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# Supermanifolds corresponding to the trivial vector bundle over torus

Mikhail Bashkin

**Abstract.** All supermanifolds whose retract  $T^{m|n}$  is determined by the trivial bundle of rank  $n$  over the torus  $T^m$  are  $\bar{0}$ -homogeneous and only  $T^{m|n}$  is homogeneous.

## 1 Preliminaries

**1.1 Split and non-split supermanifolds** The ground field is  $\mathbb{C}$ .

A *complex supermanifold* of dimension  $m|n$  is a  $\mathbb{Z}/2$ -graded ringed space of the form  $\mathcal{M} := (M, \mathcal{O})$ , where  $M$  is a topological space and  $\mathcal{O}$  is a sheaf of associative commutative superalgebras with unit on  $M$ , which is locally isomorphic to a superdomain in  $\mathbb{C}^{m|n}$ . For details, see [M], [O3], [BGLS\*]. A *superdomain* in  $\mathbb{C}^{m|n}$  is a pair  $(U, \bigwedge_{\mathcal{F}}(\xi_1, \dots, \xi_n))$ , where  $U$  is an open subset of  $\mathbb{C}^m$ , and  $\mathcal{F}$  is the sheaf of holomorphic functions on  $\mathbb{C}^m$ . The coordinates  $x_1, \dots, x_m$  in  $U \subset \mathbb{C}^m$  and generators  $\xi_1, \dots, \xi_n$  of the Grassmann algebra are identified with some sections of the sheaf  $\mathcal{O}|_U$ . They are called *local coordinates even* and *odd*, respectively.

Let  $(M, \mathcal{F})$  be a complex manifold and  $\mathcal{E}$  be a locally free analytic sheaf on it, i.e.,  $\mathcal{E}$  is a sheaf of holomorphic sections of some holomorphic vector bundle  $\mathbf{E} \rightarrow M$ . Then,  $(M, \mathcal{O}_{\text{gr}})$ , where  $\mathcal{O}_{\text{gr}} = \bigwedge_{\mathcal{F}} \mathcal{E}$ , is a complex supermanifold. A supermanifold is called *split* if it is isomorphic to a supermanifold of this form and is called *non-split* otherwise.

Let us show that every supermanifold is a deformation of a split supermanifold. Consider the subsheaf of ideals  $\mathcal{J} = (\mathcal{O}_{\bar{1}})$ , generated by odd elements. Denote  $\mathcal{F} := \mathcal{O}/\mathcal{J}$ . Then,  $\mathcal{M}_{\text{rd}} = (M, \mathcal{F})$  is a complex manifold called the *odd reduction* of  $(M, \mathcal{O})$ . The powers of  $\mathcal{J}$  determine the following filtration:

$$\mathcal{O} = \mathcal{J}^0 \supset \mathcal{J}^1 \supset \mathcal{J}^2 \supset \dots \supset \mathcal{J}^{n+1} = 0. \quad (1)$$

The associated sheaf of graded algebras,  $\text{gr } \mathcal{O} = \bigoplus_{0 \leq p \leq n} \text{gr}_p \mathcal{O}$ , where  $\text{gr}_p \mathcal{O} = \mathcal{J}^p / \mathcal{J}^{p+1}$ , is an analytic sheaf on the reduction  $\mathcal{M}_{\text{rd}}$ . Actually,  $\text{gr } \mathcal{O} \simeq \bigwedge_{\mathcal{F}} \mathcal{E}$ , where  $\mathcal{E} = \text{gr}_1 \mathcal{O}$

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is locally free sheaf. Clearly,  $(M, \text{gr } \mathcal{O}) = (M, \mathcal{O}_{\text{gr}})$  is a split supermanifold of the same dimension as  $(M, \mathcal{O})$ . We call it the *retract* of the supermanifold  $(M, \mathcal{O})$ . Obviously, a given supermanifold is split if and only if it is isomorphic to its retract.

Let  $\pi_p : \mathcal{J}^p \rightarrow \text{gr}_p \mathcal{O}$  be the canonical projection. Then, there is the exact sequence of sheaves

$$0 \longrightarrow \mathcal{J}^{p+1} \longrightarrow \mathcal{J}^p \xrightarrow{\pi_p} \text{gr}_p \mathcal{O} \longrightarrow 0. \quad (2)$$

A supermanifold  $(M, \mathcal{O})$  is split if and only if there exists an isomorphism of superalgebra sheaves  $h : \text{gr } \mathcal{O} \rightarrow \mathcal{O}$ , whose restriction  $h_p : \text{gr}_p \mathcal{O} \rightarrow \mathcal{J}^p$  splits the sequence (2), i.e., satisfies the condition  $\pi_p \circ h_p = \text{id}$ . In general, this splitting exists in a neighborhood of any point in  $M$ . It can be given by means of local coordinates.

**1.2 The tangent sheaf** For an arbitrary supermanifold  $(M, \mathcal{O})$  denote by  $\mathcal{T} := \mathcal{D}er \mathcal{O}$  its *tangent sheaf* (or the sheaf of vector fields). It is the sheaf of derivations (over  $\mathbb{C}$ ) of the structure sheaf  $\mathcal{O}$ . Note that the tangent sheaf is a sheaf of  $\mathbb{Z}/2$ -graded left  $\mathcal{O}$ -modules, and also a sheaf of Lie superalgebras. The sections of the tangent sheaf are called *holomorphic vector fields* on  $(M, \mathcal{O})$ . They form the Lie superalgebra  $\mathfrak{v}(M, \mathcal{O})$  of vector fields on  $(M, \mathcal{O})$ .

Note that the filtration (1) determines the filtration of the tangent sheaf

$$\mathcal{T} = \mathcal{T}_{(-1)} \supset \mathcal{T}_{(0)} \supset \cdots \supset \mathcal{T}_{(n)} \supset \mathcal{T}_{(n+1)} = 0, \quad (3)$$

where

$$\mathcal{T}_{(p)} = \{v \in \mathcal{T} \mid v(\mathcal{O}) \subset \mathcal{J}^p, v(\mathcal{J}) \subset \mathcal{J}^{p+1}\}, p \geq 0.$$

Since  $(M, \mathcal{O}_{\text{gr}})$  is split, its tangent sheaf  $\mathcal{T}_{\text{gr}}$  is a  $\mathbb{Z}$ -graded sheaf of Lie superalgebras

$$\mathcal{T}_{\text{gr}} = \bigoplus_{-1 \leq p \leq n} (\mathcal{T}_{\text{gr}})_p,$$

where

$$(\mathcal{T}_{\text{gr}})_p := \mathcal{D}er_p \mathcal{O}_{\text{gr}} = \{v \in \mathcal{T}_{\text{gr}} \mid v((\mathcal{O}_{\text{gr}})_q) \subset (\mathcal{O}_{\text{gr}})_{q+p}, q \in \mathbb{Z}\}.$$

This grading is compatible with the  $\mathbb{Z}/2$ -grading. The Lie superalgebra  $\mathfrak{v}(M, \mathcal{O}_{\text{gr}})$  of vector fields is a graded algebra with the  $\mathbb{Z}$ -grading compatible with the  $\mathbb{Z}/2$ -grading.

Since  $\mathcal{F} \subset \mathcal{O}_{\text{gr}}$ , the tangent sheaf  $\mathcal{T}_{\text{gr}}$  is a  $\mathbb{Z}$ -graded analytic sheaf on  $M$ . This sheaf is locally free (see [O3]), and hence it is the sheaf of holomorphic sections of a  $\mathbb{Z}$ -graded holomorphic vector bundle  $\mathbf{ST}$  over  $M$  (the *supertangent bundle*).

**1.3 Sheaves of automorphisms and the classification theorem** Let  $(M, \mathcal{O})$  be an complex supermanifold. Denote by  $\text{Aut}(M, \mathcal{O})$  the group of automorphisms of  $(M, \mathcal{O})$ . By definition,  $F \in \text{Aut}(M, \mathcal{O})$  is a pair  $(f, \varphi)$ , where  $f : M \rightarrow M$  belongs to group  $\text{Bih } M$  of biholomorphic transformations of the manifold  $M$  and  $\varphi$  is an automorphism of the superalgebra sheaf  $\mathcal{O}$  over  $f$ . Denote by  $\mathcal{A}ut \mathcal{O}$  the sheaf of automorphisms of the structure sheaf  $\mathcal{O}$  (mapping every stalk  $\mathcal{O}_x$ , where  $x \in M$ , onto itself). Moreover, for any  $F = (f, \varphi) \in \text{Aut}(M, \mathcal{O})$  the map  $\text{Int } F : a \mapsto \varphi \circ a \circ \varphi^{-1}$  is an automorphism of the

group sheaf  $\mathcal{A}ut \mathcal{O}$ . Hence, we get the action  $\text{Int}$  of the group  $\text{Aut}(M, \mathcal{O})$  on  $\mathcal{A}ut \mathcal{O}$  by automorphisms. The subsheaf

$$\mathcal{A}ut_{(2)}\mathcal{O} = \{a \in \mathcal{A}ut \mathcal{O} \mid a(f) - f \in \mathcal{J}^2, f \in \mathcal{O}\} \tag{4}$$

is invariant under this action.

Let  $\mathbf{E}$  be a holomorphic vector bundle over  $(M, \mathcal{F})$  and  $\text{Aut } \mathbf{E}$  the group of its automorphisms. Clearly, any element of this group gives rise to an automorphism of the split supermanifold  $(M, \mathcal{O}_{gr})$  corresponding to  $\mathbf{E}$  which preserves the  $\mathbb{Z}$ -grading of the structure sheaf. Hence, we can identify  $\text{Aut } \mathbf{E}$  with a subgroup of  $\text{Aut}(M, \mathcal{O}_{gr})$ , consisting of automorphisms that preserve the  $\mathbb{Z}$ -grading. So the  $\text{Aut}(M, \mathcal{O}_{gr})$ -sheaves  $\mathcal{A}ut \mathcal{O}_{gr}$  and  $\mathcal{A}ut_{(2)}\mathcal{O}_{gr}$  are also  $\text{Aut } \mathbf{E}$ -sheaves.

**1.1 Theorem ([G]).** *Any supermanifold  $(M, \mathcal{O})$  corresponds to an element of the set of 1-cohomology  $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{gr})$ . This correspondence gives rise to a bijection between the isomorphism classes of supermanifolds satisfying the above condition, and the orbits of the group  $\text{Aut } \mathbf{E}$  on  $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{gr})$  under the natural  $\text{Aut } \mathbf{E}$ -action.*

Let us describe the correspondence mentioned in Theorem 1.1. Let  $(M, \mathcal{O})$  be a supermanifold with retract  $(M, \mathcal{O}_{gr})$ . Then, we can choose an open cover  $\mathfrak{U} = (U_i)_{i \in I}$  of  $M$  such that there exist isomorphisms  $h_i : \mathcal{O}_{gr}|_{U_i} \rightarrow \mathcal{O}|_{U_i}$ , where  $i \in I$ , with conditions  $\pi_p \circ (h_i)_p = \text{id}$  on  $(\mathcal{O}_{gr})_p|_{U_i}$  (see (2)). Setting  $z_{ij} = h_i^{-1}h_j$ , we get a 1-cocycle  $z = (z_{ij}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\mathcal{O}_{gr})$ . Its class  $\zeta \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{gr})$  does not depend on the choice of  $h_i$  and corresponds to  $(M, \mathcal{O})$ .

**1.4 A non-abelian complex** Recall the construction of a non-abelian complex (see [O3], [O2]) which allows to express  $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{gr})$  in terms of differential forms. Let  $\Phi^{p,q}$  be the sheaf of smooth differential  $(p, q)$ -forms on  $M$ . First, we construct the Dolbeault–Serre resolution of the sheaf  $\mathcal{O}_{gr}$  :

$$\widehat{\Phi} := \bigoplus_{p,q \geq 0} \widehat{\Phi}^{p,q}, \quad \widehat{\Phi}^{p,q} := \Phi^{0,q} \otimes (\mathcal{O}_{gr})_p,$$

$$\bar{\partial}(\varphi \otimes u) = (\bar{\partial}\varphi) \otimes u \text{ for any } \varphi \in \Phi^{0,q}, u \in (\mathcal{O}_{gr})_p.$$

Then, regarding  $\widehat{\Phi}$  as a sheaf of graded superalgebras with respect to the total degree  $p+q$ , we get a sheaf of graded Lie superalgebras  $\widehat{\mathcal{T}} = \mathcal{D}er \widehat{\Phi}$ . The sheaf  $\widehat{\mathcal{T}}$  has the derivation  $\bar{D} = \text{ad}_{\bar{\partial}}$  of degree 1 (and of bidegree  $(0, 1)$ ). Denote

$$\mathcal{S} = \{u \in \widehat{\mathcal{T}} \mid u(\bar{f}) = u(d\bar{f}) = 0 \text{ for any } f \in \mathcal{F}\}.$$

This is a subsheaf of bigraded subalgebras, and  $\bar{D}(\mathcal{S}) \subset \mathcal{S}$ . As it was shown in [O3], the subsheaf  $\mathcal{S}_{p,q}$  is naturally identified with the sheaf  $\Phi^{0,q} \otimes (\mathcal{T}_{gr})_p$  of  $(0, q)$ -forms with values in the vector bundle  $\mathbf{ST}_p$ , and  $\bar{D} : \mathcal{S}_{p,q} \rightarrow \mathcal{S}_{p,q+1}$  goes over to the operator  $\bar{\partial} : \varphi \otimes v \mapsto \bar{\partial}\varphi \otimes v$ , where  $\varphi \in \Phi^{0,q}$ , and  $v \in (\mathcal{T}_{gr})_p$ . Hence, the sequence

$$0 \longrightarrow \mathcal{T}_{gr} \xrightarrow{i} \mathcal{S}_{*,0} \xrightarrow{\bar{D}} \mathcal{S}_{*,1} \xrightarrow{\bar{D}} \dots, \tag{5}$$

where  $i$  is a natural inclusion, is identified with the Dolbeault–Serre resolution of  $\mathcal{T}_{gr}$ . Set  $S_{p,q} := \Gamma(M, \mathcal{S}_{p,q})$  and  $S := \bigoplus_{p,q \geq 0} S_{p,q}$ . Then, the bigraded Lie superalgebras  $H^*(M, \mathcal{T})$  and  $H(S, \bar{D})$  are isomorphic.

The desired non-abelian complex is the non-linear complex associated to the differential bigraded Lie superalgebra  $(S, \bar{D})$ . More precisely, denote by  $\mathcal{F}^\infty$  the sheaf of differentiable complex-valued functions on  $M$ . Consider the sheaves  $\mathcal{O}_{gr}^\infty := \mathcal{F}^\infty \otimes \mathcal{O}_{gr}$  and the group

$$\text{PAut}_{(2)}\mathcal{O}_{gr}^\infty := \{a \in \text{Aut } \mathcal{O}_{gr}^\infty \mid a(u) - u \in \bigoplus_{k \geq 2} (\mathcal{O}_{gr}^\infty)_k, u \in \mathcal{O}_{gr}^\infty\}.$$

The non-abelian complex is the triple  $K = (K^0, K^1, K^2)$ , where

$$K^0 := \text{PAut}_{(2)}\mathcal{O}_{gr}^\infty, \quad K^q := \bigoplus_{k \geq 1} S_{2k,q} \quad \text{for } q = 1, 2,$$

with the coboundary operators  $\delta_i : K^i \rightarrow K^{i+1}$  for  $i = 0, 1$ , given by

$$\begin{aligned} \delta_0(a) &= \bar{\partial} - a\bar{\partial}a^{-1} \quad \text{for any } a \in K^0, \\ \delta_1(u) &= \bar{D}u - \frac{1}{2}[u, u] = -\frac{1}{2}[u - \bar{\partial}, u - \bar{\partial}] \quad \text{for any } u \in K^1. \end{aligned}$$

The gauge action  $\rho$  of  $K^0$  on  $K^1$  is given by

$$\rho(a)(u) = a(u - \bar{\partial})a^{-1} + \bar{\partial} \quad \text{for any } a \in K^0, u \in K^1.$$

Define  $Z^1(K) := \{u \in K^1 \mid \delta_1 u = 0\}$  and  $H^1(K) := Z^1(K)/\rho$ . In [O3], it is proved that there is an isomorphism of pointed sets

$$\mu : H^1(K) \longrightarrow H^1(M, \text{Aut}_{(2)}\mathcal{O}_{gr}).$$

In order to describe this isomorphism, take a cocycle  $w \in Z^1(K_{(1)})$  and an open cover  $\mathfrak{U} = (U_i)$  on  $M$  such that  $w = \delta_0(a_i)$ , where  $a_i \in \Gamma(U_i, \text{Aut}_{(2)}\mathcal{O}_{gr}^\infty)$ . Then, we get the Čech cocycle  $z = (z_{ij}) \in Z^1(\mathfrak{U}, \text{Aut}_{(2)}\mathcal{O}_{gr})$ , where  $z_{ij} = a_i^{-1}a_j$ . We have  $\mu(w) = \zeta$ , where  $\zeta \in H^1(M, \text{Aut}_{(2)}\mathcal{O}_{gr})$  and  $\omega \in H^1(K)$  are the cohomology classes of the cocycles  $w$  and  $z$ , respectively.

Note that the group  $\text{Aut } \mathbf{E}$  acts on the complex  $K$  and on  $H^1(K)$  in a natural way.

Using eq. (5), we can also construct a fine resolution of the tangent sheaf of any supermanifold with retract  $(M, \mathcal{O}_{gr})$ . Consider the supermanifold  $(M, \mathcal{O})$  with retract  $(M, \mathcal{O}_{gr})$  that corresponds to the cohomology classes  $\omega$  and  $\zeta$  of cocycles  $w \in Z^1(K)$  and  $z = (z_{ij})$ , as above. Twisting eq. (5) by  $z$ , we get the fine resolution

$$0 \longrightarrow \mathcal{T}_{gr}^{\text{Int}z} \xrightarrow{i} \mathcal{S}_{*,0}^{\text{Int}z} \xrightarrow{\bar{D}} \mathcal{S}_{*,1}^{\text{Int}z} \xrightarrow{\bar{D}} \dots \tag{6}$$

Here any  $v \in \mathcal{T}_{gr}^{\text{Int}z}$  is a family  $v = (v^i)$ , where  $v^i \in \Gamma(U_i, \mathcal{T}_{gr})$  and  $v^i = z_{ij} \circ v^j \circ z_{ij}^{-1}$  in  $U_i \cap U_j$ . In the same way we express the sections of the sheaves  $\mathcal{S}_{*,q}^{\text{Int}z}$ .

The correspondence  $(v^i) \mapsto (h_i \circ v^i \circ h_i^{-1})$  gives an isomorphism  $\mathcal{T}_{\text{gr}}^{\text{Int}z} \simeq \mathcal{T}$ , and the correspondence  $(v^i) \mapsto (a_i \circ v^i \circ a_i^{-1})$  gives an isomorphism  $\mathcal{S}_{*,p}^{\text{Int}z} \simeq \mathcal{S}_{*,p}$  for  $p \geq 0$ . Then, eq. (6) gives the following fine resolution of  $\mathcal{T} = \mathcal{D}er \mathcal{O}$ :

$$0 \longrightarrow \mathcal{T} \xrightarrow{\tau} \mathcal{S}_{*,0} \xrightarrow{\overline{D}^w} \mathcal{S}_{*,1} \xrightarrow{\overline{D}^w} \dots, \tag{7}$$

where  $\overline{D}^w := \overline{D} - \text{ad}_w = \text{ad}_{\overline{\partial}-w}$ . Considering global sections, we get a complex  $(S, \overline{D}^w)$  for calculating cohomology with values in the sheaf  $\mathcal{T}$ .

We give an explicit expression of  $\tau$ . As we have seen in Subsection 1.3, the cocycle  $z = (z_{ij}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$  of the cover  $\mathfrak{U}$  can be represented in the form  $z_{ij} = h_i^{-1}h_j$ . But it can also be represented in the form  $z_{ij} = a_i^{-1}a_j$ . Then, we have  $h_i^{-1}h_j = a_i^{-1}a_j$  in  $U_i \cap U_j$ , and  $\varrho = a_i h_i^{-1} = a_j h_j^{-1}$  is an injective homomorphism  $\mathcal{O} \rightarrow \mathcal{O}_{\text{gr}}^\infty$ . It follows that  $\tau : \mathcal{T} = \mathcal{D}er \mathcal{O} \rightarrow \mathcal{S}_{*,0}$  is expressed by the formula  $v \mapsto \varrho v \varrho^{-1}$ .

**1.2. Theorem.** *The mapping  $\tau : v \mapsto \varrho v \varrho^{-1}$  is an isomorphism of the graded Lie super-algebra  $H^*(M, \mathcal{T})$  onto  $H^*(S, \overline{D}^w)$ .*

*In particular, we get the isomorphism  $\tau : \mathfrak{v}(M, \mathcal{O}) \rightarrow \text{Ker } \overline{D}^w \subset S_{*,0}$ .*

**1.5 An application of the Hodge theory** Suppose that  $M$  is compact. Then, we can develop the standard Hodge theory in the complex  $(S, \overline{D})$  regarding it as the complex of  $(0, *)$ -forms with values in the bundle  $\mathbf{ST}$ , see [O3]. Endow  $M$  and  $\mathbf{E}$  with smooth Hermitian metrics and consider the corresponding Hermitian metric on  $\mathbf{ST}$ . Denote by  $\overline{D}^*$  the operator conjugate to  $\overline{D}$  and by  $\square := [\overline{D}, \overline{D}^*]$  the Beltrami–Laplace operator. Their bidegrees are  $(0, -1)$  and  $(0, 0)$ , respectively. Then, we have the orthogonal decomposition

$$S = \mathbf{H} \oplus \overline{D}S \oplus \overline{D}^*S, \tag{8}$$

where  $\mathbf{H} = \text{Ker } \square$  is the bigraded subspace of harmonic elements. Moreover,

$$\text{id} = H + \square G = H + \overline{D} \overline{D}^* G + \overline{D}^* \overline{D} G,$$

where  $H$  is the projection onto  $\mathbf{H}$  in eq. (8) and  $G$  is the Green operator. It is well known that

$$\mathbf{H}_{p,q} \simeq H^{p,q}(S, \overline{D}) \simeq H^q(M, (\mathcal{T}_{\text{gr}})_p) \text{ for any } p, q \geq 0. \tag{9}$$

Consider now the nonlinear complex  $K$ . Denote

$$\begin{aligned} \mathbf{H}_{(1)} &:= \bigoplus_{p \geq 1} \mathbf{H}_{2p,1}, \\ L_1 &:= \text{Ker } \overline{D}^* \cap K^1, \quad \mathbf{K} := Z^1(K) \cap L_1, \end{aligned}$$

and define also the subset  $\mathbf{K}_0 \subset K^1$  consisting of the  $u$  such that

$$u - \frac{1}{2} \overline{D}^* G[u, u] = Hu. \tag{10}$$

**1.3 Theorem ([O3]).** *We have  $\mathbf{K} \subset \mathbf{K}_0 \subset L_1$ . The mapping  $H : \mathbf{K}_0 \rightarrow \mathbf{H}_{(1)}$  is a bijection and maps  $\mathbf{K}$  onto the connected algebraic subset  $\mathbf{V} \subset \mathbf{H}_{(1)} \simeq \bigoplus_{p \geq 1} H^1(M, (\mathcal{T}_{gr})_{2p})$  given by the equation*

$$H[\varphi(h), \varphi(h)] = 0,$$

where  $\varphi : \mathbf{H}_{(1)} \rightarrow L_1$  is inverse to  $H$ .

The natural mapping  $\mathbf{K} \rightarrow H^1(K) \simeq H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{gr})$  is onto.

The set  $\mathbf{K}$  is an analogue of the *Kuranishi family* of complex structures on a compact manifold. By Theorem 1.3 we can see that this family cuts every cohomology class, and hence it can be used for classification of supermanifolds with retract  $(M, \mathcal{O}_{gr})$ .

**1.6 Actions on supermanifolds** Let  $(M, \mathcal{O})$  be an arbitrary supermanifold. An *action* of a (real or complex) Lie group  $G$  on  $(M, \mathcal{O})$  is a homomorphism  $\Psi : G \rightarrow \text{Aut}(M, \mathcal{O})$ . For any  $g \in G$  we have  $\Psi(g) = (f(g), \psi(g))$ , where  $f : g \mapsto f(g) \in \text{Bih } M$  is an (analytic) action of the group  $G$  on the complex manifold  $M$  and  $\psi(g)$  is an automorphism of the sheaf  $\mathcal{O}$  over  $f(g)$ .

Let  $\mathbf{E}$  be a holomorphic vector bundle over a complex manifold  $M$  and  $G$  a Lie group. Suppose that  $\mathbf{E}$  has a structure of the  $G$ -bundle, i.e., a homomorphism  $\Phi : G \rightarrow \text{Aut } \mathbf{E}$  satisfying the natural conditions of analyticity is given. Using the inclusion of  $\text{Aut } \mathbf{E}$  into  $\text{Aut}(M, \mathcal{O})$ , we may consider  $\Phi$  as an action on the split supermanifold  $(M, \mathcal{O}_{gr})$  corresponding to the bundle  $\mathbf{E}$ . This action is  $\mathbb{Z}$ -graded, i.e., all  $\varphi(g)$ , where  $g \in G$ , preserve the  $\mathbb{Z}$ -grading of the structure sheaf. Conversely, any  $\mathbb{Z}$ -graded action of the group  $G$  on  $(M, \mathcal{O}_{gr})$  extends an action on the vector bundle  $\mathbf{E}$ .

Let again  $(M, \mathcal{O})$  be an arbitrary complex supermanifold,  $(M, \mathcal{O}_{gr})$  its retract and  $\mathbf{E}$  the corresponding vector bundle.

If  $F = (f, \psi) \in \text{Aut}(M, \mathcal{O})$ , then the automorphism  $\psi$  of  $\mathcal{O}$  over  $f$  preserves a filtration (1), and hence determines an automorphism  $\varphi$  of the  $\mathbb{Z}$ -graded sheaf  $\mathcal{O}_{gr}$  over  $f$ . Here,  $\varphi$  is uniquely determined by the relation  $\pi_p \circ \psi = \varphi \circ \pi_p$  on  $\mathcal{J}^p$ .

Define  $\bar{F} = (f, \varphi) \in \text{Aut}(M, \mathcal{O}_{gr})$  for every  $F = (f, \psi) \in \text{Aut}(M, \mathcal{O})$ . Thus, we get a homomorphism  $\text{Aut}(M, \mathcal{O}) \rightarrow \text{Aut}(M, \mathbf{E})$ . It follows that any action  $\Psi : G \rightarrow \text{Aut}(M, \mathcal{O})$  induces a  $\mathbb{Z}$ -graded action  $\Phi : G \rightarrow \text{Aut}(M, \mathcal{O}_{gr})$ . In this case, we say that the action  $\Phi$  *lifts to the action  $\Psi$  on  $(M, \mathcal{O})$* .

There is the following lifting criterion:

**1.4 Theorem ([O2]).** *Let  $G$  be a compact Lie group and suppose an analytic  $\mathbb{Z}$ -graded action  $\Psi$  of  $G$  on a split supermanifold  $(M, \mathcal{O}_{gr})$  be given. Let  $(M, \mathcal{O})$  be the supermanifold corresponding to a given class  $\zeta \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{gr})$  by Theorem 1.1. Then, the following conditions are equivalent:*

- (i) *the action  $\Psi$  lifts to  $(M, \mathcal{O})$ ;*
- (ii) *the class  $\zeta$  contains a  $G$ -invariant cocycle  $z \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\mathcal{O}_{gr})$  where  $\mathfrak{U}$  is an open  $G$ -cover of  $M$ ;*
- (iii) *the class  $\mu_1^{-1}(\zeta) \in H^1(K)$  (see Theorem 1.1) contains a  $G$ -invariant cocycle.*

Now we give definitions of homogeneous and  $\bar{0}$ -homogeneous supermanifolds. Let  $(M, \mathcal{O})$  be a complex supermanifold. For any  $x \in M$  we can define the *tangent space*  $T_x(M, \mathcal{O}) := (m_x/m_x^2)^*$ , where  $m_x$  is the maximal ideal of the local superalgebra  $\mathcal{O}_x$ .

There is a natural even linear mapping  $ev_x : \mathfrak{v}(M, \mathcal{O}) \rightarrow T_x(M, \mathcal{O})$ . Namely, every  $v \in \mathfrak{v}(M, \mathcal{O})$  determines a linear mapping  $m_x \rightarrow \mathcal{O}_x$  with  $v(m_x^2) \subset m_x$ , and hence a linear mapping

$$m_x/m_x^2 \rightarrow \mathcal{O}_x/m_x = \mathbb{C},$$

i.e., an element  $ev_x(v) \in (m_x/m_x^2)^*$ .

The subalgebra  $\mathfrak{g} \subset \mathfrak{v}(M, \mathcal{O})$  is called *transitive* if  $ev_x : \mathfrak{g} \rightarrow T_x(M, \mathcal{O})$  is surjective for all  $x \in M$  and if  $ev_x : \mathfrak{g}_{\bar{0}} \rightarrow T_x(M, \mathcal{O})_{\bar{0}} = T_x(M)$  is surjective for all  $x \in M$ , then it is called  $\bar{0}$ -*transitive*. A supermanifold  $(M, \mathcal{O})$  is called *homogeneous* ( $\bar{0}$ -*homogeneous*) if there is a transitive ( $\bar{0}$ -transitive) subalgebra  $\mathfrak{g} \subset \mathfrak{v}(M, \mathcal{O})$  of finite dimension. In the case when  $M$  is a compact we can replace  $\mathfrak{g}$  by  $\mathfrak{v}(M, \mathcal{O})$ .

**1.5 Theorem ([OP]).** *If a supermanifold  $(M, \mathcal{O})$  is homogeneous ( $\bar{0}$ -homogeneous), then  $(M, \text{gr } \mathcal{O})$  is homogeneous ( $\bar{0}$ -homogeneous).*

## 2 Supermanifolds associated with the complex torus

**2.1 Complex tori** Let  $\Gamma \subset \mathbb{C}^m$  be a discrete subgroup of rank  $2m$ . Then, the manifold  $T = \mathbb{C}^m/\Gamma$  is a *complex torus* of dimension  $m$ . Note that  $T$  is a compact complex commutative Lie group. There is a local coordinate system in a neighborhood of any point of the manifold  $T$  formed by the standard coordinates  $z_1, \dots, z_m$  in  $\mathbb{C}^m$ . Let us denote these coordinates on  $T$  also by  $z_1, \dots, z_m$ . The differential forms  $dz_1, \dots, dz_m$  are defined on  $T$  globally, since they are not changed if we add a complex number to the variable. Using duality between differential forms and vector fields, we get the vector fields  $\partial_{z_1}, \dots, \partial_{z_m}$  which are defined globally, too. The tangent and the cotangent bundles over  $T$  are trivial, and  $dz_i, \partial_{z_i}$  are basis sections of these vector bundles.

**2.1 Proposition ([GH]).** *Let  $M$  be a compact Kähler manifold. A form  $\alpha \in \Gamma(M, \Phi^{0,q})$  is harmonic if and only if  $\alpha$  is an antiholomorphic form, i.e.,  $\partial\alpha = 0$ .*

It is well known that  $T = \mathbb{C}^m/\Gamma$  is a compact Kähler manifold with flat metrics induced by Hermitian metrics in  $\mathbb{C}^m$  (see [GH]). We can represent any  $\alpha \in \Gamma(M, \Phi^{0,q})$  in a form

$$\alpha = \sum_{1 \leq i_1 < \dots < i_q \leq m} a_{i_1 \dots i_q}(z, \bar{z}) d\bar{z}_{i_1} \dots d\bar{z}_{i_q}, \tag{11}$$

where  $a_{i_1 \dots i_q}(z, \bar{z})$  are smooth global defined functions,  $z = (z_1, \dots, z_m)$ ,  $\bar{z} = (\bar{z}_1, \dots, \bar{z}_m)$ . Since any antiholomorphic function on  $T$  is constant, we have

**2.2. Proposition.** *A form  $\alpha \in \Gamma(M, \Phi^{0,q})$  is harmonic if and only if*

$$\alpha = \sum_{1 \leq i_1 < \dots < i_q \leq m} a_{i_1 \dots i_q} d\bar{z}_{i_1} \dots d\bar{z}_{i_q}, \quad \text{where } a_{i_1 \dots i_q} \in \mathbb{C}.$$



A form on  $T$  is called  $T$ -invariant if it is invariant under the action of the group  $T$  on itself by translations.

**2.3. Proposition.** *The spaces of harmonic and  $T$ -invariant  $(0, q)$ -forms on  $T$  coincide.*

*Proof.* Clearly, the forms  $d\bar{z}_{i_1} \dots d\bar{z}_{i_q}$  are  $T$ -invariant. It follows that the form (11) is  $T$ -invariant if and only if  $a_{i_1 \dots i_q} \in \mathbb{C}$ . Then, we apply Proposition 2.2.  $\square$

**2.2 Supermanifolds corresponding to the trivial bundle over the complex torus** Let  $\mathbf{E} = T \times \mathbb{C}^n$  be a trivial holomorphic vector bundle of rank  $n$  over  $T$  and  $\xi_1, \dots, \xi_n$  be the standard basis of  $\mathbb{C}^n$ . Denote by  $T^{m|n} = (T, \mathcal{O}_{gr})$  the split supermanifold corresponding to the bundle  $\mathbf{E}$ . The structure sheaf  $\mathcal{O}_{gr}$  has the form  $\mathcal{F} \otimes \wedge(\xi_1, \dots, \xi_n)$ . The local coordinates  $z_1, \dots, z_m$  on  $T$  are even coordinates on  $T^{m|n}$ , and  $\xi_1, \dots, \xi_n$  are odd ones.

Consider the tangent sheaf  $\mathcal{T}_{gr} = \mathcal{D}er \mathcal{O}_{gr}$ . This sheaf is free over  $\mathcal{F}$ , or, equivalently, the bundle  $\mathbf{ST}$  is trivial, and the basis of its sections is

$$\xi_{i_1} \dots \xi_{i_k} \partial_{z_j}, \quad \xi_{i_1} \dots \xi_{i_k} \partial_{\xi_l}, \quad \text{where} \quad (12)$$

$$1 \leq i_1 < \dots < i_k \leq n, \quad j = 1, \dots, m, \quad l = 1, \dots, n.$$

Hence, the elements of  $S_{p,q}$  have the form

$$\alpha = \sum_{i_1 < \dots < i_q} \left( \sum_{\substack{j_1 < \dots < j_p \\ i=1, \dots, m}} a_{j_1, \dots, j_p}^{i, i_1, \dots, i_q}(z, \bar{z}) \xi_{j_1} \dots \xi_{j_p} \partial_{z_i} + \sum_{\substack{j_1 < \dots < j_{p+1} \\ j=1, \dots, n}} b_{j_1, \dots, j_{p+1}}^{j, i_1, \dots, i_q}(z, \bar{z}) \xi_{j_1} \dots \xi_{j_{p+1}} \partial_{\xi_j} \right) d\bar{z}_{i_1} \dots d\bar{z}_{i_q},$$

where  $a_{j_1, \dots, j_p}^{i, i_1, \dots, i_q}(z, \bar{z})$  and  $b_{j_1, \dots, j_{p+1}}^{j, i_1, \dots, i_q}(z, \bar{z})$  are smooth globally defined functions on  $T$ . Thus, from Proposition 2.2 we get

**2.4. Proposition.** *The form  $\alpha \in S_{p,q}$  is harmonic if and only if*

$$\alpha = \sum_{i_1 < \dots < i_q} \left( \sum_{\substack{j_1 < \dots < j_p \\ i=1, \dots, m}} a_{j_1, \dots, j_p}^{i, i_1, \dots, i_q} \xi_{j_1} \dots \xi_{j_p} \partial_{z_i} + \sum_{\substack{j_1 < \dots < j_{p+1} \\ j=1, \dots, n}} b_{j_1, \dots, j_{p+1}}^{j, i_1, \dots, i_q} \xi_{j_1} \dots \xi_{j_{p+1}} \partial_{\xi_j} \right) d\bar{z}_{i_1} \dots d\bar{z}_{i_q}, \quad (13)$$

where  $a_{j_1, \dots, j_p}^{i, i_1, \dots, i_q}, b_{j_1, \dots, j_{p+1}}^{j, i_1, \dots, i_q} \in \mathbb{C}$ .

**2.5. Corollary.** *If  $\alpha, \beta \in \mathbf{H}$ , then  $[\alpha, \beta] \in \mathbf{H}$ .*

Assigning to every cohomology class from  $H^q(M, (\mathcal{T}_{gr})_p)$  the correspondent harmonic form from  $\mathbf{H}_{p,q}$  (see (9)), we get an isomorphism of graded Lie superalgebras  $H(M, \mathcal{T}_{gr})$  onto the subalgebra  $\mathbf{H} \subset S$ .

Since  $\xi_{j_1} \dots \xi_{j_p} \partial_{z_i}$  and  $\xi_{j_1} \dots \xi_{j_{p+1}} \partial_{\xi_j}$  are  $T$ -invariant, from Proposition 2.3 we get

**2.6. Proposition.** *Any harmonic form from  $S$  is  $T$ -invariant, and the other way round.*

**2.7. Theorem.** *We have  $K_0 = \mathbf{H}_{(1)}$  and*

$$\mathbf{K} = \mathbf{V} = \{w \in \mathbf{H}_{(1)} \mid [w, w] = 0\}.$$

*Proof.* Take  $w \in \mathbf{K}_0$  and denote  $h = Hw$ . We write  $h = \sum_{k \geq 1} h_{2k}$ , and  $w = \sum_{k \geq 1} w_{2k}$ , where  $h_{2k} \in \mathbf{H}_{2k,1}$ ,  $w_{2k} \in S_{2k,1}$ . From (10) we get the following equations:

$$\begin{aligned} w_2 &= h_2, \\ w_4 - \frac{1}{2} \overline{D}^* G[w_2, w_2] &= h_4, \\ \dots \\ w_{2k} - \frac{1}{2} \overline{D}^* G \sum_{1 \leq s \leq k-1} [w_{2s}, w_{2(k-s)}] &= h_{2k}, \\ \dots \end{aligned}$$

We prove that  $w_{2k} = h_{2k}$  by induction on  $k$ . For  $k = 1$  this follows from the first equation. Suppose that  $w_{2i} = h_{2i}$  for  $1 \leq i \leq k - 1$ . By Corollary 2.5 we see that

$$h' = \sum_{1 \leq s \leq k-1} [w_{2s}, w_{2(k-s)}] \in \mathbf{H}.$$

Since  $\overline{D}^*$  and  $G$  commute and  $\overline{D}^* h' = 0$ , we get  $w_{2k} = h_{2k}$ .

So we have proved that  $w = h \in \mathbf{H}_{(1)}$ . By Theorem 3,  $\mathbf{K}_0 = \mathbf{H}_{(1)}$ , and  $\varphi = \text{id}$ . Therefore,  $\mathbf{K} = \mathbf{V} = \{w \in \mathbf{H}_{(1)} \mid [w, w] = 0\}$ .  $\square$

**2.3 Lie superalgebras of vector fields on supermanifolds with retract  $T^{m|n}$**

Consider holomorphic vector fields on the split supermanifold  $T^{m|n}$ . It is clear that any  $v \in \mathfrak{v}_p(T^{m|n})$  is a linear combination of the fields

$$\begin{aligned} &\xi_{j_1} \dots \xi_{j_p} \partial_{z_i} \text{ for } j_1 < \dots < j_p \text{ and } i = 1, \dots, m, \\ &\xi_{j_1} \dots \xi_{j_{p+1}} \partial_{\xi_j} \text{ for } j_1 < \dots < j_{p+1} \text{ and } j = 1, \dots, n \text{ (see (12))} \end{aligned}$$

with holomorphic coefficients. Since any holomorphic function on  $T$  is constant,

$$v = \sum_{1 \leq i \leq m} \sum_{j_1 < \dots < j_p} a_{j_1, \dots, j_p}^i \xi_{j_1} \dots \xi_{j_p} \partial_{z_i} + \sum_{1 \leq j \leq n} \sum_{j_1 < \dots < j_{p+1}} b_{j_1, \dots, j_{p+1}}^j \xi_{j_1} \dots \xi_{j_{p+1}} \partial_{\xi_j}, \quad (14)$$

where  $a_{j_1, \dots, j_p}^i, b_{j_1, \dots, j_{p+1}}^j \in \mathbb{C}$ . Since

$$\mathfrak{v}(T^{m|n}) = \mathbf{H}_{p,0} = \{v \in S_{p,0} \mid \overline{D}v = 0\},$$

we see that (14) is a special case of the formula (13).

By Theorem 1.3, any supermanifold with retract  $T^{m|n}$  can be described by a form from the Kuranishi family  $\mathbf{K}$ . By Theorem 2.7,  $\mathbf{K} = \mathbf{V}$  consists of harmonic elements.

**2.8. Theorem.** *Let  $(T, \mathcal{O})$  be a supermanifold with retract  $T^{m|n}$  given by an element  $w$  in  $\mathbf{V}$ . Then, the mapping  $\tau$  from (1.7) determines an isomorphism*

$$\mathfrak{v}(T, \mathcal{O}) \rightarrow \{v \in S_{*,0} \mid \overline{D}v = [w, v] = 0\} = \{v \in \mathfrak{v}_p(T^{m|n}) \mid [w, v] = 0\}.$$

*Proof.* We can write  $w = w_2 + w_4 + \dots$ , where  $w_{2k} \in \mathbf{H}_{2k,1}$ . Let  $v \in S_{*,0}$  and  $\overline{D}v = [w, v]$ . Then,  $v = v_{-1} + v_0 + v_1 + \dots$ , where  $v_i \in S_{i,0}$ . The equation  $\overline{D}v = [w, v]$  gives the finite system of equations:

$$\begin{aligned} \overline{D}v_{-1} &= 0, \\ \overline{D}v_0 &= 0, \\ \overline{D}v_1 &= [w_2, v_{-1}], \\ \overline{D}v_2 &= [w_2, v_0], \\ \overline{D}v_3 &= [w_2, v_1] + [w_4, v_{-1}], \\ \overline{D}v_4 &= [w_2, v_2] + [w_4, v_0], \\ &\dots \end{aligned} \tag{15}$$

Let us prove that  $\overline{D}v_k = 0$  for  $k = -1, 0, \dots$ , by induction on  $k$ . For  $k = -1, 0$  this follows from the first and the second equations of (15). If  $\overline{D}v_{-1} = \dots = \overline{D}v_{k-1} = 0$ , then, using system (15), we see that  $\overline{D}v_k$  is a sum of commutators of the fields  $v_{-1}, \dots, v_{k-2}$  with the forms  $w_{2l}$ . Since  $w_{2l} \in \mathbf{H}_{2l,1}$ , Corollary 2.5 shows that  $\overline{D}v_k \in \mathbf{H}_{p,1}$ . Hence  $\overline{D}v_k = 0$ .

Thus, we proved that the kernel of  $\overline{D}_w = \overline{D} - \text{ad } w$  in  $S_{*,0}$  coincides with the subalgebra  $\{v \in \mathfrak{v}_p(T^{m|n}) \mid [w, v] = 0\} \subset \mathfrak{v}_p(T^{m|n})$ . Now our statement follows from Theorem 1.2.  $\square$

## 2.4 Homogeneous supermanifolds with retract $T^{m|n}$

**2.9. Theorem.** *Any supermanifold  $(T, \mathcal{O})$  with retract  $T^{m|n}$  is  $\bar{0}$ -homogeneous. It is homogeneous if and only if  $(T, \mathcal{O}) \simeq T^{m|n}$ .*

*Proof.* Let  $(T, \mathcal{O})$  be an arbitrary supermanifold with retract  $T^{m|n}$ . From Theorem 2.7 we see that it is determined by a harmonic form  $w \in \mathbf{V}$ . By Proposition 2.6  $w$  is invariant under the natural action of the group  $T$ . Hence, by Theorem 1.4 the action of the group  $T$  on  $T^{m|n}$  lifts to an action on  $(T, \mathcal{O})$ . So  $(T, \mathcal{O})$  is  $\bar{0}$ -homogeneous.

From eq. (12) we see that the Lie superalgebra  $\mathfrak{v}(T^{m|n})$  is transitive. Then,  $T^{m|n}$  is homogeneous. Let  $(T, \mathcal{O})$  be the supermanifold with retract  $T^{m|n}$  determined by a co-cycle  $w \in \mathbf{V} \subset \mathbf{H}_{(1)}$ . Let  $(T, \mathcal{O})$  be homogeneous. Take a point  $x_0 \in T$ , and denote by  $\xi_1, \dots, \xi_n$  the odd local coordinates in the neighborhood  $U$  of  $x_0$  which correspond to the coordinates  $\xi_1, \dots, \xi_n$  on  $T^{m|n}$  by the local splitting  $h_U : \mathcal{O}_{\text{gr}}|_U \rightarrow \mathcal{O}|_U$ . Since  $\text{ev}_{x_0} : \mathfrak{v}(T, \mathcal{O})_{\bar{1}} \rightarrow T_{x_0}(T, \mathcal{O})_{\bar{1}}$  is surjective, then for any  $j$  such that  $1 \leq j \leq n$ , there exists a field  $v_j \in \mathfrak{v}(T, \mathcal{O})_{\bar{1}}$  such that  $v_j = \partial_{\xi_j} + v'_j$  in  $U$ , where  $(v'_j)_{x_0} \in m_{x_0} \mathcal{T}_{x_0}$ . We can assume that the neighborhood  $U = U_i$  is included into the cover  $(U_i)$  which we used in the description of  $\tau$  in Subsection 1.4. As was shown in Subsection 1.4, we have

$$\tau(v_j) = a_i(h_i^{-1} \partial_{\xi_j} h_i) a_i^{-1} + (a_i h_i^{-1}) v'_j (h_i a_i^{-1}).$$

The sheaf  $\mathcal{S}_{*,0} = \mathcal{T}_{\text{gr}}^\infty = \mathcal{D}er \mathcal{O}_{\text{gr}}^\infty$  has a filtration similar to (3):

$$\mathcal{T}_{\text{gr}}^\infty = \mathcal{T}_{\text{gr}(-1)}^\infty \subset \mathcal{T}_{\text{gr}(0)}^\infty \subset \mathcal{T}_{\text{gr}(1)}^\infty \subset \dots$$

Since  $a_i \in \Gamma(U_i, \mathcal{A}ut_{(2)} \mathcal{O}_{\text{gr}}^\infty)$ , we have

$$a_i(h_i^{-1} \partial_{\xi_j} h_i) a_i^{-1} = a_i \partial_{\xi_j} a_i^{-1} = \partial_{\xi_j} + u_j,$$

where  $u_j \in \Gamma(U_i, \mathcal{T}_{\text{gr}(1)}^\infty)$ . Clearly,

$$(a_i h_i^{-1}) v_j' (h_i a_i^{-1}) = v_j''$$

satisfies  $(v_j'')_{x_0} \in m_{x_0}^\infty(\mathcal{T}_{\text{gr}}^\infty)_{x_0}$ , where  $m_{x_0}^\infty$  is the maximal ideal of  $(\mathcal{O}_{\text{gr}}^\infty)_{x_0}$ . Hence in  $U_i$  we have

$$\tau(v_j) = \partial_{\xi_j} + u_j + v_j'',$$

where  $(u_j + v_j'')_{x_0} \in m_{x_0}^\infty(\mathcal{T}_{\text{gr}}^\infty)_{x_0}$ . By Theorem 2.8  $\tau(v_j) \in \mathfrak{v}(T^{m|n})_{\bar{1}}$ .

So, if  $\tilde{v}_j = \tau(v_j) - \partial_{\xi_j} \in \mathfrak{v}(T^{m|n})_{\bar{1}}$ , then  $\tilde{v}_{x_0} \in m_{x_0}(\mathcal{T}_{\text{gr}})_{x_0}$ , and therefore,

$$\tilde{v}_j \in \bigoplus_{k \geq 0} \mathfrak{v}(T^{m|n})_{2k+1}.$$

By Theorem 2.8

$$[w, \partial_{\xi_j} + \tilde{v}_j] = 0 \quad \text{for } j = 1, \dots, n. \tag{16}$$

We write  $w$  as  $w = w_2 + w_4 + \dots$ , where  $w_{2k} \in \mathbf{H}_{2k,1}$ , and prove that  $w_{2k} = 0$  for  $k = 1, 2, \dots$ , by induction on  $k$ .

Considering the component of degree 1 of the left part of formula (16), we see that  $[w_2, \partial_{\xi_j}] = 0$ . Hence,

$$\begin{aligned} [w_2, \partial_{\xi_j}](z_r) &= \partial_{\xi_j}(w_2(z_r)) = 0 \quad \text{for } r = 1, \dots, m, \\ [w_2, \partial_{\xi_j}](\xi_s) &= \partial_{\xi_j}(w_2(\xi_s)) = 0 \quad \text{for } s = 1, \dots, n. \end{aligned}$$

From (13) we see that  $w_2 = 0$ .

Suppose we have proved that  $w_2 = w_4 = \dots = w_{2k-2} = 0$ . Considering the component of degree  $2k - 1$  of the left part of the formula (16), we get  $[w_{2k}, \partial_{\xi_j}] = 0$  for all  $j = 1, \dots, n$ . As above, we can prove that  $w_{2k} = 0$ .

Thus,  $w = 0$ . Hence,  $(T, \mathcal{O}) \simeq T^{m|n}$ . □

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