



**HAL**  
open science

# Local exact controllability to the steady states of a parabolic system with coupled nonlinear boundary conditions

Kuntal Bhandari, Franck Boyer

► **To cite this version:**

Kuntal Bhandari, Franck Boyer. Local exact controllability to the steady states of a parabolic system with coupled nonlinear boundary conditions. 2023. hal-03737472v2

**HAL Id: hal-03737472**

**<https://hal.science/hal-03737472v2>**

Preprint submitted on 6 Jul 2023 (v2), last revised 21 Sep 2023 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# LOCAL EXACT CONTROLLABILITY TO THE STEADY STATES OF A PARABOLIC SYSTEM WITH COUPLED NONLINEAR BOUNDARY CONDITIONS

KUNTAL BHANDARI\* AND FRANCK BOYER†

**Abstract.** In this article, we study the boundary local exact controllability to any steady state of a one-dimensional parabolic system with coupled nonlinear boundary conditions by means of only one control. The significant point is that the state components are interacting only at the boundary points in terms of some nonlinear terms. We consider two cases : either the control function is acting through a *mixed nonlinear boundary condition on the first component* or through a *Neumann condition on the second component*. The results are slightly different in the two cases.

To study this problem, we first consider the associated linearized systems around the given steady state. The method of moments let us to prove its controllability and to obtain a suitable estimate of the control cost of the form  $Me^{M/T}$ . To this end, we need to develop a precise spectral analysis of a non self-adjoint operator.

Thanks to those preliminary results, we can use the source term method developed in [29], followed by the Banach fixed point argument, to obtain the small-time local boundary exact controllability to the steady state for the original system.

**Key words.** Parabolic systems, boundary controllability, method of moments, source term method, fixed-point argument.

**AMS subject classifications.** 35K20 - 35K58 - 93B05 - 93B60.

## 1. Introduction.

**1.1. The system under study.** This paper is concerned with the local exact controllability to the equilibrium states of some  $2 \times 2$  parabolic system by means of only one boundary control where the state components are only coupled through nonlinear boundary conditions. More precisely, we consider the following parabolic system (without any control for the moment), given by

$$(1.1) \quad \begin{cases} \partial_t z_1 - \partial_x^2 z_1 + \alpha_1 z_1 = 0 & \text{in } (0, T) \times (0, 1), \\ \partial_t z_2 - \partial_x^2 z_2 + \alpha_2 z_2 = 0 & \text{in } (0, T) \times (0, 1), \\ z_1(0, \cdot) = z_{0,1} & \text{in } (0, 1), \\ z_2(0, \cdot) = z_{0,2} & \text{in } (0, 1), \end{cases}$$

with the boundary conditions given by

$$(1.2) \quad \begin{cases} \partial_x z_1(\cdot, 1) = 0 & \text{in } (0, T), \\ \partial_x z_2(\cdot, 1) - f_r(z_1(\cdot, 1)) = 0 & \text{in } (0, T), \end{cases}$$

at the right boundary point and

$$(1.3) \quad \begin{cases} \partial_x z_1(\cdot, 0) + f_\ell(z_2(\cdot, 0)) = 0 & \text{in } (0, T), \\ \partial_x z_2(\cdot, 0) = 0 & \text{in } (0, T), \end{cases}$$

at the left boundary point, where  $f_\ell, f_r$  are some nonlinear functions,  $\alpha_1, \alpha_2 \in \mathbb{R}$  are some parameters and  $z_0 := (z_{0,1}, z_{0,2}) \in (L^2(0, 1))^2$  is the given initial data.

Our main assumption concerning  $f_\ell$  and  $f_r$  is

$$(1.4) \quad f_\ell, f_r \in W^{2,\infty}(\mathbb{R}).$$

Such systems with boundary interactions appear in several biological or chemical models. A prototype of system (1.1)–(1.2)–(1.3) has been considered for instance in [33] where the authors studied the equilibrium states and oscillations of localized two-enzyme kinetics. In their work, the states  $z_1$  and  $z_2$  represent the concentrations of the effectors of two enzymes localized at  $x = 0$  and  $x = 1$  affecting

---

\*Institute of Mathematics of the Czech Academy of Sciences, Žitná 25, 11567 Praha 1, Czech Republic (email: bhandari@math.cas.cz).

†Institut de Mathématiques de Toulouse & Institut Universitaire de France, UMR 5219, Université Paul Sabatier, CNRS, 31062 Toulouse Cedex 09, France (email: franck.boyer@math.univ-toulouse.fr).

each other only through the end points, and in that work, the authors consider some particular type of nonlinear functions  $f_\ell, f_r$  according to the role of the enzymes (activation/ inhibition) such as

$$(1.5) \quad s \mapsto \frac{s^2}{\gamma^2 + s^2}, \text{ (activation)} \quad \text{or} \quad s \mapsto \frac{\gamma^2}{\gamma^2 + s^2} \text{ (inhibition),}$$

where  $\gamma > 0$  is a fixed parameter. For more details, we refer to [33, Sections 1 & 2].

We consider now a steady-state  $z^* := (z_1^*, z_2^*) \in (L^2(0, 1))^2$  for the system (1.1)–(1.2)–(1.3) that will be fixed all along this work. We will assume that

$$(1.6) \quad \beta_\ell := f'_\ell(z_2^*(0)) \neq 0, \quad \text{and} \quad \beta_r := f'_r(z_1^*(1)) \neq 0.$$

REMARK 1.1. *A simple computation helps us to find the steady state of the system (1.1)–(1.2)–(1.3); namely*

$$\begin{aligned} z_1^*(x) &= a_1 \cosh(\sqrt{\alpha_1}(1-x)), \quad \forall x \in [0, 1], \\ z_2^*(x) &= a_2 \cosh(\sqrt{\alpha_2}x), \quad \forall x \in [0, 1], \end{aligned}$$

where  $a_1$  and  $a_2$  satisfy

$$\begin{aligned} a_2 \sqrt{\alpha_2} \sinh(\sqrt{\alpha_2}) &= f_r(a_1), \\ a_1 \sqrt{\alpha_1} \sinh(\sqrt{\alpha_1}) &= f_\ell(a_2). \end{aligned}$$

Depending on the particular forms of  $f_\ell$  and  $f_r$  and the values of  $\alpha_1, \alpha_2$ , it may exist one or many solutions  $(a_1, a_2)$  to this nonlinear system, each of them leading to a steady-state of our parabolic problem, as discussed for instance in [33].

In system (1.1)–(1.2)–(1.3), we observe that the coupling between the two components only occurs at the boundary points since there is no internal coupling in the system. The main mathematical challenge that we will deal with, is to analyse the controllability of such system by only one boundary control.

**Main goal.** In this paper, we are going to study the controllability properties of the system (1.1)–(1.2) by a single boundary control  $v : (0, T) \rightarrow \mathbb{R}$  acting at the left boundary point either in the case

$$(1.8) \quad \begin{cases} \partial_x z_1(\cdot, 0) + f_\ell(z_2(\cdot, 0)) = v & \text{in } (0, T), \\ \partial_x z_2(\cdot, 0) = 0 & \text{in } (0, T), \end{cases}$$

or in the case

$$(1.9) \quad \begin{cases} \partial_x z_1(\cdot, 0) + f_\ell(z_2(\cdot, 0)) = 0 & \text{in } (0, T), \\ \partial_x z_2(\cdot, 0) = v & \text{in } (0, T). \end{cases}$$

To be more precise, we look for a control  $v \in L^2(0, T)$  acting through the boundary condition (1.8) or (1.9), such that both components can be driven to the steady state  $z^* := (z_1^*, z_2^*)$  (at least locally, that is if the initial data is close enough from  $z^*$ ) at any given time  $T > 0$ , that is to say,

$$z_1(T, \cdot) = z_1^*, \quad z_2(T, \cdot) = z_2^*.$$

**1.2. Bibliographic comments and motivations.** The controllability of a system of partial differential equations with less number of control(s) than equations is gaining genuine interests to the control community. In light of this, we first refer some pioneer works. The authors in [3, 4] established some generalized Kalman rank conditions which are necessary and sufficient for the distributed null-controllability of a class of linear parabolic systems. The boundary controllability of such system is much more intricate and most of the results are restricted to the 1-D case since the very powerful Carleman technique is often inefficient in this context. Indeed, the boundary controllability of a system of PDEs when there is less controls than equations, is no more equivalent to the distributed one unlike the scalar case, see for instance [19]. The authors in [19] also proved a necessary and sufficient condition for the boundary null-controllability of a  $2 \times 2$  coupled parabolic system with a scalar Dirichlet control.

A generalization of this result for the system of  $n$  parabolic equations with  $m < n$  controls has been achieved in [5]. In the multi-dimensional case, we quote [2,8], where controllability results are obtained in particular cylindrical geometries by exploiting on the one hand a sharp estimate of the control for the associated 1D problem and on the other hand spectral Lebeau-Robbiano inequalities ([27]) for any space dimensions, see also the discussion in [9, Section 2.4.2]. We also mention [1] where symmetric parabolic coupled systems are analyzed in any dimension, provided that the control region satisfies the Geometric Control Condition (GCC).

The above cases mainly dealt with internal coupling. Concerning the controllability with boundary coupling, we mention the book [14] and the survey paper [6] where the authors studied the controllability of some wave, heat and Schrödinger systems on metric graphs. We also refer to [10] where the boundary null-controllability of some coupled parabolic systems has been addressed where the boundary coupling is chosen by means of a Kirchhoff-type condition. In most of the known cases, the boundary conditions are linearly posed. Thus, dealing with the systems (1.1)–(1.2)–(1.8)/(1.9) is naturally more interesting since the concerned boundary couplings are nonlinear.

In the context of controllability of nonlinear systems, we mention first [21, Chapter I, Sec. 4] by Fursikov and Imanuvilov where a small-time local null-controllability of semilinear heat equations has been proved using a perturbation argument. In 2000, Barbu [7], and independently Fernández-Cara and Zuazua [20], proved the small-time global null-controllability of semilinear heat equation where the growth of nonlinearities is slower than  $|s| \ln^{3/2}(1 + |s|)$ . In fact, the large-time global null-controllability of semilinear heat equations has recently been obtained in [26] for nonlinearities  $F$  such that  $s \mapsto F(s)/(s|\ln^\alpha(1 + |s|))$  tends to zero at  $\pm\infty$  for some  $\alpha > 0$ , with either  $F(s) > 0$  for  $s > 0$  or  $F(s) < 0$  for  $s < 0$  and  $1/F \in L^1([0, +\infty))$ . Furthermore, in the context of controllability for evolutionary systems with nonlinear (interior) couplings, we mention, for instance, the works [12,24,25].

In the present work, we deal with the controllability of the concerned models (1.1)–(1.2)–(1.8)/(1.9) where the couplings arise on the boundary points in terms of the nonlinear functions (1.4) verifying (1.6).

**1.3. Auxiliary control problems.** We shall first make a change of unknowns to transform our control problem into a null-control problem, and then write its linearization around zero.

To this end, we introduce the following functions, that depend on the considered steady state  $z^*$ :

$$\begin{aligned}\tilde{f}_r(s) &:= f_r(s + z_1^*(1)) - f_r(z_1^*(1)), \quad \forall s \in \mathbb{R}, \\ \tilde{f}_\ell(s) &:= f_\ell(s + z_2^*(0)) - f_\ell(z_2^*(0)), \quad \forall s \in \mathbb{R}.\end{aligned}$$

Since  $z^*$  is a steady state solution of our uncontrolled system, we see that the study of the aforementioned local exact controllability problem is equivalent to the following local null-controllability problem for the new unknown  $y = z - z^*$ ,

$$(1.10) \quad \begin{cases} \partial_t y_1 - \partial_x^2 y_1 + \alpha_1 y_1 = 0 & \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x^2 y_2 + \alpha_2 y_2 = 0 & \text{in } (0, T) \times (0, 1), \\ y_1(0, \cdot) = y_{0,1} & \text{in } (0, 1), \\ y_2(0, \cdot) = y_{0,2} & \text{in } (0, 1), \end{cases}$$

$$(1.11) \quad \begin{cases} \partial_x y_1(\cdot, 1) = 0 & \text{in } (0, T), \\ \partial_x y_2(\cdot, 1) - \tilde{f}_r(y_1(\cdot, 1)) = 0 & \text{in } (0, T), \end{cases}$$

with a control acting either through the mixed condition

$$(1.12) \quad \begin{cases} \partial_x y_1(\cdot, 0) + \tilde{f}_\ell(y_2(\cdot, 0)) = v & \text{in } (0, T), \\ \partial_x y_2(\cdot, 0) = 0 & \text{in } (0, T), \end{cases}$$

or, through the Neumann condition of  $y_2$ ,

$$(1.13) \quad \begin{cases} \partial_x y_1(\cdot, 0) + \tilde{f}_\ell(y_2(\cdot, 0)) = 0 & \text{in } (0, T), \\ \partial_x y_2(\cdot, 0) = v & \text{in } (0, T), \end{cases}$$

where the initial data  $y_0 := (y_{0,1}, y_{0,2})$  is simply defined by  $y_{0,1} = z_{0,1} - z_1^*$  and  $y_{0,2} = z_{0,2} - z_2^*$ .

If we linearize the nonlinear systems (1.10)–(1.11)–(1.12)/(1.13) around 0, we obtain the same set of linear PDEs (1.10) along with the linearized boundary conditions

$$(1.14) \quad \begin{cases} \partial_x y_1(\cdot, 1) = 0 & \text{in } (0, T), \\ \partial_x y_2(\cdot, 1) - \beta_r y_1(\cdot, 1) = 0 & \text{in } (0, T), \end{cases}$$

at the right boundary and one of the following two situations on the left boundary: either

$$(1.15) \quad \begin{cases} \partial_x y_1(\cdot, 0) + \beta_\ell y_2(\cdot, 0) = v & \text{in } (0, T), \\ \partial_x y_2(\cdot, 0) = 0 & \text{in } (0, T), \end{cases}$$

or,

$$(1.16) \quad \begin{cases} \partial_x y_1(\cdot, 0) + \beta_\ell y_2(\cdot, 0) = 0 & \text{in } (0, T), \\ \partial_x y_2(\cdot, 0) = v & \text{in } (0, T). \end{cases}$$

We recall that the parameters  $\beta_\ell$  and  $\beta_r$  have been introduced in (1.6). They only depends on  $f_\ell, f_r$  and  $z^*$  and are assumed to be non zero as mentioned in (1.6).

The study of this linear system will be central in this paper.

**1.4. Main results.** Let us state the main result concerning our nonlinear systems. We will set  $X = (L^2(0, 1))^2$ .

**THEOREM 1.2.** *Let  $f_\ell, f_r$  satisfying (1.4) and  $z^* \in X$  be a steady state of the system (1.1)–(1.2)–(1.3), that satisfies (1.6).*

*Then, we have the following local controllability results.*

1. *The system (1.1)–(1.2)–(1.8) is small time locally exact controllable to the equilibrium, that is to say, for any given time  $T > 0$ , there is a  $\delta > 0$  such that for any initial data  $z_0 \in X$  with  $\|z_0 - z^*\|_X \leq \delta$ , there exists a control  $v \in L^2(0, T)$  such that the associated solution  $z$  satisfies  $z(T) = z^*$ .*
2. *There exists a non-empty discrete set  $\mathcal{R} \subset \mathbb{R}$  such that if  $\beta_\ell \beta_r \notin \mathcal{R}$ , then the system (1.1)–(1.2)–(1.9) is also small time locally exact controllable to the equilibrium.*

The set  $\mathcal{R}$  will be explicitly specified later, namely in Lemma 4.1.

As we discussed earlier, the proof of Theorem 1.2 is equivalent to prove a local null-controllability result for the systems (1.10)–(1.11)–(1.12)/(1.13), and the first step in the analysis of the nonlinear systems is the study of the linearized systems (1.10)–(1.14)–(1.15)/(1.16). More precisely, we shall first prove the following theorem.

**THEOREM 1.3.** *Let  $\alpha_1, \alpha_2, \beta_\ell, \beta_r \in \mathbb{R}$  with  $\beta_\ell \beta_r \neq 0$ .*

1. *For any initial data  $y_0 \in X$ , and any time  $T > 0$ , there exists a control  $v \in L^2(0, T)$  such that the solution of (1.10)–(1.14)–(1.15) satisfies  $y(T) = 0$ .*
2. *There exists a non-empty discrete set  $\mathcal{R} \subset \mathbb{R}$  such that if  $\beta_\ell \beta_r \notin \mathcal{R}$ , then for any initial data  $y_0 \in X$ , and any time  $T > 0$ , there is a control  $v \in L^2(0, T)$  such that the solution of (1.10)–(1.14)–(1.16) satisfies  $y(T) = 0$ .*

*In both cases, the control can be chosen to satisfy the following estimate*

$$(1.17) \quad \|v\|_{L^2(0, T)} \leq M e^{M/T} \|y_0\|_X,$$

where the constant  $M > 0$  neither depend on  $T$  nor on  $y_0$  but may depend on  $\alpha_1, \alpha_2, \beta_\ell$  and  $\beta_r$ .

**Outline of the paper.** The paper is organized as follows.

- Section 2 is devoted to study the well-posedness results of the linearized control systems (1.10)–(1.14)–(1.15)/(1.16) by virtue of transposition method. Moreover, we formulate the null-control problems for the concerned models.
- The global boundary null-controllability results of the associated linear models is established in Section 4 using the so-called method of moments. To this end, we develop a concrete spectral analysis of the adjoint operator (which is non-self-adjoint) to the corresponding linear control systems, and this is described in Section 3.

A proper estimate of the linear control cost (precisely  $M e^{M/T} \|y_0\|_X$ ) is proved in subsection 4.4 which is crucial to deduce the controllability results for the non-linear models.

- In Section 5, we prove the local null-controllability results for the systems (1.10)–(1.11)–(1.12)/(1.13).

To do so, we apply the source term method introduced in [29]; more precisely, we prove the null-controllability of our linearized models with additional boundary terms in  $L^2(0, T)$  (exponentially decreasing while  $t \rightarrow T^-$ ) on the boundary points where the non-linearities appear. Then, we use the Banach fixed-point argument to obtain the local (boundary) null-controllability for the concerned non-linear models.

- We conclude our paper by mentioning several remarks in Section 6 and finally in Appendix A, we present the proofs of some technical lemmas which are useful for our spectral study.

**Notations.** Throughout the paper,  $C > 0$  denotes a generic constant that may vary line to line and may depend on  $\alpha_1, \alpha_2, \beta_\ell, \beta_r$  but neither depend on  $T$  nor on the initial data  $y_0$ . By the notation

$$\gamma_1(\cdot) = O(\gamma_2(\cdot)),$$

for some functions  $\gamma_1, \gamma_2$ , we mean

$$|\gamma_1(\cdot)| \leq C|\gamma_2(\cdot)|, \quad \text{for some constant } C := C(\alpha_1, \alpha_2, \beta_\ell, \beta_r).$$

The set of all non-zero real numbers is denoted by  $\mathbb{R}^*$ . For any  $z \in \mathbb{C}$ , the real and complex parts are denoted by  $\operatorname{Re} z$  and  $\operatorname{Im} z$  respectively. Moreover, in the whole paper we denote by  $\sqrt{\cdot}$  the principal determination of the square root in the complex plane.

**2. Well-posedness and formulation of the control problems.** In this section, we shall discuss about the existence of the weak solutions to our linearized systems and formulate the associated null-control problems.

**2.1. Existence of the semigroup.** As mentioned before, we will work in the Hilbert space  $X = (L^2(0, 1))^2$ . We introduce the following elliptic operator  $\mathcal{A}$  associated to the system (1.10)–(1.14) with (1.15) or (1.16),

$$(2.1) \quad \mathcal{A} = \begin{pmatrix} -\partial_x^2 + \alpha_1 & 0 \\ 0 & -\partial_x^2 + \alpha_2 \end{pmatrix},$$

with its domain

$$(2.2) \quad D(\mathcal{A}) := \left\{ \phi := (\phi_1, \phi_2) \in (H^2(0, 1))^2 \mid \begin{aligned} &\phi_1'(0) + \beta_\ell \phi_2(0) = 0, \phi_2'(0) = 0, \\ &\phi_1'(1) = 0, \phi_2'(1) - \beta_r \phi_1(1) = 0 \end{aligned} \right\}.$$

It is clear that  $(\mathcal{A}, D(\mathcal{A}))$  is a non-self-adjoint operator due to the presence of such coupled boundary conditions. More precisely, we find that, the adjoint operator  $\mathcal{A}^*$  of  $\mathcal{A}$  has the same formal expression as (2.1), yet with a different domain given by

$$(2.3) \quad D(\mathcal{A}^*) = \left\{ u := (u_1, u_2) \in (H^2(0, 1))^2 \mid \begin{aligned} &u_1'(0) = 0, u_2'(0) + \beta_\ell u_1(0) = 0, \\ &u_1'(1) - \beta_r u_2(1) = 0, u_2'(1) = 0 \end{aligned} \right\}.$$

**PROPOSITION 2.1.** *The operator  $(-\mathcal{A}^*, D(\mathcal{A}^*))$  defined by (2.1)–(2.3), generates an analytic semigroup in  $X$ , denoted by  $(e^{-t\mathcal{A}^*})_{t \geq 0}$ .*

A similar proof shows that the same property holds true for the operator  $(-\mathcal{A}, D(\mathcal{A}))$  defined in (2.1)–(2.2).

*Proof.* We consider the following densely defined sesquilinear form  $h$ ; for all  $u := (u_1, u_2)$ ,  $\psi := (\psi_1, \psi_2) \in (H^1(0, 1))^2$ ,

$$h(u, \psi) := \int_0^1 \left( u_1'(x) \overline{\psi_1'(x)} + u_2'(x) \overline{\psi_2'(x)} \right) dx + \int_0^1 \left( \alpha_1 u_1(x) \overline{\psi_1(x)} + \alpha_2 u_2(x) \overline{\psi_2(x)} \right) dx \\ - \beta_\ell u_1(0) \overline{\psi_2(0)} - \beta_r u_2(1) \overline{\psi_1(1)}.$$

It is clear that  $h$  is continuous in  $(H^1(0, 1))^2$  and moreover, we have

$$|h(u, \psi)| \leq \kappa \|u\|_{(H^1(0,1))^2} \|\psi\|_{(H^1(0,1))^2},$$

where  $\kappa > 0$  depends on the parameters  $\alpha_1, \alpha_2, \beta_\ell$  and  $\beta_r$ . Denote by  $(\tilde{\mathcal{A}}, D(\tilde{\mathcal{A}}))$  the operator associated with the form  $h$ , which is by definition given by

$$\begin{cases} D(\tilde{\mathcal{A}}) := \left\{ u \in (H^1(0, 1))^2 \mid \exists w \in X \text{ s.t. } h(u, \psi) = (w, \psi)_X, \quad \forall \psi \in (H^1(0, 1))^2 \right\}, \\ \tilde{\mathcal{A}}u := w \in X. \end{cases}$$

Since any  $u \in (H^1(0, 1))^2$  satisfies

$$\operatorname{Re}(h(u, u)) \geq \kappa_1 \|u\|_{(H^1(0,1))^2}^2 - \kappa_2 \|u\|_X^2,$$

for some constants  $\kappa_1, \kappa_2 > 0$ , we know by [31, Proposition 1.51 and Theorem 1.52] that  $-\tilde{\mathcal{A}}$  generates an analytic semigroup in  $X$  of angle  $\pi/2 - \arctan \kappa$  for some  $\kappa > 0$ .

It remains to prove that  $(\tilde{\mathcal{A}}, D(\tilde{\mathcal{A}}))$  is indeed equal to  $(\mathcal{A}^*, D(\mathcal{A}^*))$ .

- First, we fix some  $u := (u_1, u_2) \in D(\mathcal{A}^*)$ . Then for all  $\psi := (\psi_1, \psi_2) \in (H^1(0, 1))^2$ , we have

$$\begin{aligned} h(u, \psi) &:= \int_0^1 \left( u_1'(x) \overline{\psi_1'(x)} + u_2'(x) \overline{\psi_2'(x)} \right) dx + \int_0^1 \left( \alpha_1 u_1(x) \overline{\psi_1(x)} + \alpha_2 u_2(x) \overline{\psi_2(x)} \right) dx \\ &\quad - \beta_\ell u_1(0) \overline{\psi_2(0)} - \beta_r u_2(1) \overline{\psi_1(1)} \\ &= \int_0^1 \left( -u_1''(x) + \alpha_1 u_1(x) \right) \overline{\psi_1(x)} dx + \int_0^1 \left( -u_2''(x) + \alpha_2 u_2(x) \right) \overline{\psi_2(x)} dx \\ &= (\mathcal{A}^* u, \psi), \end{aligned}$$

where we performed an integration by parts using the boundary conditions satisfied by  $u \in D(\mathcal{A}^*)$ . Thus, for the chosen  $u \in D(\mathcal{A}^*)$ , there is a  $w = \mathcal{A}^* u \in X$  such that  $h(u, \psi) = (w, \psi)_X$ , for all  $\psi \in (H^1(0, 1))^2$ , which concludes the inclusion  $D(\mathcal{A}^*) \subseteq D(\tilde{\mathcal{A}})$  and the fact that  $\tilde{\mathcal{A}} = \mathcal{A}^*$  on  $D(\mathcal{A}^*)$ .

- Conversely, let  $u \in D(\tilde{\mathcal{A}})$ . By definition, there exists some  $w \in X$  such that  $h(u, \psi) = (w, \psi)_X$  with  $\tilde{\mathcal{A}}u = w$ , for all  $\psi \in (H^1(0, 1))^2$ , and accordingly

$$\begin{aligned} \int_0^1 \left( u_1'(x) \overline{\psi_1'(x)} + u_2'(x) \overline{\psi_2'(x)} \right) dx + \int_0^1 \left( \alpha_1 u_1(x) \overline{\psi_1(x)} + \alpha_2 u_2(x) \overline{\psi_2(x)} \right) dx \\ - \beta_\ell u_1(0) \overline{\psi_2(0)} - \beta_r u_2(1) \overline{\psi_1(1)} = \int_0^1 \left( w_1(x) \overline{\psi_1(x)} + w_2(x) \overline{\psi_2(x)} \right) dx. \end{aligned}$$

In particular, by considering any  $\psi = (\psi_1, 0) \in (H_0^1(0, 1))^2$ , then  $\psi = (0, \psi_2) \in (H_0^1(0, 1))^2$  we conclude that

$$\begin{aligned} w_1 &= -u_1'' + \alpha_1 u_1, \\ w_2 &= -u_2'' + \alpha_2 u_2. \end{aligned}$$

In particular, since  $w_i \in L^2(0, 1)$  ( $i = 1, 2$ ), we have  $u_1, u_2 \in H^2(0, 1)$  and thus an integration by parts yields

$$\begin{aligned} \int_0^1 \left( -u_1''(x) + \alpha_1 u_1(x) \right) \overline{\psi_1(x)} dx + \int_0^1 \left( -u_2''(x) + \alpha_2 u_2(x) \right) \overline{\psi_2(x)} dx \\ - \beta_\ell u_1(0) \overline{\psi_2(0)} - \beta_r u_2(1) \overline{\psi_1(1)} + u_1'(1) \overline{\psi_1(1)} - u_1'(0) \overline{\psi_1(0)} + u_2'(1) \overline{\psi_2(1)} - u_2'(0) \overline{\psi_2(0)} \\ = \int_0^1 \left( w_1(x) \overline{\psi_1(x)} + w_2(x) \overline{\psi_2(x)} \right) dx, \end{aligned}$$

for all  $\psi \in (H^1(0, 1))^2$ . We eventually obtain

$$\begin{aligned} u_1'(0) &= 0, & u_2'(0) + \beta_\ell u_1(0) &= 0, \\ u_2'(1) &= 0, & u_1'(1) - \beta_r u_2(1) &= 0. \end{aligned}$$

which are exactly the boundary conditions of an element of  $D(\mathcal{A}^*)$  and thus  $D(\tilde{\mathcal{A}}) \subseteq D(\mathcal{A}^*)$ . The proof is complete.  $\square$

**2.2. The backward in time adjoint system.** The adjoint problem (backward in time) of our control systems (1.10)–(1.14) with (1.15) or (1.16) is

$$(2.4) \quad \begin{cases} -\partial_t q_1 - \partial_x^2 q_1 + \alpha_1 q_1 = \tilde{w}_1 & \text{in } (0, T) \times (0, 1), \\ -\partial_t q_2 - \partial_x^2 q_2 + \alpha_2 q_2 = \tilde{w}_2 & \text{in } (0, T) \times (0, 1), \\ \partial_x q_1(\cdot, 0) = 0 & \text{in } (0, T), \\ \partial_x q_2(\cdot, 0) + \beta_\ell q_1(\cdot, 0) = 0 & \text{in } (0, T), \\ \partial_x q_1(\cdot, 1) - \beta_r q_2(\cdot, 1) = 0 & \text{in } (0, T), \\ \partial_x q_2(\cdot, 1) = 0 & \text{in } (0, T), \\ q_1(T, \cdot) = \zeta_1 & \text{in } (0, 1), \\ q_2(T, \cdot) = \zeta_2 & \text{in } (0, 1), \end{cases}$$

where  $\tilde{w} := (\tilde{w}_1, \tilde{w}_2) \in L^2(0, T; X)$  is some given right hand side and  $\zeta = (\zeta_1, \zeta_2)$  a final data.

**PROPOSITION 2.2.** *For any given  $\zeta \in X$  and  $\tilde{w} \in L^2(0, T; X)$ , there exists a unique weak solution  $q := (q_1, q_2) \in \mathcal{C}^0([0, T]; X) \cap L^2(0, T; (H^1(0, 1))^2)$  to (2.4) that satisfies the following energy estimate*

$$\|q\|_{\mathcal{C}^0([0, T]; X)} + \|q\|_{L^2(0, T; (H^1(0, 1))^2)} + \|\partial_t q\|_{L^2(0, T; (H^{-1}(0, 1))^2)} \leq C e^{CT} (\|\zeta\|_X + \|\tilde{w}\|_{L^2(0, T; X)}),$$

where the constant  $C > 0$  does not depend on  $T > 0$ .

*Proof.* We just give a short sketch of the proof. For  $\zeta \in D(\mathcal{A}^*)$  and  $\tilde{w} \in \mathcal{C}^1([0, T]; X)$  we indeed have the existence of a strong solution  $q \in \mathcal{C}^1([0, T]; X) \cap \mathcal{C}^0([0, T]; D(\mathcal{A}^*))$  to (2.4). It remains to prove claimed the estimate to obtain the result by the usual density argument.

Let us test the first and second equation of (2.4) by  $q_1$  and  $q_2$  respectively, which provides

$$(2.5) \quad -\frac{1}{2} \frac{d}{dt} \|q(t)\|_X^2 + \|\partial_x q(t)\|_X^2 + \alpha_1 \|q_1(t)\|_{L^2(0, 1)}^2 + \alpha_2 \|q_2(t)\|_{L^2(0, 1)}^2 \\ + \beta_\ell q_1(t, 0) q_2(t, 0) - \beta_r q_1(t, 1) q_2(t, 1) \leq \frac{1}{2} (\|\tilde{w}(t)\|_X^2 + \|q(t)\|_X^2).$$

Using the following trace inequality

$$|\phi(0)| + |\phi(1)| \leq 2 \|\phi\|_{L^2(0, 1)}^{\frac{1}{2}} \|\phi\|_{H^1(0, 1)}^{\frac{1}{2}}, \forall \phi \in H^1(0, 1),$$

we get from (2.5) that

$$-\frac{1}{2} \frac{d}{dt} \|q(t)\|_X^2 + \|q(t)\|_{(H^1(0, 1))^2}^2 \leq C \epsilon \|q(t)\|_{(H^1(0, 1))^2}^2 + C_\epsilon \|q(t)\|_X^2 + \|\tilde{w}(t)\|_X^2,$$

for some constant  $C > 0$  that does not depend on  $\epsilon$ .

Choosing a fixed and small enough  $\epsilon > 0$  and by using Gronwall's lemma we deduce that

$$\|q\|_{\mathcal{C}^0([0, T]; X)}^2 \leq C e^{CT} (\|\zeta\|_X^2 + \|\tilde{w}\|_{L^2(0, T; X)}^2).$$

The other estimates can be proved in a standard fashion, more details can be found in [28, Chapter 4] (see also [16, Chapter 7]).  $\square$

**2.3. A non homogeneous forward system.** Let us consider the following non homogeneous system

$$(2.6) \quad \begin{cases} \partial_t y_1 - \partial_x^2 y_1 + \alpha_1 y_1 = F_1 & \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x^2 y_2 + \alpha_2 y_2 = F_2 & \text{in } (0, T) \times (0, 1), \\ \partial_x y_1(\cdot, 0) + \beta_\ell y_2(\cdot, 0) = G_1 & \text{in } (0, T), \\ \partial_x y_2(\cdot, 0) = G_2 & \text{in } (0, T), \\ \partial_x y_1(\cdot, 1) = G_3 & \text{in } (0, T), \\ \partial_x y_2(\cdot, 1) - \beta_r y_1(\cdot, 1) = G_4 & \text{in } (0, T), \\ y_1(0, \cdot) = y_{0,1} & \text{in } (0, 1), \\ y_2(0, \cdot) = y_{0,2} & \text{in } (0, 1), \end{cases}$$



where  $F_1, F_2 \in L^2(0, T; L^2(0, 1))$  and  $G_j \in L^2(0, T; \mathbb{R})$  for  $j = 1, 2, 3, 4$ .

Recall that the adjoint operator  $(-\mathcal{A}^*, D(\mathcal{A}^*))$  given by (2.1)–(2.3), defines an analytic semigroup in  $X$ , thanks to Proposition 2.1. Therefore, the existence of a unique weak solution to (2.6) can be shown using the method of transposition; see [13, 34]. More precisely, we test the PDEs in (2.6) against the solution  $q$  to the adjoint system (2.4) with  $\tilde{w}_1 = \tilde{w}_2 = 0$ . We express this below.

**THEOREM 2.3.** *For any given  $y_0 := (y_{0,1}, y_{0,2}) \in X$ ,  $F := (F_1, F_2) \in L^2(0, T; X)$  and  $G := (G_1, G_2, G_3, G_4) \in L^2(0, T; \mathbb{R}^4)$ , there exists a unique weak solution  $y$  to (2.6) belonging to the space  $C^0([0, T]; X) \cap L^2(0, T; (H^1(0, 1))^2)$  in the following sense: for any  $t \in [0, T]$  and  $\zeta := (\zeta_1, \zeta_2) \in X$ , we have*

$$\begin{aligned} (y(t), \zeta)_X &= (y_0, e^{-t\mathcal{A}^*} \zeta)_X + \int_0^t (F(s), e^{-(t-s)\mathcal{A}^*} \zeta)_X ds \\ &\quad - \int_0^t \left\langle \begin{pmatrix} G_1(s) \\ G_2(s) \end{pmatrix}, \left( e^{-(t-s)\mathcal{A}^*} \zeta \right) \Big|_{\{x=0\}} \right\rangle_{\mathbb{R}^2} ds + \int_0^t \left\langle \begin{pmatrix} G_3(s) \\ G_4(s) \end{pmatrix}, \left( e^{-(t-s)\mathcal{A}^*} \zeta \right) \Big|_{\{x=1\}} \right\rangle_{\mathbb{R}^2} ds. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \|y\|_{C^0([0, T]; X)} + \|y\|_{L^2(0, T; (H^1(0, 1))^2)} + \|\partial_t y\|_{L^2(0, T; (H^{-1}(0, 1))^2)} \\ \leq C e^{CT} (\|y_0\|_X + \|F\|_{L^2(0, T; X)} + \|G\|_{L^2(0, T; \mathbb{R}^4)}). \end{aligned}$$

**2.4. The control problems.** First, observe that the existence of a unique weak solutions to our control systems (1.10)–(1.14) with (1.15) or (1.16) is clear from Theorem 2.3. Now, let us prescribe the formulation of the control problems in both cases. We hereby introduce the observation operators  $\mathcal{B}_1^*$  and  $\mathcal{B}_2^*$  associated with the control problems (2.8) and (2.9) respectively as follows,

$$(2.7a) \quad \mathcal{B}_1^* = \mathbf{1}_{\{x=0\}} \begin{pmatrix} 1 & 0 \end{pmatrix} : (H^1(0, 1))^2 \rightarrow \mathbb{R},$$

$$(2.7b) \quad \mathcal{B}_2^* = \mathbf{1}_{\{x=0\}} \begin{pmatrix} 0 & 1 \end{pmatrix} : (H^1(0, 1))^2 \rightarrow \mathbb{R}.$$

**PROPOSITION 2.4.** *Let any  $y_0 \in X$ ,  $(\alpha_1, \alpha_2) \in \mathbb{R}^2$  and  $T > 0$  be given, and assume that  $\beta_\ell \beta_r \neq 0$ .*

1. *A function  $v \in L^2(0, T)$  is a null-control for the system (1.10) with the boundary conditions (1.14)–(1.15) if and only if it satisfies: for any  $\zeta \in X$ ,*

$$(2.8) \quad (y_0, e^{-T\mathcal{A}^*} \zeta)_X = \int_0^T v(t) \mathcal{B}_1^* e^{-(T-t)\mathcal{A}^*} \zeta dt.$$

2. *A function  $v \in L^2(0, T)$  is a null-control for the system (1.10) with the boundary conditions (1.14)–(1.16) if and only if it satisfies: for any  $\zeta \in X$ ,*

$$(2.9) \quad (y_0, e^{-T\mathcal{A}^*} \zeta)_X = \int_0^T v(t) \mathcal{B}_2^* e^{-(T-t)\mathcal{A}^*} \zeta dt.$$

**3. Spectral analysis.** Let us write the eigenvalue problem  $\mathcal{A}^* u = \lambda u$ , for  $\lambda \in \mathbb{C}$  as follows

$$(3.1) \quad \begin{cases} -u_1'' + \alpha_1 u_1 = \lambda u_1 & \text{in } (0, 1), \\ -u_2'' + \alpha_2 u_2 = \lambda u_2 & \text{in } (0, 1), \\ u_1'(0) = 0, \quad u_2'(0) + \beta_\ell u_1(0) = 0, \\ u_1'(1) - \beta_r u_2(1) = 0, \quad u_2'(1) = 0. \end{cases}$$

with the parameters  $(\alpha_1, \alpha_2) \in \mathbb{R}^2$  and  $\beta_\ell \beta_r \neq 0$ .

**3.1.  $\mathcal{A}^*$  has compact resolvent.** Let us denote the spectrum (resp. the resolvent set) of  $\mathcal{A}^*$  by  $\sigma(\mathcal{A}^*)$  (resp. by  $\rho(\mathcal{A}^*)$ ). The sesquilinear form associated with the operator  $\mathcal{A}^* - \xi \text{Id}$  for any  $\xi \in \rho(\mathcal{A}^*)$  is

$$\begin{aligned} h(u, \phi) &:= \int_0^1 \left( u_1'(x) \overline{\phi_1'(x)} + u_2'(x) \overline{\phi_2'(x)} \right) dx + (\alpha_1 - \xi) \int_0^1 u_1(x) \overline{\phi_1(x)} dx \\ &\quad + (\alpha_2 - \xi) \int_0^1 u_2(x) \overline{\phi_2(x)} dx - \beta_\ell u_1(0) \overline{\phi_2(0)} - \beta_r u_2(1) \overline{\phi_1(1)}, \end{aligned}$$

for all  $\phi := (\phi_1, \phi_2) \in (H^1(0, 1))^2$ .

One can show that  $h$  is continuous and that there exists constants  $\gamma_1, \gamma_2 > 0$  such that we have

$$(3.2) \quad \operatorname{Re}(h(u, u)) \geq \gamma_1 \|u'\|_{(L^2(0,1))^2}^2 + (-\operatorname{Re}(\xi) - \gamma_2) \|u\|_X^2.$$

From (3.2), it follows that for  $\xi \in \mathbb{C}$  with  $\operatorname{Re}(\xi) \leq -\gamma_2 - 1$ , the operator  $(\mathcal{A}^* - \xi \operatorname{Id})$  is invertible. In particular,

$$(-\infty, -\gamma_2 - 1] \subset \rho(\mathcal{A}^*).$$

Finally, we have that  $D(\mathcal{A}^*) \hookrightarrow (H^1(0, 1))^2 \hookrightarrow X$  with compact embedding and so  $R_\xi := (\mathcal{A}^* - \xi \operatorname{Id})^{-1} : X \rightarrow X$  is a compact operator in  $X$  for any  $\xi \in \mathbb{C}$  with  $\operatorname{Re}(\xi) \leq -\gamma_2 - 1$  and consequently for any  $\xi \in \rho(\mathcal{A}^*)$ .

The above analysis confirms that the spectrum  $\sigma(\mathcal{A}^*)$  is discrete and contains only eigenvalues of the operator  $\mathcal{A}^*$ .

**3.2. Computation of the eigenfunctions of  $\mathcal{A}^*$ .** We first observe that all the eigenvalues are geometrically simple. Indeed, assume that for some  $\lambda \in \sigma(\mathcal{A}^*)$ , there exists two linearly independent solutions  $\phi = (\phi_1, \phi_2)$  and  $\psi = (\psi_1, \psi_2)$  of the eigenvalue problem (3.1). We consider  $u = a\phi + b\psi$ , where  $a, b \in \mathbb{C}^*$  are chosen so that  $u_1(0) = 0$  (recall that  $\phi_1(0) \neq 0$  and  $\psi_1(0) \neq 0$ ). But we also have  $u'_1(0) = 0$  from the boundary conditions in (3.1) which leads that  $u_1 \equiv 0$  in  $[0, 1]$ . Finally, we immediately get from the set of equations (3.1) that  $u_2 \equiv 0$  which is a contradiction. This proves that the geometric multiplicity of any eigenvalue is 1.

Then, we can see that  $\lambda = \alpha_1$  or  $\alpha_2$  cannot be an eigenvalue for  $\mathcal{A}^*$ . Indeed, by putting  $\lambda = \alpha_1$  or  $\alpha_2$  in the set of equations (3.1), we find that  $u_1$  or  $u_2$  is affine and then using the boundary conditions, one can show that  $u_1 = u_2 = 0$ .

In the sequel we will use the notation  $\theta := \alpha_2 - \alpha_1 \in \mathbb{R}$ . Let us now consider the variable  $\mu \in \mathbb{C}$  defined by

$$\mu = \sqrt{\lambda - \alpha_1},$$

where we recall that  $\sqrt{\cdot}$  is the principal determination of the square root in the complex plane. We also set  $\nu = \sqrt{\mu^2 - \theta} = \sqrt{\lambda - \alpha_2}$  to simplify the presentation. By the discussion above, we know that  $\mu \neq 0$  and  $\nu \neq 0$ .

We will now find out an explicit formula for the eigenfunctions of  $\mathcal{A}^*$  as well as the transcendental equation satisfied by the eigenvalues. To begin with, we rewrite the equations of  $u_1$  and  $u_2$  as follows

$$\begin{aligned} u_1'' + \mu^2 u_1 &= 0, \\ u_2'' + \nu^2 u_2 &= 0, \end{aligned}$$

along with the boundary conditions as given in (3.1).

Since  $u'_1(0) = u'_2(1) = 0$ , we look for solutions in the following form

$$(3.3) \quad \begin{cases} u_1(x) = K_1 \cos(\mu x), & x \in [0, 1], \\ u_2(x) = K_2 \cos(\nu(1-x)), & x \in [0, 1], \end{cases}$$

for some constants  $K_1, K_2 \in \mathbb{C}$ , to be determined. Using the other two boundary conditions at  $x = 0$ , we have

$$\begin{pmatrix} \beta_\ell & \nu \sin \nu \\ -\mu \sin \mu & -\beta_r \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This system has non trivial solutions if and only if the determinant of the coefficient matrix is zero. This yields the following equation for  $\mu \in \mathbb{C}$

$$(3.4) \quad h(\mu) := \frac{1}{\beta_\ell \beta_r} \mu \sqrt{\mu^2 - \theta} \sin(\mu) \sin(\sqrt{\mu^2 - \theta}) - 1 = 0.$$

Note that, even though the square-root is not an entire function, the function  $h$  is actually entire. Indeed, if we introduce the entire function

$$\psi(z) = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)!} z^{n+1},$$

we see that  $\psi(z^2) = z \sin(z)$ , and thus  $h$  can be written

$$h(\mu) = \frac{1}{\beta_\ell \beta_r} \psi(\mu^2) \psi(\mu^2 - \theta) - 1, \quad \forall \mu \in \mathbb{C}.$$

Note that  $h$  is an even function, so that for any solution  $\mu \neq 0$  of (3.4),  $-\mu$  is also a solution, but those two solutions lead to the same eigenvalue  $\lambda$  through the formula  $\lambda = \alpha_1 + \mu^2$ .

Before analyzing the set of eigenvalues of  $\mathcal{A}^*$ , let us determine the solution to (3.1) associated with every such eigenvalue. Let  $\mu$  be a solution to (3.4). In (3.3), we set  $K_1 = 1$  and then we get

$$K_2 = -\frac{\mu \sin \mu}{\beta_r}.$$

It follows that the solution to (3.1), denoted by  $\Phi_\lambda$ , is given by

$$(3.5) \quad \Phi_\lambda(x) := \begin{pmatrix} \cos(\mu x) \\ -\frac{\mu \sin(\mu)}{\beta_r} \cos(\sqrt{\mu^2 - \theta}(1-x)) \end{pmatrix}, \quad x \in [0, 1],$$

associated with  $\lambda = \mu^2 + \alpha_1$ , where  $\theta = \alpha_2 - \alpha_1$  and  $\mu \in \mathbb{C}$  satisfies the equation (3.4).

**3.3. Localization of the eigenvalues.** We are now in position to describe precisely the structure of the spectrum of  $\mathcal{A}^*$ .

**3.3.1. Preliminaries.** We begin with the following set of lemmas, whose proofs are given in Appendix A.

LEMMA 3.1. *There exist  $\mu_0 > 0$  depending only on  $\theta$  such that, for every  $\mu \in \mathbb{C}$  satisfying  $|\mu| > \mu_0$ , we have*

$$|\sqrt{\mu^2 - \theta}| \geq \frac{1}{2}|\mu|,$$

and

$$|\sin(\sqrt{\mu^2 - \theta})| \geq \frac{3}{4}|\sin \mu| - \frac{1}{4}|\cos \mu|.$$

We set  $\beta_* = \beta_\ell \beta_r$ , which satisfies  $\beta_* \neq 0$  by the assumption (1.6), and then we introduce the function

$$h_0(\mu) := \frac{1}{\beta_*} \mu \sqrt{\mu^2 - \theta} \sin(\mu) \sin(\sqrt{\mu^2 - \theta}),$$

that appears in the equation (3.4).

LEMMA 3.2. *There exists  $\vartheta_0 > 0$ , depending only on  $\theta$  and  $\beta_*$ , such that for any  $\mu \in \mathbb{C}$ , satisfying  $|\operatorname{Im} \mu| \geq \vartheta_0$ , we have*

$$|h_0(\mu)| > 1.$$

LEMMA 3.3. *There exists  $k_0 \in \mathbb{N}^*$ , depending only on  $\theta$  and  $\beta_*$ , such that for any  $\mu \in \mathbb{C}$ , with  $\operatorname{Re} \mu = (k + 1/2)\pi$ ,  $k \geq k_0$ , we have*

$$|h_0(\mu)| > 1.$$

LEMMA 3.4. *Let  $\gamma \in \mathbb{C}$ , and two sequences complex numbers  $(\delta_k)_k$  and  $(\varepsilon_k)_k$ . We assume that  $(\delta_k)_k$  is bounded, and  $(\varepsilon_k)_k$  tends to 0.*

*Then we have*

$$(3.6) \quad \begin{aligned} \sqrt{(k\pi + \delta_k)^2 + \gamma} &= k\pi + \delta_k + \frac{\gamma}{2k\pi} + O\left(\frac{1}{k^2}\right), \\ \sqrt{k^2\pi^2 + \gamma + \varepsilon_k} &= k\pi + \frac{\gamma + \varepsilon_k}{2k\pi} - \frac{\gamma^2}{8k^3\pi^3} + o\left(\frac{1}{k^3}\right), \\ h_0(\sqrt{k^2\pi^2 + \gamma}) &= \frac{\gamma(\gamma - \theta)}{4\beta_*} + O\left(\frac{1}{k^2}\right), \end{aligned}$$

$$(3.7) \quad h_0(\sqrt{k^2\pi^2 + \theta/2 + \varepsilon_k}) = -\frac{\theta^2}{16\beta_*} \left(1 - \frac{4\varepsilon_k^2}{\theta^2} - \frac{\theta^2}{48k^2\pi^2} + o\left(\frac{1}{k^2}\right)\right).$$

We are now ready to give a quite precise description of the eigenvalues of  $\mathcal{A}^*$ .

**3.3.2. Eigenvalues with large real parts.** For any integer  $k$  we introduce the strip of the complex plane defined by

$$(3.8) \quad R_k := \{z \in \mathbb{C}, \quad (k - 1/2)\pi \leq \operatorname{Re} z \leq (k + 1/2)\pi\}.$$

Assuming for the moment that  $h$  has at least one root in each  $R_k$  - this will be proved below - we can first obtain the following result.

**PROPOSITION 3.5.** *Let  $(\mu_k)_{k \geq k_0}$  be a sequence of complex numbers such that  $\mu_k \in R_k$  for every  $k \geq k_0$  and  $h(\mu_k) = 0$ . Then we have the following properties:*

1. *The sequence  $\delta_k = \mu_k - k\pi$  tends to zero as  $k$  goes to infinity.*
2. *The sequence  $\gamma_k = k\pi\delta_k = k\pi(\mu_k - k\pi)$  is bounded and satisfies*

$$(3.9) \quad \gamma_k^2 - \frac{\theta}{2}\gamma_k \xrightarrow[k \rightarrow \infty]{} \beta_*.$$

*Proof.* 1. By definition of  $R_k$  and [Lemma 3.2](#), we already know that  $(\delta_k)_k$  lies into the compact set  $K := \{z \in \mathbb{C}, |\operatorname{Re} z| \leq \pi/2, |\operatorname{Im} z| \leq \vartheta_0\}$ .

By using [Lemma 3.4](#) and the equation satisfied by  $\mu_k = k\pi + \delta_k$  we obtain

$$\begin{aligned} \beta_* &= (k\pi + \delta_k)\sqrt{(k\pi + \delta_k)^2 - \theta} \sin(k\pi + \delta_k) \sin\left(\sqrt{(k\pi + \delta_k)^2 - \theta}\right) \\ &= (k\pi + \delta_k)(k\pi + \delta_k + O(1/k))(-1)^k \sin(\delta_k)(-1)^k (\sin(\delta_k) + O(1/k)). \end{aligned}$$

This leads to

$$\frac{\beta_*}{k^2\pi^2} = (\sin \delta_k)^2 + O(1/k),$$

and finally to

$$(\sin \delta_k)^2 = O(1/k),$$

which implies that  $\delta_k \rightarrow 0$  since 0 is the unique root of the sine function in the compact  $K$ .

2. By a similar reasoning, we find that

$$\beta_* = (k\pi)^2(1 + O(1/k^2))\delta_k(1 + O(\delta_k^2)) \left( \delta_k - \frac{\theta}{2k\pi} + O(\delta_k^3) + O(1/k^2) \right),$$

which leads to

$$\beta_* = (1 + O(1/k^2))\gamma_k(1 + O(\delta_k^2)) \left( \gamma_k - \frac{\theta}{2} + O(\gamma_k\delta_k^2) + O(1/k) \right).$$

This clearly implies that  $(\gamma_k)_k$  is bounded and that [\(3.9\)](#) holds.  $\square$

Our main result of this section is the following.

**THEOREM 3.6.** *There exists an integer  $k_1 \geq 1$ , depending only on  $\alpha_1, \alpha_2$  and  $\beta_*$ , such that for all  $k \geq k_1$ , there exist exactly two distinct solutions  $\mu_{k,+}$  and  $\mu_{k,-}$  of the equation [\(3.4\)](#) in the strip  $R_k$  defined in [\(3.8\)](#).*

*Moreover, they satisfy  $|\mu_{k,\pm}| \leq \vartheta_0$  for every  $k \geq k_1$  ( $\vartheta_0$  is introduced in [Lemma 3.2](#)), as well as the following asymptotics:*

- *If  $16\beta_* + \theta^2 \neq 0$ , there exist two distinct non zero complex numbers  $\gamma_{\pm}$  such that*

$$(3.10) \quad \mu_{k,\pm} = k\pi + \frac{\gamma_{\pm}}{k\pi} + O\left(\frac{1}{k^2}\right)$$

- *If  $16\beta_* + \theta^2 = 0$ , there exist  $\gamma \in \mathbb{R}^*$  and  $\xi \in \mathbb{R}^*$  such that*

$$\mu_{k,\pm} = k\pi + \frac{\gamma}{k\pi} \pm i\frac{\xi}{k^2\pi^2} + O\left(\frac{1}{k^3}\right).$$

*To simplify the presentation, we will also set  $\gamma_+ = \gamma_- = \gamma$  in that case, in such a way that [\(3.10\)](#) is still valid.*

Note that the values of  $\gamma_{\pm}$ ,  $\gamma$  and  $\xi$ , are explicit in function of the parameters, as we will see in the proof. Moreover, if  $16\beta_* + \theta^2 > 0$  then  $\mu_{k,\pm}$  (and  $\gamma_{\pm}$ ) are real, whereas if  $16\beta_* + \theta^2 \leq 0$ , they are complex conjugate.

By the property (3.9), we see that the equation

$$(3.11) \quad \gamma^2 - \frac{\theta}{2}\gamma = \beta_*,$$

will play a role in the analysis and since  $\beta_* \neq 0$ , the solutions of this equation are not zero.

The reasoning will be slightly different depending on the sign of the discriminant of this equation which is, up to a positive factor, the quantity  $16\beta_* + \theta^2$ .

*Proof.* We first observe that the set of roots of  $h_0$  in the complex plane is exactly given by

$$\left\{ \pm k\pi, \pm \sqrt{k^2\pi^2 + \theta}, k \in \mathbb{N} \right\}.$$

Hence, for  $k_1$  large enough and  $k \geq k_1$ , we see that  $h_0$  has exactly two roots (counted with multiplicities) in the strip  $R_k$ .

Moreover, by Lemmas 3.2 and 3.3 we know that  $|h_0| > 1$  on the boundary of  $R_k$  and that  $|h_0| > 1$  on  $R_k \cap \{\mu \in \mathbb{C}, |\operatorname{Im} \mu| \geq \vartheta_0\}$ . Therefore, by Rouché's theorem, we deduce that  $h$  has exactly two roots in  $R_k$ , counted with multiplicities, and that their imaginary part is bounded by  $\vartheta_0$ .

Finally, we observe that  $\overline{h(\mu)} = h(\overline{\mu})$  for every  $\mu$ . Therefore if one of the root of  $h$  in  $R_k$  is not real, the other root is necessarily its complex conjugate.

- Case 1 : assume that

$$(3.12) \quad 16\beta_* + \theta^2 < 0.$$

In that case, the solutions of (3.11) are two conjugate complex numbers

$$\gamma_{\pm} = \frac{\theta \pm i\sqrt{|16\beta_* + \theta^2|}}{4}.$$

In particular we deduce that the solutions of  $h(\mu) = 0$  in  $R_k$ , for  $k$  large enough, cannot be real and therefore, they are necessarily two conjugate complex numbers. Those two roots are denoted by  $\mu_{k,+}$  and  $\mu_{k,-} = \overline{\mu_{k,+}}$ , with  $\operatorname{Im} \mu_{k,+} > 0$  and they are the unique roots of  $h$  in the strip  $R_k$ .

From Proposition 3.5, we also deduce that

$$\mu_{k,\pm} = k\pi + \frac{\gamma_{\pm}}{k\pi} + o\left(\frac{1}{k}\right).$$

- Case 2 : assume that

$$16\beta_* + \theta^2 > 0.$$

By (3.6), we know that for any  $\gamma \in \mathbb{R}$ , we have

$$(3.13) \quad h(\sqrt{k^2\pi^2 + \gamma}) = \frac{\gamma(\gamma - \theta)}{4\beta_*} - 1 + O\left(\frac{1}{k^2}\right).$$

- If  $\beta_* > 0$ , we can find some  $\gamma_1$  and  $\gamma_2$  such that  $\gamma_1 < 0 < \gamma_2$  and with

$$\frac{\gamma_i(\gamma_i - \theta)}{4\beta_*} - 1 > 0, \text{ for } i = 1, 2.$$

It follows that for  $k$  large enough we have

$$h(k\pi) = -1 < 0, \quad h(\sqrt{k^2\pi^2 + \gamma_1}) > 0, \quad \text{and} \quad h(\sqrt{k^2\pi^2 + \gamma_2}) > 0.$$

This implies that  $h$  (which is real valued on  $\mathbb{R}$ ) has exactly two distinct real roots satisfying

$$\sqrt{k^2\pi^2 + \gamma_1} < \mu_{k,-} < k\pi < \mu_{k,+} < \sqrt{k^2\pi^2 + \gamma_2}.$$

Introducing  $\gamma_{k,\pm} = k\pi(\mu_{k,\pm} - k\pi)$ , we have that

$$\gamma_{k,+} > 0, \text{ and } \gamma_{k,-} < 0.$$

It follows from [Proposition 3.5](#) that  $\gamma_{k,-}$  necessarily converges towards the unique negative solution of [\(3.11\)](#) and that  $\gamma_{k,+}$  converges towards the unique positive solution of [\(3.11\)](#). In other words we have

$$(3.14) \quad \mu_{k,\pm} = k\pi + \frac{\theta \pm \sqrt{16\beta_* + \theta^2}}{4k\pi} + O\left(\frac{1}{k^2}\right).$$

– If  $\beta_* < 0$ , then we can take  $\gamma = \theta/2$  in [\(3.13\)](#) so that

$$\frac{\gamma(\gamma - \theta)}{4\beta_*} - 1 = -\frac{\theta^2}{16\beta_*} - 1 > 0,$$

by [\(3.12\)](#). It follows that for  $k$  large enough we have

$$h(k\pi) = -1 < 0, h(\sqrt{k^2\pi^2 + \theta/2}) > 0, \text{ and } h(\sqrt{k^2\pi^2 + \theta}) = -1 < 0.$$

This implies that  $h$  has also two distinct real roots satisfying

$$\mu_{k,-} < \sqrt{k^2\pi^2 + \theta/2} < \mu_{k,+}.$$

It follows that

$$\gamma_{k,-} < k\pi \left( \sqrt{k^2\pi^2 + \theta/2} - k\pi \right) < \gamma_{k,+},$$

and since  $k\pi \left( \sqrt{k^2\pi^2 + \theta/2} - k\pi \right) \rightarrow \theta/4$ , we see that  $\gamma_{k,-}$  necessarily converges towards the unique solution of [\(3.11\)](#) which is less than  $\theta/4$  and  $\gamma_{k,+}$  necessarily converges towards the unique solution of [\(3.11\)](#) which is higher than  $\theta/4$ .

It follows that [\(3.14\)](#) still holds.

- Case 3 : assume that

$$(3.15) \quad 16\beta_* + \theta^2 = 0.$$

In this case, the equation [\(3.11\)](#) has only one double (real) root which is  $\gamma = \theta/4$  and therefore we have

$$\mu_k = k\pi + \frac{\theta}{4k\pi} + O\left(\frac{1}{k^2}\right).$$

We need to go one step further in the asymptotic expansion to determine the behavior of this sequence. To this end, we define a complex number  $\varepsilon_k$  by

$$\varepsilon_k = \mu_k^2 - k^2\pi^2 - \frac{\theta}{2},$$

which tends to 0 by the above asymptotic expansion of  $\mu_k$ . Using [\(3.7\)](#), we deduce that

$$\begin{aligned} h_0(\mu_k) &= h_0\left(\sqrt{k^2\pi^2 + \frac{\theta}{2} + \varepsilon_k}\right) \\ &= -\frac{\theta^2}{16\beta_*} \left(1 - \frac{4\varepsilon_k^2}{\theta^2} - \frac{\theta^2}{48k^2\pi^2} + o\left(\frac{1}{k^2}\right)\right). \end{aligned}$$

By assumption we have  $h_0(\mu_k) = 1$  and  $16\beta_* = -\theta^2$ , so that we end up with the equality

$$0 = \frac{4\varepsilon_k^2}{\theta^2} + \frac{\theta^2}{48k^2\pi^2} + o\left(\frac{1}{k^2}\right),$$

that is

$$\varepsilon_k^2 k^2 \pi^2 \xrightarrow[k \rightarrow \infty]{} -\frac{\theta^4}{192}.$$

We have that  $\theta \neq 0$  (since  $\beta_* \neq 0$  and  $16\beta_* + \theta^2 = 0$ ), and thus the limit above implies that, for  $k$  large enough,  $\varepsilon_k$  cannot be a real number and therefore  $\mu_k$  also is necessarily not real. The same reasoning as before shows that we have two complex conjugate solutions  $\mu_{k,+}$  and  $\mu_{k,-} = \overline{\mu_{k,+}}$ , with  $\text{Im } \mu_{k,+} > 0$  and that the corresponding sequences  $\varepsilon_{k,+}$  and  $\varepsilon_{k,-}$  satisfy

$$\varepsilon_{k,\pm} = \pm i \frac{\theta^2}{8\sqrt{3}k\pi} + o\left(\frac{1}{k}\right).$$

By Lemma 3.4 and the fact that  $\mu_{k,\pm} = \sqrt{k^2\pi^2 + \frac{\theta}{2}} + \varepsilon_{k,\pm}$ , we deduce

$$\mu_{k,\pm} = k\pi + \frac{\theta}{4k\pi} \pm i \frac{\theta^2}{16\sqrt{3}k^2\pi^2} + o\left(\frac{1}{k^2}\right).$$

The proof of the theorem is complete.  $\square$

The previous theorem gives information on the solutions of the equation  $h(\mu) = 0$  from which we can deduce the following properties concerning the eigenvalues of our operator  $\mathcal{A}^*$ , away from the origin.

**COROLLARY 3.7.** *We can choose  $k_1$  large enough in the previous theorem in such a way that all the eigenvalues of  $\mathcal{A}^*$  lying outside the disk  $D(\alpha_1, R)$ , with  $R = (k_1 - 1/2)^2\pi^2$  are exactly given by*

$$\lambda_{k,\pm} = \mu_{k,\pm}^2 + \alpha_1, \quad \text{with } k \geq k_1.$$

Moreover, those eigenvalues and the associated eigenfunctions given by (3.5) (and denoted here by  $\Phi_{k,\pm}$  to lighten the notation) satisfy the following asymptotics:

- If  $16\beta_* + \theta^2 \neq 0$ , we have

$$\lambda_{k,\pm} = k^2\pi^2 + 2\gamma_{\pm} + \alpha_1 + O\left(\frac{1}{k}\right),$$

and

$$(3.16) \quad \Phi_{k,\pm}(x) = \begin{pmatrix} 1 \\ -\frac{\gamma_{\pm}}{\beta_r} \end{pmatrix} \cos(k\pi x) + O\left(\frac{1}{k}\right).$$

- If  $16\beta_* + \theta^2 = 0$ , we have

$$\lambda_{k,\pm} = k^2\pi^2 + 2\gamma + \alpha_1 \pm 2i \frac{\xi}{k\pi} + O\left(\frac{1}{k^2}\right),$$

$$(3.17) \quad \Phi_{k,\pm}(x) = \begin{pmatrix} 1 \\ -\frac{\gamma}{\beta_r} \end{pmatrix} \cos(k\pi x) + O\left(\frac{1}{k}\right).$$

Moreover, in both cases, if we set

$$(3.18) \quad \Psi_k = \frac{\Phi_{k,+} - \Phi_{k,-}}{\lambda_{k,+} - \lambda_{k,-}},$$

we have

$$(3.19) \quad \Psi_k(x) = \begin{pmatrix} 0 \\ -\frac{1}{2\beta_r} \end{pmatrix} \cos(k\pi x) + O\left(\frac{1}{k}\right).$$

In the properties above, the  $O(\cdot)$  terms are uniform with respect to  $x \in (0, 1)$ .

*Proof.* • The asymptotics for the eigenvalues is just a consequence of the one obtained for  $\mu_{k,\pm}$  and of the relation  $\lambda_{k,\pm} = \mu_{k,\pm}^2 + \alpha_1$ .

- Using the expression (3.5) and the asymptotics on  $\mu_{k,\pm}$  we obtain immediately

$$\begin{aligned}\cos(\mu_{k,\pm}x) &= \cos(k\pi x) + O\left(\frac{1}{k}\right), \\ \cos(\sqrt{\mu_{k,\pm}^2 - \theta}(1-x)) &= (-1)^k \cos(k\pi x) + O\left(\frac{1}{k}\right), \\ \mu_{k,\pm} \sin(\mu_{k,\pm}) &= (-1)^k \gamma_{\pm} + O\left(\frac{1}{k}\right).\end{aligned}$$

Those estimates imply (3.16) and (3.17).

- In the case  $16\beta_* + \theta^2 \neq 0$ , we have that  $\lambda_{k,+} - \lambda_{k,-} \rightarrow 2(\gamma_+ - \gamma_-)$  which gives (3.19).
- In the case  $16\beta_* + \theta^2 = 0$ , (3.19) is not as straightforward since  $\lambda_{k,+} - \lambda_{k,-} \rightarrow 0$ . More precisely, we have

$$\lambda_{k,+} - \lambda_{k,-} = i\frac{4\xi}{k\pi} + O\left(\frac{1}{k^2}\right).$$

- Using the order expansions for  $\mu_{k,\pm}$  given in Theorem 3.6, we get

$$\mu_{k,\pm} \sin(\mu_{k,\pm}) = (-1)^k \left( \frac{\theta}{4} \pm i\frac{\xi}{k\pi} + O\left(\frac{1}{k^2}\right) \right),$$

and thus

$$\mu_{k,+} \sin(\mu_{k,+}) - \mu_{k,-} \sin(\mu_{k,-}) = (-1)^k 2i\frac{\xi}{k\pi} + O\left(\frac{1}{k^2}\right).$$

It follows that

$$\frac{\mu_{k,+} \sin(\mu_{k,+}) - \mu_{k,-} \sin(\mu_{k,-})}{\lambda_{k,+} - \lambda_{k,-}} = \frac{(-1)^k}{2} + O\left(\frac{1}{k}\right)$$

- Using again the third order asymptotics of  $\mu_{k,\pm}$  we obtain

$$\begin{aligned}\cos(\mu_{k,+}x) - \cos(\mu_{k,-}x) &= 2 \sin\left(\frac{\mu_{k,+} - \mu_{k,-}}{2}x\right) \sin\left(\frac{\mu_{k,+} + \mu_{k,-}}{2}x\right) \\ &= 2 \sin\left(i\frac{\xi x}{k^2\pi^2} + O(1/k^3)\right) \sin\left(k\pi x + O\left(\frac{1}{k}\right)\right) \\ &= O\left(\frac{1}{k^2}\right)\end{aligned}$$

which proves in particular that

$$\frac{\cos(\mu_{k,+}x) - \cos(\mu_{k,-}x)}{\lambda_{k,+} - \lambda_{k,-}} = O\left(\frac{1}{k}\right).$$

The very same estimate holds with  $\nu_{k,\pm}$  and  $1-x$  instead of  $x$ , by using that

$$\nu_{k,\pm} = \sqrt{\mu_{k,\pm}^2 - \theta} = k\pi - \frac{\theta}{4k\pi} \pm i\frac{\xi}{k^2\pi^2} + O\left(\frac{1}{k^3}\right).$$

Combining all the above estimates, we obtain that (3.19) also holds in that case.  $\square$

Note that, in the case  $16\beta_* + \theta^2 = 0$ , the eigenvalues condensate (in the sense that  $\lambda_{k,+} - \lambda_{k,-}$  tends to 0 as  $k$  goes to infinity), and the associated eigenfunctions also condensate. However, the property (3.19) shows that both phenomenon somehow compensate.



**3.3.3. Conclusions.** By [Corollary 3.7](#), the spectrum of  $\mathcal{A}^*$  can be split into two disjoint parts: one containing a finite number of possibly multiple eigenvalues, and the other one that contains only simple eigenvalues with positive real part whose asymptotics is known. More precisely, we will write

$$(3.20) \quad \sigma(\mathcal{A}^*) := \Lambda_0 \cup \Lambda_\infty.$$

with

$$\Lambda_0 := \left\{ \lambda = \mu^2 + \alpha_1 \mid \mu \in D(0, (k_1 - 1/2)\pi) \text{ satisfying } h(\mu) = 0 \right\},$$

$$\Lambda_\infty := \bigcup_{k \geq k_1} \{ \lambda_{k,+}, \lambda_{k,-} \}.$$

Note that  $\Lambda_0$  is finite and contained into the disk  $D(\alpha_1, R)$ , with  $R = (k_1 - 1/2)^2 \pi^2$ .

### 3.4. Completeness of the set of root vectors of $\mathcal{A}^*$ .

**3.4.1. Notations.** Let  $\mathcal{B}$  be an operator with compact resolvent in  $X$ . For any eigenvalue  $\lambda \in \mathbb{C}$  of  $\mathcal{B}$ , we consider a circle  $C(\lambda, r)$  in the complex plane, positively oriented, centered at  $\lambda$  and with a radius  $r > 0$  small enough such that the disk  $\bar{D}(\lambda, r)$  does not contain any other element of  $\sigma(\mathcal{A}^*)$  than  $\lambda$ . Then, we know that

$$P_{\mathcal{B}, \lambda} = \frac{1}{2i\pi} \int_{C(\lambda, r)} (\xi \text{Id} - \mathcal{B})^{-1} d\xi,$$

is the spectral projector corresponding to the  $\lambda$ . Its range is spanned by all the root vectors of  $\mathcal{B}$  associated with  $\lambda$ .

Similarly, if  $\Lambda \subset \sigma(\mathcal{B})$  is a finite set of eigenvalues, the spectral projector corresponding to  $\Lambda$  is given by

$$P_{\mathcal{B}, \Lambda} = \sum_{\lambda \in \Lambda} P_{\mathcal{B}, \lambda}.$$

**3.4.2. Generalized eigenvectors of  $\mathcal{A}^*$ .** For each eigenvalue  $\lambda \in \sigma(\mathcal{A}^*)$ , the eigenfunction  $\Phi_\lambda$  given by [\(3.5\)](#) spans the associated eigenspace, where we recall that  $\mu = \sqrt{\lambda - \alpha_1}$ .

We will see that the eigenvalues in  $\Lambda_\infty$  are algebraically simple. However, it may happen that some eigenvalues in  $\Lambda_0$  are not simple. Hence, we associate to each of them a Jordan chain of length  $n_\lambda \geq 1$ , denoted by  $\Phi_\lambda^0 = \Phi_\lambda, \Phi_\lambda^1, \dots, \Phi_\lambda^{n_\lambda-1}$ . By definition, this chain spans the range of the projector  $P_{\mathcal{A}^*, \lambda}$  and satisfies

$$(\mathcal{A}^* - \lambda)\Phi_\lambda^i = \Phi_\lambda^{i-1}, \quad \forall i \in \{1, \dots, n_\lambda - 1\}.$$

We gather all those Jordan chains in the finite family

$$\mathcal{F}_0 = \{ \Phi_\lambda^i, \lambda \in \Lambda_0, i \in \{0, \dots, n_\lambda - 1\} \}.$$

By definition this family spans the range of the spectral projector  $P_{\mathcal{A}^*, \Lambda_0}$ , corresponding the eigenvalues in  $\Lambda_0$ . Finally the eigenfunctions corresponding to the eigenvalues in  $\Lambda_\infty$  are gathered into the family

$$\mathcal{F}_\infty = \{ \Phi_\lambda, \lambda \in \Lambda_\infty \} = \bigcup_{k \geq k_1} \{ \Phi_{k,-}, \Phi_{k,+} \}.$$

In the next sections, our goal is to show that the family

$$(3.21) \quad \mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_\infty,$$

is complete in  $X$ .

**3.4.3. A known Riesz basis for  $X$ .** Let us consider the functions defined by

$$(3.22) \quad \Phi_{k,\pm}^0(x) := \begin{pmatrix} 1 \\ -\frac{\gamma_\pm}{\beta_r} \end{pmatrix} \cos(k\pi x), \quad \forall k \geq 0,$$

$$(3.23) \quad \Psi_k^0(x) := \begin{pmatrix} 0 \\ -\frac{1}{2\beta_r} \end{pmatrix} \cos(k\pi x), \quad \forall k \geq 0.$$

then we introduce the family

$$(3.24) \quad \mathcal{G} := \bigcup_{k \geq 0} \{\Phi_{k,+}^0, \Psi_k^0\}.$$

It is clear that there exists a constant matrix  $M$  such that

$$M\Phi_{k,+}^0(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(k\pi x), \text{ and } M\Psi_k^0(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(k\pi x),$$

which proves that the family  $\mathcal{G}$  given by (3.24)-(3.22)-(3.23) forms a Riesz basis of  $X$ .

We split  $\mathcal{G}$  into two parts as follows

$$\mathcal{G}_0 = \bigcup_{0 \leq k \leq k_1-1} \{\Phi_{k,+}^0, \Psi_k^0\}, \text{ and } \mathcal{G}_\infty = \bigcup_{k \geq k_1} \{\Phi_{k,+}^0, \Psi_k^0\}.$$

We shall use a result from [32, Corollary 11.4, Chapter II-§11] (see also [23, Lemma 6.2]), that we state here with slightly different notation.

**LEMMA 3.8.** *Let  $\{\phi_n^0\}_{n \geq 0}$  be a Riesz basis of the Hilbert space  $X$  and  $\{\phi_n\}_{n \geq N}$  (for  $N \geq 0$ ) be another sequence in  $X$  such that*

$$\sum_{n \geq N} \|\phi_n - \phi_n^0\|_H^2 < +\infty.$$

*Then, there exists an  $M \geq N$  such that the family  $\{\phi_n^0\}_{n=1}^M \cup \{\phi_n\}_{n \geq M+1}$  forms a Riesz basis of  $X$ .*

**3.4.4. Main result.** We are now ready to prove the main result of this section. It follows quite standard arguments as given in [15, Lemma 5, XIX.2.5 and Lemma 5, XIX.5.5] that we specialize to our problem.

**PROPOSITION 3.9.** *The family  $\mathcal{F}$  made of the root vectors of  $\mathcal{A}^*$  is complete in  $X$ .*

*Proof.* We first observe, by (3.18), that the family  $\mathcal{F}_\infty$  spans exactly the same space as

$$\tilde{\mathcal{F}}_\infty = \bigcup_{k \geq k_1} \{\Phi_{k,+}, \Psi_k\}.$$

Therefore we are reduced to show that the new family  $\tilde{\mathcal{F}} = \mathcal{F}_0 \cup \tilde{\mathcal{F}}_\infty$  is complete in  $X$ .

Thanks to (3.16), (3.17) and (3.19), we can use Lemma 3.8 to ensure that, choosing a larger value of  $k_1$  if necessary, the family

$$\tilde{\mathcal{G}} = \mathcal{G}_0 \cup \tilde{\mathcal{F}}_\infty,$$

forms a Riesz basis of  $X$ . We will now fix such a value of  $k_1$ .

Let us set  $\mathcal{H} = \overline{\text{Span}(\mathcal{F})} = \overline{\text{Span}(\tilde{\mathcal{F}})} \subset X$ . Our goal is to show that  $\mathcal{H} = X$ .

- Let us first show that  $\mathcal{H}^\perp$  is finite dimensional, and more precisely that  $\dim \mathcal{H}^\perp \leq 2k_1$ . Indeed if this is not the case, we can find a non trivial element  $\Phi$  in  $\mathcal{H}^\perp$  that satisfies the following additional orthogonality properties

$$(\Phi, \Psi) = 0, \quad \forall \Psi \in \mathcal{G}_0,$$

since the cardinal of  $\mathcal{G}_0$  is  $2k_1$ . Since  $\Phi$  is orthogonal to  $\mathcal{H}$ , that is to the elements of  $\tilde{\mathcal{F}}$ , we deduce that  $\Phi$  is in fact orthogonal to each element of the Riesz basis  $\tilde{\mathcal{G}}$ . This implies that  $\Phi = 0$  which is a contradiction.

- Let  $\mu \in \rho(\mathcal{A})$ . We observe that  $\mathcal{H}^\perp$  is stable by the resolvent  $(\mu \text{Id} - \mathcal{A})^{-1}$ . Indeed, if  $x \in \mathcal{H}^\perp$  and  $\Phi$  is any element in  $\mathcal{F}$ , we have

$$((\mu \text{Id} - \mathcal{A})^{-1}x, \Phi) = (x, (\bar{\mu} \text{Id} - \mathcal{A}^*)^{-1}\Phi) = 0,$$

since  $(\bar{\mu} \text{Id} - \mathcal{A}^*)^{-1}\Phi \in \mathcal{F} \subset \mathcal{H}$ .

- We prove now that  $\mathcal{H}^\perp = \{0\}$ . Indeed, if it is not the case, the operator  $(\mu \text{Id} - \mathcal{A})^{-1}$  has an eigenvector  $x$  in the non trivial finite dimensional space  $\mathcal{H}^\perp$ . It follows that  $x \in D(\mathcal{A})$  and that it is an eigenvector of  $\mathcal{A}$ . We write  $\mathcal{A}x = \lambda x$  for some  $\lambda \in \sigma(\mathcal{A}^*)$  and thus

$$x = P_{\mathcal{A}, \bar{\lambda}} x,$$

which leads to

$$\|x\|^2 = (P_{\mathcal{A}, \bar{\lambda}} x, x) = (x, P_{\mathcal{A}^*, \lambda} x) = 0,$$

since  $P_{\mathcal{A}^*, \lambda} x \in \mathcal{H}$  by definition of  $\mathcal{H}$  and  $x \perp \mathcal{H}$ . This is a contradiction.

We thus have proved that  $\mathcal{H} = X$ , which was our claim.  $\square$

REMARK 3.10. *Even though this fact will not be used in the sequel, it can be seen from the discussion above that the cardinal of  $\mathcal{F}_0$  is the same as the one of  $\mathcal{G}_0$ , that is  $2k_1$ .*

**4. Boundary controllability of the linearized systems.** This section is devoted to prove the boundary null-controllability of the linear systems associated to our nonlinear models.

**4.1. Approximate controllability.** Let us first discuss about the approximate controllability of the linearized systems in the space  $X$  at any time  $T > 0$ .

LEMMA 4.1. 1. *The system (1.10)–(1.14)–(1.15) is approximately controllable in  $X$  at any given time  $T > 0$  and the observation terms satisfy*

$$(4.1) \quad |\mathcal{B}_1^* \Phi_\lambda| = 1, \quad \forall \lambda \in \sigma(\mathcal{A}^*).$$

2. *We introduce the set*

$$(4.2) \quad \mathcal{R} := \left\{ (-1)^k \left(k + \frac{1}{2}\right) \pi \sqrt{\left(k + \frac{1}{2}\right)^2 \pi^2 + \theta} \sin \left( \sqrt{\left(k + \frac{1}{2}\right)^2 \pi^2 + \theta} \right); k \geq 0 \right\} \cap \mathbb{R}^*,$$

where we recall that  $\theta = \alpha_2 - \alpha_1$ .

Then, the system (1.10)–(1.14)–(1.16) is approximately controllable in  $X$  at any time  $T > 0$  if and only if  $\beta_* = \beta_\ell \beta_r \notin \mathcal{R}$ . In that case, there exists a constant  $c_0 > 0$  such that the observation terms satisfy

$$(4.3) \quad |\mathcal{B}_2^* \Phi_\lambda| \geq c_0, \quad \forall \lambda \in \sigma(\mathcal{A}^*).$$

*Proof.* Recall the observation operators  $\mathcal{B}_1^*$ ,  $\mathcal{B}_2^*$  respectively defined by (2.7a), (2.7b) and the explicit expressions of the eigenfunctions  $\Phi_\lambda$  from (3.5).

1. It is easy to see that

$$\mathcal{B}_1^* \Phi_\lambda = 1, \quad \forall \lambda \in \sigma(\mathcal{A}^*).$$

So, by using Fattorini-Hautus test (see [17], [30]), the linear control system (1.10)–(1.14)–(1.15) is approximately controllable in  $X$  at any time  $T > 0$ .

2. Let us compute from (3.5) that

$$\mathcal{B}_2^* \Phi_\lambda = -\frac{\mu \sin \mu}{\beta_r} \cos \left( \sqrt{\mu^2 - \theta} \right), \quad \forall \lambda \in \sigma(\mathcal{A}^*),$$

where, by construction  $\lambda = \mu^2 + \alpha_1$ .

Assume that  $\mathcal{B}_2^* \Phi_\lambda = 0$  for some eigenvalue  $\lambda$ . Since  $\mu \sin \mu \neq 0$  (if so, then the eigenvalue equation (3.4) is invalid), this yields  $\cos \left( \sqrt{\mu^2 - \theta} \right) = 0$  and thus  $\mu^2 = \left(k + \frac{1}{2}\right)^2 \pi^2 + \theta$ , for some  $k \geq 0$ .

But we remember that  $\mu$  also satisfies the equation (3.4). This is possible only when

$$\frac{1}{\beta_*} \left(k + \frac{1}{2}\right) \pi \sqrt{\left(k + \frac{1}{2}\right)^2 \pi^2 - \theta} \sin \left( \left(k + \frac{1}{2}\right) \pi \right) \sin \left( \sqrt{\left(k + \frac{1}{2}\right)^2 \pi^2 - \theta} \right) = 1.$$

Such a  $k$  exists if and only if  $\beta_* \in \mathcal{R}$ . Hence, the Fattorini-Hautus criterion confirms the approximate controllability of the system (1.10)–(1.14)–(1.16), if and only if  $\beta_* \notin \mathcal{R}$ .

It remains to show (4.3), under that condition. Since  $\mathcal{B}^*\Phi_\lambda \neq 0$  for any  $\lambda \in \sigma(\mathcal{A}^*)$  and since  $\sigma(\mathcal{A}^*) = \Lambda_0 \cup \Lambda_\infty$  where  $\Lambda_0$  is finite, we only need to show that

$$\liminf_{k \rightarrow \infty} |\mathcal{B}^*\Phi_{k,\pm}| > 0.$$

This property is clear from (3.16), because  $\gamma_+$  and  $\gamma_-$  are not zero.  $\square$

#### 4.2. Bounds on the (generalized) eigenfunctions.

We have the following result.

LEMMA 4.2. *There exists some constant  $C > 0$  depending on the parameters  $\alpha_1, \alpha_2, \beta_\ell, \beta_r$  such that*

$$(4.4) \quad \|\Phi\|_X \leq C, \quad \forall \Phi \in \mathcal{F},$$

where  $\mathcal{F}$  is the set of eigenfunctions and generalized eigenfunctions of  $\mathcal{A}^*$  introduced in (3.21).

*Proof.* Using (3.21) and the fact that  $\mathcal{F}_0$  is finite, we see that we just need to show that

$$\sup_{k \geq k_1} \|\Phi_{k,\pm}\|_X < +\infty,$$

but this is a straightforward consequence of Corollary 3.7.  $\square$

From now on we will assume that the generalized eigenfunctions satisfy the property

$$(4.5) \quad \mathcal{B}_1^*\Phi_\lambda^j = 0, \quad \forall \lambda \in \Lambda_0, \forall j \in \{1, \dots, n_\lambda - 1\},$$

in the case where the control is given by (1.15), or the property

$$(4.6) \quad \mathcal{B}_2^*\Phi_\lambda^j = 0, \quad \forall \lambda \in \Lambda_0, \forall j \in \{1, \dots, n_\lambda - 1\},$$

in the case (1.16), when  $\beta_* \notin \mathcal{R}$ . This is always possible by using Lemma 4.1 and the fact that one can add any multiple of the eigenfunction to each generalized eigenfunction.

We recall that the eigenvalues in  $\Lambda_\infty$  are all algebraically simple. Thus, to simplify the presentation in the sequel, we will conventionally set the multiplicity  $n_\lambda = 1$  for every  $\lambda \in \Lambda_\infty$ . We will finally set

$$n_{\max} = \sup_{\lambda \in \sigma(\mathcal{A}^*)} n_\lambda,$$

which is clearly finite.

**4.3. The moments problem.** Since the family  $\mathcal{F}$  is complete in  $X$  (Proposition 3.9), it is enough to check the equations (2.8) and (2.9) for all  $\zeta \in \mathcal{F}$ .

This gives us the following characterization of the controls.

- For any  $y_0 \in X$  and  $T > 0$ , a function  $v \in L^2(0, T)$  is a boundary null-control for the system (1.10)–(1.14)–(1.15) if and only if, we have

$$(4.7) \quad \frac{e^{-T\lambda}}{\mathcal{B}_1^*\Phi_\lambda} \left( y_0, \sum_{k=0}^j \frac{(-T)^k}{k!} \Phi_\lambda^{j-k} \right)_X = \int_0^T v(t) \frac{(-(T-t))^k}{k!} e^{-(T-t)\lambda} dt,$$

$$\forall \lambda \in \sigma(\mathcal{A}^*), \forall j \in \{0, \dots, n_\lambda - 1\}.$$

- Similarly, assuming that  $\beta_* \notin \mathcal{R}$ , for any  $y_0 \in X$  and  $T > 0$  a function  $v \in L^2(0, T)$  is a boundary null-control for the system (1.10)–(1.14)–(1.16) if and only if, we have

$$(4.8) \quad \frac{e^{-T\lambda}}{\mathcal{B}_2^*\Phi_\lambda} \left( y_0, \sum_{k=0}^j \frac{(-T)^k}{k!} \Phi_\lambda^{j-k} \right)_X = \int_0^T v(t) \frac{(-(T-t))^k}{k!} e^{-(T-t)\lambda} dt,$$

$$\forall \lambda \in \sigma(\mathcal{A}^*), \forall j \in \{0, \dots, n_\lambda - 1\}.$$

Here, we have used (4.5), (4.6) and the fact that

$$e^{-t\mathcal{A}^*} \Phi_\lambda^j = e^{-\lambda t} \sum_{k=0}^j \frac{(-t)^k}{k!} \Phi_\lambda^{j-k}, \quad \forall \lambda \in \sigma(\mathcal{A}^*), \forall j \in \{0, \dots, n_\lambda - 1\}.$$

The above set of equations are the moment problems for our linear models and we shall solve these in the next paragraphs.

**4.4. Solving the moment problem.** Let us show how to solve the moment problems introduced before. The resolution of the moment problem relies on the existence and estimates of biorthogonal families to exponential functions in  $L^2(0, T)$ . This topic is a long story starting from the pioneering work [18] in the framework of control theory, up to recent developments where sharper estimates were obtained under a quite general set of hypothesis. More precisely, in this paper, we will use the results given in [11, Theorem V.4.16 and Corollary V.4.25], that are quite similar to [8, Theorem 1.5] or [22, Theorem 1.2] but with a simplified set of assumptions.

In order to use those theorems, we need to show that the set of eigenvalues  $\sigma(\mathcal{A}^*)$ , defined by (3.20), belongs to some sector of the complex half-plane, satisfies a uniform gap property and some asymptotic conditions on the counting function.

**REMARK 4.3.** *Without loss of generality, we assume that all the eigenvalues of  $\mathcal{A}^*$  have positive real parts. If not, one could choose some  $\alpha_0 > 0$  such that  $\lambda + \alpha_0 > 0$  for any  $\lambda \in \sigma(\mathcal{A}^*)$  and then consider the new unknown function  $\tilde{y} = e^{-\alpha_0 t} y$  that satisfies a similar problem with  $\mathcal{A}^*$  replaced by  $\mathcal{A}^* + \alpha_0$ . That will just change the estimate of the cost of the control by a factor  $e^{T\alpha_0}$ , without consequence on our analysis.*

- **The sector condition.**

For any  $\eta > 0$ , we define the sector

$$\mathcal{S}_\eta := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0, \text{ and } |\operatorname{Im} z| < (\sinh \eta) \operatorname{Re} z\}.$$

Thanks to Corollary 3.7, we know that all the eigenvalues of  $\mathcal{A}^*$  have an imaginary part which is bounded; thanks to Remark 4.3, it is thus clear that there exists some  $\eta > 0$  such that

$$(4.9) \quad \sigma(\mathcal{A}^*) \subset \mathcal{S}_\eta.$$

- **Asymptotics of the counting function.**

Let  $\mathcal{N}$  be the counting function associated with the set of eigenvalues  $\sigma(\mathcal{A}^*)$ , defined by

$$\mathcal{N}(r) := \#\{\lambda \in \sigma(\mathcal{A}^*), \text{ s.t. } |\lambda| \leq r\}, \quad \forall r > 0.$$

Our goal is to show that there exists some  $\kappa_0 > 0$  independent in the set of eigenvalues such that

$$(4.10a) \quad \mathcal{N}(r) \leq \kappa_0 r^{1/2}, \quad \forall r > 0,$$

$$(4.10b) \quad |\mathcal{N}(r) - \mathcal{N}(s)| \leq \kappa_0 \left(1 + |r - s|^{1/2}\right), \quad \forall r, s > 0.$$

First of all, we observe from Section 3.3.3, that we have

$$\sigma(\mathcal{A}^*) = \Lambda_0 \cup \Lambda_{\infty,+} \cup \Lambda_{\infty,-},$$

with  $\Lambda_{\infty,\pm} = \{\lambda_{k,\pm}, k \geq k_1\}$ .

The result will come if we are able to prove that the counting function of each of the three sets  $\Lambda_0$ ,  $\Lambda_{\infty,+}$  and  $\Lambda_{\infty,-}$  satisfy (4.10), as shown for instance in the proof of [11, Lemma V.4.20]. Those estimates are straightforward for  $\Lambda_0$  which is a finite set that does not contain 0 by Remark 4.3. Let us show them for  $\Lambda_{\infty,+}$  since the reasoning is similar for  $\Lambda_{\infty,-}$ . We denote by  $\mathcal{N}_+$  the corresponding counting function.

– Let  $r > 0$  be fixed. We set  $n = \mathcal{N}_+(r)$ . By definition, we have

$$|\lambda_{k_1+n-1,+}| \leq r,$$

and thus,

$$|\mu_{k_1+n-1,+}| \leq \sqrt{r + |\alpha_1|}.$$

Since  $\mu_{k_1+n-1,+}$  belongs to the strip  $R_{k_1+n-1}$  we deduce

$$(n + k_1 - 3/2)\pi \leq \sqrt{r + |\alpha_1|},$$

which gives

$$n \leq C(1 + \sqrt{r}),$$

for some  $C > 0$  depending only on  $\alpha_1$  and  $k_1$ . This gives (4.10a) for  $\mathcal{N}_+$  since this function vanishes for  $r$  small enough.

- Let  $0 < s < r$  be given. We assume that  $s > \inf \mathcal{N}_+$  since if it is not the case, we have  $\mathcal{N}_+(s) = 0$  and thus (4.10b) follows from (4.10a). We set  $n = \mathcal{N}_+(r)$  and  $m = \mathcal{N}_+(s)$ . If  $m = n$  there is nothing to prove, so we will assume that  $n > m$ . By definition of the counting function we have

$$|\lambda_{k_1+n-1,+}| \leq r, \text{ and } |\lambda_{k_1+m,+}| > s.$$

By using that  $\mu_{k,+}^2 = \lambda_{k,+} - \alpha_1$  and the fact that the square root function is sublinear, we get

$$|\mu_{k_1+n-1,+}| \leq \sqrt{r} + \sqrt{|\alpha_1|}, \text{ and } |\mu_{k_1+m,+}| \geq \sqrt{s} - \sqrt{|\alpha_1|}.$$

Moreover, by Theorem 3.6, we know that  $|\operatorname{Im} \mu_{k,+}| \leq \vartheta_0$  for every  $k \geq k_1$ . It comes

$$\operatorname{Re} \mu_{k_1+n-1,+} \leq \sqrt{r} + \sqrt{|\alpha_1|} + \vartheta_0, \text{ and } \operatorname{Re} \mu_{k_1+m,+} \geq \sqrt{s} - \sqrt{|\alpha_1|} - \vartheta_0,$$

which, in turn, leads to

$$(4.11) \quad \operatorname{Re} \mu_{k_1+n-1,+} - \operatorname{Re} \mu_{k_1+m,+} \leq \sqrt{r} - \sqrt{s} + 2\sqrt{|\alpha_1|} + 2\vartheta_0 \leq \sqrt{r-s} + 2\sqrt{|\alpha_1|} + 2\vartheta_0.$$

Moreover, by the asymptotics (3.10), we know that there exists  $c > 0$  such that for any  $k \geq k_1$  we have  $\operatorname{Re} \mu_{k+1,+} - \operatorname{Re} \mu_{k,+} \geq c$ , and thus

$$(4.12) \quad \operatorname{Re} \mu_{k_1+n-1,+} - \operatorname{Re} \mu_{k_1+m,+} \geq (n-m-1)c.$$

Combining (4.11) and (4.12) proves the claim.

- **The (weak) gap condition.**

Using the asymptotic expressions given in Corollary 3.7 for the eigenvalues in  $\Lambda_\infty$ , one has the following: there exists some  $\rho_1 > 0$  depending only on  $\alpha_1, \alpha_2, \beta_\ell$  and  $\beta_r$ , such that

$$|\lambda_{k,\pm} - \lambda_{k+1,\pm}| \geq \rho_1 k \quad \text{for } k \geq k_1,$$

- In the case  $\theta^2 + 16\beta_* \neq 0$ , we deduce still from Corollary 3.7 that, possibly changing the value of  $\rho_1$ , we have

$$|\lambda_{k,+} - \lambda_{k,-}| \geq \rho_1, \quad \text{for } k \geq k_1.$$

Since  $\Lambda_0$  is finite, we can finally conclude that there is some  $\rho > 0$  such that

$$(4.13) \quad |\lambda - \tilde{\lambda}| \geq \rho, \quad \lambda, \tilde{\lambda} \in \sigma(\mathcal{A}^*) \text{ with } \lambda \neq \tilde{\lambda},$$

which is the uniform spectral gap property for our operator.

- In the case  $\theta^2 + 16\beta_* = 0$ , the situation is quite different since, we know from Corollary 3.7 that

$$|\lambda_{k,+} - \lambda_{k,-}| \sim \frac{C}{k} \sim \frac{C_2}{\sqrt{\operatorname{Re} \lambda_{k,+}}},$$

so that the uniform gap property (4.13) does not hold anymore. However, we see that the following weak gap condition holds (see [11, Definition V.4.18])

$$(4.14) \quad \#(\sigma(\mathcal{A}^*) \cap B(\mu, \rho/2)) \leq 2, \quad \forall \mu \in \mathbb{C},$$

for some  $\rho > 0$ . And more precisely, the only case where two eigenvalues of  $\mathcal{A}^*$  can be close one from each other is the case where  $\lambda_{k,+}$  is close from  $\lambda_{k,-}$ , indeed we can choose  $\rho > 0$  small enough so that, if  $\mu \in \mathbb{C}$  is such that  $\#(\sigma(\mathcal{A}^*) \cap B(\mu, \rho/2)) = 2$  then there exists  $k \geq k_1$  such that

$$\sigma(\mathcal{A}^*) \cap B(\mu, \rho/2) = \{\lambda_{k,+}, \lambda_{k,-}\}.$$

- As a conclusion, in both cases we have the weak gap condition (4.14) as well as the estimate

$$(4.15) \quad |\lambda - \tilde{\lambda}| \geq \frac{C}{\sqrt{\operatorname{Re} \lambda}}, \quad \forall \lambda, \tilde{\lambda} \in \sigma(\mathcal{A}^*), \text{ s.t. } 0 < |\lambda - \tilde{\lambda}| \leq \rho.$$

Moreover, if  $|\lambda - \tilde{\lambda}| \leq \rho$ , we know that  $\lambda, \tilde{\lambda} \in \Lambda_\infty$  and in particular, those are simple eigenvalues of the operator, see Section 3.3.3.

The four assumptions (4.9), (4.10), (4.14) and (4.15) being satisfied we know, according for instance to [11, Theorem V.4.26 and Corollary V.4.27] (in the case  $\theta^2 + 16\beta_* \neq 0$ , [11, Theorem V.4.16] can also be used), that there exists a family  $\{q_{\lambda,T}^j\}_{\substack{\lambda \in \sigma(\mathcal{A}^*) \\ 0 \leq j < n_{\max}}} \subset L^2(0, T)$  such that

$$(4.16) \quad \int_0^T q_{\lambda,T}^j(t) \frac{(t-T)^k}{k!} e^{-(T-t)\mu} dt = \delta_{\lambda,\mu} \delta_{j,k}, \quad \forall \lambda, \mu \in \sigma(\mathcal{A}^*), \forall j, k \in \{0, \dots, n_{\max} - 1\},$$

with the following estimate

$$(4.17) \quad \|q_{\lambda,T}^j\|_{L^2(0,T)} \leq C \sqrt{\operatorname{Re} \lambda} e^{C\sqrt{\operatorname{Re} \lambda} + \frac{C}{T}}, \quad \forall \lambda \in \sigma(\mathcal{A}^*), \forall j \in \{0, \dots, n_{\max} - 1\}$$

where the constant  $C > 0$  depends on  $\eta, \rho, \kappa_0$  but not on  $\lambda$  and  $T$ . Note that the prefactor  $\sqrt{\operatorname{Re} \lambda}$  comes from (4.15) but it can immediately be integrated in the exponential term since  $\sqrt{\operatorname{Re} \lambda} \leq e^{\sqrt{\operatorname{Re} \lambda}}$ .

We can now solve the set of moments equations (4.7) and (4.8) to construct boundary controls for our linear systems.

**Proof of Theorem 1.3.**

We will first consider the control problem (1.10)–(1.14)–(1.15).

We suppose given any  $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ ,  $\beta_* = \beta_\ell \beta_r \neq 0$ . For any initial data  $y_0 \in X$  we consider

$$v(t) = \sum_{\lambda \in \sigma(\mathcal{A}^*)} \sum_{j=0}^{n_\lambda-1} v_{\lambda,j}(t), \quad \forall t \in [0, T],$$

with

$$v_{\lambda,j}(t) = \frac{e^{-T\lambda}}{\mathcal{B}_1^* \Phi_\lambda} \left( y_0, \sum_{k=0}^j \frac{(-T)^k}{k!} \Phi_\lambda^{j-k} \right)_X q_{\lambda,T}^j(t), \quad \forall t \in [0, T], \text{ and } \lambda \in \sigma(\mathcal{A}^*).$$

Observe that, thanks to (4.16), this choice of  $v$  formally satisfies the moments equation (4.7).

Using the bi-orthogonal estimate (4.17), lower bounds of the observation terms (4.1) and the upper bounds (4.4), we get

$$(4.19) \quad \|v_{\lambda,j}\|_{L^2(0,T)} \leq C e^{C/T} e^{-T(\operatorname{Re} \lambda) + C\sqrt{\operatorname{Re} \lambda}} \|y_0\|_X, \quad \forall \lambda \in \sigma(\mathcal{A}^*), \forall j \in \{0, \dots, n_\lambda - 1\}.$$

Applying Young's inequality, we get

$$C\sqrt{\operatorname{Re}(\lambda)} \leq \frac{T}{2} \operatorname{Re}(\lambda) + \frac{C^2}{T}, \quad \forall \lambda \in \sigma(\mathcal{A}^*),$$

that we use in (4.19). Taking sum over  $\lambda$  and  $j$ , we obtain that the series defining  $v$  is convergent as well as the required estimate □

$$\|v\|_{L^2(0,T)} = \sum_{\lambda \in \sigma(\mathcal{A}^*)} \sum_{j=0}^{n_\lambda-1} \|v_{\lambda,j}\|_{L^2(0,T)} \leq M e^{M/T} \|y_0\|_X,$$

here we have used the asymptotics of the eigenvalues given by Corollary 3.7.

The reasoning is exactly the same for the control problem (1.10)–(1.14)–(1.16) as soon as we have the additional assumption  $\beta_* \notin \mathcal{R}$ .

**5. Boundary controllability of the the nonlinear system.** In the previous section, we obtained that the control cost for the linear systems (1.10)–(1.14)–(1.15)/(1.16) is  $M e^{M/T}$ . Using this we shall deal with the null-controllability of the nonlinear systems (1.10)–(1.11)–(1.12)/(1.13) in the current section.

We introduce the new nonlinear functions

$$(5.1) \quad g_\ell(s) := \tilde{f}_\ell(s) - \beta_\ell s = f_\ell(z_2^*(0) + s) - f_r(z_2^*(0)) - \beta_\ell s,$$

$$(5.2) \quad g_r(s) := \tilde{f}_r(s) - \beta_r s = f_r(z_1^*(1) + s) - f_r(z_1^*(1)) - \beta_r s,$$

in such a way that, by definition and by (1.4), we have

$$(5.3) \quad \begin{cases} g_\ell(0) = g'_\ell(0) = 0, \text{ and } |g_\ell(s_1) - g_\ell(s_2)| \leq \|f_\ell\|_{W^{2,\infty}} |s_1 - s_2| (|s_1| + |s_2|), \forall s_1, s_2 \in \mathbb{R}, \\ g_r(0) = g'_r(0) = 0, \text{ and } |g_r(s_1) - g_r(s_2)| \leq \|f_r\|_{W^{2,\infty}} |s_1 - s_2| (|s_1| + |s_2|), \forall s_1, s_2 \in \mathbb{R}, \end{cases}$$

Therefore, we see that the nonlinear boundary conditions in (1.11), (1.12) and (1.13) can be written, respectively, as follows

$$\begin{aligned} \partial_x y_2(\cdot, 1) - \beta_r y_1(\cdot, 1) &= g_r(y_1(\cdot, 1)), \\ \partial_x y_1(\cdot, 0) + \beta_\ell y_2(\cdot, 0) &= v - g_\ell(y_2(\cdot, 0)), \\ \partial_x y_1(\cdot, 0) + \beta_\ell y_2(\cdot, 0) &= -g_\ell(y_2(\cdot, 0)), \end{aligned}$$

where the terms in  $g_\ell$  and  $g_r$  are quadratic and can then be seen as source terms in the control problem that can be dealt with using the method developed in [29].

**5.1. Source term method.** Let us apply the source term method (see [29]) to our case. We suppose given two constants  $p > 0$ ,  $q > 1$  in such a way that

$$(5.4) \quad 1 < q < \sqrt{2}, \quad \text{and } p > \frac{q^2}{2 - q^2}.$$

We recall that  $M > 0$  denotes the constant appearing in the control estimate (1.17) for the linearized problem. We now define the functions

$$\begin{cases} \rho_0(t) = e^{-\frac{pM}{(q-1)(T-t)}}, \\ \rho_S(t) = e^{-\frac{(1+p)q^2 M}{(q-1)(T-t)}}, \end{cases} \quad \forall t \in \left[ T \left( 1 - \frac{1}{q^2} \right), T \right],$$

extended in  $[0, T(1 - 1/q^2)]$  in a constant way such that the functions  $\rho_0$  and  $\rho_S$  are continuous and non-increasing in  $[0, T]$  with  $\rho_0(T) = \rho_S(T) = 0$ .

REMARK 5.1. *We compute that*

$$\frac{\rho_0^2(t)}{\rho_S(t)} = e^{\frac{q^2 M + pM(q^2 - 2)}{(q-1)(T-t)}}, \quad \forall t \in \left[ T \left( 1 - \frac{1}{q^2} \right), T \right].$$

Due to the choices of  $p, q$  in (5.4), we have  $q^2 + p(q^2 - 2) < 0$ ,  $(q-1) > 0$  and therefore we can conclude that

$$\frac{\rho_0^2(t)}{\rho_S(t)} \leq 1, \quad \forall t \in [0, T].$$

With these functions, we define the following weighted spaces (recall that  $X = (L^2(0, 1))^2$ ),

$$(5.5a) \quad \mathcal{S} := \left\{ S \in L^2(0, T) \mid \frac{S}{\rho_S} \in L^2(0, T) \right\},$$

$$(5.5b) \quad \mathcal{Y} := \left\{ y = (y_1, y_2) \in L^2(0, T; X) \mid \frac{y}{\rho_0} \in L^2(0, T; X) \right\},$$

$$(5.5c) \quad \mathcal{V} := \left\{ v \in L^2(0, T) \mid \frac{v}{\rho_0} \in L^2(0, T) \right\}.$$

We introduce the inner products in the spaces  $\mathcal{S}$  and  $\mathcal{V}$  respectively by

$$\langle S, \tilde{S} \rangle_{\mathcal{S}} := \int_0^T \rho_S^{-2} S(t) \tilde{S}(t) dt \quad \text{and} \quad \langle v, \tilde{v} \rangle_{\mathcal{V}} := \int_0^T \rho_0^{-2} v(t) \tilde{v}(t) dt,$$

for any  $S, \tilde{S} \in \mathcal{S}$  and  $v, \tilde{v} \in \mathcal{V}$ . The corresponding norms in those spaces are

$$(5.6) \quad \|S\|_{\mathcal{S}} := \left( \int_0^T \left| \frac{S(t)}{\rho_S(t)} \right|^2 dt \right)^{1/2}, \quad \text{and} \quad \|v\|_{\mathcal{V}} := \left( \int_0^T \left| \frac{v(t)}{\rho_0(t)} \right|^2 dt \right)^{1/2}.$$



We consider the following system

$$(5.7) \quad \begin{cases} \partial_t y_1 - \partial_x^2 y_1 + \alpha_1 y_1 = 0 & \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x^2 y_2 + \alpha_2 y_2 = 0 & \text{in } (0, T) \times (0, 1), \\ y_1(0, \cdot) = y_{0,1} & \text{in } (0, 1), \\ y_2(0, \cdot) = y_{0,2} & \text{in } (0, 1), \end{cases}$$

along with the boundary conditions at  $x = 1$  given by

$$(5.8) \quad \begin{cases} \partial_x y_1(\cdot, 1) = 0 & \text{in } (0, T), \\ \partial_x y_2(\cdot, 1) - \beta_r y_1(\cdot, 1) = S_r & \text{in } (0, T), \end{cases}$$

and at  $x = 0$ , we consider either

$$(5.9) \quad \begin{cases} \partial_x y_1(\cdot, 0) + \beta_\ell y_2(\cdot, 0) = v + S_\ell & \text{in } (0, T), \\ \partial_x y_2(\cdot, 0) = 0 & \text{in } (0, T), \end{cases}$$

or,

$$(5.10) \quad \begin{cases} \partial_x y_1(\cdot, 0) + \beta_\ell y_2(\cdot, 0) = S_\ell & \text{in } (0, T), \\ \partial_x y_2(\cdot, 0) = v & \text{in } (0, T), \end{cases}$$

for any given  $y_0 \in X$ ,  $S_\ell, S_r \in L^2(0, T)$  and  $v \in L^2(0, T)$ .

Our goal is to prove the following result using the technique developed in [29].

**PROPOSITION 5.2.** *We suppose given the parameters  $\alpha_1, \alpha_2, \beta_\ell$  and  $\beta_r$  as well as the time  $T > 0$ .*

1. *There exists a linear map*

$$(y_0, S_\ell, S_r) \in X \times L^2(0, T) \times L^2(0, T) \longmapsto (y, v) \in \mathcal{Y} \times \mathcal{V},$$

*such that  $(y, v)$  solves the set of equations (5.7)–(5.8)–(5.9).*

2. *If  $\beta_\ell \beta_r \notin \mathcal{R}$ , as defined in (4.2), then there exists a linear map*

$$(y_0, S_\ell, S_r) \in X \times L^2(0, T) \times L^2(0, T) \longmapsto (y, v) \in \mathcal{Y} \times \mathcal{V},$$

*such that  $(y, v)$  solves the set of equations (5.7)–(5.8)–(5.10)*

*In both cases, we have the following estimate*

$$(5.11) \quad \left\| \frac{y}{\rho_0} \right\|_{C^0(0, T; X)} + \left\| \frac{y}{\rho_0} \right\|_{L^2(0, T; (H^1(0, 1))^2)} + \left\| \frac{v}{\rho_0} \right\|_{L^2(0, T)} \leq C e^{C(T + \frac{1}{T})} (\|y_0\|_X + \|S_\ell\|_S + \|S_r\|_S),$$

*where the constant  $C > 0$  neither depends on  $y_0$  nor on  $T$ .*

*Proof.* Let us define the sequence  $\{T_k\}_{k \geq 0}$  with

$$T_k := T - \frac{T}{q^k}, \quad \forall k \geq 0,$$

for given  $T > 0$ . With this  $T_k$ , we have the following relation between  $\rho_0$  and  $\rho_S$

$$(5.12) \quad \rho_0(T_{k+2}) = \rho_S(T_k) e^{\frac{M}{T_{k+2} - T_{k+1}}}, \quad \forall k \geq 0.$$

We also define a sequence  $\{a_k\}_{k \geq 0}$  with

$$(5.13) \quad a_0 = y_0 \in X, \quad a_{k+1} = \tilde{y}(T_{k+1}^-), \quad \forall k \geq 0,$$

where  $\tilde{y} := (\tilde{y}_1, \tilde{y}_2) \in \mathcal{C}^0([T_k, T_{k+1}]; X) \cap L^2(T_k, T_{k+1}; (H^1(0, 1))^2)$  is the unique weak solution to the following system for every  $k \geq 0$ ,

$$(5.14) \quad \begin{cases} \partial_t \tilde{y}_1 - \partial_x^2 \tilde{y}_1 + \alpha_1 \tilde{y}_1 = 0 & \text{in } (T_k, T_{k+1}) \times (0, 1), \\ \partial_t \tilde{y}_2 - \partial_x^2 \tilde{y}_2 + \alpha_2 \tilde{y}_2 = 0 & \text{in } (T_k, T_{k+1}) \times (0, 1), \\ \partial_x \tilde{y}_1(\cdot, 0) + \beta_\ell \tilde{y}_2(\cdot, 0) = S_\ell & \text{in } (T_k, T_{k+1}), \\ \partial_x \tilde{y}_1(\cdot, 0) = 0 & \text{in } (T_k, T_{k+1}), \\ \partial_x \tilde{y}_1(\cdot, 1) = 0 & \text{in } (T_k, T_{k+1}), \\ \partial_x \tilde{y}_2(\cdot, 1) - \beta_r \tilde{y}_1(\cdot, 1) = S_r & \text{in } (T_k, T_{k+1}), \\ \tilde{y}_1(T_k^+, \cdot) = 0 & \text{in } (0, 1), \\ \tilde{y}_2(T_k^+, \cdot) = 0 & \text{in } (0, 1). \end{cases}$$

Moreover, using [Theorem 2.3](#), we have

$$\|\tilde{y}\|_{\mathcal{C}^0([T_k, T_{k+1}]; X)} + \|\tilde{y}\|_{L^2(T_k, T_{k+1}; (H^1(0, 1))^2)} \leq C e^{CT} (\|S_\ell\|_{L^2(T_k, T_{k+1})} + \|S_r\|_{L^2(T_k, T_{k+1})}).$$

In particular,

$$(5.15) \quad \|a_{k+1}\|_X \leq C e^{CT} (\|S_\ell\|_{L^2(T_k, T_{k+1})} + \|S_r\|_{L^2(T_k, T_{k+1})}), \quad \forall k \geq 0,$$

recall the definition [\(5.13\)](#).

*Weighted estimate of the control.* For every  $k \geq 0$ , we consider the following system (introduce  $\hat{y} := (\hat{y}_1, \hat{y}_2)$ )

$$(5.16) \quad \begin{cases} \partial_t \hat{y}_1 - \partial_x^2 \hat{y}_1 + \alpha_1 \hat{y}_1 = 0 & \text{in } (T_k, T_{k+1}) \times (0, 1), \\ \partial_t \hat{y}_2 - \partial_x^2 \hat{y}_2 + \alpha_2 \hat{y}_2 = 0 & \text{in } (T_k, T_{k+1}) \times (0, 1), \\ \partial_x \hat{y}_1(\cdot, 1) = 0 & \text{in } (T_k, T_{k+1}), \\ \partial_x \hat{y}_2(\cdot, 1) - \beta_r \hat{y}_1(\cdot, 1) = 0 & \text{in } (T_k, T_{k+1}), \\ (\hat{y}_1(T_k^+, \cdot), \hat{y}_2(T_k^+, \cdot)) = a_k & \text{in } (0, 1), \end{cases}$$

with a control  $v_k$  acting through either

$$(5.17) \quad \begin{cases} \partial_x \hat{y}_1(\cdot, 0) + \beta_\ell \hat{y}_2(\cdot, 0) = v_k & \text{in } (T_k, T_{k+1}), \\ \partial_x \hat{y}_2(\cdot, 0) = 0 & \text{in } (T_k, T_{k+1}), \end{cases}$$

or, we consider

$$(5.18) \quad \begin{cases} \partial_x \hat{y}_1(\cdot, 0) + \beta_\ell \hat{y}_2(\cdot, 0) = 0 & \text{in } (T_k, T_{k+1}), \\ \partial_x \hat{y}_2(\cdot, 0) = v_k & \text{in } (T_k, T_{k+1}). \end{cases}$$

- In the first case [\(5.16\)](#)–[\(5.17\)](#), we have the existence of a null-control  $v_k \in L^2(T_k, T_{k+1})$  for any given set of parameters  $(\alpha_1, \alpha_2) \in \mathbb{R}^2$  and  $\beta_\ell, \beta_r \neq 0$  by [Theorem 1.3–Item 1](#).
- In the second case [\(5.16\)](#)–[\(5.18\)](#), for any given set of parameters  $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ , if  $\beta_\ell \beta_r \notin \mathcal{R}$ , then there exists a null-control  $v_k \in L^2(T_k, T_{k+1})$  for the concerned system as per [Theorem 1.3–Item 2](#).

In both cases, we have the following estimate

$$(5.19) \quad \|v_k\|_{L^2(T_k, T_{k+1})} \leq M e^{\frac{M}{T_{k+1}-T_k}} \|a_k\|_X, \quad \forall k \geq 0,$$

and the associated solutions satisfy

$$(5.20) \quad \hat{y}(T_{k+1}^-, \cdot) = 0, \quad \forall k \geq 0.$$

Now, combining [\(5.15\)](#) and [\(5.19\)](#), we have for every  $k \geq 0$

$$\begin{aligned} \|v_{k+1}\|_{L^2(T_{k+1}, T_{k+2})} &\leq M e^{\frac{M}{T_{k+2}-T_{k+1}}} \|a_{k+1}\|_X \\ &\leq C e^{CT} M e^{\frac{M}{T_{k+2}-T_{k+1}}} \rho_S(T_k) \left( \left\| \frac{S_\ell}{\rho_S} \right\|_{L^2(T_k, T_{k+1})} + \left\| \frac{S_r}{\rho_S} \right\|_{L^2(T_k, T_{k+1})} \right), \end{aligned}$$

since  $\rho_S$  is an non-increasing function in  $(T_k, T_{k+1})$ .

Using the relation (5.12), we obtain for every  $k \geq 0$

$$\|v_{k+1}\|_{L^2(T_{k+1}, T_{k+2})} \leq C e^{CT} \rho_0(T_{k+2}) \left( \left\| \frac{S_\ell}{\rho_S} \right\|_{L^2(T_k, T_{k+1})} + \left\| \frac{S_r}{\rho_S} \right\|_{L^2(T_k, T_{k+1})} \right),$$

for some  $C > 0$ . But the function  $\rho_0$  being non-increasing, we deduce that

$$(5.21) \quad \left\| \frac{v_{k+1}}{\rho_0} \right\|_{L^2(T_{k+1}, T_{k+2})} \leq C e^{CT} \left( \left\| \frac{S_\ell}{\rho_S} \right\|_{L^2(T_k, T_{k+1})} + \left\| \frac{S_r}{\rho_S} \right\|_{L^2(T_k, T_{k+1})} \right), \quad \forall k \geq 0.$$

Let us define

$$v := \sum_{k \geq 0} v_k \mathbf{1}_{(T_k, T_{k+1})}.$$

From the estimate (5.19), we have (since  $\rho_0(T_1) = e^{-\frac{pM}{T} \frac{q}{(q-1)}}$ )

$$(5.22) \quad \begin{aligned} \|v_0\|_{L^2(0, T_1)} &\leq M e^{\frac{M}{T_1}} \|a_0\|_X \\ &= M e^{\frac{1}{T} \frac{q(1+p)M}{(q-1)}} \rho_0(T_1) \|y_0\|_X \\ &\leq C e^{\frac{C}{T}} \rho_0(T_1) \|y_0\|_X, \end{aligned}$$

where  $C = \frac{M(1+p)q}{(q-1)}$ . Since  $\rho_0$  is non-increasing function in  $(0, T_1)$  we get

$$(5.23) \quad \left\| \frac{v_0}{\rho_0} \right\|_{L^2(0, T_1)} \leq C e^{C/T} \|y_0\|_X.$$

Now, using the estimates (5.23) and (5.21), we have

$$(5.24) \quad \left\| \frac{v}{\rho_0} \right\|_{L^2(0, T)} \leq C e^{C(T+\frac{1}{T})} \left( \|y_0\|_X + \left\| \frac{S_\ell}{\rho_S} \right\|_{L^2(0, T)} + \left\| \frac{S_r}{\rho_S} \right\|_{L^2(0, T)} \right).$$

*Weighted estimate of the solution.* Let us set  $y = \tilde{y} + \hat{y}$ . Then, for every  $k \geq 0$ ,  $y$  satisfies

$$\begin{cases} \partial_t y_1 - \partial_x^2 y_1 + \alpha_1 y_1 = 0 & \text{in } (T_k, T_{k+1}) \times (0, 1), \\ \partial_t y_2 - \partial_x^2 y_2 + \alpha_2 y_2 = 0 & \text{in } (T_k, T_{k+1}) \times (0, 1), \\ \partial_x y_1(\cdot, 1) = 0 & \text{in } (T_k, T_{k+1}), \\ \partial_x y_2(\cdot, 1) - \beta_r y_1(\cdot, 1) = S_r & \text{in } (T_k, T_{k+1}), \\ (y_1(T_k, \cdot), y_2(T_k, \cdot)) = a_k & \text{in } (0, 1), \end{cases}$$

where the control  $v_k$  acts either through the condition

$$\begin{cases} \partial_x y_1(\cdot, 0) + \beta_\ell y_2(\cdot, 0) = S_\ell + v_k & \text{in } (T_k, T_{k+1}), \\ \partial_x y_2(\cdot, 0) = 0 & \text{in } (T_k, T_{k+1}), \end{cases}$$

or through the condition

$$\begin{cases} \partial_x y_1(\cdot, 0) + \beta y_2(\cdot, 0) = S_\ell & \text{in } (T_k, T_{k+1}), \\ \partial_x y_2(\cdot, 0) = v_k & \text{in } (T_k, T_{k+1}). \end{cases}$$

Here,  $y(T_0) = a_0 = y_0$ , and we have

$$y(T_k^-) = \tilde{y}(T_k^-) + \hat{y}(T_k^-) = a_k + 0 = \hat{y}(T_k^+) + \tilde{y}(T_k^+) = y(T_k^+), \quad \forall k \geq 1,$$

thanks to the facts (5.13), (5.20) and the chosen initial data in the systems (5.14) and (5.16). Thus,  $y$  is continuous at  $T_k$  for all  $k \geq 0$ .

Using [Theorem 2.3](#), we have the following estimate for  $y$  (for every  $k \geq 0$ ):

$$\begin{aligned} & \|y\|_{C^0([T_k, T_{k+1}]; X)} + \|y\|_{L^2(T_k, T_{k+1}; (H^1(0,1))^2)} \\ & \leq C e^{CT} \left( \|a_k\|_X + \|v_k\|_{L^2(T_k, T_{k+1})} + \|S_\ell\|_{L^2(T_k, T_{k+1})} + \|S_r\|_{L^2(T_k, T_{k+1})} \right). \end{aligned}$$

Start with  $k \geq 1$ ; using the estimates of  $a_k$  and  $v_k$ , respectively from [\(5.15\)](#) and [\(5.19\)](#), we deduce that

$$\begin{aligned} & \|y\|_{C^0([T_k, T_{k+1}]; X)} + \|y\|_{L^2(T_k, T_{k+1}; (H^1(0,1))^2)} \leq C e^{CT} e^{\frac{M}{T_{k+1}-T_k}} \left( \|S_\ell\|_{L^2(T_{k-1}, T_k)} + \|S_r\|_{L^2(T_{k-1}, T_k)} \right) \\ & \quad + C e^{CT} \left( \|S_\ell\|_{L^2(T_k, T_{k+1})} + \|S_r\|_{L^2(T_k, T_{k+1})} \right) \\ & \leq C e^{CT} e^{\frac{M}{T_{k+1}-T_k}} \rho_S(T_{k-1}) \left( \left\| \frac{S_\ell}{\rho_S} \right\|_{L^2(T_{k-1}, T_{k+1})} + \left\| \frac{S_r}{\rho_S} \right\|_{L^2(T_{k-1}, T_{k+1})} \right) \quad \forall k \geq 1, \end{aligned}$$

since  $\rho_S$  is a non-increasing function. Now, thanks to the relation [\(5.12\)](#), one has

$$\begin{aligned} & \|y\|_{C^0([T_k, T_{k+1}]; X)} + \|y\|_{L^2(T_k, T_{k+1}; (H^1(0,1))^2)} \\ & \leq C e^{CT} \rho_0(T_{k+1}) \left( \left\| \frac{S_\ell}{\rho_S} \right\|_{L^2(T_{k-1}, T_{k+1})} + \left\| \frac{S_r}{\rho_S} \right\|_{L^2(T_{k-1}, T_{k+1})} \right), \quad \forall k \geq 1. \end{aligned}$$

Now, using the fact that  $\rho_0$  is a non-increasing function, we deduce

$$\begin{aligned} (5.25) \quad & \left\| \frac{y}{\rho_0} \right\|_{C^0([T_k, T_{k+1}]; X)} + \left\| \frac{y}{\rho_0} \right\|_{L^2(T_k, T_{k+1}; (H^1(0,1))^2)} \\ & \leq C e^{CT} \left( \left\| \frac{S_\ell}{\rho_S} \right\|_{L^2(T_{k-1}, T_{k+1})} + \left\| \frac{S_r}{\rho_S} \right\|_{L^2(T_{k-1}, T_{k+1})} \right), \quad \forall k \geq 1. \end{aligned}$$

For  $k = 0$ , we use the estimate of  $v_0$  from [\(5.22\)](#) to deduce

$$\begin{aligned} & \|y\|_{C^0([0, T_1]; X)} + \|y\|_{L^2(0, T_1; (H^1(0,1))^2)} \leq C e^{CT} \left( \|a_0\|_X + \|v_0\|_{L^2(0, T_1)} + \|S_\ell\|_{L^2(0, T_1)} + \|S_r\|_{L^2(0, T_1)} \right) \\ & \leq C e^{C(T+\frac{1}{T})} \rho_0(T_1) \left( \|y_0\|_X + \|S_\ell\|_{L^2(0, T_1)} + \|S_r\|_{L^2(0, T_1)} \right). \end{aligned}$$

Using the easy observation that  $\|S\|_{L^2(0, T)} \leq \left\| \frac{S}{\rho_S} \right\|_{L^2(0, T)}$ , we get

$$\begin{aligned} (5.26) \quad & \left\| \frac{y}{\rho_0} \right\|_{C^0([0, T_1]; X)} + \left\| \frac{y}{\rho_0} \right\|_{L^2(0, T_1; (H^1(0,1))^2)} \\ & \leq C e^{C(T+\frac{1}{T})} \left( \|y_0\|_X + \left\| \frac{S_\ell}{\rho_S} \right\|_{L^2(0, T_1)} + \left\| \frac{S_r}{\rho_S} \right\|_{L^2(0, T_1)} \right). \end{aligned}$$

So, the estimates [\(5.25\)](#) and [\(5.26\)](#) conclude that

$$\begin{aligned} (5.27) \quad & \left\| \frac{y}{\rho_0} \right\|_{C^0([0, T]; X)} + \left\| \frac{y}{\rho_0} \right\|_{L^2(0, T; (H^1(0,1))^2)} \\ & \leq C e^{C(T+\frac{1}{T})} \left( \|y_0\|_X + \left\| \frac{S_\ell}{\rho_S} \right\|_{L^2(0, T)} + \left\| \frac{S_r}{\rho_S} \right\|_{L^2(0, T)} \right). \end{aligned}$$

Finally, using the definitions of norms [\(5.6\)](#) (of the weighted spaces) in the estimates [\(5.24\)](#) and [\(5.27\)](#), we have the required estimate [\(5.11\)](#) of our proposition.  $\square$

**5.2. Application of the fixed point argument.** In this section, we prove the main theorem regarding the local null-controllability of our systems.

We assume the initial data  $y_0 \in X$  such that  $\|y_0\|_X \leq \delta$  for some  $\delta > 0$  which will be determined later. We also introduce the set

$$\mathcal{S}_\delta := \{S \in \mathcal{S} \times \mathcal{S} \mid \|S\|_{\mathcal{S} \times \mathcal{S}} \leq \delta\},$$

where the space  $\mathcal{S}$  is defined in (5.5a).

Now, recall from Proposition 5.2 that for any given source term  $S := (S_\ell, S_r) \in \mathcal{S} \times \mathcal{S}$ , there exists a control  $v$  such that the trajectory  $y = (y_1, y_2)$  of (5.7)–(5.8) with (5.9) or (5.10) (depending on the position of the control as per Proposition 5.2) satisfies the estimate (5.11). In what follows, we define an operator  $\mathcal{N}$  acting on  $\mathcal{S}_\delta$  by

$$\mathcal{N}(S) = \begin{pmatrix} -g_\ell(y_2(\cdot, 0)) \\ g_r(y_1(\cdot, 1)) \end{pmatrix},$$

where  $g_\ell$  and  $g_r$  have been introduced in (5.1) and (5.2).

**Proof of Theorem 1.2.** We start by showing that  $\mathcal{N}$  is a contraction map from  $\mathcal{S}_\delta$  onto itself for some suitable choice of  $\delta > 0$ .

**Step 1.** Let us show that  $\mathcal{S}_\delta$  is invariant under the map  $\mathcal{N}$  provided  $\delta > 0$  is small enough.

Thanks to the properties of  $g_\ell$  and  $g_r$  (recalled in (5.3)) and the trace theorem, we have

$$\begin{aligned} \left| \frac{\mathcal{N}(S)(t)}{\rho_S(t)} \right| &\leq \frac{C}{|\rho_S(t)|} (|y_2(t, 0)|^2 + |y_1(t, 1)|^2) \\ &\leq \frac{C}{|\rho_S(t)|} (\|y_2(t)\|_{L^2(0,1)} \|y_2(t)\|_{H^1(0,1)} + \|y_1(t)\|_{L^2(0,1)} \|y_1(t)\|_{H^1(0,1)}), \end{aligned}$$

and it follows that

$$(5.28) \quad \begin{aligned} \left| \frac{\mathcal{N}(S)(t)}{\rho_S(t)} \right|^2 &\leq \frac{C}{|\rho_S(t)|^2} \|y(t)\|_X^2 \|y(t)\|_{(H^1(0,1))^2}^2 \\ &\leq C \frac{|\rho_0(t)|^4}{|\rho_S(t)|^2} \left\| \frac{y(t)}{\rho_0(t)} \right\|_X^2 \left\| \frac{y(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2}^2. \end{aligned}$$

- Now, using Remark 5.1 we have that the quantity  $\frac{\rho_0^2(t)}{\rho_S(t)}$  is bounded by 1 irrespective on  $t$ .
- On the other hand, since we have  $\|y_0\|_X \leq \delta$  and  $S = (S_\ell, S_r) \in \mathcal{S}_\delta$ , using the result (5.11), we deduce that

$$(5.29) \quad \left\| \frac{y}{\rho_0} \right\|_{C^0([0,T];X)} + \left\| \frac{y}{\rho_0} \right\|_{L^2(0,T;(H^1(0,1))^2)} \leq C e^{C(T+\frac{1}{T})} \delta.$$

Using the bound (5.29) in (5.28), we get

$$\left\| \frac{\mathcal{N}(S)}{\rho_S} \right\|_{L^2(0,T)} \leq C \left\| \frac{y}{\rho_0} \right\|_{C^0([0,T];X)} \left\| \frac{y}{\rho_0} \right\|_{L^2(0,T;(H^1(0,1))^2)} \leq C e^{C(T+\frac{1}{T})} \delta^2.$$

Thus, for  $\delta > 0$  small enough, the map  $\mathcal{N}$  stabilizes  $\mathcal{S}_\delta$ .

**Step 2.** In this step, we prove that  $\mathcal{N}$  is a contraction map for  $\delta > 0$  small enough.

For any  $S, \tilde{S} \in \mathcal{S}_\delta$ , we denote the trajectories respectively by  $y = (y_1, y_2)$  and  $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)$  associated with the controls  $v$  and  $\tilde{v}$  of the system (5.7)–(5.8)–(5.9) (or (5.10)), by means of Proposition 5.2.

Then, still using (5.3), it follows that

$$\begin{aligned}
 \left| \frac{\mathcal{N}(S)(t) - \mathcal{N}(\tilde{S})(t)}{\rho_S(t)} \right| &\leq \frac{C}{|\rho_S(t)|} \left( |y_2(t, 0) - \tilde{y}_2(t, 0)| (|y_2(t, 0)| + |\tilde{y}_2(t, 0)|) \right. \\
 &\quad \left. + |y_1(t, 1) - \tilde{y}_1(t, 1)| (|y_1(t, 1)| + |\tilde{y}_1(t, 1)|) \right) \\
 &\leq \frac{C}{|\rho_S(t)|} \|y(t) - \tilde{y}(t)\|_X^{\frac{1}{2}} \|y(t) - \tilde{y}(t)\|_{(H^1(0,1))^2}^{\frac{1}{2}} \\
 &\quad \times \left( \|y(t)\|_X^{\frac{1}{2}} \|y(t)\|_{(H^1(0,1))^2}^{\frac{1}{2}} + \|\tilde{y}(t)\|_X^{\frac{1}{2}} \|\tilde{y}(t)\|_{(H^1(0,1))^2}^{\frac{1}{2}} \right) \\
 &\leq \frac{C\rho_0^2(t)}{\rho_S(t)} \left\| \frac{y - \tilde{y}}{\rho_0} \right\|_{C^0([0,T];X)}^{\frac{1}{2}} \left\| \frac{y(t) - \tilde{y}(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2}^{\frac{1}{2}} \\
 &\quad \times \left( \left\| \frac{y}{\rho_0} \right\|_{C^0([0,T];X)}^{\frac{1}{2}} \left\| \frac{y(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2}^{\frac{1}{2}} + \left\| \frac{\tilde{y}}{\rho_0} \right\|_{C^0([0,T];X)}^{\frac{1}{2}} \left\| \frac{\tilde{y}(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2}^{\frac{1}{2}} \right).
 \end{aligned}$$

Now, as per Proposition 5.2 the solution map is linear and thus  $y - \tilde{y}$  is the trajectory of the set of equations (5.7)–(5.8)–(5.9) (instead, (5.7)–(5.8)–(5.10)) with the source terms  $S - \tilde{S}$ , the control  $v - \tilde{v}$  and the initial data  $y(0, \cdot) - \tilde{y}(0, \cdot) = 0$ . Then, thanks to the estimate (5.11),  $y - \tilde{y}$  satisfies the following:

$$\begin{aligned}
 (5.30) \quad \left\| \frac{y - \tilde{y}}{\rho_0} \right\|_{C^0([0,T];X)} + \left\| \frac{y - \tilde{y}}{\rho_0} \right\|_{L^2(0,T;(H^1(0,1))^2)} &\leq C e^{C(T+\frac{1}{T})} (\|S_\ell - \tilde{S}_\ell\|_S + \|S_r - \tilde{S}_r\|_S) \\
 &\leq C e^{C(T+\frac{1}{T})} \|S - \tilde{S}\|_{S \times S}.
 \end{aligned}$$

Also, we recall the estimate (5.29) for  $y$  (similarly for  $\tilde{y}$ ) since we started with initial data  $\|y_0\| \leq \delta$  and  $S$  (or,  $\tilde{S}$ ) in  $\mathcal{S}_\delta$ . Beside this, we have by Remark 5.1 that  $\frac{\rho_0^2(t)}{\rho_S(t)} \leq 1, \forall t \in [0, T]$ . As a consequence, the inequality (11) yields to

$$\begin{aligned}
 \left| \frac{\mathcal{N}(S)(t) - \mathcal{N}(\tilde{S})(t)}{\rho_S(t)} \right| &\leq C_T \delta^{\frac{1}{2}} \|S - \tilde{S}\|_{\mathcal{S} \times \mathcal{S}}^{\frac{1}{2}} \left\| \frac{y(t) - \tilde{y}(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2}^{\frac{1}{2}} \\
 &\quad \times \left( \left\| \frac{y(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2}^{\frac{1}{2}} + \left\| \frac{\tilde{y}(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2}^{\frac{1}{2}} \right),
 \end{aligned}$$

with  $C_T := C e^{C(T+\frac{1}{T})}$ . This implies

$$\begin{aligned}
 \left\| \frac{\mathcal{N}(S) - \mathcal{N}(\tilde{S})}{\rho_S} \right\|_{(L^2(0,T))^2}^2 &\leq C_T \delta \|S - \tilde{S}\|_{S \times S} \\
 &\quad \times \int_0^T \left\| \frac{y(t) - \tilde{y}(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2} \left( \left\| \frac{y(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2} + \left\| \frac{\tilde{y}(t)}{\rho_0(t)} \right\|_{(H^1(0,1))^2} \right) \\
 &\leq C_T \delta \|S - \tilde{S}\|_{S \times S} \left\| \frac{y - \tilde{y}}{\rho_0} \right\|_{L^2(0,T;(H^1(0,1))^2)} \\
 &\quad \times \left( \left\| \frac{y}{\rho_0} \right\|_{L^2(0,T;(H^1(0,1))^2)} + \left\| \frac{\tilde{y}}{\rho_0} \right\|_{L^2(0,T;(H^1(0,1))^2)} \right) \\
 &\leq C_T \delta^2 \|S - \tilde{S}\|_{S \times S}^2,
 \end{aligned}$$

where we have used the estimates (5.29) and (5.30).

So, we eventually proved that

$$\|\mathcal{N}(S) - \mathcal{N}(\tilde{S})\|_{S \times S} \leq C_T \delta \|S - \tilde{S}\|_{S \times S},$$

for  $\delta > 0$  chosen small enough and this ensures that the map  $\mathcal{N}$  is a contraction in the closed ball  $\mathcal{S}_\delta$ .

Hence, by applying Banach fixed-point argument, there exists unique fixed point of the map  $\mathcal{N}$  denoted by  $S^* := (S_\ell^*, S_r^*)$  in the ball  $\mathcal{S}_\delta$ .

Thanks to [Proposition 5.2](#), the above  $S^* \in \mathcal{S}_\delta$  is such that there exists a control  $v \in \mathcal{V}$  such that the solution  $y := (y_1, y_2)$  to [\(5.7\)](#)–[\(5.8\)](#)–[\(5.9\)](#) or [\(5.7\)](#)–[\(5.8\)](#)–[\(5.10\)](#), along with the initial condition  $y_0$ , satisfies the estimates [\(5.11\)](#).

The property  $\lim_{t \rightarrow T^-} \rho_0(t) = 0$  forces that  $y(T, \cdot) = 0$  and thus the local null-controllability holds for the concerned nonlinear systems [\(1.10\)](#)–[\(1.11\)](#) with [\(1.12\)](#) or [\(1.13\)](#).

This indeed proves the local exact controllability result [Theorem 1.2](#) of the main control systems [\(1.1\)](#)–[\(1.2\)](#)–[\(1.8\)](#)/[\(1.9\)](#).  $\square$

**6. Further remarks and conclusions.** In this article, we studied the local exact controllability result to a given steady state for a  $2 \times 2$  parabolic system with coupled nonlinear boundary conditions by means of only one control function acting at the left end of the boundary (through a mixed condition or Neumann condition). Let us now address some other problems related to this study.

- **Possible generalizations.**

The proof we developed in this paper is strongly based on a quite precise spectral analysis for the linearized system. For this reason it does not seem straightforward for instance to obtain similar results, at least with the same approach, if one replaces the Laplace operator  $-\partial_x^2$  in the system by a more general elliptic operator, possibly different for the two components of the system.

- **Local exact controllability to arbitrary trajectories. Global controllability.**

One can be also interested in the controllability to any trajectory  $z^* := (z_1^*, z_2^*)$  for the system [\(1.1\)](#)–[\(1.2\)](#)–[\(1.3\)](#) other than steady state, but the situation is much more delicate to handle. Indeed, the linearized models are associated with the following boundary conditions (in comparison with [\(1.14\)](#)–[\(1.15\)](#))

$$\begin{cases} \partial_x y_1(\cdot, 1) = 0 & \text{in } (0, T), \\ \partial_x y_2(\cdot, 1) - f_r'(z_1^*(\cdot, 1))y_1(\cdot, 1) = 0 & \text{in } (0, T), \\ \\ \partial_x y_1(\cdot, 0) + f_\ell'(z_2^*(\cdot, 0))y_2(\cdot, 0) = v & \text{in } (0, T), \\ \partial_x y_2(\cdot, 0) = 0 & \text{in } (0, T), \end{cases}$$

thus, it is not possible to rely on spectral methods for the associated adjoint operator since the parameters  $\beta_\ell = f_\ell'(z_2^*(\cdot, 0))$  and  $\beta_r = f_r'(z_1^*(\cdot, 1))$  are now time dependent. In particular, it is not clear how to obtain the controllability of the linearized systems in that case.

Note that the same issue arises even if we assume that  $f_\ell$  and  $f_r$  are globally Lipschitz, in which case we could expect for global controllability results.

- **Relaxing the conditions  $\beta_\ell \beta_r \neq 0$  or  $\beta_\ell \beta_r \notin \mathcal{R}$ .**

In this paper we assumed [\(1.6\)](#) and it can be interesting to see whether or not this is a necessary condition for controllability. In the case where  $\beta_\ell = \beta_r = 0$ , it is clear that the linearized systems are not controllable anymore with one single control since the two equations are not coupled at all. On the other hand, in the cases when  $\beta_\ell = 0, \beta_r \neq 0$ , or  $\beta_\ell \neq 0, \beta_r = 0$ , then the linearized models [\(1.10\)](#)–[\(1.14\)](#)–[\(1.15\)](#)/[\(1.16\)](#) may not be controllable. Indeed, in the case:

$\beta_\ell = 0, \beta_r \neq 0$ , we can compute from the eigenvalue problem [\(3.1\)](#) that  $\left\{ \begin{pmatrix} \cos(k\pi x) \\ 0 \end{pmatrix} \right\}_{k \geq 0}$  is

one set of eigenfunctions for the associated adjoint operator, and thus it appears that when a control acts through the Neumann condition [\(1.16\)](#), then the concerned system is not even approximately controllable in  $X$ .

Furthermore, according to [Theorem 1.3](#), when  $\beta_\ell \beta_r \in \mathcal{R}$ , the linearized system [\(1.10\)](#)–[\(1.14\)](#)–[\(1.16\)](#) also fails to be approximately controllable in  $X$ .

Therefore, in those cases, at least with the method we are using, we cannot conclude anything about the local exact-controllability to the steady state for our nonlinear system.

Studying the controllability of the nonlinear system in such cases should thus be attacked with another approach(es) which we do not consider in this paper. Other techniques (such as, for

instance, Coron's return method [13, Chapter 6]) could be applied to study the local controllability of the nonlinear systems in the cases where we do not have the null-controllability of the linearized models, but this would need further investigations.

**Acknowledgements.** The work of the first author is partially supported by the French government research program "Investissements d'Avenir" through the IDEX-ISITE initiative 16-IDEX-0001 (CAP 20-25). The authors warmly thanks the anonymous referees whose comments have permitted many improvements of the initial version of this work.

### Appendix A. Proofs of technical lemmas.

*Proof of Lemma 3.1.* Let  $\mu \in \mathbb{C}$  such that  $|\mu|^2 > |\theta|$ . We can write

$$\sqrt{\mu^2 - \theta} = \pm \mu \sqrt{1 - \frac{\theta}{\mu^2}},$$

where the sign indetermination comes from the fact that we use the principal determination of the square root. This indetermination will have no impact on the following computations due to parity arguments.

Setting  $\mu_1 = \sqrt{\frac{4|\theta|}{3}}$ , we immediately get that

$$|\sqrt{\mu^2 - \theta}| \geq \frac{1}{2}|\mu|, \quad \forall |\mu| \geq \mu_1.$$

For  $z \in \mathbb{C}$ ,  $|z| < 1$ , we write

$$\sqrt{1 - z} = 1 + z\zeta(z),$$

where  $\zeta$  is holomorphic in the unit disk.

We can now obtain

$$\begin{aligned} \pm \sin(\sqrt{\mu^2 - \theta}) &= \sin\left(\mu + \frac{\theta}{\mu}\zeta\left(\frac{\theta}{\mu^2}\right)\right) \\ &= \sin(\mu) \cos\left(\frac{\theta}{\mu}\zeta\left(\frac{\theta}{\mu^2}\right)\right) + \cos(\mu) \sin\left(\frac{\theta}{\mu}\zeta\left(\frac{\theta}{\mu^2}\right)\right). \end{aligned}$$

We can now find some  $\mu_2 > \sqrt{|\theta|}$  depending only on  $\theta$  such that

$$\left|\cos\left(\frac{\theta}{\mu}\zeta\left(\frac{\theta}{\mu^2}\right)\right)\right| \geq 3/4, \text{ and } \left|\sin\left(\frac{\theta}{\mu}\zeta\left(\frac{\theta}{\mu^2}\right)\right)\right| \leq 1/4, \text{ for every } \mu \text{ s.t. } |\mu| > \mu_2.$$

It follows that for every  $\mu$  such that  $|\mu| > \mu_2$  we have

$$|\sin(\sqrt{\mu^2 - \theta})| \geq \frac{3}{4}|\sin \mu| - \frac{1}{4}|\cos \mu|.$$

Taking  $\mu_0 = \max(\mu_1, \mu_2)$  gives the claim. □

*Proof of Lemma 3.2.* Let us assume that  $|\operatorname{Im} \mu| \geq \mu_0$ . By the previous lemma we know that

$$|\sin(\sqrt{\mu^2 - \theta})| \geq \frac{3}{4}|\sin \mu| - \frac{1}{4}|\cos \mu|.$$

Moreover we have

$$\begin{aligned} |\sin \mu| &= \frac{|e^{i\mu} - e^{-i\mu}|}{2} \geq \frac{e^{|\operatorname{Im} \mu|} - e^{-|\operatorname{Im} \mu|}}{2} = \sinh |\operatorname{Im} \mu|, \\ |\cos \mu| &= \frac{|e^{i\mu} + e^{-i\mu}|}{2} \leq \frac{e^{|\operatorname{Im} \mu|} + e^{-|\operatorname{Im} \mu|}}{2} = \cosh |\operatorname{Im} \mu|. \end{aligned}$$

It follows that, for some  $\vartheta_0$  large enough that we choose satisfying  $\vartheta_0 \geq \mu_0$ , we have

$$|\cos \mu| \leq 2|\sin \mu|, \quad \forall \mu, \text{ s.t. } |\operatorname{Im} \mu| \geq \vartheta_0.$$



Finally we obtain for all such values of  $\mu$

$$|\sin(\sqrt{\mu^2 - \theta})| \geq \frac{1}{4} |\sin \mu| = \sinh |\operatorname{Im} \mu|.$$

Going back to the definition of  $h_0$  we obtain

$$|h_0(\mu)| \geq \frac{1}{\beta_*} |\mu| \sqrt{\mu^2 - \theta} \sinh^2 |\operatorname{Im} \mu| \geq \frac{1}{2\beta_*} |\operatorname{Im} \mu|^2 \sinh^2 |\operatorname{Im} \mu| \geq \frac{1}{2\beta_*^2} \vartheta_0^2 \sinh^2 \vartheta_0.$$

Choosing  $\vartheta_0$  large enough gives the claim.  $\square$

*Proof of Lemma 3.3.* We first choose  $k_1$  any integer such that  $k_1 \geq \frac{\mu_0}{\pi}$ . Let  $\mu$  such that  $\operatorname{Re} \mu = (k + 1/2)\pi$  with  $k \geq k_1$ .

By standard trigonometric properties, we have for such values of  $\mu$ , the equalities

$$|\sin \mu| = \cosh \operatorname{Im} \mu, \quad \text{and} \quad |\cos \mu| = |\sinh \operatorname{Im} \mu|,$$

and in particular we get

$$|\sin \mu| \geq 1, \quad |\cos \mu| \leq |\sin \mu|.$$

By the choice of  $k_1$ , we have in particular  $|\mu| \geq \mu_0$  and therefore by Lemma 3.1

$$|\sin(\sqrt{\mu^2 - \theta})| \geq \frac{3}{4} |\sin \mu| - \frac{1}{4} |\cos \mu|,$$

and the inequalities above give

$$|\sin(\sqrt{\mu^2 - \theta})| \geq \frac{1}{2} |\sin \mu| \geq \frac{1}{2}.$$

All in all, we have for such values of  $\mu$ ,

$$|h_0(\mu)| \geq \frac{1}{4\beta_*} |\mu|^2 \geq \frac{\pi^2}{4\beta_*} k^2,$$

and this quantity is greater than 1 for any  $k$  large enough.

*Proof of Lemma 3.4.* The proof is straightforward by using usual asymptotic expansions. We first get

$$\begin{aligned} \sqrt{(k\pi + \delta_k)^2 + \gamma} &= k\pi \sqrt{1 + \frac{2\delta_k}{k\pi} + \frac{\delta_k^2 + \gamma}{k^2\pi^2}} \\ &= k\pi \left( 1 + \frac{\delta_k}{k\pi} + \frac{1}{2} \frac{\delta_k^2 + \gamma}{k^2\pi^2} - \frac{1}{8} \frac{4\delta_k^2}{k^2\pi^2} + O\left(\frac{1}{k^3}\right) \right) \\ &= k\pi + \delta_k + \frac{\gamma}{2k\pi} + O\left(\frac{1}{k^2}\right). \end{aligned}$$

Then, in a similar way, it follows

$$\begin{aligned} \sqrt{k^2\pi^2 + \gamma + \varepsilon_k} &= k\pi \sqrt{1 + \frac{\gamma + \varepsilon_k}{k^2\pi^2}} \\ &= k\pi \left( 1 + \frac{\gamma + \varepsilon_k}{2k^2\pi^2} - \frac{(\gamma + \varepsilon_k)^2}{8k^4\pi^4} + O\left(\frac{1}{k^6}\right) \right) \\ &= k\pi + \frac{\gamma + \varepsilon_k}{2k\pi} - \frac{(\gamma + \varepsilon_k)^2}{8k^3\pi^3} + O\left(\frac{1}{k^5}\right) \\ &= k\pi + \frac{\gamma + \varepsilon_k}{2k\pi} - \frac{\gamma^2}{8k^3\pi^3} + o\left(\frac{1}{k^3}\right). \end{aligned}$$

From this last equality, we deduce that

$$\begin{aligned}\sin \sqrt{k^2 \pi^2 + \gamma + \varepsilon_k} &= (-1)^k \sin \left( \frac{\gamma + \varepsilon_k}{2k\pi} - \frac{\gamma^2}{8k^3 \pi^3} + o\left(\frac{1}{k^3}\right) \right) \\ &= (-1)^k \left( \frac{\gamma + \varepsilon_k}{2k\pi} - \frac{\gamma^2}{8k^3 \pi^3} - \frac{\gamma^3}{48k^3 \pi^3} + o\left(\frac{1}{k^3}\right) \right)\end{aligned}$$

and therefore we have

$$\begin{aligned}\sqrt{k^2 \pi^2 + \gamma + \varepsilon_k} \sin \sqrt{k^2 \pi^2 + \gamma + \varepsilon_k} &= (-1)^k \left( k\pi + \frac{\gamma}{2k\pi} + o\left(\frac{1}{k}\right) \right) \\ &\quad \times \left( \frac{\gamma + \varepsilon_k}{2k\pi} - \frac{\gamma^2}{8k^3 \pi^3} - \frac{\gamma^3}{48k^3 \pi^3} + o\left(\frac{1}{k^3}\right) \right) \\ &= (-1)^k \left( 1 + \frac{\gamma}{2k^2 \pi^2} + o\left(\frac{1}{k^2}\right) \right) \\ &\quad \times \left( \frac{\gamma + \varepsilon_k}{2} - \frac{\gamma^2}{8k^2 \pi^2} - \frac{\gamma^3}{48k^2 \pi^2} + o\left(\frac{1}{k^2}\right) \right) \\ &= (-1)^k \left( \frac{\gamma + \varepsilon_k}{2} + \frac{\gamma^2}{8k^2 \pi^2} - \frac{\gamma^3}{48k^2 \pi^2} + o\left(\frac{1}{k^2}\right) \right) \\ &= (-1)^k \frac{\gamma}{2} \left( 1 + \frac{\varepsilon_k}{\gamma} + \frac{\gamma}{4k^2 \pi^2} - \frac{\gamma^2}{24k^2 \pi^2} + o\left(\frac{1}{k^2}\right) \right).\end{aligned}$$

Applying the previous result twice (once with  $\gamma$  and once with  $\gamma$  replaced by  $\gamma - \theta$ ), we obtain

$$\begin{aligned}h_0(\sqrt{k^2 \pi^2 + \gamma + \varepsilon_k}) &= \frac{\gamma(\gamma - \theta)}{4\beta_*} \left( 1 + \frac{\varepsilon_k}{\gamma} + \frac{\gamma}{4k^2 \pi^2} - \frac{\gamma^2}{24k^2 \pi^2} + o\left(\frac{1}{k^2}\right) \right) \\ &\quad \times \left( 1 + \frac{\varepsilon_k}{\gamma - \theta} + \frac{\gamma - \theta}{4k^2 \pi^2} - \frac{(\gamma - \theta)^2}{24k^2 \pi^2} + o\left(\frac{1}{k^2}\right) \right) \\ &= \frac{\gamma(\gamma - \theta)}{4\beta_*} \left( 1 + \frac{\varepsilon_k(\varepsilon_k + 2\gamma - \theta)}{\gamma(\gamma - \theta)} + \frac{2\gamma - \theta}{4k^2 \pi^2} - \frac{\gamma^2 + (\gamma - \theta)^2}{24k^2 \pi^2} + o\left(\frac{1}{k^2}\right) \right).\end{aligned}$$

Taking  $\varepsilon_k = 0$  in this formula, leads to (3.6), whereas taking  $\gamma = \theta/2$  leads to (3.7).  $\square$

## REFERENCES

- [1] F. Alabau-Boussouira and M. Léautaud. Indirect controllability of locally coupled wave-type systems and applications. *J. Math. Pures Appl.* (9), 99(5):544–576, 2013.
- [2] D. Allonsius and F. Boyer. Boundary null-controllability of semi-discrete coupled parabolic systems in some multi-dimensional geometries. *Math. Control Relat. Fields*, 10(2):217–256, 2020.
- [3] F. Ammar-Khodja, A. Benabdallah, C. Dupaix, and M. González-Burgos. A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems. *Differ. Equ. Appl.*, 1(3):427–457, 2009.
- [4] F. Ammar-Khodja, A. Benabdallah, C. Dupaix, and M. González-Burgos. A Kalman rank condition for the localized distributed controllability of a class of linear parabolic systems. *J. Evol. Equ.*, 9(2):267–291, 2009.
- [5] F. Ammar-Khodja, A. Benabdallah, M. González-Burgos, and L. de Teresa. The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials. *J. Math. Pures Appl.* (9), 96(6):555–590, 2011.
- [6] S. Avdonin. Control problems on quantum graphs. In *Analysis on graphs and its applications*, volume 77 of *Proc. Sympos. Pure Math.*, pages 507–521. Amer. Math. Soc., Providence, RI, 2008.
- [7] V. Barbu. Exact controllability of the superlinear heat equation. *Appl. Math. Optim.*, 42(1):73–89, 2000.
- [8] A. Benabdallah, F. Boyer, M. González-Burgos, and G. Olive. Sharp estimates of the one-dimensional boundary control cost for parabolic systems and application to the  $N$ -dimensional boundary null controllability in cylindrical domains. *SIAM Journal on Control and Optimization*, 52(5):2970–3001, jan 2014.
- [9] K. Bhandari and F. Boyer. Boundary null-controllability of coupled parabolic systems with Robin conditions. *Evol. Equ. Control Theory*, 10(1):61–102, 2021.
- [10] K. Bhandari, F. Boyer, and V. Hernández-Santamaría. Boundary null-controllability of 1-D coupled parabolic systems with Kirchhoff-type conditions. *Math. Control Signals Systems*, 33(3):413–471, 2021.
- [11] F. Boyer. Controllability of linear parabolic equations and systems. Lecture Notes <https://hal.archives-ouvertes.fr/hal-02470625v4>, July 2023.
- [12] F. W. Chaves-Silva and S. Guerrero. A controllability result for a chemotaxis-fluid model. *J. Differential Equations*, 262(9):4863–4905, 2017.

- [13] J.-M. Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.
- [14] R. Dáger and E. Zuazua. *Wave propagation, observation and control in 1-d flexible multi-structures*, volume 50 of *Mathématiques & Applications (Berlin)*. Springer-Verlag, Berlin, 2006.
- [15] N. Dunford and J. T. Schwartz. *Linear operators. Part III*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1988. Spectral operators, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1971 original, A Wiley-Interscience Publication.
- [16] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.
- [17] H. O. Fattorini. Some remarks on complete controllability. *SIAM J. Control*, 4:686–694, 1966.
- [18] H. O. Fattorini and D. L. Russell. Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations. *Quart. Appl. Math.*, 32:45–69, 1974/75.
- [19] E. Fernández-Cara, M. González-Burgos, and L. de Teresa. Boundary controllability of parabolic coupled equations. *J. Funct. Anal.*, 259(7):1720–1758, 2010.
- [20] E. Fernández-Cara and E. Zuazua. Null and approximate controllability for weakly blowing up semilinear heat equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 17(5):583–616, 2000.
- [21] A. V. Fursikov and O. Yu. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.
- [22] Manuel González-Burgos and Lydia Ouaili. Sharp estimates for biorthogonal families to exponential functions associated to complex sequences without gap conditions. working paper or preprint, January 2021.
- [23] B.-Z. Guo. Riesz basis approach to the stabilization of a flexible beam with a tip mass. *SIAM J. Control Optim.*, 39(6):1736–1747, 2001.
- [24] K. Le Balc’h. Controllability of a  $4 \times 4$  quadratic reaction-diffusion system. *J. Differential Equations*, 266(6):3100–3188, 2019.
- [25] K. Le Balc’h. Null-controllability of two species reaction-diffusion system with nonlinear coupling: a new duality method. *SIAM J. Control Optim.*, 57(4):2541–2573, 2019.
- [26] K. Le Balc’h. Global null-controllability and nonnegative-controllability of slightly superlinear heat equations. *J. Math. Pures Appl. (9)*, 135:103–139, 2020.
- [27] G. Lebeau and L. Robbiano. Contrôle exact de l’équation de la chaleur. *Comm. Partial Differential Equations*, 20(1-2):335–356, 1995.
- [28] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. II*. Springer-Verlag, New York-Heidelberg, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 182.
- [29] Y. Liu, T. Takahashi, and M. Tucsnak. Single input controllability of a simplified fluid-structure interaction model. *ESAIM Control Optim. Calc. Var.*, 19(1):20–42, 2013.
- [30] G. Olive. Boundary approximate controllability of some linear parabolic systems. *Evol. Equ. Control Theory*, 3(1):167–189, 2014.
- [31] E. M. Ouhabaz. *Analysis of heat equations on domains*, volume 31 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2005.
- [32] I. Singer. *Bases in Banach spaces. I*. Die Grundlehren der mathematischen Wissenschaften, Band 154. Springer-Verlag, New York-Berlin, 1970.
- [33] Jr. H. D. Thames and A. D. Elster. Equilibrium states and oscillations for localized two-enzyme kinetics: a model for circadian rhythms. *J. Theoret. Biol.*, 59(2):415–427, 1976.
- [34] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.