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Supermanifolds corresponding to the trivial vector bundle over torus

Mikhail Bashkin

Abstract. All supermanifolds whose retract $T^{m|n}$ is determined by the trivial bundle of rank n over the torus T^m are $\bar{0}$ -homogeneous if and only if $T^{m|n}$ is homogeneous.

1 Preliminaries

1.1 Split and non-split supermanifolds The ground field is \mathbb{C} .

A *complex supermanifold* of dimension $m|n$ is a $\mathbb{Z}/2$ -graded ringed space of the form $\mathcal{M} := (M, \mathcal{O})$, where M is a topological space and \mathcal{O} is a sheaf of associative commutative superalgebras with unit on M , which is locally isomorphic to a superdomain in $\mathbb{C}^{m|n}$. For details, see [M], [O3], [BGLS*]. A *superdomain* in $\mathbb{C}^{m|n}$ is a pair $(U, \bigwedge_{\mathcal{F}}(\xi_1, \dots, \xi_n))$, where U is an open subset of \mathbb{C}^m , and \mathcal{F} is the sheaf of holomorphic functions on \mathbb{C}^m . The coordinates x_1, \dots, x_m in $U \subset \mathbb{C}^m$ and generators ξ_1, \dots, ξ_n of the Grassmann algebra are identified with some sections of the sheaf $\mathcal{O}|_U$. They are called *local coordinates even* and *odd*, respectively.

Let (M, \mathcal{F}) be a complex manifold and \mathcal{E} be a locally free analytic sheaf on it, i.e., \mathcal{E} is a sheaf of holomorphic sections of some holomorphic vector bundle $\mathbf{E} \rightarrow M$. Then, $(M, \mathcal{O}_{\text{gr}})$, where $\mathcal{O}_{\text{gr}} = \bigwedge_{\mathcal{F}} \mathcal{E}$, is a complex supermanifold. A supermanifold is called *split* if it is isomorphic to a supermanifold of this form and is called *non-split* otherwise.

Let us show that every supermanifold is a deformation of a split supermanifold. Consider the subsheaf of ideals $\mathcal{J} = (\mathcal{O}_{\bar{1}})$, generated by odd elements. Denote $\mathcal{F} := \mathcal{O}/\mathcal{J}$. Then, $\mathcal{M}_{\text{rd}} = (M, \mathcal{F})$ is a complex manifold. It is called the *odd reduction* of (M, \mathcal{O}) and hereafter is often denoted by M . The powers of \mathcal{J} determine the following filtration:

$$\mathcal{O} = \mathcal{J}^0 \supset \mathcal{J}^1 \supset \mathcal{J}^2 \supset \dots \supset \mathcal{J}^{n+1} = 0. \quad (1)$$

The associated sheaf of graded algebras

$$\text{gr } \mathcal{O} = \bigoplus_{0 \leq p \leq n} \text{gr}_p \mathcal{O}, \quad \text{where } \text{gr}_p \mathcal{O} = \mathcal{J}^p / \mathcal{J}^{p+1},$$

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is an analytic sheaf on the reduction \mathcal{M}_{rd} . Actually, $\text{gr } \mathcal{O} \simeq \bigwedge_{\mathcal{F}} \mathcal{E}$, where $\mathcal{E} = \text{gr}_1 \mathcal{O}$ is locally free sheaf. Clearly, $(M, \text{gr } \mathcal{O}) = (M, \mathcal{O}_{\text{gr}})$ is a split supermanifold of the same dimension as (M, \mathcal{O}) . We call it the *retract* of the supermanifold (M, \mathcal{O}) . Obviously, a given supermanifold is split if and only if it is isomorphic to its retract.

Let $\pi_p : \mathcal{J}^p \rightarrow \text{gr}_p \mathcal{O}$ be the canonical projection. Then, there is the exact sequence of sheaves

$$0 \longrightarrow \mathcal{J}^{p+1} \longrightarrow \mathcal{J}^p \xrightarrow{\pi_p} \text{gr}_p \mathcal{O} \longrightarrow 0. \quad (2)$$

A supermanifold (M, \mathcal{O}) is split if and only if there exists an isomorphism of superalgebra sheaves $h : \text{gr } \mathcal{O} \rightarrow \mathcal{O}$, whose restriction $h_p : \text{gr}_p \mathcal{O} \rightarrow \mathcal{J}^p$ splits the sequence (2), i.e., satisfies the condition $\pi_p \circ h_p = \text{id}$. In general, this splitting exists in a neighborhood of any point in M . It can be given by means of local coordinates.

1.2 The tangent sheaf For an arbitrary supermanifold (M, \mathcal{O}) denote by $\mathcal{T} := \mathcal{D}er \mathcal{O}$ its *tangent sheaf* (or the sheaf of vector fields). It is the sheaf of derivations (over \mathbb{C}) of the structure sheaf \mathcal{O} . Note that the tangent sheaf is a sheaf of $\mathbb{Z}/2$ -graded left \mathcal{O} -modules, and also a sheaf of Lie superalgebras. The sections of the tangent sheaf are called *holomorphic vector fields* on (M, \mathcal{O}) . They form the Lie superalgebra $\mathfrak{v}(M, \mathcal{O})$ of vector fields on (M, \mathcal{O}) .

Note that the filtration (1) determines the filtration of the tangent sheaf

$$\mathcal{T} = \mathcal{T}_{(-1)} \supset \mathcal{T}_{(0)} \supset \cdots \supset \mathcal{T}_{(n)} \supset \mathcal{T}_{(n+1)} = 0, \quad (3)$$

where

$$\mathcal{T}_{(p)} = \{v \in \mathcal{T} \mid v(\mathcal{O}) \subset \mathcal{J}^p, v(\mathcal{J}) \subset \mathcal{J}^{p+1}\}, p \geq 0.$$

Since $(M, \mathcal{O}_{\text{gr}})$ is split, its tangent sheaf \mathcal{T}_{gr} is a \mathbb{Z} -graded sheaf of Lie superalgebras

$$\mathcal{T}_{\text{gr}} = \bigoplus_{-1 \leq p \leq n} (\mathcal{T}_{\text{gr}})_p,$$

where

$$(\mathcal{T}_{\text{gr}})_p := \mathcal{D}er_p \mathcal{O}_{\text{gr}} = \{v \in \mathcal{T}_{\text{gr}} \mid v((\mathcal{O}_{\text{gr}})_q) \subset (\mathcal{O}_{\text{gr}})_{q+p}, q \in \mathbb{Z}\}.$$

This grading is compatible with the $\mathbb{Z}/2$ -grading. The Lie superalgebra $\mathfrak{v}(M, \mathcal{O}_{\text{gr}})$ of vector fields is a graded algebra with the \mathbb{Z} -grading compatible with the $\mathbb{Z}/2$ -grading.

Since $\mathcal{F} \subset \mathcal{O}_{\text{gr}}$, the tangent sheaf \mathcal{T}_{gr} is a \mathbb{Z} -graded analytic sheaf on M . This sheaf is locally free (see [O3]), and hence it is the sheaf of holomorphic sections of a \mathbb{Z} -graded holomorphic vector bundle \mathbf{ST} over M (the *supertangent bundle*).

1.3 Sheaves of automorphisms and the classification theorem Let (M, \mathcal{O}) be an complex supermanifold. Denote by $\text{Aut}(M, \mathcal{O})$ the group of automorphisms of (M, \mathcal{O}) . By definition, $F \in \text{Aut}(M, \mathcal{O})$ is a pair (f, φ) , where $f : M \rightarrow M$ belongs to group $\text{Bih } M$ of biholomorphic transformations of the manifold M and φ is an automorphism of the superalgebra sheaf \mathcal{O} over f . Denote by $\mathcal{A}ut \mathcal{O}$ the sheaf of automorphisms of the structure sheaf \mathcal{O} (mapping every stalk \mathcal{O}_x , where $x \in M$, onto itself). Moreover, for any $F = (f, \varphi) \in \text{Aut}(M, \mathcal{O})$ the map $\text{Int } F : a \mapsto \varphi \circ a \circ \varphi^{-1}$ is an automorphism of the group sheaf $\mathcal{A}ut \mathcal{O}$. Hence, we get the action Int of the group $\text{Aut}(M, \mathcal{O})$ on $\mathcal{A}ut \mathcal{O}$ by automorphisms. The subsheaf

$$\mathcal{A}ut_{(2)} \mathcal{O} = \{a \in \mathcal{A}ut \mathcal{O} \mid a(f) - f \in \mathcal{J}^2, f \in \mathcal{O}\} \quad (4)$$

is invariant under this action.

Let \mathbf{E} be a holomorphic vector bundle over (M, \mathcal{F}) and $\text{Aut } \mathbf{E}$ the group of its automorphisms. Clearly, any element of this group gives rise to an automorphism of the split supermanifold $(M, \mathcal{O}_{\text{gr}})$ corresponding to \mathbf{E} which preserves the \mathbb{Z} -grading of the structure sheaf. Hence, we can identify $\text{Aut } \mathbf{E}$ with a subgroup of $\text{Aut}(M, \mathcal{O}_{\text{gr}})$, consisting of automorphisms that preserve the \mathbb{Z} -grading. So the $\text{Aut}(M, \mathcal{O}_{\text{gr}})$ -sheaves $\mathcal{A}ut_{\mathcal{O}_{\text{gr}}}$ and $\mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}}$ are also $\text{Aut } \mathbf{E}$ -sheaves.

1.3.1. Theorem ([G]). *Any supermanifold (M, \mathcal{O}) corresponds to an element of the set of 1-cohomology $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$. This correspondence gives rise to a bijection between the isomorphism classes of supermanifolds satisfying the above condition, and the orbits of the group $\text{Aut } \mathbf{E}$ on $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ under the natural $\text{Aut } \mathbf{E}$ -action.*

Let us describe the correspondence mentioned in Theorem 1.3.1. Let (M, \mathcal{O}) be a supermanifold with retract $(M, \mathcal{O}_{\text{gr}})$. Then, we can choose an open cover $\mathfrak{U} = (U_i)_{i \in I}$ of M such that there exist isomorphisms $h_i : \mathcal{O}_{\text{gr}}|_{U_i} \rightarrow \mathcal{O}|_{U_i}$, where $i \in I$, with conditions $\pi_p \circ (h_i)_p = \text{id}$ on $(\mathcal{O}_{\text{gr}})_p|_{U_i}$ (see (2)). Setting $z_{ij} = h_i^{-1}h_j$, we get a 1-cocycle $z = (z_{ij}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$. Its class $\zeta \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ does not depend on the choice of h_i and corresponds to (M, \mathcal{O}) .

1.4 A non-abelian complex Recall the construction of a non-abelian complex (see [O3], [O2]) which allows to express $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\text{gr}})$ in terms of differential forms. Let $\Phi^{p,q}$ be the sheaf of smooth differential (p, q) -forms on M . First, we construct the Dolbeault–Serre resolution of the sheaf \mathcal{O}_{gr} :

$$\widehat{\Phi} := \bigoplus_{p,q \geq 0} \widehat{\Phi}^{p,q}, \quad \widehat{\Phi}^{p,q} := \Phi^{0,q} \otimes (\mathcal{O}_{\text{gr}})_p,$$

$$\bar{\partial}(\varphi \otimes u) = (\bar{\partial}\varphi) \otimes u \quad \text{for any } \varphi \in \Phi^{0,q}, u \in (\mathcal{O}_{\text{gr}})_p.$$

Then, regarding $\widehat{\Phi}$ as a sheaf of graded superalgebras with respect to the total degree $p + q$, we get a sheaf of graded Lie superalgebras $\widehat{\mathcal{T}} = \text{Der } \widehat{\Phi}$. The sheaf $\widehat{\mathcal{T}}$ has the derivation $\bar{D} = \text{ad}_{\bar{\partial}}$ of degree 1 (and of bidegree $(0, 1)$). Denote

$$\mathcal{S} = \{u \in \widehat{\mathcal{T}} \mid u(\bar{f}) = u(d\bar{f}) = 0 \text{ for any } f \in \mathcal{F}\}.$$

This is a subsheaf of bigraded subalgebras, and $\bar{D}(\mathcal{S}) \subset \mathcal{S}$. As it was shown in [O3], the subsheaf $\mathcal{S}_{p,q}$ is naturally identified with the sheaf $\Phi^{0,q} \otimes (\mathcal{T}_{\text{gr}})_p$ of $(0, q)$ -forms with values in the vector bundle $\mathbf{S}\mathcal{T}_p$, and $\bar{D} : \mathcal{S}_{p,q} \rightarrow \mathcal{S}_{p,q+1}$ goes over to the operator $\bar{\partial} : \varphi \otimes v \mapsto \bar{\partial}\varphi \otimes v$, where $\varphi \in \Phi^{0,q}$, and $v \in (\mathcal{T}_{\text{gr}})_p$. Hence, the sequence

$$0 \longrightarrow \mathcal{T}_{\text{gr}} \xrightarrow{i} \mathcal{S}_{*,0} \xrightarrow{\bar{D}} \mathcal{S}_{*,1} \xrightarrow{\bar{D}} \dots, \quad (5)$$

where i is a natural inclusion, is identified with the Dolbeault–Serre resolution of \mathcal{T}_{gr} . Set $S_{p,q} := \Gamma(M, \mathcal{S}_{p,q})$ and $S := \bigoplus_{p,q \geq 0} S_{p,q}$. Then, the bigraded Lie superalgebras $H^*(M, \mathcal{T})$ and $H(S, \bar{D})$ are isomorphic.

The desired non-abelian complex is the non-linear complex associated to the differential bigraded Lie superalgebra (S, \bar{D}) . More precisely, denote by \mathcal{F}^∞ the sheaf of differentiable complex-valued functions on M . Consider the sheaves $\mathcal{O}_{\text{gr}}^\infty := \mathcal{F}^\infty \otimes \mathcal{O}_{\text{gr}}$ and the group

$$\text{PAut}_{(2)}\mathcal{O}_{\text{gr}}^\infty := \{a \in \text{Aut } \mathcal{O}_{\text{gr}}^\infty \mid a(u) - u \in \bigoplus_{k \geq 2} (\mathcal{O}_{\text{gr}}^\infty)_k, u \in \mathcal{O}_{\text{gr}}^\infty\}.$$

The non-abelian complex is the triple $K = (K^0, K^1, K^2)$, where

$$K^0 := \text{PAut}_{(2)}\mathcal{O}_{\text{gr}}^\infty, \quad K^q := \bigoplus_{k \geq 1} S_{2k,q} \quad \text{for } q = 1, 2,$$

with the coboundary operators $\delta_i : K^i \rightarrow K^{i+1}$ for $i = 0, 1$, given by

$$\begin{aligned} \delta_0(a) &= \bar{\partial} - a\bar{\partial}a^{-1} \quad \text{for any } a \in K^0, \\ \delta_1(u) &= \bar{D}u - \frac{1}{2}[u, u] = -\frac{1}{2}[u - \bar{\partial}, u - \bar{\partial}] \quad \text{for any } u \in K^1. \end{aligned}$$

The gauge action ρ of K^0 on K^1 is given by

$$\rho(a)(u) = a(u - \bar{\partial})a^{-1} + \bar{\partial} \quad \text{for any } a \in K^0, u \in K^1.$$

Define $Z^1(K) := \{u \in K^1 \mid \delta_1 u = 0\}$ and $H^1(K) := Z^1(K)/\rho$. In [O3], it is proved that there is an isomorphism of pointed sets

$$\mu : H^1(K) \longrightarrow H^1(M, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}}).$$

To describe this isomorphism, take a cocycle $w \in Z^1(K_{(1)})$ and an open cover $\mathfrak{U} = (U_i)$ on M such that $w = \delta_0(a_i)$, where $a_i \in \Gamma(U_i, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}}^\infty)$. Then, we get the Čech cocycle $z = (z_{ij}) \in Z^1(\mathfrak{U}, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}})$, where $z_{ij} = a_i^{-1}a_j$. We have $\mu(w) = \zeta$, where $\zeta \in H^1(M, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}})$ and $\omega \in H^1(K)$ are the cohomology classes of the cocycles w and z , respectively.

Note that the group $\text{Aut } \mathbf{E}$ acts on the complex K and on $H^1(K)$ in a natural way.

Using eq. (5), we can also construct a fine resolution of the tangent sheaf of any supermanifold with retract $(M, \mathcal{O}_{\text{gr}})$. Consider the supermanifold (M, \mathcal{O}) with retract $(M, \mathcal{O}_{\text{gr}})$ that corresponds to the cohomology classes ω and ζ of cocycles $w \in Z^1(K)$ and $z = (z_{ij})$, as above. Twisting eq. (5) by z , we get the fine resolution

$$0 \longrightarrow \mathcal{T}_{\text{gr}}^{\text{Int}z} \xrightarrow{i} \mathcal{S}_{*,0}^{\text{Int}z} \xrightarrow{\bar{D}} \mathcal{S}_{*,1}^{\text{Int}z} \xrightarrow{\bar{D}} \dots \quad (6)$$

Here any $v \in \mathcal{T}_{\text{gr}}^{\text{Int}z}$ is a family $v = (v^i)$, where $v^i \in \Gamma(U_i, \mathcal{T}_{\text{gr}})$ and $v^i = z_{ij} \circ v^j \circ z_{ij}^{-1}$ in $U_i \cap U_j$. In the same way we express the sections of the sheaves $\mathcal{S}_{*,q}^{\text{Int}z}$.

The correspondence $(v^i) \mapsto (h_i \circ v^i \circ h_i^{-1})$ gives an isomorphism $\mathcal{T}_{\text{gr}}^{\text{Int}z} \simeq \mathcal{T}$, and the correspondence $(v^i) \mapsto (a_i \circ v^i \circ a_i^{-1})$ gives an isomorphism $\mathcal{S}_{*,p}^{\text{Int}z} \simeq \mathcal{S}_{*,p}$ for $p \geq 0$. Then, eq. (6) gives the following fine resolution of $\mathcal{T} = \text{Der } \mathcal{O}$:

$$0 \longrightarrow \mathcal{T} \xrightarrow{\tau} \mathcal{S}_{*,0} \xrightarrow{\bar{D}^w} \mathcal{S}_{*,1} \xrightarrow{\bar{D}^w} \dots, \quad (7)$$

where $\bar{D}^w := \bar{D} - \text{ad}_w = \text{ad}_{\bar{\partial}-w}$. Considering global sections, we get a complex (S, \bar{D}^w) for calculating cohomology with values in the sheaf \mathcal{T} .

We give an explicit expression of τ . As we have seen in Subsection 1.3, the cocycle $z = (z_{ij}) \in Z^1(\mathfrak{U}, \text{Aut}_{(2)}\mathcal{O}_{\text{gr}})$ of the cover \mathfrak{U} can be represented in the form $z_{ij} = h_i^{-1}h_j$. But it can also be represented in the form $z_{ij} = a_i^{-1}a_j$. Then, in $U_i \cap U_j$ we have $h_i^{-1}h_j = a_i^{-1}a_j$, and $\varrho = a_i h_i^{-1} = a_j h_j^{-1}$ is an injective homomorphism $\mathcal{O} \rightarrow \mathcal{O}_{\text{gr}}^\infty$. It follows that $\tau : \mathcal{T} = \text{Der } \mathcal{O} \rightarrow \mathcal{S}_{*,0}$ is expressed by the formula $v \mapsto \varrho v \varrho^{-1}$.

1.4.1. Theorem. *The mapping $\tau : v \mapsto \varrho v \varrho^{-1}$ is an isomorphism of the graded Lie superalgebra $H^*(M, \mathcal{T})$ onto $H^*(S, \bar{D}^w)$.*

In particular, we get the isomorphism $\tau : \mathfrak{v}(M, \mathcal{O}) \rightarrow \text{Ker } \bar{D}^w \subset S_{,0}$.*

1.5 An application of the Hodge theory Suppose that M is compact. Then, we can develop the standard Hodge theory in the complex (S, \bar{D}) regarding it as the complex of $(0, *)$ -forms with values in the bundle \mathbf{ST} , see [O3]. Endow M and \mathbf{E} with smooth Hermitian metrics and consider the corresponding Hermitian metric on \mathbf{ST} . Denote by \bar{D}^* the operator conjugate to \bar{D} and by $\square := [\bar{D}, \bar{D}^*]$ the Beltrami–Laplace operator. Their bidegrees are $(0, -1)$ and $(0, 0)$, respectively. Then, we have the orthogonal decomposition

$$S = \mathbf{H} \oplus \bar{D}S \oplus \bar{D}^*S, \quad (8)$$

where $\mathbf{H} = \text{Ker } \square$ is the bigraded subspace of harmonic elements. Moreover,

$$\text{id} = H + \square G = H + \bar{D} \bar{D}^* G + \bar{D}^* \bar{D} G,$$

where H is the projection onto \mathbf{H} in eq. (8) and G is the Green operator. It is well known that

$$\mathbf{H}_{p,q} \simeq H^{p,q}(S, \bar{D}) \simeq H^q(M, (\mathcal{T}_{\text{gr}})_p) \text{ for any } p, q \geq 0. \quad (9)$$

Consider now the nonlinear complex K . Denote

$$\begin{aligned} \mathbf{H}_{(1)} &:= \bigoplus_{p \geq 1} \mathbf{H}_{2p,1}, \\ L_1 &:= \text{Ker } \bar{D}^* \cap K^1, \quad \mathbf{K} := Z^1(K) \cap L_1, \end{aligned}$$

and define also the subset $\mathbf{K}_0 \subset K^1$ consisting of the u such that

$$u - \frac{1}{2} \bar{D}^* G[u, u] = Hu. \quad (10)$$

1.5.1. Theorem ([O3]). *We have $\mathbf{K} \subset \mathbf{K}_0 \subset L_1$. The mapping $H : \mathbf{K}_0 \rightarrow \mathbf{H}_{(1)}$ is a bijection and maps \mathbf{K} onto the connected algebraic subset $\mathbf{V} \subset \mathbf{H}_{(1)} \simeq \bigoplus_{p \geq 1} H^1(M, (\mathcal{T}_{\text{gr}})_{2p})$ given by the equation*

$$H[\varphi(h), \varphi(h)] = 0,$$

where $\varphi : \mathbf{H}_{(1)} \rightarrow L_1$ is inverse to H .

The natural mapping $\mathbf{K} \rightarrow H^1(K) \simeq H^1(M, \text{Aut}_{(2)} \mathcal{O}_{\text{gr}})$ is onto.

The set \mathbf{K} is an analogue of the *Kuranishi family* of complex structures on a compact manifold. By Theorem 1.5.1 we can see that this family cuts every cohomology class, and hence it can be used for classification of supermanifolds with retract $(M, \mathcal{O}_{\text{gr}})$.

1.6 Actions on supermanifolds Let (M, \mathcal{O}) be an arbitrary supermanifold. An *action* of a (real or complex) Lie group G on (M, \mathcal{O}) is a homomorphism $\Psi : G \rightarrow \text{Aut}(M, \mathcal{O})$. For any $g \in G$ we have $\Psi(g) = (f(g), \psi(g))$, where $f : g \mapsto f(g) \in \text{Bih } M$ is an (analytic) action of the group G on the complex manifold M and $\psi(g)$ is an automorphism of the sheaf \mathcal{O} over $f(g)$.

Let \mathbf{E} be a holomorphic vector bundle over a complex manifold M and G a Lie group. Suppose that \mathbf{E} has a structure of the G -bundle, i.e., a homomorphism $\Phi : G \rightarrow \text{Aut } \mathbf{E}$ satisfying the natural conditions of analyticity is given. Using the inclusion $\text{Aut } \mathbf{E} \subset \text{Aut}(M, \mathcal{O})$, we may consider Φ as an action on the split supermanifold $(M, \mathcal{O}_{\text{gr}})$ corresponding to the bundle \mathbf{E} . This action is

\mathbb{Z} -graded, i.e., all $\varphi(g)$, where $g \in G$, preserve the \mathbb{Z} -grading of the structure sheaf. Conversely, any \mathbb{Z} -graded action of the group G on (M, \mathcal{O}_{gr}) extends an action on the vector bundle \mathbf{E} .

Let again (M, \mathcal{O}) be an arbitrary complex supermanifold, (M, \mathcal{O}_{gr}) its retract and \mathbf{E} the corresponding vector bundle.

If $F = (f, \psi) \in \text{Aut}(M, \mathcal{O})$, then the automorphism ψ of \mathcal{O} over f preserves a filtration (1), and hence determines an automorphism φ of the \mathbb{Z} -graded sheaf \mathcal{O}_{gr} over f . Here, φ is uniquely determined by the relation $\pi_p \circ \psi = \varphi \circ \pi_p$ on \mathcal{J}^p .

Define $\bar{F} = (f, \varphi) \in \text{Aut}(M, \mathcal{O}_{gr})$ for every $F = (f, \psi) \in \text{Aut}(M, \mathcal{O})$. Thus, we get a homomorphism $\text{Aut}(M, \mathcal{O}) \rightarrow \text{Aut}(M, \mathcal{O}_{gr})$. It follows that any action $\Psi : G \rightarrow \text{Aut}(M, \mathcal{O})$ induces a \mathbb{Z} -graded action $\Phi : G \rightarrow \text{Aut}(M, \mathcal{O}_{gr})$. In this case, we say that the action Φ *lifts to the action* Ψ *on* (M, \mathcal{O}) .

There is the following lifting criterion:

1.6.1. Theorem ([O2]). *Let G be a compact Lie group and suppose an analytic \mathbb{Z} -graded action Ψ of G on a split supermanifold (M, \mathcal{O}_{gr}) be given. Let (M, \mathcal{O}) be the supermanifold corresponding to a given class $\zeta \in H^1(M, \text{Aut}_{(2)}\mathcal{O}_{gr})$ by Theorem 1.3.1. Then, the following conditions are equivalent:*

- (i) *the action Ψ lifts to (M, \mathcal{O}) ;*
- (ii) *the class ζ contains a G -invariant cocycle $z \in Z^1(\mathfrak{U}, \text{Aut}_{(2)}\mathcal{O}_{gr})$ where \mathfrak{U} is an open G -cover of M ;*
- (iii) *the class $\mu_1^{-1}(\zeta) \in H^1(K)$ (see Theorem 1.3.1) contains a G -invariant cocycle.*

Now we give definitions of homogeneous and $\bar{0}$ -homogeneous supermanifolds. Let (M, \mathcal{O}) be a complex supermanifold. For any $x \in M$ we can define the *tangent space* $T_x(M, \mathcal{O}) := (m_x/m_x^2)^*$, where m_x is the maximal ideal of the local superalgebra \mathcal{O}_x .

There is a natural even linear mapping $\text{ev}_x : \mathfrak{v}(M, \mathcal{O}) \rightarrow T_x(M, \mathcal{O})$. Namely, every $v \in \mathfrak{v}(M, \mathcal{O})$ determines a linear mapping $m_x \rightarrow \mathcal{O}_x$ with $v(m_x^2) \subset m_x$, and hence a linear mapping

$$m_x/m_x^2 \rightarrow \mathcal{O}_x/m_x = \mathbb{C},$$

i.e., an element $\text{ev}_x(v) \in (m_x/m_x^2)^*$.

The subalgebra $\mathfrak{g} \subset \mathfrak{v}(M, \mathcal{O})$ is called *transitive* if $\text{ev}_x : \mathfrak{g} \rightarrow T_x(M, \mathcal{O})$ is surjective for all $x \in M$ and if $\text{ev}_x : \mathfrak{g}_{\bar{0}} \rightarrow T_x(M, \mathcal{O})_{\bar{0}} = T_x(M)$ is surjective for all $x \in M$, then it is called *$\bar{0}$ -transitive*. A supermanifold (M, \mathcal{O}) is called *homogeneous* (*$\bar{0}$ -homogeneous*) if there is a transitive ($\bar{0}$ -transitive) subalgebra $\mathfrak{g} \subset \mathfrak{v}(M, \mathcal{O})$ of finite dimension. In the case when M is a compact we can replace \mathfrak{g} by $\mathfrak{v}(M, \mathcal{O})$.

1.6.2. Theorem ([OP]). *If a supermanifold (M, \mathcal{O}) is homogeneous ($\bar{0}$ -homogeneous), then $(M, \text{gr } \mathcal{O})$ is homogeneous ($\bar{0}$ -homogeneous).*

2 Supermanifolds associated with the complex torus

2.1 Complex tori Let $\Gamma \subset \mathbb{C}^m$ be a discrete subgroup of rank $2m$. Then, the manifold $T = \mathbb{C}^m/\Gamma$ is a *complex torus* of dimension m . Note that T is a compact complex commutative Lie group. There is a local coordinate system in a neighborhood of any point of the manifold T formed by the standard coordinates z_1, \dots, z_m in \mathbb{C}^m . Let us denote these coordinates on T also by z_1, \dots, z_m . The differential forms dz_1, \dots, dz_m are defined on T globally, since they are not changed if we add a complex number to the variable. Using duality between differential forms and

vector fields, we get the vector fields $\partial_{z_1}, \dots, \partial_{z_m}$ which are defined globally, too. The tangent and the cotangent bundles over T are trivial, and dz_i, ∂_{z_i} are basis sections of these vector bundles.

2.1.1. Proposition ([GH]). *Let M be a compact Kähler manifold. A form $\alpha \in \Gamma(M, \Phi^{0,q})$ is harmonic if and only if α is an antiholomorphic form, i.e., $\partial\alpha = 0$.*

It is well known that $T = \mathbb{C}^m/\Gamma$ is a compact Kähler manifold with flat metrics induced by Hermitian metrics in \mathbb{C}^m (see [GH]). We can represent any $\alpha \in \Gamma(M, \Phi^{0,q})$ in a form

$$\alpha = \sum_{1 \leq i_1 < \dots < i_q \leq m} a_{i_1 \dots i_q}(z, \bar{z}) d\bar{z}_{i_1} \dots d\bar{z}_{i_q}, \quad (11)$$

where $a_{i_1 \dots i_q}(z, \bar{z})$ are smooth global defined functions, $z = (z_1, \dots, z_m)$, $\bar{z} = (\bar{z}_1, \dots, \bar{z}_m)$. Since any antiholomorphic function on T is constant, we have

2.1.2. Proposition. *A form $\alpha \in \Gamma(M, \Phi^{0,q})$ is harmonic if and only if*

$$\alpha = \sum_{1 \leq i_1 < \dots < i_q \leq m} a_{i_1 \dots i_q} d\bar{z}_{i_1} \dots d\bar{z}_{i_q}, \quad \text{where } a_{i_1 \dots i_q} \in \mathbb{C}.$$

A form on T is called T -invariant if it is invariant under the action of the group T on itself by translations.

2.1.3. Proposition. *The spaces of harmonic and T -invariant $(0, q)$ -forms on T coincide.*

Proof. Clearly, the forms $d\bar{z}_{i_1} \dots d\bar{z}_{i_q}$ are T -invariant. It follows that the form (11) is T -invariant if and only if $a_{i_1 \dots i_q} \in \mathbb{C}$. Then, we apply Proposition 2.1.2. \square

2.2 Supermanifolds corresponding to the trivial bundle over the complex torus Let $\mathbf{E} = T \times \mathbb{C}^n$ be a trivial holomorphic vector bundle of rank n over T and ξ_1, \dots, ξ_n be the standard basis of \mathbb{C}^n . Denote by $T^{m|n} = (T, \mathcal{O}_{gr})$ the split supermanifold corresponding to the bundle \mathbf{E} . The structure sheaf \mathcal{O}_{gr} has the form $\mathcal{F} \otimes \wedge(\xi_1, \dots, \xi_n)$. The local coordinates z_1, \dots, z_m on T are even coordinates on $T^{m|n}$, and ξ_1, \dots, ξ_n are odd ones.

Consider the tangent sheaf $\mathcal{T}_{gr} = \mathcal{D}er \mathcal{O}_{gr}$. This sheaf is free over \mathcal{F} , or, equivalently, the bundle \mathbf{ST} is trivial, and the basis of its sections is

$$\xi_{i_1} \dots \xi_{i_k} \partial_{z_j}, \quad \xi_{i_1} \dots \xi_{i_k} \partial_{\xi_l}, \quad \text{where} \quad (12)$$

$$1 \leq i_1 < \dots < i_k \leq n, \quad j = 1, \dots, m, \quad l = 1, \dots, n.$$

Hence, the elements of $S_{p,q}$ have the form

$$\alpha = \sum_{i_1 < \dots < i_q} \left(\sum_{\substack{j_1 < \dots < j_p \\ i=1, \dots, m}} a_{j_1, \dots, j_p}^{i, i_1, \dots, i_q}(z, \bar{z}) \xi_{j_1} \dots \xi_{j_p} \partial_{z_i} + \sum_{\substack{j_1 < \dots < j_{p+1} \\ j=1, \dots, n}} b_{j_1, \dots, j_{p+1}}^{j, i_1, \dots, i_q}(z, \bar{z}) \xi_{j_1} \dots \xi_{j_{p+1}} \partial_{\xi_j} \right) d\bar{z}_{i_1} \dots d\bar{z}_{i_q},$$

where $a_{j_1, \dots, j_p}^{i, i_1, \dots, i_q}(z, \bar{z})$ and $b_{j_1, \dots, j_{p+1}}^{j, i_1, \dots, i_q}(z, \bar{z})$ are smooth globally defined functions on T . Thus, from Proposition 2.1.2 we get

2.2.1. Proposition. *The form $\alpha \in S_{p,q}$ is harmonic if and only if*

$$\alpha = \sum_{i_1 < \dots < i_q} \left(\sum_{\substack{j_1 < \dots < j_p \\ i=1, \dots, m}} a_{j_1, \dots, j_p}^{i, i_1, \dots, i_q} \xi_{j_1} \dots \xi_{j_p} \partial_{z_i} + \sum_{\substack{j_1 < \dots < j_{p+1} \\ j=1, \dots, n}} b_{j_1, \dots, j_{p+1}}^{j, i_1, \dots, i_q} \xi_{j_1} \dots \xi_{j_{p+1}} \partial_{\xi_j} \right) d\bar{z}_{i_1} \dots d\bar{z}_{i_q}, \quad (13)$$

where $a_{j_1, \dots, j_p}^{i, i_1, \dots, i_q}, b_{j_1, \dots, j_{p+1}}^{j, i_1, \dots, i_q} \in \mathbb{C}$.

2.2.1a. Corollary. *If $\alpha, \beta \in \mathbf{H}$, then $[\alpha, \beta] \in \mathbf{H}$.*

Assigning to every cohomology class from $H^q(M, (\mathcal{T}_{\text{gr}})_p)$ the correspondent harmonic form from $\mathbf{H}_{p,q}$ (see (9)), we get an isomorphism of graded Lie superalgebras $H(M, \mathcal{T}_{\text{gr}})$ onto the subalgebra $\mathbf{H} \subset S$.

Since $\xi_{j_1} \dots \xi_{j_p} \partial_{z_i}$ and $\xi_{j_1} \dots \xi_{j_{p+1}} \partial_{\xi_j}$ are T -invariant, from Proposition 2.1.3 we get

2.2.2. Proposition. *Any harmonic form from S is T -invariant, and the other way round.*

2.2.3. Theorem. *We have $\mathbf{K}_0 = \mathbf{H}_{(1)}$ and*

$$\mathbf{K} = \mathbf{V} = \{w \in \mathbf{H}_{(1)} \mid [w, w] = 0\}.$$

Proof. Take $w \in \mathbf{K}_0$ and denote $h = Hw$. We write $h = \sum_{k \geq 1} h_{2k}$, and $w = \sum_{k \geq 1} w_{2k}$, where $h_{2k} \in \mathbf{H}_{2k,1}$, $w_{2k} \in S_{2k,1}$. From (10) we get the following equations:

$$\begin{aligned} w_2 &= h_2, \\ w_4 - \frac{1}{2} \overline{D}^* G[w_2, w_2] &= h_4, \\ \dots \\ w_{2k} - \frac{1}{2} \overline{D}^* G \sum_{1 \leq s \leq k-1} [w_{2s}, w_{2(k-s)}] &= h_{2k}, \\ \dots \end{aligned}$$

We prove that $w_{2k} = h_{2k}$ by induction on k . For $k = 1$ this follows from the first equation. Suppose that $w_{2i} = h_{2i}$ for $1 \leq i \leq k-1$. By Corollary 2.2.1a we see that

$$h' = \sum_{1 \leq s \leq k-1} [w_{2s}, w_{2(k-s)}] \in \mathbf{H}.$$

Since \overline{D}^* and G commute and $\overline{D}^* h' = 0$, we get $w_{2k} = h_{2k}$.

So we have proved that $w = h \in \mathbf{H}_{(1)}$. By Theorem 3, $\mathbf{K}_0 = \mathbf{H}_{(1)}$, and $\varphi = \text{id}$. Therefore, $\mathbf{K} = \mathbf{V} = \{w \in \mathbf{H}_{(1)} \mid [w, w] = 0\}$. \square

2.3 Lie superalgebras of vector fields on supermanifolds with retract $T^{m|n}$ Consider holomorphic vector fields on the split supermanifold $T^{m|n}$. Clearly, any $v \in \mathfrak{v}_p(T^{m|n})$ is a linear combination of the fields

$$\begin{aligned} &\xi_{j_1} \dots \xi_{j_p} \partial_{z_i} \text{ for } j_1 < \dots < j_p \text{ and } i = 1, \dots, m, \\ &\xi_{j_1} \dots \xi_{j_{p+1}} \partial_{\xi_j} \text{ for } j_1 < \dots < j_{p+1} \text{ and } j = 1, \dots, n \text{ (see (12))} \end{aligned}$$

with holomorphic coefficients. Since any holomorphic function on T is constant,

$$v = \sum_{1 \leq i \leq m} \sum_{j_1 < \dots < j_p} a_{j_1, \dots, j_p}^i \xi_{j_1} \dots \xi_{j_p} \partial_{z_i} + \sum_{1 \leq j \leq n} \sum_{j_1 < \dots < j_{p+1}} b_{j_1, \dots, j_{p+1}}^j \xi_{j_1} \dots \xi_{j_{p+1}} \partial_{\xi_j}, \quad (14)$$

where $a_{j_1, \dots, j_p}^i, b_{j_1, \dots, j_{p+1}}^j \in \mathbb{C}$. Since

$$\mathfrak{v}(T^{m|n}) = \mathbf{H}_{p,0} = \{v \in S_{p,0} \mid \overline{D}v = 0\},$$

we see that (14) is a special case of the formula (13).

By Theorem 1.5.1, any supermanifold with retract $T^{m|n}$ can be described by a form from the Kuranishi family \mathbf{K} . By Theorem 2.2.3, $\mathbf{K} = \mathbf{V}$ consists of harmonic elements.

2.3.1. Theorem. *Let (T, \mathcal{O}) be a supermanifold with retract $T^{m|n}$ given by an element $w \in \mathbf{V}$. Then, the mapping τ from (1.7) determines an isomorphism*

$$\mathfrak{v}(T, \mathcal{O}) \rightarrow \{v \in S_{*,0} \mid \bar{D}v = [w, v] = 0\} = \{v \in \mathfrak{v}_p(T^{m|n}) \mid [w, v] = 0\}.$$

Proof. We can write $w = w_2 + w_4 + \dots$, where $w_{2k} \in \mathbf{H}_{2k,1}$. Let $v \in S_{*,0}$ and $\bar{D}v = [w, v]$. Then, $v = v_{-1} + v_0 + v_1 + \dots$, where $v_i \in S_{i,0}$. The equation $\bar{D}v = [w, v]$ gives the finite system of equations:

$$\begin{aligned} \bar{D}v_{-1} &= 0, \\ \bar{D}v_0 &= 0, \\ \bar{D}v_1 &= [w_2, v_{-1}], \\ \bar{D}v_2 &= [w_2, v_0], \\ \bar{D}v_3 &= [w_2, v_1] + [w_4, v_{-1}], \\ \bar{D}v_4 &= [w_2, v_2] + [w_4, v_0], \\ &\dots \end{aligned} \tag{15}$$

Let us prove that $\bar{D}v_k = 0$ for $k = -1, 0, \dots$, by induction on k . For $k = -1, 0$ this follows from the first and the second equations of (15). Suppose that $\bar{D}v_{-1} = \dots = \bar{D}v_{k-1} = 0$. Then, using system (15), we see that $\bar{D}v_k$ is a sum of commutators of the fields v_{-1}, \dots, v_{k-2} with the forms w_{2l} . Since $w_{2l} \in \mathbf{H}_{2l,1}$, Corollary 2.2.1a shows that $\bar{D}v_k \in \mathbf{H}_{p,1}$. Hence $\bar{D}v_k = 0$.

Thus, we proved that the kernel of $\bar{D}_w = \bar{D} - \text{ad } w$ in $S_{*,0}$ coincides with the subalgebra $\{v \in \mathfrak{v}_p(T^{m|n}) \mid [w, v] = 0\} \subset \mathfrak{v}_p(T^{m|n})$. Now our statement follows from Theorem 1.4.1. \square

2.4 Homogeneous supermanifolds with retract $T^{m|n}$

2.4.1. Theorem. *Any supermanifold (T, \mathcal{O}) with retract $T^{m|n}$ is $\bar{0}$ -homogeneous. It is homogeneous if and only if $(T, \mathcal{O}) \simeq T^{m|n}$.*

Proof. Let (T, \mathcal{O}) be an arbitrary supermanifold with retract $T^{m|n}$. From Theorem 2.2.3 we see that it is determined by a harmonic form $w \in \mathbf{V}$. By Proposition 2.2.2 w is invariant under the natural action of the group T . Hence, by Theorem 1.6.1 the action of the group T on $T^{m|n}$ lifts to an action on (T, \mathcal{O}) . So (T, \mathcal{O}) is $\bar{0}$ -homogeneous.

From eq. (12) we see that the Lie superalgebra $\mathfrak{v}(T^{m|n})$ is transitive. Then, $T^{m|n}$ is homogeneous. Let (T, \mathcal{O}) be the supermanifold with retract $T^{m|n}$ determined by a cocycle $w \in \mathbf{V} \subset \mathbf{H}_{(1)}$. Let (T, \mathcal{O}) be homogeneous. Take a point $x_0 \in T$, and denote by ξ_1, \dots, ξ_n the odd local coordinates in the neighborhood U of x_0 which correspond to the coordinates ξ_1, \dots, ξ_n on $T^{m|n}$ by the local splitting $h_U : \mathcal{O}_{\text{gr}}|_U \rightarrow \mathcal{O}|_U$. Since $\text{ev}_{x_0} : \mathfrak{v}(T, \mathcal{O})_{\bar{1}} \rightarrow T_{x_0}(T, \mathcal{O})_{\bar{1}}$ is surjective, then for any j such that $1 \leq j \leq n$, there exists a field $v_j \in \mathfrak{v}(T, \mathcal{O})_{\bar{1}}$ such that $v_j = \partial_{\xi_j} + v'_j$ in U , where $(v'_j)_{x_0} \in m_{x_0} \mathcal{T}_{x_0}$. We can assume that the neighborhood $U = U_i$ is included into the cover (U_i) which we used in the description of τ in Subsection 1.4. As was shown in Subsection 1.4, we have

$$\tau(v_j) = a_i(h_i^{-1} \partial_{\xi_j} h_i) a_i^{-1} + (a_i h_i^{-1}) v'_j (h_i a_i^{-1}).$$

The sheaf $\mathcal{S}_{*,0} = \mathcal{T}_{\text{gr}}^\infty = \mathcal{D}er \mathcal{O}_{\text{gr}}^\infty$ has a filtration similar to (3):

$$\mathcal{T}_{\text{gr}}^\infty = \mathcal{T}_{\text{gr}(-1)}^\infty \subset \mathcal{T}_{\text{gr}(0)}^\infty \subset \mathcal{T}_{\text{gr}(1)}^\infty \subset \dots$$

Since $a_i \in \Gamma(U_i, \mathcal{A}ut_{(2)} \mathcal{O}_{\text{gr}}^\infty)$, we have

$$a_i(h_i^{-1} \partial_{\xi_j} h_i) a_i^{-1} = a_i \partial_{\xi_j} a_i^{-1} = \partial_{\xi_j} + u_j,$$

where $u_j \in \Gamma(U_i, \mathcal{T}_{\text{gr}(1)}^\infty)$. Clearly,

$$(a_i h_i^{-1})v'_j(h_i a_i^{-1}) = v''_j$$

satisfies $(v''_j)_{x_0} \in m_{x_0}^\infty(\mathcal{T}_{\text{gr}}^\infty)_{x_0}$, where $m_{x_0}^\infty$ is the maximal ideal of $(\mathcal{O}_{\text{gr}}^\infty)_{x_0}$. Hence in U_i we have

$$\tau(v_j) = \partial_{\xi_j} + u_j + v''_j,$$

where $(u_j + v''_j)_{x_0} \in m_{x_0}^\infty(\mathcal{T}_{\text{gr}}^\infty)_{x_0}$. By Theorem 2.3.1 $\tau(v_j) \in \mathfrak{v}(T^{m|n})_{\bar{1}}$.

So, if $\tilde{v}_j = \tau(v_j) - \partial_{\xi_j} \in \mathfrak{v}(T^{m|n})_{\bar{1}}$, then $\tilde{v}_{x_0} \in m_{x_0}(\mathcal{T}_{\text{gr}})_{x_0}$. Therefore, $\tilde{v}_j \in \bigoplus_{k \geq 0} \mathfrak{v}(T^{m|n})_{2k+1}$.

By Theorem 2.3.1

$$[w, \partial_{\xi_j} + \tilde{v}_j] = 0 \quad \text{for } j = 1, \dots, n. \quad (16)$$

We write w as $w = w_2 + w_4 + \dots$, where $w_{2k} \in \mathbf{H}_{2k,1}$, and prove that $w_{2k} = 0$ for $k = 1, 2, \dots$, by induction on k .

Considering the component of degree 1 of the left part of formula (16), we see that $[w_2, \partial_{\xi_j}] = 0$. Hence,

$$\begin{aligned} [w_2, \partial_{\xi_j}](z_r) &= \partial_{\xi_j}(w_2(z_r)) = 0 \quad \text{for } r = 1, \dots, m, \\ [w_2, \partial_{\xi_j}](\xi_s) &= \partial_{\xi_j}(w_2(\xi_s)) = 0 \quad \text{for } s = 1, \dots, n. \end{aligned}$$

From (13) we see that $w_2 = 0$.

Suppose we have proved that $w_2 = w_4 = \dots = w_{2k-2} = 0$. Considering the component of degree $2k - 1$ of the left part of the formula (16), we get $[w_{2k}, \partial_{\xi_j}] = 0$ for all $j = 1, \dots, n$. As above, we can prove that $w_{2k} = 0$.

Thus, $w = 0$. Hence, $(T, \mathcal{O}) \simeq T^{m|n}$. \square

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