# Approximation of dissipative systems by elastic chains: Numerical evidence

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**Abstract**
An old and debated problem in Mechanics concerns the capacity of finite dimensional Lagrangian systems to describe dissipation phenomena. It is true that Helmholtz conditions determine not-always verifiable conditions establishing when a system of $n$ second-order ordinary differential equations in normal form (nODEs) be the Lagrange equations deriving from an $n$th dimensional Lagrangian. However, it is also true that one could conjecture that, given nODEs it is possible to find a $(n+k)$th dimensional Lagrangian such that the evolution of suitably chosen $n$ Lagrangian parameters allows for the approximation of the solutions of the nODEs. In fact, while it is well known that the ordinary differential equations (ODEs) usually introduced for describing some dissipation phenomena do not verify Helmholtz conditions, in this paper, we give some preliminary evidence for a positive answer to the conjecture that a dissipative system having $n$ degrees of freedom (DOFs) can be approximated, in a finite time interval and in a suitable norm, by an extended Lagrangian system, having a greater number of DOFs. The theoretical foundation necessary to formulate such a conjecture is here laid and three different examples of extended Lagrangians are shown. Finally, we give some computational results, which encourage to deepen the study of the theoretical aspects of the problem.

**Keywords**
Lagrangian formalism, dissipative systems, oscillatory systems

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## 1. Introduction

In Physics, and in Mechanics in particular, a general conceptual scheme (a metaphysical principle, using the nomenclature by Gabrio Piola, see dell'Isola and colleagues [1,2]) is searched in which one can hope to frame the formulation of every model used for predicting the evolution of the states of any considered physical systems. Such a general conceptual scheme not only is intended to reflect the intrinsic unity of Nature, a statement that can be considered rather vague although very suggestive [3], but also (and more effectively) to supply a secure guidance in the formulation of novel methods aimed to describe novel phenomenology.

The Principle of Least Action, as formulated by D'Alembert, Lagrange and Hamilton seems to supply such a "meta-theoretical" conceptual scheme. It has been longly discussed if it is really comprehensive enough. We believe, in fact, that a careful historical analysis (see, for example, Rojo and Bloch [4]) seems to definitively indicate that almost the totality of novel theories has been formulated, at first, by scholars who were guided mainly by the Principle of Least Action.

The debates are often also focusing on the concerns about the capacity of finite dimensional Lagrangian systems (i.e., those systems governed by Lagrangian formulation of the Principle of Least Action) to describe dissipation phenomena. Many scholars claim, in fact, that Lagrangian formalism is not capable to predict the phenomenology of dissipation.

Even more often the principle of authority is invoked to prove such an incapability statement; in other words, it is stated that the most gifted scientists in history of Physics believed that dissipation cannot be Lagrangian. Very often it even is recalled that Richard Feynman is among the just cited prominent scientists. However, a careful student, in perusing his Lecture Notes in modern Physics, understands immediately that instead Feynman strongly believes that the Principle of Least Action is indeed the most powerful heuristic and meta-theoretical tool in the mathematical modeling of Nature.

In fact, every scholar in modern Physics is aware of the fact that in Feynman's PhD thesis, the concepts of Path Integrals are introduced in Quantum Mechanics, that were inspired by an innovative interpretation (but in many aspects rather orthodox) of Lagrangian Action in such a more general context.

It is enough, for establishing the truth in this controversy, to read carefully what Feynman himself writes:

> ***I have been saying that we get Newton's law [from Least Action Principle]. That is not quite true, because Newton's law includes nonconservative forces like friction. Newton said that ma is equal to any F. But the principle of least action only works for conservative systems-where all forces can be gotten from a potential function. You know, however, that on a microscopic level-on the deepest level of physics-there are no nonconservative forces. Nonconservative forces, like friction, appear only because we neglect microscopic complications-there are just too many particles to analyze. But the fundamental laws can be put in the form of a principle of least action.*** [3, p. 19.7]

Clearly, Feynman's words which we quote in italic bold are not read too often.

Feynman clearly states that non-conservative forces appear in our mathematical models only when we neglect to model some (many) degrees of freedom (DOFs) in which energy can be trapped producing the phenomenology that we call "friction" or "dissipation." Feynman's authoritative opinion, therefore, can be stated as follows: Lagrangian formalism can be used to describe dissipation by suitably increasing the DOFs in the proposed model.

Moreover, it has been observed that some specific types of dissipation phenomena can be described without increasing the DOFs but introducing "non-natural" Lagrangians; the existence of both aforementioned kinds of Lagrangian systems suitably introduced to describe non-conservative phenomena (see Bersani and Caressa [5]) suggests the idea of looking more deeply in the relationships between conservative and non-conservative systems.

This topic is also very important in the study and design of many peculiar physical and engineering models: those where it is needed to transfer energy from a conservative sub-system to a dissipative one,

and vice versa. For instance, in Koç et al. [6], it is proposed the concept of "energy sink," i.e., a set of oscillators which absorb and retain energy from a vibrating structure, without dissipating it as heat in the classical sense; an energy sink is designed to suitably introduce a controlled dissipation for a "master" system by means of the addition of a "slave" system. In this note, we start the analysis of a specific, correlated, topic in the study of the relationship between dissipative and conservative systems.

In particular, we ask the following:

*Question 1.1.* To what extent and when it is possible to approximate a dissipative system with a conservative one having more DOFs?

We put forth in this note some hints toward a rigorous understanding of this question, largely based on computations and numerical simulations in special cases.

Consider, for example, the paradigmatic and overused example of the damped harmonic oscillator, which is well known and exactly solved [5]

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0. \tag{1}$$

We ask whether it is possible to approximate such a non-conservative system by a conservative one having more DOFs, up to a fixed but arbitrarily great precision.[1]

Namely, let us consider the solution $x(t)$ of the Cauchy problem

$$\begin{cases} \ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0, \\ x(0) = x_0, \\ \dot{x}(0) = \dot{x}_0. \end{cases} \tag{2}$$

We ask if, given an arbitrary $\varepsilon > 0$ and an arbitrary time interval $[0, T]$, it is possible to find out a set of Lagrangian coordinates $\mathbf{q}$ and a Lagrangian conservative system

$$\begin{cases} \mathbf{q} = \mathbf{F}(t, \mathbf{q}, \dot{\mathbf{q}}), \\ \mathbf{q}(0) = \mathbf{q}_0, \\ \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0. \end{cases} \tag{3}$$

such that, if $q_0$ is the first component of its solution $\mathbf{q}(\mathbf{t})$, then

$$\|x(t) - q_0(t)\| < \varepsilon, \tag{4}$$

where $\|\cdot\|$ is a suitable norm (typically the uniform norm) in the space of coordinates.

In other words, does it exist a Lagrangian system with $N$ DOFs whose motion equations are given by the conservative system of ordinary differential equations (ODEs) (3) and which approximates the dissipative system as precisely as wanted?

We provide hereinafter some computational and numerical evidences for a positive answer to this question. The paper is structured in the following way. In section 2, we consider an extended Lagrangian system in which we have a set of coupled harmonic oscillators and determine, by means of optimization techniques, the optimal coupling and elastic constants which allow to approximate the motion of two different damped systems, related to the classical damped oscillator and to a pendulum subject to a quadratic dissipation, respectively, with the central coupled oscillator of the extended Lagrangian system. Numerical computations are illustrated which hint for a positive answer to the question of approximating the two damped systems. In section 3, we recall the pioneering model of an infinite chain of coupled harmonic oscillators introduced by Erwin Schrödinger [7,8], and we discuss its solutions. In section 4, we recall some further generalizations of the Schrödinger chain, together with their solutions. In section 5, we adapt the Schrödinger model to the case of a finite chain of coupled oscillators; this model is very interesting due to the structure of the system of equations, which is simpler than the classical

coupling discussed in section 2. Moreover, we compare the numerical optimization results in both cases. According to these preliminary simulations, the former system appears to approximate the damped system more efficiently than the latter. In section 6, conclusion and some perspectives for future research are proposed. In the Supplemental Appendix, we report the numerical code we used to implement the optimizations described in this paper.

We explicitly remark here that the effectiveness of the approximation of the linear dissipative equations from which we are starting via the extended Lagrangian system does not depend on the initial data of the considered motions. In fact, because of the equation linearity, it is enough to verify such effectiveness for unit displacement and zero velocity (or unit velocity and zero displacement) initial data. Such a circumstance makes evident the difficulties to be confronted when the dissipative equations to be approximated by an extended Lagrangian are not linear.

## 2. Approximation via elastic coupling: computations and motivations

A first hint to our Question 1.1 was proposed in 1931 by Bateman [9] as an objection to Bauer's paper [10], in which it was claimed that a linear dissipative set of differential equations with constant coefficients cannot be derived from a variational principle. To disprove this claim, Bateman considered the damped oscillator (1) and increased the DOFs by one, mirroring in some sense the original system by adding to it the equation

$$\ddot{y} - 2\gamma\dot{y} + \omega^2 y = 0.$$

The $(x, y)$ system stems from the Lagrangian

$$L = \dot{x}\dot{y} + \gamma(x\dot{y} - y\dot{x})m - \omega^2 xy,$$

which allows to retrieve the original equation as the first of the corresponding Lagrange equations.[2]

Although this procedure is somewhat artificial, it raises the interesting suggestion that, within a fixed finite interval of time, the energy lost by the original damped system may be taken on by the other damped system, so that the total energy balance of the mirrored system remains constant. One could look at this suggestion the other way round: take two (or more) conservative harmonic oscillators and couple them so that, in a finite interval of time, one of them displays a damped behavior, loosing energy which is taken on by the other oscillators.

Let us consider $2N + 1$ harmonic oscillators

$$\begin{cases} m_{-N}\ddot{\xi}_{-N} + \omega_{-N}^2 \xi_{-N} = 0, \\ m_{-N+1}\ddot{\xi}_{-N+1} + \omega_{-N+1}^2 \xi_{-N+1} = 0, \\ \vdots \\ m_k \ddot{\xi}_k + \omega_k^2 \xi_k = 0, \\ \vdots \\ m_N \ddot{\xi}_N + \omega_N^2 \xi_N = 0, \end{cases}$$

and let us couple them pairwise: this means adding to the Lagrangian of the previous system, which is just the sum of all Lagrangians of single oscillator equations, elastic terms as follows (see, for example, Goldstein [11, §10])

$$L = \frac{1}{2}\sum_{k=-N}^{N}\left(m_k\dot{\xi}_k^2 - \omega_k^2\xi_k^2\right) + \sum_{k=-N}^{N-1}\omega_{k,k+1}\xi_k\xi_{k+1}.$$

The corresponding Lagrange equations of the coupled system are

$$\begin{cases} m_{-N}\ddot{\xi}_{-N} = -\omega_{-N}^2\xi_{-N} + \omega_{-N,-N+1}\xi_{-N+1}, \\ m_{-N+1}\ddot{\xi}_{-N+1} = \omega_{-N,-N+1}\xi_{-N} - \omega_{-N+1}^2\xi_{-N+1} + \omega_{-N+1,-N+2}\xi_{-N+2}, \\ \vdots \\ m_k\ddot{\xi}_k = \omega_{k-1,k}\xi_{k-1} - \omega_k^2\xi_k + \omega_{k,k+1}\xi_{k+1}, \\ \vdots \\ m_{N-1}\ddot{\xi}_{N-1} = \omega_{N-2,N-1}\xi_{N-2} - \omega_{N-1}^2\xi_{N-1} + \omega_{N-1,N}\xi_N, \\ m_N\ddot{\xi}_N = \omega_{N-1,N}\xi_{N-1} - \omega_N^2\xi_N. \end{cases} \tag{5}$$

Let us introduce as usual variables $p_k = m_k\dot{\xi}_k$ to get a $2(2N+1) = 4N+2$ first-order system and let us focus on the "central" equation, the one corresponding to $k = 0$

$$\dot{p}_0 = \omega_{-1,0}\xi_{-1} - \omega_0^2\xi_0 + \omega_{0,1}\xi_1. \tag{6}$$

We ask whether the motion of this single mass inside the undamped system can somehow replicate, on a finite time interval, a dissipative system such as the damped oscillator. The question is meaningful since the total energy of the coupled system, which is constant, is not equally partitioned between the $2N+1$ masses during the motion.

*Example 2.1.* To corroborate the previous ideas, we performed some numerical computations for the paradigmatic damped harmonic motion. As a first step, we considered the previous elastic coupling where all $\omega_k$ are equal (let us call $\omega_0$ this common parameter to optimize). We optimized[3] this single $\omega_0$ and the elastic constants $\omega_{-N,-N+1}, ..., \omega_{N-1,N}$ in such a way that the solution of the central equation (6) approximates the solution of the damped equation (1).

In all our numerical computations, we considered the distance between two curves to be the uniform norm

$$\|f - g\|_\infty := \sup_{t\in[0,T]} |f(t) - g(t)|.$$

The optimization method we used is the double simulated annealing one [12]. The result is shown in Figure 1, where the central solutions for $N = 1, 2, ..., 10$, corresponding to systems with $3, 5, ..., 21$ DOFs, are plotted (we considered all masses equal to one, and the following parameters: $\gamma = 0.5$ and $\omega = 4$). The optimal value found is $\omega_0 = 3.84$.

It is seen that the damping trend of the central trajectory may be observed already in the $N = 1$ case, with three equations: as the DOFs increase, the error in approximating the trajectory of the damped

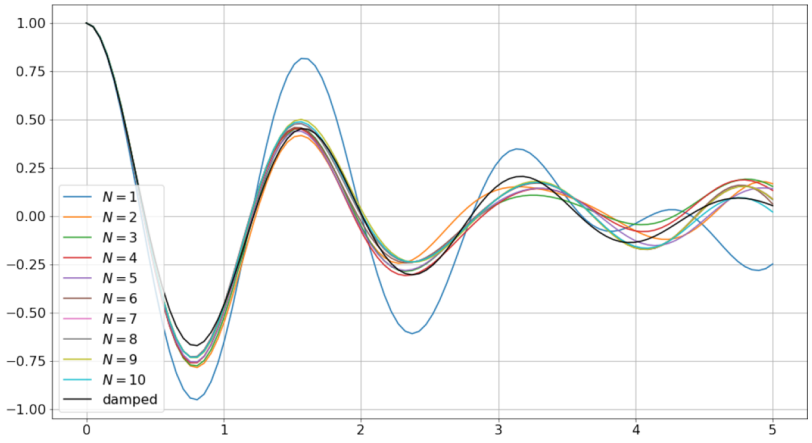



**Figure 1.** Approximation of a damped harmonic motion via an undamped one coupled with other 2*N* harmonic oscillators with the same frequency.

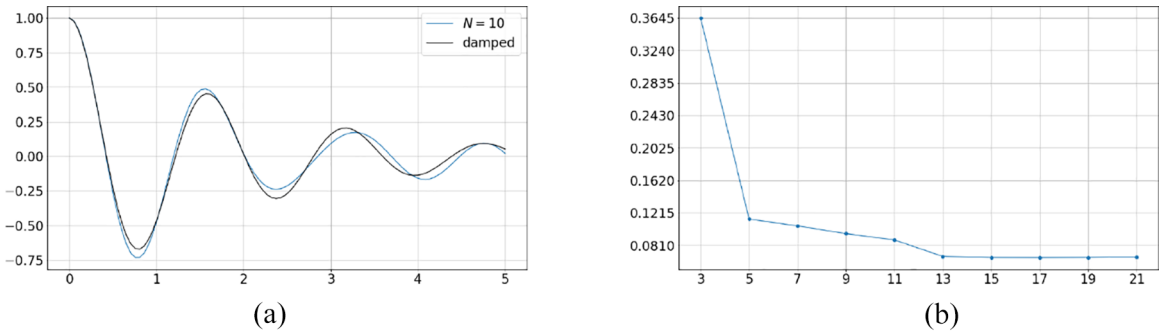



**Figure 2.** (a) Approximation of a damped harmonic motion via an undamped one coupled with other 20 harmonic oscillators with the same frequency. (b) The approximation error decreases as the degrees of freedom increase (other parameters being equal).

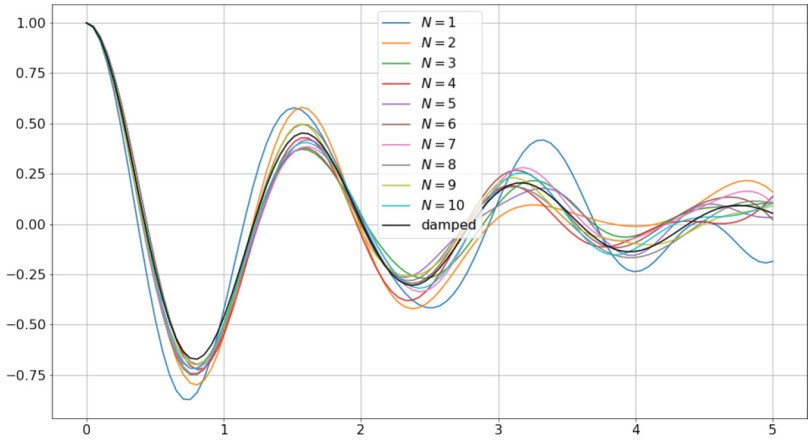



**Figure 3.** Approximation of a damped harmonic motion via an undamped one coupled with other 2*N* harmonic oscillators with different frequencies.

motion by the undamped central one decreases, as expected (Figure 2(b)): the best trajectory, obtained for $N = 10$, is plotted against the damped one in Figure 2(a).

The previous example shows that, even for the simple case when all oscillators have the same frequency, the central component of the coupled system looses energy in the specified time interval, approximating in a very satisfactory way the damped solution. Of course, a more costly optimization, with the same parameters as before but optimizing $2N + 1$ different frequencies, provides better results.

The result is shown in Figure 3 where the central solutions for $N = 1, 2, ..., 10$, corresponding to systems with $3, 5, ..., 21$ DOFs, are plotted (also in this case, we considered all masses equal to one, and the following parameters: $\gamma = 0.5$ and $\omega = 4$). The optimal value found is $\omega_0 = 3.84$, somewhat "near" to the corresponding parameter of the damped system.

It is worth to remark that also in this case, it is seen that the damping trend of the central trajectory may be observed already in the $N = 1$ case, with three equations. The best trajectory, obtained for $N = 10$, is plotted against the damped one in Figure 4(a).

Although optimizing $4N + 1$ instead of $2N + 1$ parameters requires much more iterations and a different parameters tuning, nevertheless, the error decreases faster than in the case of oscillators with the same frequency and a general decreasing trend may be conjectured notwithstanding these numerical fluctuations.

*Example 2.2.* In the case of an elastic coupling of harmonic oscillators with the same frequency, the simulations resulted in a value near to the frequency of the damped oscillator to approximate; this suggested

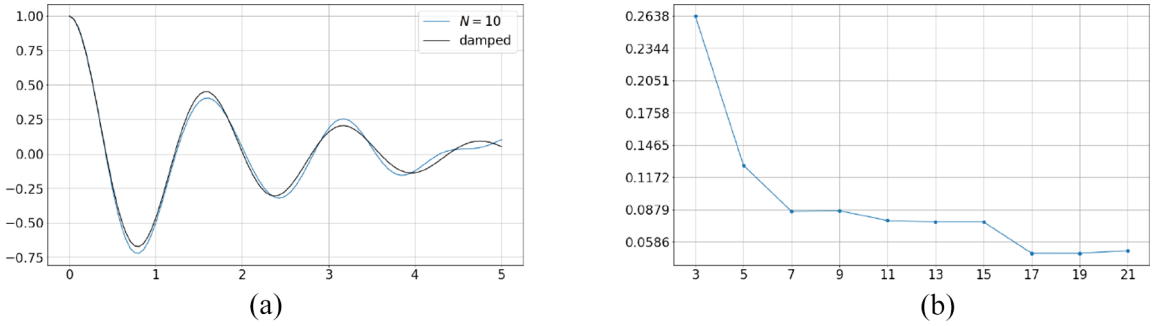



**Figure 4.** (a) Approximation of a damped harmonic motion via an undamped one coupled with other 20 harmonic oscillators with different frequencies. (b) The approximation error decreases as the degrees of freedom increase (other parameters being equal) up to ending numerical fluctuations.

to try an optimization where the frequencies of the elastically coupled oscillators are all equal to the frequency of the damped oscillator, and only the coupling constants are optimized, starting from values depending on $\gamma$, since for $\gamma = 0$ the damped system would reduce to (anyone of) the uncoupled harmonic oscillators.

The results shown in Figure 5 display a trend similar to the other cases, of course, with a worse approximation, but nevertheless the error seems to be decreasing in the number of DOFs.

*Example 2.3.* We also tried to approximate with the same method (and the same parameters) a quadratic damped nonlinear equation, namely, the motion of a pendulum immersed in a medium which exerts a force proportional to the square of the velocity

$$\ddot{x} + 2\gamma\dot{x}|\dot{x}| + \omega^2 \sin x = 0. \tag{7}$$

(This equation is studied, for example, in Stoker [13].)

Of course, due to the non-linearity of the equation, using the same parameters as in the linear case of viscous damping, the convergence of the approximation is slower; nevertheless, as shown in Figure 6 (same frequencies) and in Figure 7 (different frequencies), the clear indication of an improvement as long as the DOFs increase can be observed. In this case, too, simulations with different frequencies (see Figure 7) display numerical fluctuations in the tail of the error curve.

Until now, we considered in our examples the approximation of damped oscillations corresponding to underdamped solutions of damped systems; it is worth to notice that our approximation also works in the case of critical and overdamped solutions; we show this for the damped harmonic oscillator approximated by a system of elastically coupled harmonic oscillators with different frequencies.

*Example 2.4.* Consider equation (1) with coefficients $\omega = 2$ and $\gamma = 2$: these values correspond to a critical damping; thus, the trajectory is not oscillatory anymore and the motion rapidly converges to the zero amplitude.

Let us observe that, in this example as in the following one, our main concern is not the search of the set of parameters which can guarantee the best approximation of the critical damped oscillator, but only to show that a general method can provide a suitable approximation of the harmonic oscillator using the same conservative system, even in the absence of oscillations. Indeed, we used the same parameters setup as in the case of the underdamped motion. This is the reason why the results shown in Figure 8 are apparently not completely satisfactory. However, even if not in a monotonic way, the error seems to show an overall decrease, as the DOFs increase. The search of the optimal parameter values would for sure give a better approximation and a clearer decrease of the error, as the DOFs increase. However, as expected, the search for the optimal parameters is, in this example as in the following one, much more computationally costly.

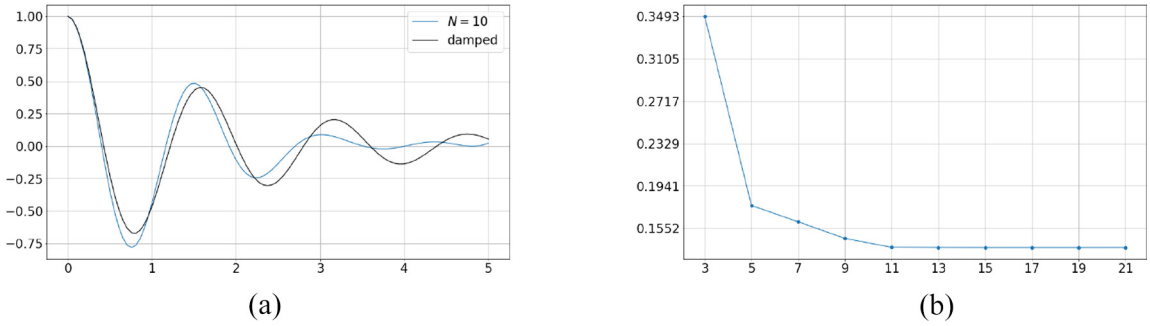



**Figure 5.** (a) Approximation of a damped harmonic motion via an undamped one coupled with other 20 harmonic oscillators with the same frequency of the damped oscillator. (b) The approximation error decreases as the degrees of freedom increase (other parameters being equal).

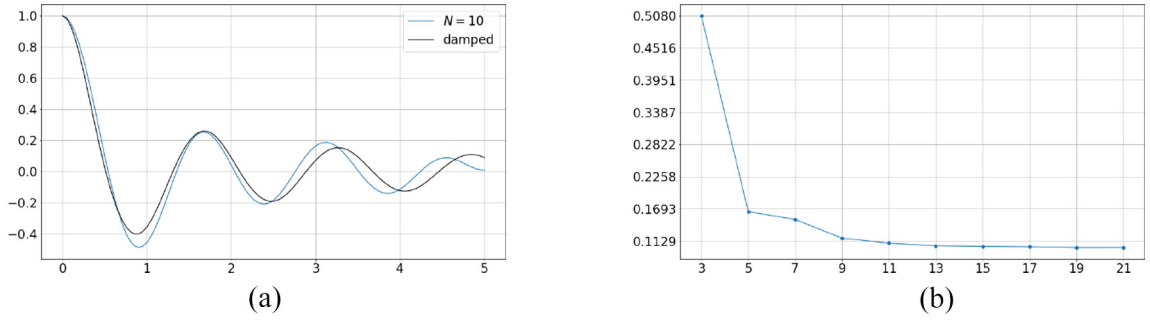



**Figure 6.** (a) Approximation of a quadratic damped harmonic motion via an undamped one coupled with other 20 harmonic oscillators with the same frequency. (b) The approximation error decreases as the degrees of freedom increase (other parameters being equal).

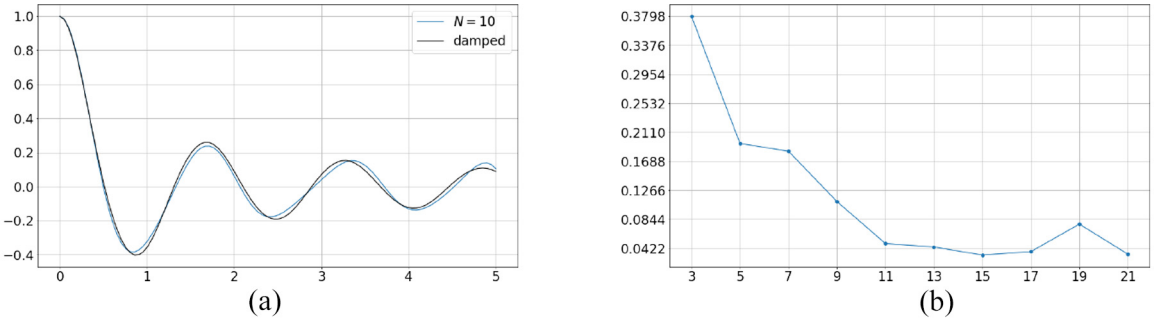
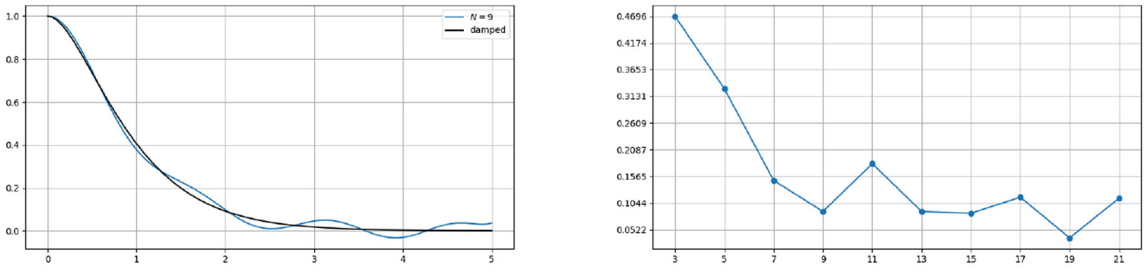



**Figure 7.** (a) Approximation of a quadratic damped harmonic motion via an undamped one coupled with other 20 harmonic oscillators with different frequencies. (b) The approximation error seems to show an overall decrease as the degrees of freedom increase (other parameters being equal) up to ending numerical fluctuations.

*Example 2.5.* Consider equation (1) with coefficients $\omega = 2$ and $\gamma = 3$: these values correspond to an overdamped solution; also in this case, as shown in Figure 9, the motion approaches zero amplitude without oscillations but more slowly than in the critical case. The same remarks as in the critical case do apply, concerning the lower performances in approximating the actual motion via the elastically coupled system and the error decrease.




(a) (b)

**Figure 8.** (a) Approximation of a critical damped harmonic motion via an undamped one coupled with other 20 harmonic oscillators with the same frequency of the damped oscillator. (b) The approximation error shows an overall decrease, as the degrees of freedom increase (other parameters being equal).

These numerical evidences encourage to investigate elastic couplings as possible solutions to our main question; in the next section, we will consider a particular case of elastic coupling, which, in our opinion, can be used for theoretical studies and which stems from an outstanding infinite dimensional system.

## 3. Schrödinger infinite chain

Behind the idea of approximating a dissipative system with a conservative one in a finite interval of time lies the fact that the behavior of the conservative system may be apparently dissipative in that interval; so far we considered examples of conservative harmonic oscillators coupled in such a way that, despite their total energy remains constant, in a finite interval, one of them looses energy which is gained by another one.

One could also look for a conservative system with infinite DOFs, which may exhibit many local behaviors, and "cut it" somehow in order to obtain a finite system which approximates, in some sense, the original non-conservative one. Loosely speaking, in this way, we would use the finite approximation of the infinite system as an approximation for the dissipative system, too.

The use of such infinite assemblies of harmonic oscillators is far from new, and an outstanding example has been recently pointed out: indeed, in 1914, Erwin Schrödinger published a noteworthy paper [8] in which he considered a system of material points elastically paired, a paper only very recently translated into English.[4] The idea of Schrödinger may be resumed as follows: let us consider a bilateral sequence of harmonic oscillators indexed by integer numbers $k \in \mathbb{Z}$, in such a way that the $k$th oscillator is coupled both with the $(k-1)$th and with the $(k+1)$th by a spring. We imagine the sequence as lying on a line, so we speak about a *chain*.

We assume that all oscillators have the same mass $m$, which we will assume to be 1, and the same frequency $\omega$ (Schrödinger uses the $f$ letter which is quite confusing in some formulas). The equation of the displacement $\xi_k$ from the equilibrium point for the $k$th oscillator is

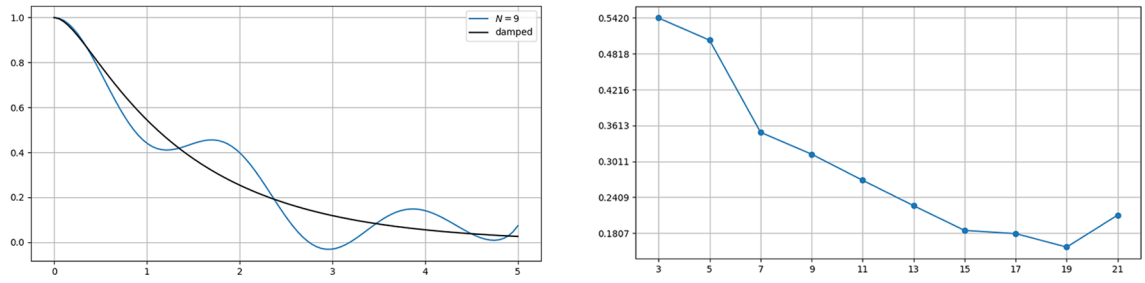
$$\ddot{\xi}_k = \omega^2(\xi_{k+1} - \xi_k) - \omega^2(\xi_k - \xi_{k-1}),$$

thus

$$\ddot{\xi}_k = \omega^2(\xi_{k+1} - 2\xi_k + \xi_{k-1}). \tag{8}$$

One may notice that this system with infinite DOFs is Lagrangian and we may write its Lagrangian function as

$$L = \frac{1}{2}\sum_{k\in\mathbb{Z}}\left(\dot{\xi}_k^2 - \omega^2(\xi_{k+1} - \xi_k)^2\right).$$




(a) (b)

**Figure 9.** (a) Approximation of an overdamped harmonic motion via an undamped one coupled with other 20 harmonic oscillators with the same frequency of the damped oscillator. (b) Also in this case, although in a less clear way, the approximation error seems to show an overall decrease, as the degrees of freedom increase (other parameters being equal).

The initial conditions considered for the system are

$$\begin{cases} \xi_k(0) = a \cdot \delta_{k0}, \\ \dot{\xi}_k(0) = 0. \end{cases}$$

(where $\delta_{k0}$ is the Kronecker delta), being $a \in \mathbb{R}$ fixed. Therefore, all oscillators are in their equilibrium position at start, with the possible exception of the 0th.

Schrödinger introduces the auxiliary variables

$$x_{2k} = \dot{\xi}_k, \quad x_{2k+1} = \omega(\xi_k - \xi_{k+1}),$$

so that

$$\begin{aligned} \dot{x}_{2k} + \omega(x_{2k+1} - x_{2k-1}) &= \ddot{\xi}_k + \omega^2(\xi_k - \xi_{k+1}) - \omega^2(\xi_{k-1} - \xi_k). \\ &= \ddot{\xi}_k + \omega^2(\xi_k - \xi_{k+1} - \xi_{k-1} + \xi_k) = 0. \end{aligned}$$

Analogously, for odd indexes, we find that

$$\dot{x}_{2k+1} + \omega(x_{2k+2} - x_{2k}) = \omega\left(\dot{\xi}_k - \dot{\xi}_{k+1}\right) + \omega\left(\dot{\xi}_{k+1} - \dot{\xi}_k\right) = 0.$$

Hence, in the new variables $x_k$, the system reduces to

$$\dot{x}_k + \omega(x_{k+1} - x_{k-1}) = 0.$$

Schrödinger built a solution of this infinite system of differential equations as follows: fix $k \in \mathbb{Z}$ and take for any $h \in \mathbb{Z}$

$$x_h = x_h(0) J_{h-k}(2\omega t),$$

being $J_{k-h} = (-1)^{k-h} J_{h-k}$ a Bessel function: this is a direct consequence of well-known facts about special functions.[5] Another way to get the same result is via generating functions as done by Razavy [14].

In any case, the general solution $x_k$ is therefore computed as a superposition of all these linearly independent solutions as $k \in \mathbb{Z}$

$$x_k = \sum_{h=-\infty}^{\infty} x_h(0) J_{k-h}(2\omega t).$$

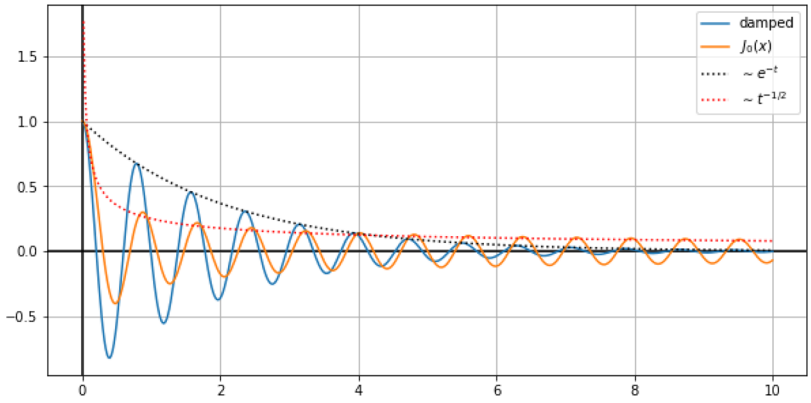



**Figure 10.** Comparison between a Bessel function and a solution of the damped harmonic oscillator: it is clearly seen that the amplitude of the former decreases as $t^{-1/2}$, while the latter decreases as $e^{-t}$.

Next, the initial conditions for $\xi_k$ imply

$$\begin{cases} x_1(0) = \omega a, \\ x_{-1}(0) = -\omega a, \\ x_k(0) = 0 \quad \text{for } k \neq -1 \text{ and } k \neq 1. \end{cases}$$

so that the Schrödinger system is solved by

$$\xi_k = aJ_{2k}(2\omega t).$$

For a different derivation of this result using generating functions we refer to Razavy's paper [15], where the same result is also obtained by solving equation (8) by a Fourier transform which leads to

$$\xi_l(t) = \frac{a}{2\pi i^{2l}} \int_0^{2\pi} \exp(2i\omega \cos\tau) \cos(2l\tau) d\tau.$$

Razavy also writes down the Hamiltonian of this system and computes its energy as

$$E = \omega^2 a^2.$$

The solution of the Schrödinger infinite system has also been given in terms of Fourier series by Kreuzer in [16]. Our interest in the Schrödinger model stems from the trend of the solutions expressed in terms of the Bessel functions $J_{2n}$, in particular for the fact that they exhibit an oscillatory and decreasing pattern. Indeed, the motion of elements of the Schrödinger chain may display a seemingly damped behavior; however, their decreasing is not exponential. Take, for example, solution $\xi_0 = aJ_2(2\omega t + \delta)$ and compare it with the trajectory of the damped motion (see Figure 10).

The crucial remark is developed by Schrödinger himself when he says [8] that asymptotically the function $J_n(2\omega t)$ behaves as

$$\sqrt{\frac{1}{\pi t}} \cos\left(2\omega t - \frac{2n+1}{4}\pi\right),$$

so that the amplitude of phases of his system decreases as $t^{-1/2}$ and not as $e^{-t}$.

## 4. Generalizations of the Schrödinger chain

Schrödinger was interested in his model since it is a discretization of the vibrating string; also subsequent authors dealt with this model, as Razavy in [15,17] or its generalizations, as Dyson [18], Ford et al. [19], Caldeira and Legett [20], Caldeira [21], Zanette [22], and so on. The interest of late authors was addressed to the quantization of such systems or to their use as heath bath interacting with a quantum particle.

The Schrödinger model appears in some classic treatises, too: in the thermodynamics book by Leontovich [23], where it is applied to the computation of the heath capacity of a solid at low temperatures, and in the Mechanics treatise by Goldstein [11], to illustrate the transition from finite to infinite DOFs, thus from discrete to continuum, in a dynamical system.

Curiously enough, none of these authors, apart from Razavy, cites the Schrödinger paper, even if his model is obviously considered as part of Statistical Mechanics folklore.[6] Moreover, a generalization of this construction to different frequencies, of course loosing the elegant simplicity of the Schrödinger model, appeared in the literature: indeed, this "non-uniform" infinite chain was considered by Dyson [18] who was interested in the case of chains with random frequencies. The same model was employed by Razavy [17], who was interested in quantization issues.

In this more general case, motion equations are

$$m_k\ddot{\xi}_k = \omega_k^2(\xi_{k+1} - \xi_k) + \omega_{k-1}^2(\xi_{k-1} - \xi_k).$$

We will assume that all masses are equal (to 1)

$$\ddot{\xi}_k = \omega_k^2(\xi_{k+1} - \xi_k) + \omega_{k-1}^2(\xi_{k-1} - \xi_k).$$

A Lagrangian function for this infinite system is

$$L = \frac{1}{2}\sum_{k\in\mathbb{Z}}\left(\dot{\xi}_k^2 - \omega_k^2(\xi_{k+1} - \xi_k)^2\right).$$

We mimic the Schrödinger reasoning for this more general system on putting

$$\tilde{\omega}_{2k+1} = \tilde{\omega}_{2k} = \omega_k,$$

and

$$\begin{cases} x_{2k} = \dot{\xi}_k, \\ x_{2k+1} = \tilde{\omega}_{2k+1}\xi_{k+1} - \tilde{\omega}_{2k}\xi_k, \end{cases}$$

from which

$$\begin{cases} x_{-1}(0) = a\tilde{\omega}_{-1}, \\ x_1(0) = -a\tilde{\omega}_0, \\ x_k(0) = 0 \quad \text{for } k \notin \{-1, 1\}. \end{cases} \tag{9}$$

Then, if we compute $\dot{x}_{2k}$, we find

$$\begin{aligned} \dot{x}_{2k} = \ddot{\xi}_k &= \omega_k^2(\xi_{k+1} - \xi_k) + \omega_{k-1}^2(\xi_{k-1} - \xi_k), \\ &= \tilde{\omega}_{2k}(\tilde{\omega}_{2k+1}\xi_{k+1} - \tilde{\omega}_{2k}\xi_k) + \tilde{\omega}_{2k-1}(\tilde{\omega}_{2k-2}\xi_{k-1} - \tilde{\omega}_{2k-1}\xi_k), \\ &= \tilde{\omega}_{2k}x_{2k+1} - \tilde{\omega}_{2k-1}x_{2k-1}. \end{aligned}$$

However, it is immediate to compute $\dot{x}_{2k+1}$

$$\dot{x}_{2k+1} = \tilde{\omega}_{2k+1}\dot{\xi}_{k+1} - \tilde{\omega}_{2k}\dot{\xi}_k = \tilde{\omega}_{2k+1}x_{2k+2} - \tilde{\omega}_{2k}x_{2k}.$$

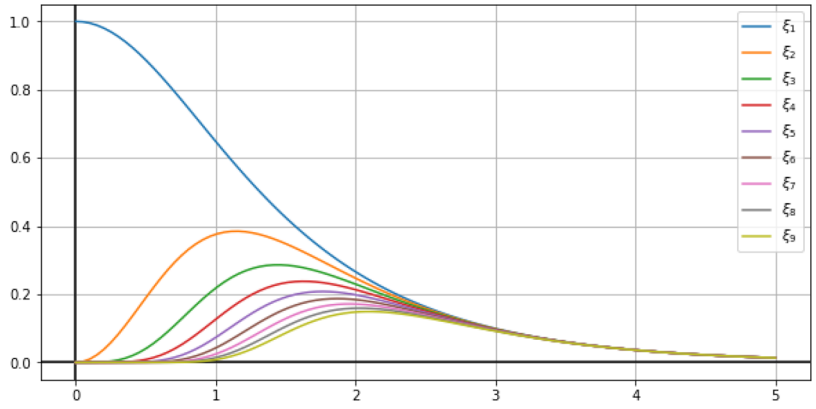



**Figure 11.** Some solutions of the Razavy chain.

Therefore, for each $k \in \mathbb{Z}$

$$\dot{x}_k = \tilde{\omega}_k x_{k+1} - \tilde{\omega}_{k-1} x_{k-1}. \tag{10}$$

This equation reduces to the Schrödinger one when all $\tilde{\omega}_k$ are equal.

Razavy [17] solves this system, using generating functions, under the assumption that $\tilde{\omega}_k = k\lambda$; in this particular case, it is not difficult to prove that the generating function of the infinite system satisfies the differential equation

$$\frac{1}{\lambda}\frac{\partial G}{\partial t} = \frac{\partial G}{\partial z} - z\frac{\partial (zG)}{\partial x},$$

where $z$ is the formal series indeterminate; the solutions of this equation lead to the following solution for the infinite non-uniform chain

$$\xi_k(t) = \frac{(\tanh \lambda t)^{2k-2}}{\cosh \lambda t}.$$

Some of these solutions are plotted in Figure 11; they all display a damped behavior, and provide a further hint to the approximation of damped motion, although in a particular case.

## 5. Cut-off of Schrödinger chains

Now let us explain why we are interested in the Schrödinger chain, both in the uniform and non-uniform versions; indeed, a finite dimensional cut-off (which may well be seen as a numerical approximation to the solution of the infinite DOFs system) of this chain results in a particular case of the elastic coupling system we considered in section 2.

To see it, let us consider an approximation of the Schrödinger infinite chain, where we impose a cut-off and consider only $2N+1$ oscillators ($k = -N, -N+1, ..., N-1, N$). Then, the system of coupled (non-uniform) Schrödinger oscillators can be written in the following way

$$\begin{cases} \ddot{\xi}_{-N} = \omega^2_{-N}(\xi_{-N+1} - \xi_{-N}), \\ \ddot{\xi}_{-N+1} = \omega^2_{-N+1}(\xi_{-N+2} - \xi_{-N+1}) + \omega^2_{-N}(\xi_{-N} - \xi_{-N+1}), \\ \ddot{\xi}_{-N+2} = \omega^2_{-N+2}(\xi_{-N+3} - \xi_{-N+2}) + \omega^2_{-N+1}(\xi_{-N+1} - \xi_{-N+2}), \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \ddot{\xi}_{N-2} = \omega^2_{N-2}(\xi_{N-1} - \xi_{N-2}) + \omega^2_{N-3}(\xi_{N-3} - \xi_{N-2}), \\ \ddot{\xi}_{N-1} = \omega^2_{N-1}(\xi_{N} - \xi_{N-1}) + \omega^2_{N-2}(\xi_{N-2} - \xi_{N-1}), \\ \ddot{\xi}_{N} = -\omega^2_{N}\xi_{N} + \omega^2_{N-1}(\xi_{N-1} - \xi_{N}). \end{cases} \tag{11}$$

We can write the Lagrangian function of this system with a finite number of DOFs

$$L = \frac{1}{2}\sum_{k=-N}^{N-1}\left(\dot{\xi}_k^2 - \omega^2(\xi_{k+1}-\xi_k)^2\right) + \frac{1}{2}\left(\dot{\xi}_N^2 - \omega_N^2\xi_N^2\right).$$

Also in this case, we mimic the Schrödinger reasoning on putting

$$\tilde{\omega}_{2k+1} = \tilde{\omega}_{2k} = \omega_k,$$

and

$$\begin{cases} x_{2k} = \dot{\xi}_k, \\ x_{2k+1} = \tilde{\omega}_{2k+1}\xi_{k+1} - \tilde{\omega}_{2k}\xi_k, \end{cases} \tag{12}$$

from which

$$\begin{cases} x_{-1}(0) = a\tilde{\omega}_{-1}, \\ x_1(0) = -a\tilde{\omega}_0, \\ x_k(0) = 0 \quad \text{for } k \notin \{-1, 1\}. \end{cases} \tag{13}$$

Again, if we compute $\dot{x}_{2k}$ and $\dot{x}_{2k+1}$, we find a system of $M = 4N+2$ equations

$$\begin{cases} \dot{x}_{-2N} = \tilde{\omega}_{-2N}x_{-2N+1}, \\ \dot{x}_{-2N+1} = \tilde{\omega}_{-2N+1}x_{-2N+2} - \tilde{\omega}_{-2N}x_{-2N}, \\ \dot{x}_{-2N+2} = \tilde{\omega}_{-2N+2}x_{-2N+3} - \tilde{\omega}_{-2N+1}x_{-2N+1}, \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \dot{x}_{2N-2} = \tilde{\omega}_{2N-2}x_{2N-1} - \tilde{\omega}_{2N-3}x_{2N-3}, \\ \dot{x}_{2N-1} = \tilde{\omega}_{2N-1}x_{2N} - \tilde{\omega}_{2N-2}x_{2N-2}, \\ \dot{x}_{2N} = \tilde{\omega}_{2N}x_{2N+1} - \tilde{\omega}_{2N-1}x_{2N-1}, \\ \dot{x}_{2N+1} = -\tilde{\omega}_{2N}x_{2N+1}. \end{cases} \tag{14}$$

Rewriting the system in matrix form

$$\dot{\mathbf{x}} = A\mathbf{x}\ ,$$

the matrix $A$ is skew-symmetric and tridiagonal

$$\begin{pmatrix} 0 & \tilde{\omega}_{-2N} & 0 & 0 & \dots & 0 & 0 & 0 \\ -\tilde{\omega}_{-2N} & 0 & \tilde{\omega}_{-2N+1} & 0 & \dots & 0 & 0 & 0 \\ 0 & -\tilde{\omega}_{-2N+1} & 0 & \tilde{\omega}_{-2N+2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \tilde{\omega}_{2N-1} & 0 \\ 0 & 0 & 0 & 0 & \dots & -\tilde{\omega}_{2N-1} & 0 & \tilde{\omega}_{2N} \\ 0 & 0 & 0 & 0 & \dots & 0 & -\tilde{\omega}_{2N} & 0 \end{pmatrix}. \tag{15}$$

This first-order system is characterized by $2N+1$ parameters $\tilde{\omega}_k$, which can be used to tune the solutions in order to approximate the trajectory of the damped harmonic motion. Now let us observe that, when the masses are all equal to 1, after transformation (12), system (11) becomes a special case of the elastically coupled system (5), setting $\omega_{k-1}^2$ instead of $\omega_{k-1,1}$ and $(\omega_{k-1}^2 + \omega_k^2)$ instead of $\omega_k^2$.

This means that when we try to approximate the original undamped oscillator with a truncated Schrödinger chain, the parameter values can vary in more restrictive ranges than in the former case. Thus, we expect that the approximation with elastically coupled oscillators will be, in general, more efficient (although not less effective) than any approximation with truncated Schrödinger chains.

However, our previous numerical evidences apply to this case as well, although, for the reasons just stated, we get in this case a worse convergence and less numerical stability (parameters being equal to

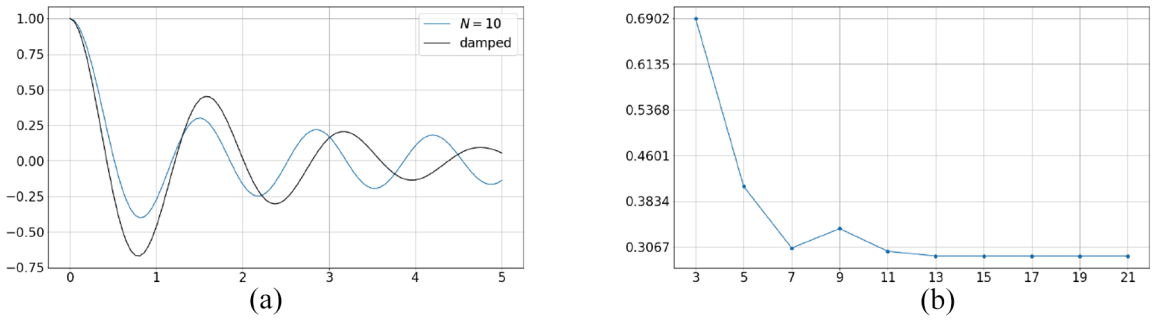



**Figure 12.** (a) Approximation of a damped harmonic motion via Schrödinger cut-off of 21 harmonic oscillators with the same frequency. (b) The approximation error decreases as the degrees of freedom increase (other parameters being equal).

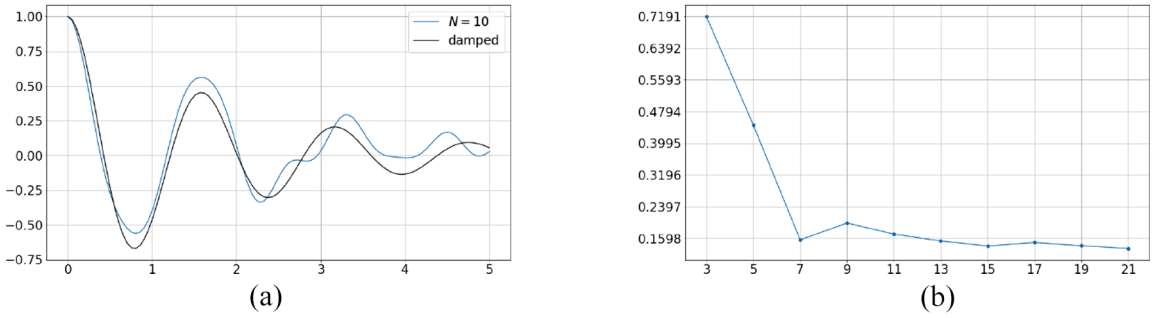



**Figure 13.** (a) Approximation of a damped harmonic motion via Schrödinger cut-off of 21 harmonic oscillators with different frequencies. (b) The approximation error decreases as the degrees of freedom increase (other parameters being equal) up to ending numerical fluctuations.

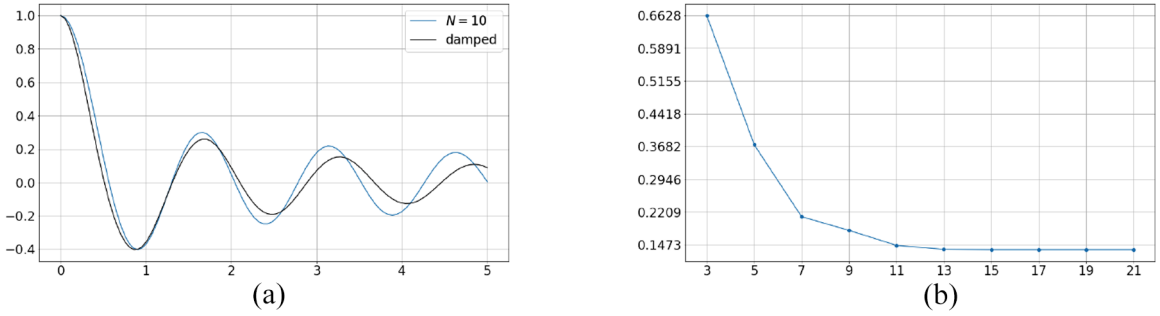



**Figure 14.** (a) Approximation of a quadratic damped harmonic motion via Schrödinger cut-off of 21 harmonic oscillators with the same frequencies. (b) The approximation error decreases as the degrees of freedom increase (other parameters being equal).

previous numerical computations) (see Figures 12 and 13). This can be observed also when approximating the quadratic damped harmonic motion, described in Example 2.3, via the Schrödinger cut-off of $2N+1$ harmonic oscillators with the same frequencies (Figure 14(a) and (b)) and via the Schrödinger cut-off of $2N+1$ harmonic oscillators with different frequencies (Figure 15(a) and (b)).

Even worse are the results shown in Figures 16 and 17 in the approximation of critical ($\omega = 2$ and $\gamma = 2$) and overdamped ($\omega = 2$ and $\gamma = 3$) cases, as expected; as a general remark, the optimizer should be fine tuned according to the specific case we want to approximate, but our concern here is to show

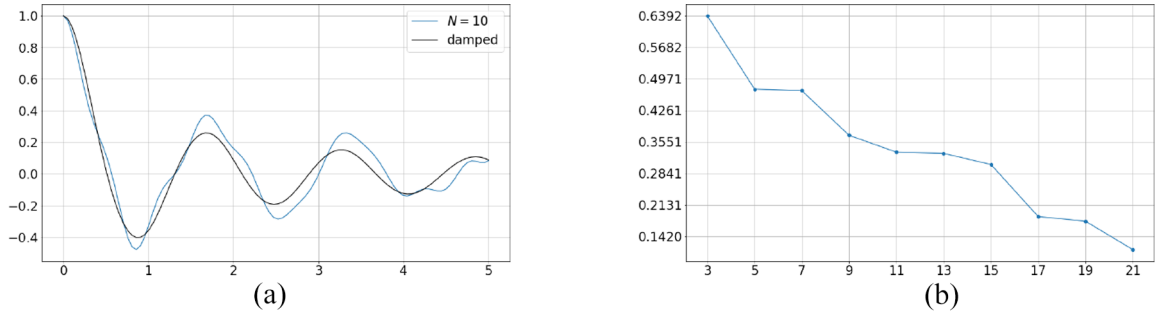



**Figure 15.** (a) Approximation of a quadratic damped harmonic motion via Schrödinger cut-off of 21 harmonic oscillators with different frequencies. (b) The approximation error decreases as the degrees of freedom increase (other parameters being equal) up to ending numerical fluctuations.

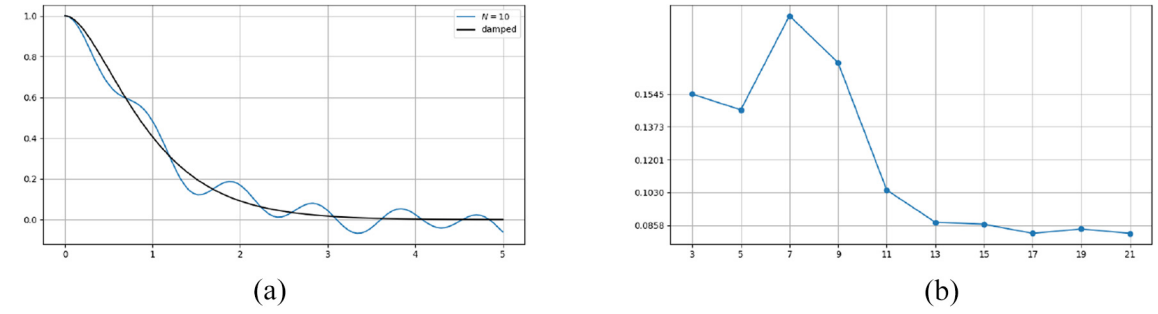



**Figure 16.** (a) Approximation of a critical damped harmonic motion via Schrödinger cut-off of 21 harmonic oscillators with different frequencies. (b) The approximation error decreases as the degrees of freedom increase (other parameters being equal) up to ending numerical fluctuations.

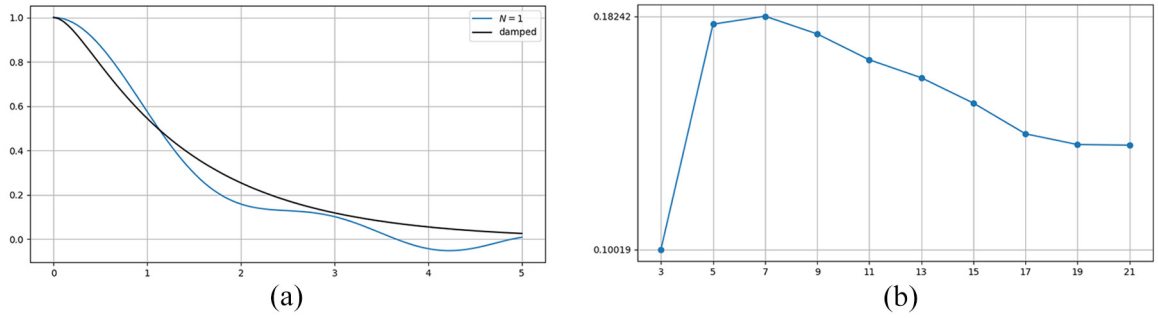



**Figure 17.** (a) Approximation of an overdamped harmonic motion via Schrödinger cut-off of 21 harmonic oscillators with different frequencies. (b) The approximation error decreases as the degrees of freedom increase (other parameters being equal) up to ending numerical fluctuations. Interestingly, mainly due to the stochasticity of the optimization method, in these simulations the error for $N = 3$ is very small.

that the approximation is feasible in general, even if, in order to get a top performance, one should have to calibrate parameters according to the specific problem to deal with.

However, the cut-off Schrödinger system is simpler than the general elastic one, so we expect it to be easier to deal with it at theoretical level: that is why still we think it is a useful tool to attack our main question. Let us also observe that, differently from the matrix related to system (5), which can be decomposed into the sum of a diagonal matrix and a skew-symmetric one, the matrix related to system (14) is skew-symmetric.

It is well known that, in general, the non-zero eigenvalues of a skew-symmetric matrix with real coefficients are couples of conjugate pure imaginary numbers (see, for example, Gantmacher [24]). This implies that the basis of the general integral is formed by oscillating functions, apart from a constant one, when the order of the matrix is odd.

Due to the simple, tridiagonal form of the matrix, one could look for explicit formulas for the eigenvalues, which could give interesting insight about the analytical structure of the solutions of system (5), beyond the numerical simulations. This topic will be deepened in a forthcoming paper.

## 6. Conclusion

In this paper, we investigated the possibility of approximating a given one-dimensional dissipative system by means of a conservative one, in such a way that, for any $\varepsilon > 0$ and any $T > 0$, there exists a component of the solution of the conservative system whose distance, in the uniform norm, from the solution of the dissipative one, computed in $[0, T]$, can be made less than $\varepsilon$.

The main goal of this paper was to give some preliminary numerical evidences toward a positive answer to the question. The simulations give a clear hint about the achievability of the approximation. The damped system can be approximated by different undamped systems, with different efficacy.

Starting from the simple and well-understood case of the damped oscillator, we followed the idea of considering a system of a finite number of coupled oscillators, in order to determine the optimal elastic and coupling constants which can guarantee that the central oscillator of the undamped system can approximate the damped oscillator in terms of minimization of the distance. It is crucial to understand how the tuning of parameters $\tilde{\omega}_k$ can influence the search for the best approximation (in the uniform norm) of the original damped harmonic oscillator. To this aim we applied numerical optimization techniques.

Our studies can be further developed, for example, by analyzing, first from a numerical point of view, further conservative systems characterized by different coupling terms (e.g., gyroscopic coupling). The basic idea we intend to pursue on the basis of obtained results is to find an algorithm capable to determine an extended Lagrangian capable to effectively approximate the motion of a given dissipative system for the largest possible class of initial data. The influence of the initial data on the approximation efficacy clearly depends on the present non-linearities: it is, however, likely that the extended DOFs may be linearly coupled to the initial DOFs and among themselves.

In terms of possible applications, the paper aims to give a deep insight into the possibility of modeling the dissipative phenomenon as an exchange of energy between the main system and a "hidden" system characterized by many DOFs that behaves as a damper. The damping property of a mechanical system is an essential feature in many applications. To provide a gallery of the potential important fields of application, it suffices to mention some examples of one-dimensional (1D) structures (see, for example, previous studies [25–28]), two-dimensional (2D) structures [29], robotic arms [30,31], building or granular materials [32–34], biosystems [35], viscoelastic materials [36–38], and wave propagation in generalized media where thermal effects are taken into account [39,40]. In these cases, the mechanical system has to be synthesized, focusing on the demand of producing a particular attitude to damping. Specifically, the microstructure of considered materials can be designed to have hidden DOFs to produce some dissipative effect at the macro level of observation. This way of thinking is becoming more and more popular nowadays and characterizes the design of those "exotic" (i.e., having unconventional properties) materials which are often called metamaterials (see, for some relevant examples, [41–50]).

The proposed approach for modeling dissipative systems can be also promising when used to mathematically model with discrete models the source of dissipation classically traced to damage and plasticity phenomena (see, for example, [51–56]). The possibility of finding a conservative multi-DOF model to "improve" a dissipative one can be very useful to simplify the analysis of the class of inverse problems studied in Turco [57]. The idea here is that the behavior of the physical system to be characterized can be in some way "fitted" by means of a conservative model that, at least in a limited time interval, is capable to describe a "macroscopic" apparent dissipation.

Moreover, encouraged by the presented first evidences, a deeper investigation about the theoretical justification of the numerical approximations has to be studied: in particular, investigating the eigenvalues of the matrix of the system of ordinary differential equations governing the undamped system, in

the case of linear equations. As observed in the previous sections, the particular structure of the tridiagonal skew-symmetric matrix related to the truncated Schrödinger chain could bring to explicit formulas for the eigenvalues, which, at present, can be determined only for some explicit cases, for $N$ sufficiently small. Such an apparently exclusively technical searched result may be of great help in the aforementioned synthesis problem of extended Lagrangians, whose importance in applications cannot be underestimated. We are addressing this specific topic in a work in preparation.

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## Supplemental material

Supplemental material for this article is available online.

## Notes

1. Of course for the damped harmonic oscillator, as it can be exactly solved, the question is quite useless in practice, but our aim is to get a better insight in the problem to prepare ourselves to more complicated systems.
2. Actually, it is possible to write Lagrangians for the single damped equation (1), see Bersani and Caressa [5].
3. See the Supplemental Appendix for details on such computations.
4. An English translation with an introduction and commentary underlying the importance of such a contribution, shadowed by his later works on wave mechanics and applications to quantum theory, has been recently published by Mülich et al. [7].
5. Recurrent formulas for the Bessel functions and related differential equations are explained, for example, in Bowman [58].
6. This circumstance justifies the recent translation published by Mülich et al. [7].

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