{(L^2(0,1))^2}^2 + (-\text{Re}(\xi) - \gamma_2) \|u\|_X^2. \quad (28)$$

From (28), it follows that for $\xi \in \mathbb{C}$ with $\text{Re}(\xi) \leq -\gamma_2 - 1$, the operator $(\mathcal{A}^* - \xi \text{Id})$ is invertible. In particular,

$$(-\infty, -\gamma_2 - 1] \subset \rho(\mathcal{A}^*).$$

Finally, we have that $D(\mathcal{A}^*) \hookrightarrow (H^1(0,1))^2 \hookrightarrow X$ with compact embedding and so $R_\xi := (\mathcal{A}^* - \xi \text{Id})^{-1} : X \rightarrow X$ is a compact operator in X for any $\xi \in \mathbb{C}$ with $\text{Re}(\xi) \leq -\gamma_2 - 1$ and consequently for any $\xi \in \rho(\mathcal{A}^*)$.

The above analysis confirms that the spectrum $\sigma(\mathcal{A}^*)$ is discrete and contains only eigenvalues of the operator \mathcal{A}^* .

3.2. Computation of the eigenfunctions of \mathcal{A}^* . We first observe that all the eigenvalues are geometrically simple. Indeed, assume that for some $\lambda \in \sigma(\mathcal{A}^*)$, there exists two linearly independent solutions $\phi = (\phi_1, \phi_2)$ and $\psi = (\psi_1, \psi_2)$ of the eigenvalue problem (27). We consider $u = a\phi + b\psi$, where $a, b \in \mathbb{C}^*$ are chosen so that $u_1(0) = 0$ (recall that $\phi_1(0) \neq 0$ and $\psi_1(0) \neq 0$). But we also have $u'_1(0) = 0$ from the boundary conditions in (27) which leads that $u_1 \equiv 0$ in $[0, 1]$. Finally, we immediately get from the set of equations (27) that $u_2 \equiv 0$ which is a contradiction. This proves that the geometric multiplicity of any eigenvalue is 1.

Then, we can see that $\lambda = \alpha_1$ or α_2 cannot be an eigenvalue for \mathcal{A}^* . Indeed, by putting $\lambda = \alpha_1$ or α_2 in the set of equations (27), we find that u_1 or u_2 is affine and then using the boundary conditions, one can show that $u_1 = u_2 = 0$.

In the sequel we will use the notation $\theta := \alpha_2 - \alpha_1 \in \mathbb{R}$. Let us now consider the variable $\mu \in \mathbb{C}$ defined by

$$\mu = \sqrt{\lambda - \alpha_1},$$

where we recall that $\sqrt{\cdot}$ is the principal determination of the square root in the complex plane. We also set $\nu = \sqrt{\mu^2 - \theta} = \sqrt{\lambda - \alpha_2}$ to simplify the presentation. By the discussion above, we know that $\mu \neq 0$ and $\nu \neq 0$.

We will now find out an explicit formula for the eigenfunctions of \mathcal{A}^* as well as the transcendental equation satisfied by the eigenvalues. To begin with, we rewrite the equations of u_1 and u_2 as follows

$$\begin{aligned} u_1'' + \mu^2 u_1 &= 0, \\ u_2'' + \nu^2 u_2 &= 0, \end{aligned}$$

along with the boundary conditions as given in (27).

Since $u'_1(0) = u'_2(1) = 0$, we look for solutions in the following form

$$\begin{cases} u_1(x) = K_1 \cos(\mu x), & x \in [0, 1], \\ u_2(x) = K_2 \cos(\nu(1-x)), & x \in [0, 1], \end{cases} \quad (29)$$

for some constants $K_1, K_2 \in \mathbb{C}$, to be determined. Using the other two boundary conditions, we have

$$\begin{pmatrix} \beta_\ell & \nu \sin \nu \\ -\mu \sin \mu & -\beta_r \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This system has non trivial solutions if and only if the determinant of the coefficient matrix is zero. This yields the following equation for $\mu \in \mathbb{C}$

$$h(\mu) := \frac{1}{\beta_\ell \beta_r} \mu \sqrt{\mu^2 - \theta} \sin(\mu) \sin\left(\sqrt{\mu^2 - \theta}\right) - 1 = 0. \quad (30)$$

Note that, even though the square-root is not an entire function, the function h is actually entire. Indeed, if we introduce the entire function

$$\psi(z) = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)!} z^{n+1},$$

we see that $\psi(z^2) = z \sin(z)$, and thus h can be written

$$h(\mu) = \frac{1}{\beta_\ell \beta_r} \psi(\mu^2) \psi(\mu^2 - \theta) - 1, \quad \forall \mu \in \mathbb{C}.$$

Note that h is an even function, so that for any solution $\mu \neq 0$ of (30), $-\mu$ is also a solution, but those two solutions lead to the same eigenvalue λ through the formula $\lambda = \alpha_1 + \mu^2$.

Before analyzing the set of eigenvalues of \mathcal{A}^* , let us determine the solution to (27) associated with every such eigenvalue. Let μ be a solution to (30). In (29), we set $K_1 = 1$ and then we get

$$K_2 = -\frac{\mu \sin \mu}{\beta_r}.$$

It follows that the solution to (27), denoted by Φ_λ , is given by

$$\Phi_\lambda(x) := \begin{pmatrix} \cos(\mu x) \\ -\frac{\mu \sin(\mu)}{\beta_r} \cos(\sqrt{\mu^2 - \theta}(1-x)) \end{pmatrix}, \quad x \in [0, 1], \quad (31)$$

associated with $\lambda = \mu^2 + \alpha_1$, where $\theta = \alpha_2 - \alpha_1$ and $\mu \in \mathbb{C}$ satisfies the equation (30).

3.3. Localization of the eigenvalues. We are now in position to describe precisely the structure of the spectrum of \mathcal{A}^* .

3.3.1. Preliminaries. We begin with the following set of lemmas, whose proofs are given in Appendix A.

Lemma 3.1. *There exist $\mu_0 > 0$ depending only on θ such that, for every $\mu \in \mathbb{C}$ satisfying $|\mu| > \mu_0$, we have*

$$|\sqrt{\mu^2 - \theta}| \geq \frac{1}{2} |\mu|,$$

and

$$|\sin(\sqrt{\mu^2 - \theta})| \geq \frac{3}{4} |\sin \mu| - \frac{1}{4} |\cos \mu|.$$

We set $\beta_* = \beta_\ell \beta_r$, which satisfies $\beta_* \neq 0$ by the assumption (6), and then we introduce the function

$$h_0(\mu) := \frac{1}{\beta_*} \mu \sqrt{\mu^2 - \theta} \sin(\mu) \sin(\sqrt{\mu^2 - \theta}), \quad (32)$$

that appears in the equation (30).

Lemma 3.2. *There exists $\vartheta_0 > 0$, depending only on θ and β_* , such that for any $\mu \in \mathbb{C}$, satisfying $|\operatorname{Im} \mu| \geq \vartheta_0$, we have*

$$|h_0(\mu)| > 1.$$

Lemma 3.3. *There exists $k_0 \in \mathbb{N}^*$, depending only on θ and β_* , such that for any $\mu \in \mathbb{C}$, with $\operatorname{Re} \mu = (k + 1/2)\pi$, $k \geq k_0$, we have*

$$|h_0(\mu)| > 1.$$

Lemma 3.4. *Let $\gamma \in \mathbb{C}$, and two sequences complex numbers $(\delta_k)_k$ and $(\varepsilon_k)_k$. We assume that $(\delta_k)_k$ is bounded, and $(\varepsilon_k)_k$ tends to 0.*

Then we have

$$\sqrt{(k\pi + \delta_k)^2 + \gamma} = k\pi + \delta_k + \frac{\gamma}{2k\pi} + O\left(\frac{1}{k^2}\right),$$

$$\sqrt{k^2\pi^2 + \gamma + \varepsilon_k} = k\pi + \frac{\gamma + \varepsilon_k}{2k\pi} - \frac{\gamma^2}{8k^3\pi^3} + o\left(\frac{1}{k^3}\right),$$

$$h_0(\sqrt{k^2\pi^2 + \gamma}) = \frac{\gamma(\gamma - \theta)}{4\beta_*} + O\left(\frac{1}{k^2}\right), \quad (33)$$

$$h_0(\sqrt{k^2\pi^2 + \theta/2 + \varepsilon_k}) = -\frac{\theta^2}{16\beta_*} \left(1 - \frac{4\varepsilon_k^2}{\theta^2} - \frac{\theta^2}{48k^2\pi^2} + o\left(\frac{1}{k^2}\right)\right). \quad (34)$$

We are now ready to give a quite precise description of the eigenvalues of \mathcal{A}^* .

3.3.2. Eigenvalues with large real parts. For any integer k we introduce the strip of the complex plane defined by

$$R_k := \{z \in \mathbb{C}, \quad (k - 1/2)\pi \leq \operatorname{Re} z \leq (k + 1/2)\pi\}. \quad (35)$$

Assuming for the moment that h has at least one root in each R_k - this will be proved below - we can first obtain the following result.

Proposition 3.5. *Let $(\mu_k)_{k \geq k_0}$ be a sequence of complex numbers such that $\mu_k \in R_k$ for every $k \geq k_0$ and $h(\mu_k) = 0$. Then we have the following properties:*

1. *The sequence $\delta_k = \mu_k - k\pi$ tends to zero as k goes to infinity.*
2. *The sequence $\gamma_k = k\pi\delta_k = k\pi(\mu_k - k\pi)$ is bounded and satisfies*

$$\gamma_k^2 - \frac{\theta}{2}\gamma_k \xrightarrow[k \rightarrow \infty]{} \beta_*. \quad (36)$$

Proof. 1. By definition of R_k and Lemma 3.2, we already know that $(\delta_k)_k$ lies into the compact set $K := \{z \in \mathbb{C}, |\operatorname{Re} z| \leq \pi/2, |\operatorname{Im} z| \leq \vartheta_0\}$.

By using Lemma 3.4 and the equation satisfied by $\mu_k = k\pi + \delta_k$ we obtain

$$\begin{aligned} \beta_* &= (k\pi + \delta_k)\sqrt{(k\pi + \delta_k)^2 - \theta} \sin(k\pi + \delta_k) \sin\left(\sqrt{(k\pi + \delta_k)^2 - \theta}\right) \\ &= (k\pi + \delta_k)(k\pi + \delta_k + O(1/k))(-1)^k \sin(\delta_k)(-1)^k (\sin(\delta_k) + O(1/k)). \end{aligned}$$

This leads to

$$\frac{\beta_*}{k^2\pi^2} = (\sin \delta_k)^2 + O(1/k),$$

and finally to

$$(\sin \delta_k)^2 = O(1/k),$$

which implies that $\delta_k \rightarrow 0$ since 0 is the unique root of the sine function in the compact K .

2. By a similar reasoning, we find that

$$\beta_* = (k\pi)^2(1 + O(1/k^2))\delta_k(1 + O(\delta_k^2)) \left(\delta_k - \frac{\theta}{2k\pi} + O(\delta_k^3) + O(1/k^2) \right),$$

which leads to

$$\beta_* = (1 + O(1/k^2))\gamma_k(1 + O(\delta_k^2)) \left(\gamma_k - \frac{\theta}{2} + O(\gamma_k\delta_k^2) + O(1/k) \right).$$

This clearly implies that $(\gamma_k)_k$ is bounded and that (36) holds. \square

Our main result of this section is the following.

Theorem 3.6. *There exists an integer $k_1 \geq 1$, depending only on α_1, α_2 and β_* , such that for all $k \geq k_1$, there exist exactly two distinct solutions $\mu_{k,+}$ and $\mu_{k,-}$ of the equation (30) in the strip R_k defined in (35).*

Moreover, they satisfy $|\operatorname{Im} \mu_{k,\pm}| \leq \vartheta_0$ for every $k \geq k_1$ (ϑ_0 is introduced in Lemma 3.2), as well as the following asymptotics:

- *If $16\beta_* + \theta^2 \neq 0$, there exist two distinct non zero complex numbers γ_{\pm} such that*

$$\mu_{k,\pm} = k\pi + \frac{\gamma_{\pm}}{k\pi} + O\left(\frac{1}{k^2}\right) \quad (37)$$

- *If $16\beta_* + \theta^2 = 0$, there exist $\gamma \in \mathbb{R}^*$ and $\xi \in \mathbb{R}^*$ such that*

$$\mu_{k,\pm} = k\pi + \frac{\gamma}{k\pi} \pm i \frac{\xi}{k^2\pi^2} + O\left(\frac{1}{k^3}\right).$$

To simplify the presentation, we will also set $\gamma_+ = \gamma_- = \gamma$ in that case, in such a way that (37) is still valid.

Note that the values of γ_{\pm} , γ and ξ , are explicit in function of the parameters, as we will see in the proof. Moreover, if $16\beta_* + \theta^2 > 0$ then $\mu_{k,\pm}$ (and γ_{\pm}) are real, whereas if $16\beta_* + \theta^2 \leq 0$, they are complex conjugate.

By the property (36), we see that the equation

$$\gamma^2 - \frac{\theta}{2}\gamma = \beta_*, \quad (38)$$

will play a role in the analysis and since $\beta_* \neq 0$, the solutions of this equation are not zero.

The reasoning will be slightly different depending on the sign of the discriminant of this equation which is, up to a positive factor, the quantity $16\beta_* + \theta^2$.

Proof. We first observe that the set of roots of h_0 in the complex plane is exactly given by

$$\left\{ \pm k\pi, \pm \sqrt{k^2\pi^2 + \theta}, k \in \mathbb{N} \right\}.$$

Hence, for k_1 large enough and $k \geq k_1$, we see that h_0 has exactly two roots (counted with multiplicities) in the strip R_k .

Moreover, by Lemmas 3.2 and 3.3 we know that $|h_0| > 1$ on the boundary of R_k and that $|h_0| > 1$ on $R_k \cap \{\mu \in \mathbb{C}, |\operatorname{Im} \mu| \geq \vartheta_0\}$. Therefore, by Rouché's theorem, we deduce that h has exactly two roots in R_k , counted with multiplicities, and that their imaginary part is bounded by ϑ_0 .

Finally, we observe that $\overline{h(\mu)} = h(\overline{\mu})$ for every μ . Therefore if one of the root of h in R_k is not real, the other root is necessarily its complex conjugate.

- Case 1 : assume that

$$16\beta_* + \theta^2 < 0. \quad (39)$$

In that case, the solutions of (38) are two conjugate complex numbers

$$\gamma_{\pm} = \frac{\theta \pm i\sqrt{|16\beta_* + \theta^2|}}{4}.$$

In particular we deduce that the solutions of $h(\mu) = 0$ in R_k , for k large enough, cannot be real and therefore, they are necessarily two conjugate complex numbers. Those two roots are denoted by $\mu_{k,+}$ and $\mu_{k,-} = \overline{\mu_{k,+}}$, with $\operatorname{Im} \mu_{k,+} > 0$ and they are the unique roots of h in the strip R_k .

From Proposition 3.5, we also deduce that

$$\mu_{k,\pm} = k\pi + \frac{\gamma_{\pm}}{k\pi} + o\left(\frac{1}{k}\right).$$

- Case 2 : assume that

$$16\beta_* + \theta^2 > 0.$$

By (33), we know that for any $\gamma \in \mathbb{R}$, we have

$$h(\sqrt{k^2\pi^2 + \gamma}) = \frac{\gamma(\gamma - \theta)}{4\beta_*} - 1 + O\left(\frac{1}{k^2}\right). \quad (40)$$

- If $\beta_* > 0$, we can find some γ_1 and γ_2 such that $\gamma_1 < 0 < \gamma_2$ and with

$$\frac{\gamma_i(\gamma_i - \theta)}{4\beta_*} - 1 > 0, \text{ for } i = 1, 2.$$

It follows that for k large enough we have

$$h(k\pi) = -1 < 0, \quad h(\sqrt{k^2\pi^2 + \gamma_1}) > 0, \quad \text{and} \quad h(\sqrt{k^2\pi^2 + \gamma_2}) > 0.$$

This implies that h (which is real valued on \mathbb{R}) has exactly two distinct real roots satisfying

$$\sqrt{k^2\pi^2 + \gamma_1} < \mu_{k,-} < k\pi < \mu_{k,+} < \sqrt{k^2\pi^2 + \gamma_2}.$$

Introducing $\gamma_{k,\pm} = k\pi(\mu_{k,\pm} - k\pi)$, we have that

$$\gamma_{k,+} > 0, \text{ and } \gamma_{k,-} < 0.$$

It follows from Proposition 3.5 that $\gamma_{k,-}$ necessarily converges towards the unique negative solution of (38) and that $\gamma_{k,+}$ converges towards the unique positive solution of (38). In other words we have

$$\mu_{k,\pm} = k\pi + \frac{\theta \pm \sqrt{16\beta_* + \theta^2}}{4k\pi} + O\left(\frac{1}{k^2}\right). \quad (41)$$

- If $\beta_* < 0$, then we can take $\gamma = \theta/2$ in (40) so that

$$\frac{\gamma(\gamma - \theta)}{4\beta_*} - 1 = -\frac{\theta^2}{16\beta_*} - 1 > 0,$$

by (39). It follows that for k large enough we have

$$h(k\pi) = -1 < 0, \quad h(\sqrt{k^2\pi^2 + \theta/2}) > 0, \quad \text{and} \quad h(\sqrt{k^2\pi^2 + \theta}) = -1 < 0.$$

This implies that h has also two distinct real roots satisfying

$$\mu_{k,-} < \sqrt{k^2\pi^2 + \theta/2} < \mu_{k,+}.$$

It follows that

$$\gamma_{k,-} < k\pi \left(\sqrt{k^2\pi^2 + \theta/2} - k\pi \right) < \gamma_{k,+},$$

and since $k\pi \left(\sqrt{k^2\pi^2 + \theta/2} - k\pi \right) \rightarrow \theta/4$, we see that $\gamma_{k,-}$ necessarily converges towards the unique solution of (38) which is less than $\theta/4$ and $\gamma_{k,+}$ necessarily converges towards the unique solution of (38) which is higher than $\theta/4$.

It follows that (41) still holds.

- Case 3 : assume that

$$16\beta_* + \theta^2 = 0. \quad (42)$$

In this case, the equation (38) has only one double (real) root which is $\gamma = \theta/4$ and therefore we have

$$\mu_k = k\pi + \frac{\theta}{4k\pi} + O\left(\frac{1}{k^2}\right).$$

We need to go one step further in the asymptotic expansion to determine the behavior of this sequence. To this end, we define a complex number ε_k by

$$\varepsilon_k = \mu_k^2 - k^2\pi^2 - \frac{\theta}{2},$$

which tends to 0 by the above asymptotic expansion of μ_k . Using (34), we deduce that

$$\begin{aligned} h_0(\mu_k) &= h_0\left(\sqrt{k^2\pi^2 + \frac{\theta}{2} + \varepsilon_k}\right) \\ &= -\frac{\theta^2}{16\beta_*} \left(1 - \frac{4\varepsilon_k^2}{\theta^2} - \frac{\theta^2}{48k^2\pi^2} + o\left(\frac{1}{k^2}\right)\right). \end{aligned}$$

By assumption we have $h_0(\mu_k) = 1$ and $16\beta_* = -\theta^2$, so that we end up with the equality

$$0 = \frac{4\varepsilon_k^2}{\theta^2} + \frac{\theta^2}{48k^2\pi^2} + o\left(\frac{1}{k^2}\right),$$

that is

$$\varepsilon_k^2 k^2 \pi^2 \xrightarrow[k \rightarrow \infty]{} -\frac{\theta^4}{192}.$$

We have that $\theta \neq 0$ (since $\beta_* \neq 0$ and $16\beta_* + \theta^2 = 0$), and thus the limit above implies that, for k large enough, ε_k cannot be a real number and therefore μ_k also is necessarily not real.

The same reasoning as before shows that we have two complex conjugate solutions $\mu_{k,+}$ and $\mu_{k,-} = \overline{\mu_{k,+}}$, with $\text{Im } \mu_{k,+} > 0$ and that the corresponding sequences $\varepsilon_{k,+}$ and $\varepsilon_{k,-}$ satisfy

$$\varepsilon_{k,\pm} = \pm i \frac{\theta^2}{8\sqrt{3}k\pi} + o\left(\frac{1}{k}\right).$$

By Lemma 3.4 and the fact that $\mu_{k,\pm} = \sqrt{k^2\pi^2 + \frac{\theta}{2} + \varepsilon_{k,\pm}}$, we deduce

$$\mu_{k,\pm} = k\pi + \frac{\theta}{4k\pi} \pm i \frac{\theta^2}{16\sqrt{3}k^2\pi^2} + o\left(\frac{1}{k^2}\right).$$

The proof of the theorem is complete. \square

The previous theorem gives information on the solutions of the equation $h(\mu) = 0$ from which we can deduce the following properties concerning the eigenvalues of our operator \mathcal{A}^* , away from the origin.

Corollary 3.7. *We can choose k_1 large enough in the previous theorem in such a way that all the eigenvalues of \mathcal{A}^* lying outside the disk $D(\alpha_1, R)$, with $R = (k_1 - 1/2)^2\pi^2$ are exactly given by*

$$\lambda_{k,\pm} = \mu_{k,\pm}^2 + \alpha_1, \quad \text{with } k \geq k_1.$$

Moreover, those eigenvalues and the associated eigenfunctions given by (31) (and denoted here by $\Phi_{k,\pm}$ to lighten the notation) satisfy the following asymptotics:

- If $16\beta_* + \theta^2 \neq 0$, we have

$$\lambda_{k,\pm} = k^2\pi^2 + 2\gamma_{\pm} + \alpha_1 + O\left(\frac{1}{k}\right),$$

and

$$\Phi_{k,\pm}(x) = \begin{pmatrix} 1 \\ -\frac{\gamma_{\pm}}{\beta_r} \end{pmatrix} \cos(k\pi x) + O\left(\frac{1}{k}\right). \quad (43)$$

- If $16\beta_* + \theta^2 = 0$, we have

$$\lambda_{k,\pm} = k^2\pi^2 + 2\gamma + \alpha_1 \pm 2i\frac{\xi}{k\pi} + O\left(\frac{1}{k^2}\right),$$

$$\Phi_{k,\pm}(x) = \begin{pmatrix} 1 \\ -\frac{\gamma}{\beta_r} \end{pmatrix} \cos(k\pi x) + O\left(\frac{1}{k}\right). \quad (44)$$

Moreover, in both cases, if we set

$$\Psi_k = \frac{\Phi_{k,+} - \Phi_{k,-}}{\lambda_{k,+} - \lambda_{k,-}}, \quad (45)$$

we have

$$\Psi_k(x) = \begin{pmatrix} 0 \\ -\frac{1}{2\beta_r} \end{pmatrix} \cos(k\pi x) + O\left(\frac{1}{k}\right). \quad (46)$$

In the properties above, the $O(\cdot)$ terms are uniform with respect to $x \in (0, 1)$.

Proof. • The asymptotics for the eigenvalues is just a consequence of the one obtained for $\mu_{k,\pm}$ and of the relation $\lambda_{k,\pm} = \mu_{k,\pm}^2 + \alpha_1$.

- Using the expression (31) and the asymptotics on $\mu_{k,\pm}$ we obtain immediately

$$\cos(\mu_{k,\pm}x) = \cos(k\pi x) + O\left(\frac{1}{k}\right),$$

$$\cos(\sqrt{\mu_{k,\pm}^2 - \theta(1-x)}) = (-1)^k \cos(k\pi x) + O\left(\frac{1}{k}\right),$$

$$\mu_{k,\pm} \sin(\mu_{k,\pm}) = (-1)^k \gamma_{\pm} + O\left(\frac{1}{k}\right).$$

Those estimates imply (43) and (44).

- In the case $16\beta_* + \theta^2 \neq 0$, we have that $\lambda_{k,+} - \lambda_{k,-} \rightarrow 2(\gamma_+ - \gamma_-)$ which gives (46).
- In the case $16\beta_* + \theta^2 = 0$, (46) is not as straightforward since $\lambda_{k,+} - \lambda_{k,-} \rightarrow 0$. More precisely, we have

$$\lambda_{k,+} - \lambda_{k,-} = i\frac{4\xi}{k\pi} + O\left(\frac{1}{k^2}\right).$$

- Using the order expansions for $\mu_{k,\pm}$ given in Theorem 3.6, we get

$$\mu_{k,\pm} \sin(\mu_{k,\pm}) = (-1)^k \left(\frac{\theta}{4} \pm i\frac{\xi}{k\pi} + O\left(\frac{1}{k^2}\right) \right),$$

and thus

$$\mu_{k,+} \sin(\mu_{k,+}) - \mu_{k,-} \sin(\mu_{k,-}) = (-1)^k 2i\frac{\xi}{k\pi} + O\left(\frac{1}{k^2}\right).$$

It follows that

$$\frac{\mu_{k,+} \sin(\mu_{k,+}) - \mu_{k,-} \sin(\mu_{k,-})}{\lambda_{k,+} - \lambda_{k,-}} = \frac{(-1)^k}{2} + O\left(\frac{1}{k}\right).$$

– Using again the third order asymptotics of $\mu_{k,\pm}$ we obtain

$$\begin{aligned} & \cos(\mu_{k,+}x) - \cos(\mu_{k,-}x) \\ &= 2 \sin\left(\frac{\mu_{k,+} - \mu_{k,-}}{2}x\right) \sin\left(\frac{\mu_{k,+} + \mu_{k,-}}{2}x\right) \\ &= 2 \sin\left(i\frac{\xi x}{k^2\pi^2} + O\left(\frac{1}{k^3}\right)\right) \sin\left(k\pi x + O\left(\frac{1}{k}\right)\right) \\ &= O\left(\frac{1}{k^2}\right), \end{aligned}$$

which proves in particular that

$$\frac{\cos(\mu_{k,+}x) - \cos(\mu_{k,-}x)}{\lambda_{k,+} - \lambda_{k,-}} = O\left(\frac{1}{k}\right).$$

The very same estimate holds with $\nu_{k,\pm}$ and $1-x$ instead of x , by using that

$$\nu_{k,\pm} = \sqrt{\mu_{k,\pm}^2 - \theta} = k\pi - \frac{\theta}{4k\pi} \pm i\frac{\xi}{k^2\pi^2} + O\left(\frac{1}{k^3}\right).$$

Combining all the above estimates, we obtain that (46) also holds in that case. \square

Note that, in the case $16\beta_* + \theta^2 = 0$, the eigenvalues condensate (in the sense that $\lambda_{k,+} - \lambda_{k,-}$ tends to 0 as k goes to infinity), and the associated eigenfunctions also condensate. However, the property (46) shows that both phenomenon somehow compensate.

3.3.3. Conclusions. By Corollary 3.7, the spectrum of \mathcal{A}^* can be split into two disjoint parts

$$\sigma(\mathcal{A}^*) := \Lambda_0 \cup \Lambda_\infty. \quad (47)$$

with

$$\Lambda_0 := \left\{ \lambda = \mu^2 + \alpha_1 \mid \mu \in D(0, (k_1 - 1/2)\pi) \text{ satisfying } h(\mu) = 0 \right\},$$

$$\Lambda_\infty := \bigcup_{k \geq k_1} \{ \lambda_{k,+}, \lambda_{k,-} \}.$$

Note that Λ_0 is finite and contained in the disk $D(\alpha_1, R)$, with $R = (k_1 - 1/2)^2\pi^2$. Moreover, as a consequence of the discussion in Section 3.4.2, it will appear that k_1 can be chosen large enough so that the eigenvalues in Λ_∞ are simple. However the eigenvalues in Λ_0 may be multiple.

3.4. Completeness of the set of root vectors of \mathcal{A}^* .

3.4.1. *Notations.* Let \mathcal{B} be an operator with compact resolvent in X . For any eigenvalue $\lambda \in \mathbb{C}$ of \mathcal{B} , we consider a circle $C(\lambda, r)$ in the complex plane, positively oriented, centered at λ and with a radius $r > 0$ small enough such that the disk $\bar{D}(\lambda, r)$ does not contain any other element of $\sigma(\mathcal{A}^*)$ than λ . Then, we know that

$$P_{\mathcal{B}, \lambda} = \frac{1}{2i\pi} \int_{C(\lambda, r)} (\xi \text{Id} - \mathcal{B})^{-1} d\xi,$$

is the spectral projector corresponding to the λ . Its range is spanned by all the root vectors of \mathcal{B} associated with λ .

Similarly, if $\Lambda \subset \sigma(\mathcal{B})$ is a finite set of eigenvalues, the spectral projector corresponding to Λ is given by

$$P_{\mathcal{B}, \Lambda} = \sum_{\lambda \in \Lambda} P_{\mathcal{B}, \lambda}.$$

3.4.2. *Generalized eigenvectors of \mathcal{A}^* .* For each eigenvalue $\lambda \in \sigma(\mathcal{A}^*)$, the eigenfunction Φ_λ given by (31) spans the associated eigenspace, where we recall that $\mu = \sqrt{\lambda - \alpha_1}$.

Since it may happen that some eigenvalues in Λ_0 are not simple, we associate to each of them a Jordan chain of length $n_\lambda \geq 1$, denoted by $\Phi_\lambda^0 = \Phi_\lambda, \Phi_\lambda^1, \dots, \Phi_\lambda^{n_\lambda-1}$. By definition, this chain spans the range of the projector $P_{\mathcal{A}^*, \lambda}$ and satisfies

$$(\mathcal{A}^* - \lambda)\Phi_\lambda^i = \Phi_\lambda^{i-1}, \quad \forall i \in \{1, \dots, n_\lambda - 1\}.$$

We gather all those Jordan chains in the finite family

$$\mathcal{F}_0 = \{\Phi_\lambda^i, \lambda \in \Lambda_0, i \in \{0, \dots, n_\lambda - 1\}\}.$$

By definition this family spans the range of the spectral projector $P_{\mathcal{A}^*, \Lambda_0}$, corresponding the eigenvalues in Λ_0 . Finally the *eigenfunctions* corresponding to the eigenvalues in Λ_∞ are gathered into the family

$$\mathcal{F}_\infty = \{\Phi_\lambda, \lambda \in \Lambda_\infty\} = \bigcup_{k \geq k_1} \{\Phi_{k,-}, \Phi_{k,+}\}.$$

The goal is to show that the family

$$\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_\infty, \tag{48}$$

is complete in X , which will be done in the next sections.

3.4.3. *A known Riesz basis for X .* Let us consider the functions defined by

$$\Phi_{k,\pm}^0(x) := \begin{pmatrix} 1 \\ -\frac{\gamma_\pm}{\beta_r} \end{pmatrix} \cos(k\pi x), \quad \forall k \geq 0, \tag{49}$$

$$\Psi_k^0(x) := \begin{pmatrix} 0 \\ -\frac{1}{2\beta_r} \end{pmatrix} \cos(k\pi x), \quad \forall k \geq 0. \tag{50}$$

Then we introduce the family

$$\mathcal{G} := \bigcup_{k \geq 0} \{\Phi_{k,+}^0, \Psi_k^0\}. \tag{51}$$

It is clear that there exists a constant matrix M such that

$$M\Phi_{k,+}^0(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(k\pi x), \quad \text{and} \quad M\Psi_k^0(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(k\pi x),$$

which proves that the family \mathcal{G} given by (51)-(49)-(50) forms a Riesz basis of X .

We split \mathcal{G} into two parts as follows

$$\mathcal{G}_0 = \bigcup_{0 \leq k \leq k_1 - 1} \{\Phi_{k,+}^0, \Psi_k^0\}, \text{ and } \mathcal{G}_\infty = \bigcup_{k \geq k_1} \{\Phi_{k,+}^0, \Psi_k^0\}.$$

We shall use a result from [32, Corollary 11.4, Chapter II-§11] (see also [23, Lemma 6.2]), that we state here with slightly different notation.

Lemma 3.8. *Let $\{\phi_n^0\}_{n \geq 0}$ be a Riesz basis of the Hilbert space X and $\{\phi_n\}_{n \geq N}$ (for $N \geq 0$) be another sequence in X such that*

$$\sum_{n \geq N} \|\phi_n - \phi_n^0\|_H^2 < +\infty.$$

Then, there exists an $M \geq N$ such that the family $\{\phi_n^0\}_{n=1}^M \cup \{\phi_n\}_{n \geq M+1}$ forms a Riesz basis of X .

3.4.4. *Main result.* We are now ready to prove the main result of this section. It follows quite standard arguments as given in [15, Lemma 5, XIX.2.5 and Lemma 5, XIX.5.5] that we specialize to our problem.

Proposition 3.9. *The family \mathcal{F} made of the root vectors of \mathcal{A}^* is complete in X .*

Proof. We first observe, by (45), that the family \mathcal{F}_∞ spans exactly the same space as

$$\tilde{\mathcal{F}}_\infty = \bigcup_{k \geq k_1} \{\Phi_{k,+}, \Psi_k\}.$$

Therefore we are reduced to show that the new family $\tilde{\mathcal{F}} = \mathcal{F}_0 \cup \tilde{\mathcal{F}}_\infty$ is complete in X .

Thanks to (43), (44) and (46), we can use Lemma 3.8 to ensure that, choosing a larger value of k_1 if necessary, the family

$$\tilde{\mathcal{G}} = \mathcal{G}_0 \cup \tilde{\mathcal{F}}_\infty,$$

forms a Riesz basis of X . We will now fix such a value of k_1 .

Let us set $\mathcal{H} = \overline{\text{Span}(\mathcal{F})} = \overline{\text{Span}(\tilde{\mathcal{F}})} \subset X$. Our goal is to show that $\mathcal{H} = X$.

- Let us first show that \mathcal{H}^\perp is finite dimensional, and more precisely that $\dim \mathcal{H}^\perp \leq 2k_1$. Indeed if this is not the case, we can find a non trivial element Φ in \mathcal{H}^\perp that satisfies the following additional orthogonality properties

$$(\Phi, \Psi) = 0, \quad \forall \Psi \in \mathcal{G}_0,$$

since the cardinal of \mathcal{G}_0 is $2k_1$. Since Φ is orthogonal to \mathcal{H} , that is to the elements of $\tilde{\mathcal{F}}$, we deduce that Φ is in fact orthogonal to each element of the Riesz basis $\tilde{\mathcal{G}}$. This implies that $\Phi = 0$ which is a contradiction.

- Let $\mu \in \rho(\mathcal{A})$. We observe that \mathcal{H}^\perp is stable by the resolvent $(\mu \text{Id} - \mathcal{A})^{-1}$. Indeed, if $x \in \mathcal{H}^\perp$ and Φ is any element in \mathcal{F} , we have

$$((\mu \text{Id} - \mathcal{A})^{-1}x, \Phi) = (x, (\bar{\mu} \text{Id} - \mathcal{A}^*)^{-1}\Phi) = 0,$$

since $(\bar{\mu} \text{Id} - \mathcal{A}^*)^{-1}\Phi \in \text{Span} \mathcal{F} \subset \mathcal{H}$.

- We prove now that $\mathcal{H}^\perp = \{0\}$. Indeed, if it is not the case, the operator $(\mu \text{Id} - \mathcal{A})^{-1}$ has an eigenvector x in the non trivial finite dimensional space \mathcal{H}^\perp . It follows that $x \in D(\mathcal{A})$ and that it is an eigenvector of \mathcal{A} . We write $\mathcal{A}x = \bar{\lambda}x$ for some $\lambda \in \sigma(\mathcal{A}^*)$ and thus

$$x = P_{\mathcal{A}, \bar{\lambda}}x,$$

which leads to

$$\|x\|^2 = (P_{\mathcal{A}, \bar{\lambda}} x, x) = (x, P_{\mathcal{A}^*, \lambda} x) = 0,$$

since $P_{\mathcal{A}^*, \lambda} x \in \mathcal{H}$ by definition of \mathcal{H} and $x \perp \mathcal{H}$. This is a contradiction.

We thus have proved that $\mathcal{H} = X$, which was our claim. \square

Remark 3.10. Even though this fact will not be used in the sequel, it can be seen from the discussion above that the cardinal of \mathcal{F}_0 is the same as the one of \mathcal{G}_0 , that is $2k_1$.

4. Boundary controllability of the linearized systems. This section is devoted to prove the boundary null-controllability of the linear systems associated to our nonlinear models.

4.1. Approximate controllability. Let us first discuss about the approximate controllability of the linearized systems in the space X at any time $T > 0$.

Lemma 4.1. *1. The system (10)–(14)–(15) is approximately controllable in X at any given time $T > 0$ and the observation terms satisfy*

$$|\mathcal{B}_1^* \Phi_\lambda| = 1, \quad \forall \lambda \in \sigma(\mathcal{A}^*). \quad (52)$$

2. We introduce the set

$$\mathcal{R} := \left\{ (-1)^k \left(k + \frac{1}{2}\right) \pi \sqrt{\left(k + \frac{1}{2}\right)^2 \pi^2 + \theta} \sin \left(\sqrt{\left(k + \frac{1}{2}\right)^2 \pi^2 + \theta} \right); k \geq 0 \right\} \cap \mathbb{R}^*, \quad (53)$$

where we recall that $\theta = \alpha_2 - \alpha_1$.

Then, the system (10)–(14)–(16) is approximately controllable in X at any time $T > 0$ if and only if $\beta_* = \beta_\ell \beta_r \notin \mathcal{R}$. In that case, there exists a constant $c_0 > 0$ such that the observation terms satisfy

$$|\mathcal{B}_2^* \Phi_\lambda| \geq c_0, \quad \forall \lambda \in \sigma(\mathcal{A}^*). \quad (54)$$

Proof. Recall the observation operators \mathcal{B}_1^* , \mathcal{B}_2^* respectively defined by (24a), (24b) and the explicit expressions of the eigenfunctions Φ_λ from (31).

1. It is easy to see that

$$\mathcal{B}_1^* \Phi_\lambda = 1, \quad \forall \lambda \in \sigma(\mathcal{A}^*).$$

So, by using Fattorini-Hautus test (see [17], [30]), the linear control system (10)–(14)–(15) is approximately controllable in X at any time $T > 0$.

2. Let us compute from (31) that

$$\mathcal{B}_2^* \Phi_\lambda = -\frac{\mu \sin \mu}{\beta_r} \cos \left(\sqrt{\mu^2 - \theta} \right), \quad \forall \lambda \in \sigma(\mathcal{A}^*),$$

where, by construction $\lambda = \mu^2 + \alpha_1$.

Assume that $\mathcal{B}_2^* \Phi_\lambda = 0$ for some eigenvalue λ . Since $\mu \sin \mu \neq 0$ (if so, then the eigenvalue equation (30) is invalid), this yields $\cos \left(\sqrt{\mu^2 - \theta} \right) = 0$ and thus $\mu^2 = \left(k + \frac{1}{2}\right)^2 \pi^2 + \theta$, for some $k \geq 0$.

But we remember that μ also satisfies the equation (30). This is possible only when

$$\frac{1}{\beta_*} \left(k + \frac{1}{2}\right) \pi \sqrt{\left(k + \frac{1}{2}\right)^2 \pi^2 - \theta} \sin \left(\left(k + \frac{1}{2}\right) \pi \right) \sin \left(\sqrt{\left(k + \frac{1}{2}\right)^2 \pi^2 - \theta} \right) = 1.$$

Such a k exists if and only if $\beta_* \in \mathcal{R}$. Hence, the Fattorini-Hautus criterion confirms the approximate controllability of the system (10)–(14)–(16), if and only if $\beta_* \notin \mathcal{R}$.

It remains to show (54), under that condition. Since $\mathcal{B}^*\Phi_\lambda \neq 0$ for any $\lambda \in \sigma(\mathcal{A}^*)$ and since $\sigma(\mathcal{A}^*) = \Lambda_0 \cup \Lambda_\infty$ where Λ_0 is finite, we only need to show that

$$\liminf_{k \rightarrow \infty} |\mathcal{B}^*\Phi_{k,\pm}| > 0.$$

This property is clear from (43), because γ_+ and γ_- are not zero. \square

4.2. Bounds on the (generalized) eigenfunctions. We have the following result.

Lemma 4.2. *There exists some constant $C > 0$ depending on the parameters $\alpha_1, \alpha_2, \beta_\ell, \beta_r$ such that*

$$\|\Phi\|_X \leq C, \quad \forall \Phi \in \mathcal{F}, \quad (55)$$

where \mathcal{F} is the set of eigenfunctions and generalized eigenfunctions of \mathcal{A}^* introduced in (48).

Proof. Using (48) and the fact that \mathcal{F}_0 is finite, we see that we just need to show that

$$\sup_{k \geq k_1} \|\Phi_{k,\pm}\|_X < +\infty,$$

but this is a straightforward consequence of Corollary 3.7. \square

From now on we will assume that the generalized eigenfunctions satisfy the property

$$\mathcal{B}_1^*\Phi_\lambda^j = 0, \quad \forall \lambda \in \Lambda_0, \forall j \in \{1, \dots, n_\lambda - 1\}, \quad (56)$$

in the case where the control is given by (15), or the property

$$\mathcal{B}_2^*\Phi_\lambda^j = 0, \quad \forall \lambda \in \Lambda_0, \forall j \in \{1, \dots, n_\lambda - 1\}, \quad (57)$$

in the case (16), when $\beta_* \notin \mathcal{R}$. This is always possible by using Lemma 4.1 and the fact that one can add any multiple of the eigenfunction to each generalized eigenfunction.

We recall that the eigenvalues in Λ_∞ are all algebraically simple. Thus, to simplify the presentation in the sequel, we will conventionally set the multiplicity $n_\lambda = 1$ for every $\lambda \in \Lambda_\infty$. We will finally set

$$n_{\max} = \sup_{\lambda \in \sigma(\mathcal{A}^*)} n_\lambda,$$

which is clearly finite.

4.3. The moments problem. Since the family \mathcal{F} is complete in X (Proposition 3.9), it is enough to check the equations (25) and (26) for all $\zeta \in \mathcal{F}$.

This gives us the following characterization of the controls.

- For any $y_0 \in X$ and $T > 0$, a function $v \in L^2(0, T)$ is a boundary null-control for the system (10)–(14)–(15) if and only if, we have

$$\frac{e^{-T\lambda}}{\mathcal{B}_1^*\Phi_\lambda} \left(y_0, \sum_{k=0}^j \frac{(-T)^k}{k!} \Phi_\lambda^{j-k} \right)_X = \int_0^T v(t) \frac{(-(T-t))^k}{k!} e^{-(T-t)\lambda} dt, \quad (58)$$

$$\forall \lambda \in \sigma(\mathcal{A}^*), \quad \forall j \in \{0, \dots, n_\lambda - 1\}.$$

- Similarly, assuming that $\beta_* \notin \mathcal{R}$, for any $y_0 \in X$ and $T > 0$ a function $v \in L^2(0, T)$ is a boundary null-control for the system (10)–(14)–(16) if and only if, we have

$$\frac{\overline{e^{-T\lambda}}}{\mathcal{B}_2^* \Phi_\lambda} \left(y_0, \sum_{k=0}^j \frac{(-T)^k}{k!} \Phi_\lambda^{j-k} \right)_X = \int_0^T v(t) \frac{(-(T-t))^k}{k!} \overline{e^{-(T-t)\lambda}} dt, \quad (59)$$

$$\forall \lambda \in \sigma(\mathcal{A}^*), \forall j \in \{0, \dots, n_\lambda - 1\}.$$

Here, we have used (56), (57) and the fact that

$$e^{-t\mathcal{A}^*} \Phi_\lambda^j = e^{-\lambda t} \sum_{k=0}^j \frac{(-t)^k}{k!} \Phi_\lambda^{j-k}, \quad \forall \lambda \in \sigma(\mathcal{A}^*), \forall j \in \{0, \dots, n_\lambda - 1\}.$$

The above set of equations are the moment problems for our linear models and we shall solve these in the next paragraphs.

4.4. Solving the moment problem. Let us show how to solve the moment problems introduced before. Its resolution relies on the existence and estimates of biorthogonal families to exponential functions in $L^2(0, T)$. This topic has a long story starting from the pioneering work [18] in the framework of control theory, up to recent developments where sharper estimates were obtained under a quite general set of hypothesis.

4.4.1. Biorthogonal families to exponentials. A general result. In this paper, we will use the following theorem whose proof is omitted since it can be deduced from the more general results [11, Theorem V.4.26 and Corollary V.4.27]. Note that those results are quite similar to [8, Theorem 1.5] or [22, Theorem 1.2] but with a simplified set of assumptions.

Theorem 4.3. *Let $\Lambda \subset \mathbb{C}$ be a set of complex numbers and $(n_\lambda)_{\lambda \in \Lambda} \subset \mathbb{N}^*$ be a family of integers such that*

$$n_{\max} := \sup_{\lambda \in \Lambda} n_\lambda < +\infty.$$

We define the counting function of Λ as follows

$$\mathcal{N}(r) := \# \{ \lambda \in \Lambda, \text{ s.t. } |\lambda| \leq r \}, \quad \forall r \geq 0,$$

and we assume the following assumptions on Λ .

- Sector condition : *There exists $\zeta > 0$ such that*

$$\Lambda \subset \{ z \in \mathbb{C} \mid \operatorname{Re} z > 0, \text{ and } |\operatorname{Im} z| < \zeta (\operatorname{Re} z) \}. \quad (60)$$

- Asymptotics of the counting function: *there exists $\kappa > 0$ such that*

$$\mathcal{N}(r) \leq \kappa r^{1/2}, \quad \forall r > 0, \quad (61a)$$

$$|\mathcal{N}(r) - \mathcal{N}(s)| \leq \kappa \left(1 + |r - s|^{1/2} \right), \quad \forall r, s > 0. \quad (61b)$$

- Weak gap condition of order 2 (see [11, Definition V.4.18]): *There exists $\rho > 0$ such that*

$$\#(\Lambda \cap D(\mu, \rho/2)) \leq 2, \quad \forall \mu \in \mathbb{C}, \quad (62)$$

where $D(\mu, \rho/2)$ is the open disk in the complex plane centered at μ and of radius $\rho/2$.

For any $\lambda \in \Lambda$, we set

$$\rho_\lambda := \min(\rho/2, d(\lambda, \Lambda \setminus \{\lambda\})),$$

and we assume that

$$n_\lambda = 1, \quad \forall \lambda \in \Lambda \text{ s.t. } \rho_\lambda < \rho/2. \quad (63)$$

Then, for any $T > 0$ there exists a family $\{q_{\lambda,T}^j\}_{\substack{\lambda \in \Lambda \\ 0 \leq j < n_\lambda}} \subset L^2(0, T)$ such that

$$\int_0^T q_{\lambda,T}^j(t) \frac{(t-T)^k}{k!} e^{-(T-t)\mu} dt = \delta_{\lambda,\mu} \delta_{j,k}, \quad \forall \lambda \in \Lambda, \forall j \in \{0, \dots, n_\lambda - 1\}, \quad (64)$$

$$\forall \mu \in \Lambda, \forall k \in \{0, \dots, n_\mu - 1\},$$

with the following estimate

$$\|q_{\lambda,T}^j\|_{L^2(0,T)} \leq C \frac{1}{\rho_\lambda} e^{C\sqrt{\operatorname{Re} \lambda} + \frac{C}{T}}, \quad \forall \lambda \in \Lambda, \forall j \in \{0, \dots, n_\lambda - 1\}. \quad (65)$$

4.4.2. Application to our particular problem. In order to use this theorem for our purpose, we first observe that the real parts of the eigenvalues of \mathcal{A}^* are bounded from below. Therefore, we can find some $\alpha_0 \geq 0$, such that the set $\Lambda := \sigma(\mathcal{A}^*) + \alpha_0$ satisfies

$$\operatorname{Re} \lambda \geq 1, \quad \forall \lambda \in \Lambda. \quad (66)$$

Moreover, for any $\lambda \in \Lambda$ we set

$$\tilde{n}_\lambda = n_{\lambda - \alpha_0}.$$

We will now show that the set Λ , as well as the multiplicities $(\tilde{n}_\lambda)_{\lambda \in \Lambda}$ satisfy the assumptions of Theorem 4.3.

- The sector condition (60) is clear by using (66) as well as the fact all the eigenvalues of \mathcal{A}^* , and thus all the elements in Λ , have a uniformly bounded imaginary part (see Corollary 3.7).
- Let us check that the counting function of Λ satisfies (61).

First of all, we observe from Section 3.3.3, that we have

$$\Lambda = \sigma(\mathcal{A}^*) + \alpha_0 = (\Lambda_0 + \alpha_0) \cup (\Lambda_{\infty,+} + \alpha_0) \cup (\Lambda_{\infty,-} + \alpha_0),$$

with $\Lambda_{\infty,\pm} = \{\lambda_{k,\pm}, k \geq k_1\}$.

The result will come if we manage to prove that the counting function of each of the three sets $\Lambda_0 + \alpha_0$, $\Lambda_{\infty,+} + \alpha_0$ and $\Lambda_{\infty,-} + \alpha_0$ satisfies (61), as shown for instance in the proof of [11, Lemma V.4.20].

Those estimates are straightforward for $\Lambda_0 + \alpha_0$ which is a finite set that does not contain 0 by (66). Let us show them for $\Lambda_{\infty,+} + \alpha_0$ since the reasoning is similar for $\Lambda_{\infty,-} + \alpha_0$. We denote by \mathcal{N}_+ the corresponding counting function.

- Let $r > 0$ be fixed. We set $n = \mathcal{N}_+(r)$. By definition, we have $|\lambda_{k_1+n-1,+} + \alpha_0| \leq r$ so that $|\lambda_{k_1+n-1,+}| \leq r + \alpha_0$, and finally

$$|\mu_{k_1+n-1,+}| \leq \sqrt{r + \tilde{\alpha}},$$

where we have set

$$\tilde{\alpha} := |\alpha_1| + \alpha_0, \quad (67)$$

Since $\mu_{k_1+n-1,+}$ belongs to the strip R_{k_1+n-1} we deduce

$$(n + k_1 - 3/2)\pi \leq \sqrt{r + \tilde{\alpha}},$$

which gives

$$n \leq C(1 + \sqrt{r}),$$

for some $C > 0$ depending only on α_0 , α_1 and k_1 . This gives (61a) for \mathcal{N}_+ since, by (66), this function vanishes for r small enough.

- Let $0 < s < r$ be given. We assume that $s > \inf \mathcal{N}_+$ since if it is not the case, we have $\mathcal{N}_+(s) = 0$ and thus (61b) follows from (61a).

We set $n = \mathcal{N}_+(r)$ and $m = \mathcal{N}_+(s)$. If $m = n$ there is nothing to prove, so we will assume that $n > m$. By definition of the counting function we have

$$|\lambda_{k_1+n-1,+}| \leq r + \alpha_0, \text{ and } |\lambda_{k_1+m,+}| > s - \alpha_0.$$

By using that $\mu_{k,+}^2 = \lambda_{k,+} - \alpha_1$ and the fact that the square root function is sublinear, we get

$$|\mu_{k_1+n-1,+}| \leq \sqrt{r} + \sqrt{\alpha}, \text{ and } |\mu_{k_1+m,+}| \geq \sqrt{s} - \sqrt{\alpha}.$$

Moreover, by Theorem 3.6, we know that $|\operatorname{Im} \mu_{k,+}| \leq \vartheta_0$ for every $k \geq k_1$. It comes

$$\begin{aligned} \operatorname{Re} \mu_{k_1+n-1,+} &\leq \sqrt{r} + \sqrt{\alpha} + \vartheta_0, \text{ and} \\ \operatorname{Re} \mu_{k_1+m,+} &\geq \sqrt{s} - \sqrt{\alpha} - \vartheta_0, \end{aligned}$$

which, in turn, leads to

$$\begin{aligned} \operatorname{Re} \mu_{k_1+n-1,+} - \operatorname{Re} \mu_{k_1+m,+} &\leq \sqrt{r} - \sqrt{s} + 2\sqrt{\alpha} + 2\vartheta_0 \\ &\leq \sqrt{r-s} + 2\sqrt{\alpha} + 2\vartheta_0. \end{aligned} \quad (68)$$

Moreover, by the asymptotics (37), we know that there exists $c > 0$ such that for any $k \geq k_1$ we have $\operatorname{Re} \mu_{k+1,+} - \operatorname{Re} \mu_{k,+} \geq c$, and thus

$$\operatorname{Re} \mu_{k_1+n-1,+} - \operatorname{Re} \mu_{k_1+m,+} \geq (n-m-1)c. \quad (69)$$

Combining (68) and (69), the claim is proved.

- It remains to prove the weak gap condition (62). Using the asymptotic expressions given in Corollary 3.7 for the eigenvalues in Λ_∞ , one has the following: there exists some $\rho_1 > 0$ depending only on $\alpha_1, \alpha_2, \beta_\ell$ and β_r , such that

$$|\lambda_{k,\pm} - \lambda_{k+1,\pm}| \geq \rho_1 k, \text{ for } k \geq k_1,$$

- In the case $\theta^2 + 16\beta_* \neq 0$, we deduce still from Corollary 3.7 that, possibly changing the value of ρ_1 , we have

$$|\lambda_{k,+} - \lambda_{k,-}| \geq \rho_1, \text{ for } k \geq k_1.$$

Since Λ_0 is finite, and the gap property is not influenced by the translation by α_0 , we can finally conclude that there is some $\rho > 0$ such that

$$|\lambda - \tilde{\lambda}| \geq \rho, \quad \lambda, \tilde{\lambda} \in \Lambda \text{ with } \lambda \neq \tilde{\lambda}, \quad (70)$$

which is the uniform gap property, which of course imply the weaker property (62).

- In the case $\theta^2 + 16\beta_* = 0$, the situation is quite different since, we know from Corollary 3.7 that

$$|\lambda_{k,+} - \lambda_{k,-}| \sim \frac{C}{k} \sim \frac{C_2}{\sqrt{\operatorname{Re} \lambda_{k,+}}},$$

so that the uniform gap property (70) does not hold anymore. More precisely, the only case where two eigenvalues of \mathcal{A}^* can be close one from each other is the case where $\lambda_{k,+}$ is close from $\lambda_{k,-}$, indeed we can choose $\rho > 0$ small enough so that, if $\mu \in \mathbb{C}$ is such that $\#(\sigma(\mathcal{A}^*) \cap B(\mu, \rho/2)) = 2$ then there exists $k \geq k_1$ such that

$$\sigma(\mathcal{A}^*) \cap B(\mu, \rho/2) = \{\lambda_{k,+}, \lambda_{k,-}\}.$$

– As a conclusion, in both cases we have the weak gap condition (62) as well as the estimate

$$|\lambda - \tilde{\lambda}| \geq \frac{C}{\sqrt{\alpha_0 + \operatorname{Re} \lambda}}, \quad \forall \lambda, \tilde{\lambda} \in \Lambda, \text{ s.t. } 0 < |\lambda - \tilde{\lambda}| \leq \rho,$$

which leads to

$$\rho \lambda \geq \frac{C}{\sqrt{\operatorname{Re} \lambda - \alpha_0}} \geq \frac{C}{\sqrt{\operatorname{Re} \lambda}}, \quad \forall \lambda \in \Lambda. \quad (71)$$

Moreover, if $|\lambda - \tilde{\lambda}| \leq \rho$, we know that $\lambda, \tilde{\lambda} \in \Lambda_\infty + \alpha_0$ and in particular, those are simple eigenvalues of the operator, see Section 3.3.3. This implies that $(\tilde{n}_\lambda)_{\lambda \in \Lambda}$ satisfies (63).

The assumptions of Theorem 4.3 being satisfied, and using (71), we deduce that there exists a family that we call $\{\tilde{q}_{\lambda,T}^j\}_{\substack{\lambda \in \Lambda \\ 0 \leq j < \tilde{n}_\lambda}} \subset L^2(0, T)$ satisfying the biorthogonality property

$$\int_0^T \tilde{q}_{\lambda,T}^j(t) \frac{(t-T)^k}{k!} e^{-(T-t)\mu} dt = \delta_{\lambda,\mu} \delta_{j,k}, \quad \forall \lambda \in \Lambda, \forall j \in \{0, \dots, \tilde{n}_\lambda - 1\}, \quad (72)$$

$$\forall \tilde{\mu} \in \Lambda, \forall k \in \{0, \dots, \tilde{n}_\mu - 1\},$$

with the following estimate

$$\|\tilde{q}_{\lambda,T}^j\|_{L^2(0,T)} \leq C \sqrt{\operatorname{Re} \lambda} e^{C\sqrt{\operatorname{Re} \lambda} + \frac{C}{T}}, \quad \forall \lambda \in \Lambda, \forall j \in \{0, \dots, \tilde{n}_\lambda - 1\},$$

where the constant $C > 0$ depends on $\zeta, \rho, \kappa, n_{\max}$ but not on λ and T . Note that the prefactor $\sqrt{\operatorname{Re} \lambda}$ comes from (71) but it can immediately be integrated in the exponential term since $\sqrt{\operatorname{Re} \lambda} \leq e^{\sqrt{\operatorname{Re} \lambda}}$. Hence, changing the value of the constant C , we end up with the estimate

$$\|\tilde{q}_{\lambda,T}^j\|_{L^2(0,T)} \leq C e^{C\sqrt{\operatorname{Re} \lambda} + \frac{C}{T}}, \quad \forall \lambda \in \Lambda, \forall j \in \{0, \dots, \tilde{n}_\lambda - 1\}. \quad (73)$$

Recalling that Λ is obtained by translation from the spectrum $\sigma(\mathcal{A}^*)$, the biorthogonality property (72) can be written as

$$\int_0^T \tilde{q}_{\lambda+\alpha_0,T}^j(t) \frac{(t-T)^k}{k!} e^{-(T-t)(\mu+\alpha_0)} dt = \delta_{\lambda,\mu} \delta_{j,k},$$

$$\forall \lambda \in \sigma(\mathcal{A}^*), \forall j \in \{0, \dots, n_\lambda - 1\},$$

$$\forall \mu \in \sigma(\mathcal{A}^*), \forall k \in \{0, \dots, n_\mu - 1\},$$

so that, if we set

$$q_{\lambda,T}^j(t) := \tilde{q}_{\lambda+\alpha_0,T}^j(t) e^{-(T-t)\alpha_0}, \quad \forall \lambda \in \sigma(\mathcal{A}^*), \forall j \in \{0, \dots, n_\lambda - 1\},$$

we end up with the biorthogonality property

$$\int_0^T q_{\lambda,T}^j(t) \frac{(t-T)^k}{k!} e^{-(T-t)\mu} dt = \delta_{\lambda,\mu} \delta_{j,k}, \quad \forall \lambda \in \sigma(\mathcal{A}^*), \forall j \in \{0, \dots, n_\lambda - 1\},$$

$$\forall \mu \in \sigma(\mathcal{A}^*), \forall k \in \{0, \dots, n_\mu - 1\},$$
(74)

and the estimate (73) becomes

$$\|q_{\lambda,T}^j\|_{L^2(0,T)} \leq C e^{\alpha_0 T} e^{C\sqrt{\alpha_0 + \operatorname{Re} \lambda} + \frac{C}{T}}, \quad \forall \lambda \in \sigma(\mathcal{A}^*), \forall j \in \{0, \dots, n_\lambda - 1\}. \quad (75)$$

4.4.3. *Conclusion.* We can now solve the set of moments equations (58) and (59) to construct boundary controls for our linear systems.

Proof of Theorem 1.3. We will first consider the control problem (10)–(14)–(15).

We suppose given any $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, $\beta_* = \beta_\ell \beta_r \neq 0$. For any initial data $y_0 \in X$ we consider

$$v(t) = \sum_{\lambda \in \sigma(\mathcal{A}^*)} \sum_{j=0}^{n_\lambda-1} v_{\lambda,j}(t), \quad \forall t \in [0, T],$$

with

$$v_{\lambda,j}(t) = \frac{e^{-T\lambda}}{\mathcal{B}_1^* \Phi_\lambda} \left(y_0, \sum_{k=0}^j \frac{(-T)^k}{k!} \Phi_\lambda^{j-k} \right)_X q_{\lambda,T}^j(t), \quad \forall t \in [0, T], \text{ and } \lambda \in \sigma(\mathcal{A}^*).$$

Observe that, thanks to (74), this choice of v formally satisfies the moments equation (58). Using the estimate (75), the lower bounds of the observation terms (52) and the upper bounds (55), we get

$$\|v_{\lambda,j}\|_{L^2(0,T)} \leq C e^{C/T + \alpha_0 T} e^{-T(\operatorname{Re} \lambda) + C\sqrt{\operatorname{Re} \lambda + \alpha_0}} \|y_0\|_X,$$

$$\forall \lambda \in \sigma(\mathcal{A}^*), \forall j \in \{0, \dots, n_\lambda - 1\}. \quad (77)$$

Applying Young's inequality, we get

$$C\sqrt{\operatorname{Re}(\lambda) + \alpha_0} \leq \frac{T}{2}(\operatorname{Re}(\lambda) + \alpha_0) + \frac{C^2}{T}, \quad \forall \lambda \in \sigma(\mathcal{A}^*),$$

that we use in (77) to obtain

$$\|v_{\lambda,j}\|_{L^2(0,T)} \leq C e^{C/T + 2\alpha_0 T} e^{-T\operatorname{Re} \lambda/2} \|y_0\|_X, \quad \forall \lambda \in \sigma(\mathcal{A}^*), \forall j \in \{0, \dots, n_\lambda - 1\}.$$

Taking sum over λ and j and using the asymptotics given in Corollary 3.7 (see also [11, Proposition A.5.39]), we obtain that the series defining v is convergent as well as the required estimate

$$\|v\|_{L^2(0,T)} = \sum_{\lambda \in \sigma(\mathcal{A}^*)} \sum_{j=0}^{n_\lambda-1} \|v_{\lambda,j}\|_{L^2(0,T)} \leq M e^{MT} e^{M/T} \|y_0\|_X,$$

with M that does not depend on T neither on y_0 .

The reasoning is exactly the same for the control problem (10)–(14)–(16) as soon as we have the additional assumption $\beta_* \notin \mathcal{R}$. \square

5. Boundary controllability of the the nonlinear system. In the previous section, we obtained that the control cost for the linear systems (10)–(14)–(15)/(16) is $Me^{M/T}$ for small-time T . Using this we shall deal with the null-controllability of the nonlinear systems (10)–(11)–(12)/(13) in the current section.

We introduce the new nonlinear functions

$$g_\ell(s) := \tilde{f}_\ell(s) - \beta_\ell s = f_\ell(z_2^*(0) + s) - f_r(z_2^*(0)) - \beta_\ell s, \quad (78)$$

$$g_r(s) := \tilde{f}_r(s) - \beta_r s = f_r(z_1^*(1) + s) - f_r(z_1^*(1)) - \beta_r s, \quad (79)$$

in such a way that, by definition and by (4), we have

$$\begin{cases} g_\ell(0) = g'_\ell(0) = 0, \text{ and } |g_\ell(s_1) - g_\ell(s_2)| \leq \|f_\ell\|_{W^{2,\infty}} |s_1 - s_2| (|s_1| + |s_2|), \\ \hspace{15em} \forall s_1, s_2 \in \mathbb{R}, \\ g_r(0) = g'_r(0) = 0, \text{ and } |g_r(s_1) - g_r(s_2)| \leq \|f_r\|_{W^{2,\infty}} |s_1 - s_2| (|s_1| + |s_2|), \\ \hspace{15em} \forall s_1, s_2 \in \mathbb{R}. \end{cases} \quad (80)$$

Therefore, we see that the nonlinear boundary conditions in (11), (12) and (13) can be written, respectively, as follows

$$\begin{aligned} \partial_x y_2(\cdot, 1) - \beta_r y_1(\cdot, 1) &= g_r(y_1(\cdot, 1)), \\ \partial_x y_1(\cdot, 0) + \beta_\ell y_2(\cdot, 0) &= v - g_\ell(y_2(\cdot, 0)), \\ \partial_x y_1(\cdot, 0) + \beta_\ell y_2(\cdot, 0) &= -g_\ell(y_2(\cdot, 0)), \end{aligned}$$

where the terms in g_ℓ and g_r are quadratic and can then be seen as source terms in the control problem that can be dealt with using the method developed in [29].

5.1. Source term method. Let us apply the source term method (see [29]) to our case. We suppose given two constants $p > 0$, $q > 1$ in such a way that

$$1 < q < \sqrt{2}, \quad \text{and} \quad p > \frac{q^2}{2 - q^2}. \quad (81)$$

We recall that $M > 0$ denotes the constant appearing in the control estimate (17) for the linearized problem. We now define the functions

$$\begin{cases} \rho_0(t) = e^{-\frac{pM}{(q-1)(T-t)}}, \\ \rho_S(t) = e^{-\frac{(1+p)q^2 M}{(q-1)(T-t)}}, \end{cases} \quad \forall t \in \left[T \left(1 - \frac{1}{q^2} \right), T \right],$$

extended in $[0, T(1 - 1/q^2)]$ in a constant way such that the functions ρ_0 and ρ_S are continuous and non-increasing in $[0, T]$ with $\rho_0(T) = \rho_S(T) = 0$.

Remark 5.1. We compute that

$$\frac{\rho_0^2(t)}{\rho_S(t)} = e^{\frac{q^2 M + pM(q^2 - 2)}{(q-1)(T-t)}}, \quad \forall t \in \left[T \left(1 - \frac{1}{q^2} \right), T \right].$$

Due to the choices of p, q in (81), we have $q^2 + p(q^2 - 2) < 0$, $(q - 1) > 0$ and therefore we can conclude that

$$\frac{\rho_0^2(t)}{\rho_S(t)} \leq 1, \quad \forall t \in [0, T].$$

With these functions, we define the following weighted spaces (recall that $X = (L^2(0, 1))^2$),

$$\mathcal{S} := \left\{ S \in L^2(0, T) \mid \frac{S}{\rho_S} \in L^2(0, T) \right\}, \quad (82a)$$

$$\mathcal{Y} := \left\{ y = (y_1, y_2) \in L^2(0, T; X) \mid \frac{y}{\rho_0} \in L^2(0, T; X) \right\}, \quad (82b)$$

$$\mathcal{V} := \left\{ v \in L^2(0, T) \mid \frac{v}{\rho_0} \in L^2(0, T) \right\}. \quad (82c)$$

We introduce the inner products in the spaces \mathcal{S} and \mathcal{V} respectively by

$$\langle S, \tilde{S} \rangle_{\mathcal{S}} := \int_0^T \rho_S^{-2} S(t) \tilde{S}(t) dt \quad \text{and} \quad \langle v, \tilde{v} \rangle_{\mathcal{V}} := \int_0^T \rho_0^{-2} v(t) \tilde{v}(t) dt,$$

for any $S, \tilde{S} \in \mathcal{S}$ and $v, \tilde{v} \in \mathcal{V}$. The corresponding norms in those spaces are

$$\|S\|_{\mathcal{S}} := \left(\int_0^T \left| \frac{S(t)}{\rho_S(t)} \right|^2 dt \right)^{1/2}, \quad \text{and} \quad \|v\|_{\mathcal{V}} := \left(\int_0^T \left| \frac{v(t)}{\rho_0(t)} \right|^2 dt \right)^{1/2}. \quad (83)$$

We consider the following system

$$\begin{cases} \partial_t y_1 - \partial_x^2 y_1 + \alpha_1 y_1 = 0 & \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x^2 y_2 + \alpha_2 y_2 = 0 & \text{in } (0, T) \times (0, 1), \\ y_1(0, \cdot) = y_{0,1} & \text{in } (0, 1), \\ y_2(0, \cdot) = y_{0,2} & \text{in } (0, 1), \end{cases} \quad (84)$$

along with the boundary conditions at $x = 1$ given by

$$\begin{cases} \partial_x y_1(\cdot, 1) = 0 & \text{in } (0, T), \\ \partial_x y_2(\cdot, 1) - \beta_r y_1(\cdot, 1) = S_r & \text{in } (0, T), \end{cases} \quad (85)$$

and at $x = 0$, we consider either

$$\begin{cases} \partial_x y_1(\cdot, 0) + \beta_\ell y_2(\cdot, 0) = v + S_\ell & \text{in } (0, T), \\ \partial_x y_2(\cdot, 0) = 0 & \text{in } (0, T), \end{cases} \quad (86)$$

or,

$$\begin{cases} \partial_x y_1(\cdot, 0) + \beta_\ell y_2(\cdot, 0) = S_\ell & \text{in } (0, T), \\ \partial_x y_2(\cdot, 0) = v & \text{in } (0, T), \end{cases} \quad (87)$$

for any given $y_0 \in X$, $S_\ell, S_r \in L^2(0, T)$ and $v \in L^2(0, T)$.

Our goal is to prove the following result using the technique developed in [29].

Proposition 5.2. *We suppose given the parameters $\alpha_1, \alpha_2, \beta_\ell$ and β_r as well as the time $T > 0$.*

1. *There exists a linear map*

$$(y_0, S_\ell, S_r) \in X \times L^2(0, T) \times L^2(0, T) \longmapsto (y, v) \in \mathcal{Y} \times \mathcal{V},$$

such that (y, v) solves the set of equations (84)–(85)–(86).

2. *If $\beta_\ell \beta_r \notin \mathcal{R}$, as defined in (53), then there exists a linear map*

$$(y_0, S_\ell, S_r) \in X \times L^2(0, T) \times L^2(0, T) \longmapsto (y, v) \in \mathcal{Y} \times \mathcal{V},$$

such that (y, v) solves the set of equations (84)–(85)–(87)

In both cases, we have the following estimate

$$\begin{aligned} \left\| \frac{y}{\rho_0} \right\|_{C^0(0, T; X)} + \left\| \frac{y}{\rho_0} \right\|_{L^2(0, T; (H^1(0, 1))^2)} + \left\| \frac{v}{\rho_0} \right\|_{L^2(0, T)} \\ \leq C e^{C(T + \frac{1}{T})} (\|y_0\|_X + \|S_\ell\|_{\mathcal{S}} + \|S_r\|_{\mathcal{S}}), \end{aligned} \quad (88)$$

where the constant $C > 0$ neither depends on y_0 nor on T .

Proof. Let us define the sequence $\{T_k\}_{k \geq 0}$ with

$$T_k := T - \frac{T}{q^k}, \quad \forall k \geq 0,$$

for given $T > 0$. With this T_k , we have the following relation between ρ_0 and ρ_S

$$\rho_0(T_{k+2}) = \rho_S(T_k) e^{\frac{M}{T_{k+2} - T_{k+1}}}, \quad \forall k \geq 0. \quad (89)$$

We also define a sequence $\{a_k\}_{k \geq 0}$ with

$$a_0 = y_0 \in X, \quad a_{k+1} = \tilde{y}(T_{k+1}^-), \quad \forall k \geq 0, \quad (90)$$

where $\tilde{y} := (\tilde{y}_1, \tilde{y}_2) \in C^0([T_k, T_{k+1}]; X) \cap L^2(T_k, T_{k+1}; (H^1(0, 1))^2)$ is the unique weak solution to the following system for every $k \geq 0$,

$$\left\{ \begin{array}{ll} \partial_t \tilde{y}_1 - \partial_x^2 \tilde{y}_1 + \alpha_1 \tilde{y}_1 = 0 & \text{in } (T_k, T_{k+1}) \times (0, 1), \\ \partial_t \tilde{y}_2 - \partial_x^2 \tilde{y}_2 + \alpha_2 \tilde{y}_2 = 0 & \text{in } (T_k, T_{k+1}) \times (0, 1), \\ \partial_x \tilde{y}_1(\cdot, 0) + \beta_\ell \tilde{y}_2(\cdot, 0) = S_\ell & \text{in } (T_k, T_{k+1}), \\ \partial_x \tilde{y}_2(\cdot, 0) = 0 & \text{in } (T_k, T_{k+1}), \\ \partial_x \tilde{y}_1(\cdot, 1) = 0 & \text{in } (T_k, T_{k+1}), \\ \partial_x \tilde{y}_2(\cdot, 1) - \beta_r \tilde{y}_1(\cdot, 1) = S_r & \text{in } (T_k, T_{k+1}), \\ \tilde{y}_1(T_k^+, \cdot) = 0 & \text{in } (0, 1), \\ \tilde{y}_2(T_k^+, \cdot) = 0 & \text{in } (0, 1). \end{array} \right. \quad (91)$$

Moreover, using Theorem 2.3, we have

$$\begin{aligned} \|\tilde{y}\|_{C^0([T_k, T_{k+1}]; X)} + \|\tilde{y}\|_{L^2(T_k, T_{k+1}; (H^1(0, 1))^2)} \\ \leq C e^{CT} (\|S_\ell\|_{L^2(T_k, T_{k+1})} + \|S_r\|_{L^2(T_k, T_{k+1})}). \end{aligned}$$

In particular,

$$\|a_{k+1}\|_X \leq C e^{CT} (\|S_\ell\|_{L^2(T_k, T_{k+1})} + \|S_r\|_{L^2(T_k, T_{k+1})}), \quad \forall k \geq 0, \quad (92)$$

recall the definition (90).

Weighted estimate of the control. For every $k \geq 0$, we consider the following system (introduce $\hat{y} := (\hat{y}_1, \hat{y}_2)$)

$$\left\{ \begin{array}{ll} \partial_t \hat{y}_1 - \partial_x^2 \hat{y}_1 + \alpha_1 \hat{y}_1 = 0 & \text{in } (T_k, T_{k+1}) \times (0, 1), \\ \partial_t \hat{y}_2 - \partial_x^2 \hat{y}_2 + \alpha_2 \hat{y}_2 = 0 & \text{in } (T_k, T_{k+1}) \times (0, 1), \\ \partial_x \hat{y}_1(\cdot, 1) = 0 & \text{in } (T_k, T_{k+1}), \\ \partial_x \hat{y}_2(\cdot, 1) - \beta_r \hat{y}_1(\cdot, 1) = 0 & \text{in } (T_k, T_{k+1}), \\ (\hat{y}_1(T_k^+, \cdot), \hat{y}_2(T_k^+, \cdot)) = a_k & \text{in } (0, 1), \end{array} \right. \quad (93)$$

with a control v_k acting through either

$$\left\{ \begin{array}{ll} \partial_x \hat{y}_1(\cdot, 0) + \beta_\ell \hat{y}_2(\cdot, 0) = v_k & \text{in } (T_k, T_{k+1}), \\ \partial_x \hat{y}_2(\cdot, 0) = 0 & \text{in } (T_k, T_{k+1}), \end{array} \right. \quad (94)$$

or, we consider

$$\left\{ \begin{array}{ll} \partial_x \hat{y}_1(\cdot, 0) + \beta_\ell \hat{y}_2(\cdot, 0) = 0 & \text{in } (T_k, T_{k+1}), \\ \partial_x \hat{y}_2(\cdot, 0) = v_k & \text{in } (T_k, T_{k+1}). \end{array} \right. \quad (95)$$

- In the first case (93)–(94), we have the existence of a null-control $v_k \in L^2(T_k, T_{k+1})$ for any given set of parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ and $\beta_\ell, \beta_r \neq 0$ by Theorem 1.3–Item 1.
- In the second case (93)–(95), for any given set of parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, if $\beta_\ell \beta_r \notin \mathcal{R}$, then there exists a null-control $v_k \in L^2(T_k, T_{k+1})$ for the concerned system as per Theorem 1.3–Item 2.

In both cases, we have the following estimate

$$\begin{aligned} \|v_k\|_{L^2(T_k, T_{k+1})} &\leq M e^{M(T_{k+1}-T_k)} e^{\frac{M}{T_{k+1}-T_k}} \|a_k\|_X \\ &\leq M e^{MT} e^{\frac{M}{T_{k+1}-T_k}} \|a_k\|_X, \quad \forall k \geq 0, \end{aligned} \quad (96)$$

and the associated solutions satisfy

$$\hat{y}(T_{k+1}^-, \cdot) = 0, \quad \forall k \geq 0. \quad (97)$$

Now, combining (92) and (96), we have for every $k \geq 0$

$$\begin{aligned} &\|v_{k+1}\|_{L^2(T_{k+1}, T_{k+2})} \\ &\leq C M e^{(C+M)T} e^{\frac{M}{T_{k+2}-T_{k+1}}} \rho_S(T_k) \left(\left\| \frac{S_\ell}{\rho_S} \right\|_{L^2(T_k, T_{k+1})} + \left\| \frac{S_r}{\rho_S} \right\|_{L^2(T_k, T_{k+1})} \right), \end{aligned}$$

since ρ_S is a non-increasing function in (T_k, T_{k+1}) .

Using the relation (89), we obtain for every $k \geq 0$

$$\|v_{k+1}\|_{L^2(T_{k+1}, T_{k+2})} \leq C e^{CT} \rho_0(T_{k+2}) \left(\left\| \frac{S_\ell}{\rho_S} \right\|_{L^2(T_k, T_{k+1})} + \left\| \frac{S_r}{\rho_S} \right\|_{L^2(T_k, T_{k+1})} \right),$$

for some new value of $C > 0$. Since the function ρ_0 being non-increasing, we deduce that

$$\left\| \frac{v_{k+1}}{\rho_0} \right\|_{L^2(T_{k+1}, T_{k+2})} \leq C e^{CT} \left(\left\| \frac{S_\ell}{\rho_S} \right\|_{L^2(T_k, T_{k+1})} + \left\| \frac{S_r}{\rho_S} \right\|_{L^2(T_k, T_{k+1})} \right), \quad \forall k \geq 0. \quad (98)$$

Let us define

$$v := \sum_{k \geq 0} v_k \mathbf{1}_{(T_k, T_{k+1})}.$$

From the estimate (96), we have (since $\rho_0(T_1) = e^{-\frac{pM}{T} \frac{q}{(q-1)}}$)

$$\begin{aligned} \|v_0\|_{L^2(0, T_1)} &\leq M e^{MT} e^{\frac{M}{T_1}} \|a_0\|_X \\ &= M e^{MT} e^{\frac{1}{T} \frac{q(1+p)M}{(q-1)}} \rho_0(T_1) \|y_0\|_X \\ &\leq C e^{CT} e^{\frac{C}{T}} \rho_0(T_1) \|y_0\|_X, \end{aligned} \quad (99)$$

for some C depending only on M , p and q . Since ρ_0 is non-increasing function in $(0, T_1)$ we get

$$\left\| \frac{v_0}{\rho_0} \right\|_{L^2(0, T_1)} \leq C e^{CT+C/T} \|y_0\|_X. \quad (100)$$

Now, using the estimates (100) and (98), we have

$$\left\| \frac{v}{\rho_0} \right\|_{L^2(0, T)} \leq C e^{C(T+\frac{1}{T})} \left(\|y_0\|_X + \left\| \frac{S_\ell}{\rho_S} \right\|_{L^2(0, T)} + \left\| \frac{S_r}{\rho_S} \right\|_{L^2(0, T)} \right). \quad (101)$$

Weighted estimate of the solution. Let us set $y = \tilde{y} + \hat{y}$. Then, for every $k \geq 0$, y satisfies

$$\begin{cases} \partial_t y_1 - \partial_x^2 y_1 + \alpha_1 y_1 = 0 & \text{in } (T_k, T_{k+1}) \times (0, 1), \\ \partial_t y_2 - \partial_x^2 y_2 + \alpha_2 y_2 = 0 & \text{in } (T_k, T_{k+1}) \times (0, 1), \\ \partial_x y_1(\cdot, 1) = 0 & \text{in } (T_k, T_{k+1}), \\ \partial_x y_2(\cdot, 1) - \beta_r y_1(\cdot, 1) = S_r & \text{in } (T_k, T_{k+1}), \\ (y_1(T_k, \cdot), y_2(T_k, \cdot)) = a_k & \text{in } (0, 1), \end{cases}$$

where the control v_k acts either through the condition

$$\begin{cases} \partial_x y_1(\cdot, 0) + \beta_\ell y_2(\cdot, 0) = S_\ell + v_k & \text{in } (T_k, T_{k+1}), \\ \partial_x y_2(\cdot, 0) = 0 & \text{in } (T_k, T_{k+1}), \end{cases}$$

or through the condition

$$\begin{cases} \partial_x y_1(\cdot, 0) + \beta y_2(\cdot, 0) = S_\ell & \text{in } (T_k, T_{k+1}), \\ \partial_x y_2(\cdot, 0) = v_k & \text{in } (T_k, T_{k+1}). \end{cases}$$

Here, $y(T_0) = a_0 = y_0$, and we have

$$y(T_k^-) = \tilde{y}(T_k^-) + \hat{y}(T_k^-) = a_k + 0 = \hat{y}(T_k^+) + \tilde{y}(T_k^+) = y(T_k^+), \quad \forall k \geq 1,$$

thanks to the facts (90), (97) and the chosen initial data in the systems (91) and (93). Thus, y is continuous at T_k for all $k \geq 0$.

Using Theorem 2.3, we have the following estimate for y (for every $k \geq 0$):

$$\begin{aligned} & \|y\|_{C^0([T_k, T_{k+1}]; X)} + \|y\|_{L^2(T_k, T_{k+1}; (H^1(0,1))^2)} \\ & \leq C e^{CT} \left(\|a_k\|_X + \|v_k\|_{L^2(T_k, T_{k+1})} + \|S_\ell\|_{L^2(T_k, T_{k+1})} + \|S_r\|_{L^2(T_k, T_{k+1})} \right). \end{aligned}$$

Start with $k \geq 1$; using the estimates of a_k and v_k , respectively from (92) and (96), we deduce that

$$\begin{aligned} & \|y\|_{C^0([T_k, T_{k+1}]; X)} + \|y\|_{L^2(T_k, T_{k+1}; (H^1(0,1))^2)} \\ & \leq C e^{CT} e^{\frac{M}{T_{k+1}-T_k}} \left(\|S_\ell\|_{L^2(T_{k-1}, T_k)} + \|S_r\|_{L^2(T_{k-1}, T_k)} \right) \\ & \quad + C e^{CT} \left(\|S_\ell\|_{L^2(T_k, T_{k+1})} + \|S_r\|_{L^2(T_k, T_{k+1})} \right) \\ & \leq C e^{CT} e^{\frac{M}{T_{k+1}-T_k}} \rho_S(T_{k-1}) \left(\left\| \frac{S_\ell}{\rho_S} \right\|_{L^2(T_{k-1}, T_{k+1})} + \left\| \frac{S_r}{\rho_S} \right\|_{L^2(T_{k-1}, T_{k+1})} \right), \quad \forall k \geq 1, \end{aligned}$$

since ρ_S is a non-increasing function. Now, thanks to the relation (89), one has

$$\begin{aligned} & \|y\|_{C^0([T_k, T_{k+1}]; X)} + \|y\|_{L^2(T_k, T_{k+1}; (H^1(0,1))^2)} \\ & \leq C e^{CT} \rho_0(T_{k+1}) \left(\left\| \frac{S_\ell}{\rho_S} \right\|_{L^2(T_{k-1}, T_{k+1})} + \left\| \frac{S_r}{\rho_S} \right\|_{L^2(T_{k-1}, T_{k+1})} \right), \quad \forall k \geq 1, \end{aligned}$$

and using the fact that ρ_0 is a non-increasing function, we deduce

$$\begin{aligned} & \left\| \frac{y}{\rho_0} \right\|_{C^0([T_k, T_{k+1}]; X)} + \left\| \frac{y}{\rho_0} \right\|_{L^2(T_k, T_{k+1}; (H^1(0,1))^2)} \\ & \leq C e^{CT} \left(\left\| \frac{S_\ell}{\rho_S} \right\|_{L^2(T_{k-1}, T_{k+1})} + \left\| \frac{S_r}{\rho_S} \right\|_{L^2(T_{k-1}, T_{k+1})} \right), \quad \forall k \geq 1. \quad (102) \end{aligned}$$

For $k = 0$, we use the estimate of v_0 from (99) to deduce

$$\begin{aligned} & \|y\|_{C^0([0,T_1];X)} + \|y\|_{L^2(0,T_1;(H^1(0,1))^2)} \\ & \leq C e^{CT} (\|a_0\|_X + \|v_0\|_{L^2(0,T_1)} + \|S_\ell\|_{L^2(0,T_1)} + \|S_r\|_{L^2(0,T_1)}) \\ & \leq C e^{C(T+\frac{1}{T})} \rho_0(T_1) (\|y_0\|_X + \|S_\ell\|_{L^2(0,T_1)} + \|S_r\|_{L^2(0,T_1)}). \end{aligned}$$

Using the easy observation that $\|S\|_{L^2(0,T)} \leq \left\| \frac{S}{\rho_S} \right\|_{L^2(0,T)}$, we get

$$\begin{aligned} & \left\| \frac{y}{\rho_0} \right\|_{C^0([0,T_1];X)} + \left\| \frac{y}{\rho_0} \right\|_{L^2(0,T_1;(H^1(0,1))^2)} \\ & \leq C e^{C(T+\frac{1}{T})} \left(\|y_0\|_X + \left\| \frac{S_\ell}{\rho_S} \right\|_{L^2(0,T_1)} + \left\| \frac{S_r}{\rho_S} \right\|_{L^2(0,T_1)} \right). \end{aligned} \quad (103)$$

So, the estimates (102) and (103) conclude that

$$\begin{aligned} & \left\| \frac{y}{\rho_0} \right\|_{C^0([0,T];X)} + \left\| \frac{y}{\rho_0} \right\|_{L^2(0,T;(H^1(0,1))^2)} \\ & \leq C e^{C(T+\frac{1}{T})} \left(\|y_0\|_X + \left\| \frac{S_\ell}{\rho_S} \right\|_{L^2(0,T)} + \left\| \frac{S_r}{\rho_S} \right\|_{L^2(0,T)} \right). \end{aligned} \quad (104)$$

Finally, using the definitions of norms (83) (of the weighted spaces) in the estimates (101) and (104), we have the required estimate (88) of our proposition. \square

5.2. Application of the fixed point argument. In this section, we prove the main theorem regarding the local null-controllability of our systems.

We assume the initial data $y_0 \in X$ such that $\|y_0\|_X \leq \delta$ for some $\delta > 0$ which will be determined later. We also introduce the set

$$\mathcal{S}_\delta := \{S \in \mathcal{S} \times \mathcal{S} \mid \|S\|_{\mathcal{S} \times \mathcal{S}} \leq \delta\},$$

where the space \mathcal{S} is defined in (82a).

Now, recall from Proposition 5.2 that for any given source term $S := (S_\ell, S_r) \in \mathcal{S} \times \mathcal{S}$, there exists a control v such that the trajectory $y = (y_1, y_2)$ of (84)–(85) with (86) or (87) (depending on the position of the control as per Proposition 5.2) satisfies the estimate (88). In what follows, we define an operator \mathcal{N} acting on \mathcal{S}_δ by

$$\mathcal{N}(S) = \begin{pmatrix} -g_\ell(y_2(\cdot, 0)) \\ g_r(y_1(\cdot, 1)) \end{pmatrix},$$

where g_ℓ and g_r have been introduced in (78) and (79).

Proof of Theorem 1.2. We start by showing that \mathcal{N} is a contraction map from \mathcal{S}_δ onto itself for some suitable choice of $\delta > 0$.

Step 1. Let us show that \mathcal{S}_δ is invariant under the map \mathcal{N} provided $\delta > 0$ is small enough.

Thanks to the properties of g_ℓ and g_r (recalled in (80)) and the trace theorem, we have

$$\begin{aligned} \left| \frac{\mathcal{N}(S)(t)}{\rho_S(t)} \right| & \leq \frac{C}{|\rho_S(t)|} (|y_2(t, 0)|^2 + |y_1(t, 1)|^2) \\ & \leq \frac{C}{|\rho_S(t)|} (\|y_2(t)\|_{L^2(0,1)} \|y_2(t)\|_{H^1(0,1)} + \|y_1(t)\|_{L^2(0,1)} \|y_1(t)\|_{H^1(0,1)}), \end{aligned}$$

and it follows that

$$\begin{aligned} \left| \frac{\mathcal{N}(S)(t)}{\rho_S(t)} \right|^2 &\leq \frac{C}{|\rho_S(t)|^2} \|y(t)\|_X^2 \|y(t)\|_{(H^1(0,1))^2}^2 \\ &\leq C \frac{|\rho_0(t)|^4}{|\rho_S(t)|^2} \left\| \frac{y(t)}{\rho_0(t)} \right\|_X^2 \left\| \frac{y(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2}^2. \end{aligned} \quad (105)$$

- Now, using Remark 5.1 we have that the quantity $\frac{\rho_0^2(t)}{\rho_S(t)}$ is bounded by 1 irrespective on t .
- On the other hand, since we have $\|y_0\|_X \leq \delta$ and $S = (S_\ell, S_r) \in \mathcal{S}_\delta$, using the result (88), we deduce that

$$\left\| \frac{y}{\rho_0} \right\|_{C^0([0,T];X)} + \left\| \frac{y}{\rho_0} \right\|_{L^2(0,T;(H^1(0,1))^2)} \leq C e^{C(T+\frac{1}{T})} \delta. \quad (106)$$

Using the bound (106) in (105), we get

$$\left\| \frac{\mathcal{N}(S)}{\rho_S} \right\|_{L^2(0,T)} \leq C \left\| \frac{y}{\rho_0} \right\|_{C^0([0,T];X)} \left\| \frac{y}{\rho_0} \right\|_{L^2(0,T;(H^1(0,1))^2)} \leq C e^{C(T+\frac{1}{T})} \delta^2.$$

Thus, for $\delta > 0$ small enough, the map \mathcal{N} stabilizes \mathcal{S}_δ .

Step 2. In this step, we prove that \mathcal{N} is a contraction map for $\delta > 0$ small enough.

For any $S, \tilde{S} \in \mathcal{S}_\delta$, we denote the trajectories respectively by $y = (y_1, y_2)$ and $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)$ associated with the controls v and \tilde{v} of the system (84)–(85)–(86) (or (87)), by means of Proposition 5.2. Then, still using (80), it follows that

$$\begin{aligned} &\left| \frac{\mathcal{N}(S)(t) - \mathcal{N}(\tilde{S})(t)}{\rho_S(t)} \right| \\ &\leq \frac{C}{|\rho_S(t)|} \left(|y_2(t,0) - \tilde{y}_2(t,0)| (|y_2(t,0)| + |\tilde{y}_2(t,0)|) \right. \\ &\quad \left. + |y_1(t,1) - \tilde{y}_1(t,1)| (|y_1(t,1)| + |\tilde{y}_1(t,1)|) \right) \\ &\leq \frac{C}{|\rho_S(t)|} \|y(t) - \tilde{y}(t)\|_X^{\frac{1}{2}} \|y(t) - \tilde{y}(t)\|_{(H^1(0,1))^2}^{\frac{1}{2}} \\ &\quad \times \left(\|y(t)\|_X^{\frac{1}{2}} \|y(t)\|_{(H^1(0,1))^2}^{\frac{1}{2}} + \|\tilde{y}(t)\|_X^{\frac{1}{2}} \|\tilde{y}(t)\|_{(H^1(0,1))^2}^{\frac{1}{2}} \right) \\ &\leq \frac{C \rho_0^2(t)}{\rho_S(t)} \left\| \frac{y - \tilde{y}}{\rho_0} \right\|_{C^0([0,T];X)}^{\frac{1}{2}} \left\| \frac{y(t) - \tilde{y}(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2}^{\frac{1}{2}} \\ &\quad \times \left(\left\| \frac{y}{\rho_0} \right\|_{C^0([0,T];X)}^{\frac{1}{2}} \left\| \frac{y(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2}^{\frac{1}{2}} + \left\| \frac{\tilde{y}}{\rho_0} \right\|_{C^0([0,T];X)}^{\frac{1}{2}} \left\| \frac{\tilde{y}(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2}^{\frac{1}{2}} \right). \end{aligned} \quad (107)$$

Now, as per Proposition 5.2 the solution map is linear and thus $y - \tilde{y}$ is the trajectory of the set of equations (84)–(85)–(86) (instead, (84)–(85)–(87)) with the source terms $S - \tilde{S}$, the control $v - \tilde{v}$ and the initial data $y(0, \cdot) - \tilde{y}(0, \cdot) = 0$. Then,

thanks to the estimate (88), $y - \tilde{y}$ satisfies the following:

$$\begin{aligned} & \left\| \frac{y - \tilde{y}}{\rho_0} \right\|_{C^0([0,T];X)} + \left\| \frac{y - \tilde{y}}{\rho_0} \right\|_{L^2(0,T;(H^1(0,1))^2)} \\ & \leq C e^{C(T+\frac{1}{T})} \left(\|S_\ell - \tilde{S}_\ell\|_{\mathcal{S}} + \|S_r - \tilde{S}_r\|_{\mathcal{S}} \right) \\ & \leq C e^{C(T+\frac{1}{T})} \|S - \tilde{S}\|_{\mathcal{S} \times \mathcal{S}}. \end{aligned} \quad (108)$$

Also, we recall the estimate (106) for y (similarly for \tilde{y}) since we started with initial data $\|y_0\| \leq \delta$ and S (or, \tilde{S}) in \mathcal{S}_δ . Beside this, we have by Remark 5.1 that $\frac{\rho_0^2(t)}{\rho_{\mathcal{S}}(t)} \leq 1, \forall t \in [0, T]$. As a consequence, the inequality (107) yields to

$$\begin{aligned} \left| \frac{\mathcal{N}(S)(t) - \mathcal{N}(\tilde{S})(t)}{\rho_{\mathcal{S}}(t)} \right| & \leq C_T \delta^{\frac{1}{2}} \|S - \tilde{S}\|_{\mathcal{S} \times \mathcal{S}}^{\frac{1}{2}} \left\| \frac{y(t) - \tilde{y}(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2}^{\frac{1}{2}} \\ & \quad \times \left(\left\| \frac{y(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2}^{\frac{1}{2}} + \left\| \frac{\tilde{y}(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2}^{\frac{1}{2}} \right), \end{aligned}$$

with $C_T := C e^{C(T+\frac{1}{T})}$. This implies

$$\begin{aligned} & \left\| \frac{\mathcal{N}(S) - \mathcal{N}(\tilde{S})}{\rho_{\mathcal{S}}} \right\|_{(L^2(0,T))^2}^2 \\ & \leq C_T \delta \|S - \tilde{S}\|_{\mathcal{S} \times \mathcal{S}} \\ & \quad \times \int_0^T \left\| \frac{y(t) - \tilde{y}(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2} \left(\left\| \frac{y(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2} + \left\| \frac{\tilde{y}(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2} \right) \\ & \leq C_T \delta \|S - \tilde{S}\|_{\mathcal{S} \times \mathcal{S}} \left\| \frac{y - \tilde{y}}{\rho_0} \right\|_{L^2(0,T;(H^1(0,1))^2)} \\ & \quad \times \left(\left\| \frac{y}{\rho_0} \right\|_{L^2(0,T;(H^1(0,1))^2)} + \left\| \frac{\tilde{y}}{\rho_0} \right\|_{L^2(0,T;(H^1(0,1))^2)} \right) \\ & \leq C_T \delta^2 \|S - \tilde{S}\|_{\mathcal{S} \times \mathcal{S}}^2, \end{aligned}$$

where we have used the estimates (106) and (108).

So, we eventually proved that

$$\|\mathcal{N}(S) - \mathcal{N}(\tilde{S})\|_{\mathcal{S} \times \mathcal{S}} \leq C_T \delta \|S - \tilde{S}\|_{\mathcal{S} \times \mathcal{S}},$$

for $\delta > 0$ chosen small enough and this ensures that the map \mathcal{N} is a contraction in the closed ball \mathcal{S}_δ .

Hence, by applying Banach fixed-point argument, there exists unique fixed point of the map \mathcal{N} denoted by $S^* := (S_\ell^*, S_r^*)$ in the ball \mathcal{S}_δ .

Thanks to Proposition 5.2, the above $S^* \in \mathcal{S}_\delta$ is such that there exists a control $v \in \mathcal{V}$ such that the solution $y := (y_1, y_2)$ to (84)–(85)–(86) or (84)–(85)–(87), along with the initial condition y_0 , satisfies the estimates (88).

The property $\lim_{t \rightarrow T^-} \rho_0(t) = 0$ forces that $y(T, \cdot) = 0$ and thus the local null-controllability holds for the concerned nonlinear systems (10)–(11) with (12) or (13).

This indeed proves the local exact controllability result Theorem 1.2 of the main control systems (1)–(2)–(8)/(9). \square

6. Further remarks and conclusions. In this article, we studied the local exact controllability to a given steady state for a 2×2 parabolic system with coupled nonlinear boundary conditions by means of only one control function acting at the left end of the boundary (through a mixed condition or Neumann condition). Let us now address some other problems related to this study.

- **Possible generalizations.**

The proof we developed in this paper is strongly based on a quite precise spectral analysis for the linearized system. For this reason it does not seem straightforward for instance to obtain similar results, at least with the same approach, if one replaces the Laplace operator $-\partial_x^2$ in the system by a more general elliptic operator, possibly different for the two components of the system.

- **Local exact controllability to arbitrary trajectories. Global controllability.**

One can be also interested in the controllability to any trajectory $z^* := (z_1^*, z_2^*)$ for the system (1)–(2)–(3) other than steady state, but the situation is much more delicate to handle. Indeed, the linearized models are associated with the following boundary conditions (in comparison with (14)–(15))

$$\begin{cases} \partial_x y_1(\cdot, 1) = 0 & \text{in } (0, T), \\ \partial_x y_2(\cdot, 1) - f'_r(z_1^*(\cdot, 1))y_1(\cdot, 1) = 0 & \text{in } (0, T), \\ \partial_x y_1(\cdot, 0) + f'_\ell(z_2^*(\cdot, 0))y_2(\cdot, 0) = v & \text{in } (0, T), \\ \partial_x y_2(\cdot, 0) = 0 & \text{in } (0, T), \end{cases}$$

thus, it is not possible to rely on spectral methods for the associated adjoint operator since the parameters $\beta_\ell = f'_\ell(z_2^*(\cdot, 0))$ and $\beta_r = f'_r(z_1^*(\cdot, 1))$ are now time dependent. In particular, it is not clear how to obtain the controllability of the linearized systems in that case.

Note that the same issue arises even if we assume that f_ℓ and f_r are globally Lipschitz, in which case we could expect for global controllability results.

- **Relaxing the conditions $\beta_\ell\beta_r \neq 0$ or $\beta_\ell\beta_r \notin \mathcal{R}$.**

In this paper we assumed (6) and it can be interesting to see whether or not this is a necessary condition for controllability. In the case where $\beta_\ell = \beta_r = 0$, it is clear that the linearized systems are not controllable anymore with one single control since the two equations are not coupled at all. On the other hand, in the cases when $\beta_\ell = 0, \beta_r \neq 0$, or $\beta_\ell \neq 0, \beta_r = 0$, then the linearized models (10)–(14)–(15)/(16) may not be controllable. Indeed, in the case: $\beta_\ell = 0, \beta_r \neq 0$, we can compute from the eigenvalue problem (27) that $\left\{ \begin{pmatrix} \cos(k\pi x) \\ 0 \end{pmatrix} \right\}_{k \geq 0}$ is one set of eigenfunctions for the associated adjoint operator, and thus it appears that when a control acts through the Neumann condition (16), then the concerned system is not even approximately controllable in X .

Furthermore, according to Theorem 1.3, when $\beta_\ell\beta_r \in \mathcal{R}$, the linearized system (10)–(14)–(16) also fails to be approximately controllable in X .

Therefore, in those cases, at least with the method we are using, we cannot conclude anything about the local exact-controllability to the steady state for our nonlinear system.

Studying the controllability of the nonlinear system in such cases should thus be attacked with another approach(es) which we do not consider in this paper. Other techniques (such as, for instance, Coron's return method [13, Chapter 6]) could be applied to study the local controllability of the nonlinear systems in the cases where we do not have the null-controllability of the linearized models, but this would need further investigations.

Appendix A. Proofs of technical lemmas.

Proof of Lemma 3.1. Let $\mu \in \mathbb{C}$ such that $|\mu|^2 > |\theta|$. We can write

$$\sqrt{\mu^2 - \theta} = \pm \mu \sqrt{1 - \frac{\theta}{\mu^2}},$$

where the sign indetermination comes from the fact that we use the principal determination of the square root. This indetermination will have no impact on the following computations due to parity arguments.

Setting $\mu_1 = \sqrt{\frac{4|\theta|}{3}}$, we immediately get that

$$|\sqrt{\mu^2 - \theta}| \geq \frac{1}{2}|\mu|, \quad \forall |\mu| \geq \mu_1.$$

For $z \in \mathbb{C}$, $|z| < 1$, we write

$$\sqrt{1 - z} = 1 + z\zeta(z),$$

where ζ is holomorphic in the unit disk.

We can now obtain

$$\begin{aligned} \pm \sin(\sqrt{\mu^2 - \theta}) &= \sin\left(\mu + \frac{\theta}{\mu}\zeta\left(\frac{\theta}{\mu^2}\right)\right) \\ &= \sin(\mu) \cos\left(\frac{\theta}{\mu}\zeta\left(\frac{\theta}{\mu^2}\right)\right) + \cos(\mu) \sin\left(\frac{\theta}{\mu}\zeta\left(\frac{\theta}{\mu^2}\right)\right). \end{aligned}$$

One can find some $\mu_2 > \sqrt{|\theta|}$ depending only on θ such that

$$\left|\cos\left(\frac{\theta}{\mu}\zeta\left(\frac{\theta}{\mu^2}\right)\right)\right| \geq 3/4, \quad \text{and} \quad \left|\sin\left(\frac{\theta}{\mu}\zeta\left(\frac{\theta}{\mu^2}\right)\right)\right| \leq 1/4, \\ \text{for every } \mu \text{ s.t. } |\mu| > \mu_2.$$

It follows that for every μ such that $|\mu| > \mu_2$ we have

$$|\sin(\sqrt{\mu^2 - \theta})| \geq \frac{3}{4}|\sin \mu| - \frac{1}{4}|\cos \mu|.$$

Taking $\mu_0 = \max(\mu_1, \mu_2)$ gives the claim. \square

Proof of Lemma 3.2. Let us assume that $|\operatorname{Im} \mu| \geq \mu_0$. By the previous lemma we know that

$$|\sin(\sqrt{\mu^2 - \theta})| \geq \frac{3}{4}|\sin \mu| - \frac{1}{4}|\cos \mu|.$$

Moreover we have

$$\begin{aligned} |\sin \mu| &= \frac{|e^{i\mu} - e^{-i\mu}|}{2} \geq \frac{e^{|\operatorname{Im} \mu|} - e^{-|\operatorname{Im} \mu|}}{2} = \sinh |\operatorname{Im} \mu|, \\ |\cos \mu| &= \frac{|e^{i\mu} + e^{-i\mu}|}{2} \leq \frac{e^{|\operatorname{Im} \mu|} + e^{-|\operatorname{Im} \mu|}}{2} = \cosh |\operatorname{Im} \mu|. \end{aligned}$$

It follows that, for some ϑ_0 large enough that we choose satisfying $\vartheta_0 \geq \mu_0$, we have

$$|\cos \mu| \leq 2|\sin \mu|, \quad \forall \mu, \text{ s.t. } |\operatorname{Im} \mu| \geq \vartheta_0.$$

Finally we obtain for all such values of μ

$$|\sin(\sqrt{\mu^2 - \theta})| \geq \frac{1}{4} |\sin \mu| = \sinh |\operatorname{Im} \mu|.$$

Going back to the definition of h_0 (see (32)) we obtain

$$\begin{aligned} |h_0(\mu)| &\geq \frac{1}{\beta_*} |\mu| |\sqrt{\mu^2 - \theta}| \sinh^2 |\operatorname{Im} \mu| \geq \frac{1}{2\beta_*} |\operatorname{Im} \mu|^2 \sinh^2 |\operatorname{Im} \mu| \\ &\geq \frac{1}{2\beta_*^2} \vartheta_0^2 \sinh^2 \vartheta_0. \end{aligned}$$

Choosing ϑ_0 large enough gives the claim. \square

Proof of Lemma 3.3. We first choose k_1 any integer such that $k_1 \geq \frac{\mu_0}{\pi}$. Let μ such that $\operatorname{Re} \mu = (k + 1/2)\pi$ with $k \geq k_1$.

By standard trigonometric properties, we have for such values of μ , the equalities

$$|\sin \mu| = \cosh \operatorname{Im} \mu, \quad \text{and} \quad |\cos \mu| = |\sinh \operatorname{Im} \mu|,$$

and in particular we get

$$|\sin \mu| \geq 1, \quad |\cos \mu| \leq |\sin \mu|.$$

By the choice of k_1 , we have in particular $|\mu| \geq \mu_0$ and therefore by Lemma 3.1

$$|\sin(\sqrt{\mu^2 - \theta})| \geq \frac{3}{4} |\sin \mu| - \frac{1}{4} |\cos \mu|,$$

and the inequalities above give

$$|\sin(\sqrt{\mu^2 - \theta})| \geq \frac{1}{2} |\sin \mu| \geq \frac{1}{2}.$$

All in all, we have for such values of μ ,

$$|h_0(\mu)| \geq \frac{1}{4\beta_*} |\mu|^2 \geq \frac{\pi^2}{4\beta_*} k^2,$$

and this quantity is greater than 1 for any k large enough. \square

Proof of Lemma 3.4. The proof is straightforward by using usual asymptotic expansions. We first get

$$\begin{aligned} \sqrt{(k\pi + \delta_k)^2 + \gamma} &= k\pi \sqrt{1 + \frac{2\delta_k}{k\pi} + \frac{\delta_k^2 + \gamma}{k^2\pi^2}} \\ &= k\pi \left(1 + \frac{\delta_k}{k\pi} + \frac{1}{2} \frac{\delta_k^2 + \gamma}{k^2\pi^2} - \frac{1}{8} \frac{4\delta_k^2}{k^2\pi^2} + O\left(\frac{1}{k^3}\right) \right) \\ &= k\pi + \delta_k + \frac{\gamma}{2k\pi} + O\left(\frac{1}{k^2}\right). \end{aligned}$$

Then, in a similar way, it follows

$$\begin{aligned}
\sqrt{k^2\pi^2 + \gamma + \varepsilon_k} &= k\pi\sqrt{1 + \frac{\gamma + \varepsilon_k}{k^2\pi^2}} \\
&= k\pi\left(1 + \frac{\gamma + \varepsilon_k}{2k^2\pi^2} - \frac{(\gamma + \varepsilon_k)^2}{8k^4\pi^4} + O\left(\frac{1}{k^6}\right)\right) \\
&= k\pi + \frac{\gamma + \varepsilon_k}{2k\pi} - \frac{(\gamma + \varepsilon_k)^2}{8k^3\pi^3} + O\left(\frac{1}{k^5}\right) \\
&= k\pi + \frac{\gamma + \varepsilon_k}{2k\pi} - \frac{\gamma^2}{8k^3\pi^3} + o\left(\frac{1}{k^3}\right).
\end{aligned}$$

From this last equality, we deduce that

$$\begin{aligned}
\sin\sqrt{k^2\pi^2 + \gamma + \varepsilon_k} &= (-1)^k \sin\left(\frac{\gamma + \varepsilon_k}{2k\pi} - \frac{\gamma^2}{8k^3\pi^3} + o\left(\frac{1}{k^3}\right)\right) \\
&= (-1)^k \left(\frac{\gamma + \varepsilon_k}{2k\pi} - \frac{\gamma^2}{8k^3\pi^3} - \frac{\gamma^3}{48k^3\pi^3} + o\left(\frac{1}{k^3}\right)\right),
\end{aligned}$$

and therefore we have

$$\begin{aligned}
&\sqrt{k^2\pi^2 + \gamma + \varepsilon_k} \sin\sqrt{k^2\pi^2 + \gamma + \varepsilon_k} \\
&= (-1)^k \left(k\pi + \frac{\gamma}{2k\pi} + o\left(\frac{1}{k}\right)\right) \\
&\quad \times \left(\frac{\gamma + \varepsilon_k}{2k\pi} - \frac{\gamma^2}{8k^3\pi^3} - \frac{\gamma^3}{48k^3\pi^3} + o\left(\frac{1}{k^3}\right)\right) \\
&= (-1)^k \left(1 + \frac{\gamma}{2k^2\pi^2} + o\left(\frac{1}{k^2}\right)\right) \\
&\quad \times \left(\frac{\gamma + \varepsilon_k}{2} - \frac{\gamma^2}{8k^2\pi^2} - \frac{\gamma^3}{48k^2\pi^2} + o\left(\frac{1}{k^2}\right)\right) \\
&= (-1)^k \left(\frac{\gamma + \varepsilon_k}{2} + \frac{\gamma^2}{8k^2\pi^2} - \frac{\gamma^3}{48k^2\pi^2} + o\left(\frac{1}{k^2}\right)\right) \\
&= (-1)^k \frac{\gamma}{2} \left(1 + \frac{\varepsilon_k}{\gamma} + \frac{\gamma}{4k^2\pi^2} - \frac{\gamma^2}{24k^2\pi^2} + o\left(\frac{1}{k^2}\right)\right).
\end{aligned}$$

Applying the previous result twice (once with γ and once with γ replaced by $\gamma - \theta$), we obtain

$$\begin{aligned}
&h_0(\sqrt{k^2\pi^2 + \gamma + \varepsilon_k}) \\
&= \frac{\gamma(\gamma - \theta)}{4\beta_*} \left(1 + \frac{\varepsilon_k}{\gamma} + \frac{\gamma}{4k^2\pi^2} - \frac{\gamma^2}{24k^2\pi^2} + o\left(\frac{1}{k^2}\right)\right) \\
&\quad \times \left(1 + \frac{\varepsilon_k}{\gamma - \theta} + \frac{\gamma - \theta}{4k^2\pi^2} - \frac{(\gamma - \theta)^2}{24k^2\pi^2} + o\left(\frac{1}{k^2}\right)\right) \\
&= \frac{\gamma(\gamma - \theta)}{4\beta_*} \left(1 + \frac{\varepsilon_k(\varepsilon_k + 2\gamma - \theta)}{\gamma(\gamma - \theta)} + \frac{2\gamma - \theta}{4k^2\pi^2} - \frac{\gamma^2 + (\gamma - \theta)^2}{24k^2\pi^2} + o\left(\frac{1}{k^2}\right)\right).
\end{aligned}$$

Taking $\varepsilon_k = 0$ in this formula, leads to (33), whereas taking $\gamma = \theta/2$ leads to (34). \square

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