Supermanifolds corresponding to the trivial vector bundle over torus

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EPIsciences

Communications in Mathematics n (2022) 00–∞
DOI: 10.46298/cm.ABCD-XYZW

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Abstract. All supermanifolds whose retract $T^{m|n}$ is determined by the trivial bundle of rank n over the torus T^m are $\overline{0}$-homogeneous if and only if $T^{m|n}$ is homogeneous.

1 Preliminaries

1.1 Split and non-split supermanifolds

The ground field is $\mathbb{C}$.

A *complex supermanifold* of dimension $m|n$ is a $\mathbb{Z}/2$-graded ringed space of the form $\mathcal{M} := (M, \mathcal{O})$, where M is a topological space and $\mathcal{O}$ is a sheaf of associative commutative superalgebras with unit on M, which is locally isomorphic to a superdomain in $\mathbb{C}^{m|n}$. For details, see [M], [O3], [BGLS*]. A *superdomain* in $\mathbb{C}^{m|n}$ is a pair $(U, \bigwedge_{\mathcal{F}}(\xi_1, \dots, \xi_n))$, where U is an open subset of $\mathbb{C}^m$, and $\mathcal{F}$ is the sheaf of holomorphic functions on $\mathbb{C}^m$. The coordinates $x_1, \dots, x_m$ in $U \subset \mathbb{C}^m$ and generators $\xi_1, \dots, \xi_n$ of the Grassmann algebra are identified with some sections of the sheaf $\mathcal{O}|_U$. They are called *local coordinates even* and *odd*, respectively.

Let $(M, \mathcal{F})$ be a complex manifold and $\mathcal{E}$ be a locally free analytic sheaf on it, i.e., $\mathcal{E}$ is a sheaf of holomorphic sections of some holomorphic vector bundle $\mathbf{E} \to M$. Then, $(M, \mathcal{O}_{\mathrm{gr}})$, where $\mathcal{O}_{\mathrm{gr}} = \bigwedge_{\mathcal{F}} \mathcal{E}$, is a complex supermanifold. A supermanifold is called *split* if it is isomorphic to a supermanifold of this form and is called *non-split* otherwise.

Let us show that every supermanifold is a deformation of a split supermanifold. Consider the subsheaf of ideals $\mathcal{J} = (\mathcal{O}_{\overline{1}})$, generated by odd elements. Denote $\mathcal{F} := \mathcal{O}/\mathcal{J}$. Then, $\mathcal{M}_{\mathrm{rd}} = (M, \mathcal{F})$ is a complex manifold. It is called the *odd reduction* of $(M, \mathcal{O})$ and hereafter is often denoted by M. The powers of $\mathcal{J}$ determine the following filtration:

$$\mathcal{O} = \mathcal{J}^0 \supset \mathcal{J}^1 \supset \mathcal{J}^2 \supset \dots \supset \mathcal{J}^{n+1} = 0. \tag{1}$$

The associated sheaf of graded algebras

$$\operatorname{gr} \mathcal{O} = \bigoplus_{0 \le p \le n} \operatorname{gr}_p \mathcal{O}, \quad \text{where} \quad \operatorname{gr}_p \mathcal{O} = \mathcal{J}^p / \mathcal{J}^{p+1},$$

MSC 2010: Primary 32C11 Secondary 81T30
Keywords: Homogeneous supermanifold, split supermanifold, supertorus.
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is an analytic sheaf on the reduction $\mathcal{M}_{\mathrm{rd}}$. Actually, $\operatorname{gr}\mathcal{O} \simeq \bigwedge_{\mathcal{F}} \mathcal{E}$, where $\mathcal{E} = \operatorname{gr}_1\mathcal{O}$ is locally free sheaf. Clearly, $(M, \operatorname{gr}\mathcal{O}) = (M, \mathcal{O}_{\mathrm{gr}})$ is a split supermanifold of the same dimension as $(M, \mathcal{O})$. We call it the *retract* of the supermanifold $(M, \mathcal{O})$. Obviously, a given supermanifold is split if and only if it is isomorphic to its retract.

Let $\pi_p : \mathcal{J}^p \to \operatorname{gr}_p\mathcal{O}$ be the canonical projection. Then, there is the exact sequence of sheaves

$$0 \longrightarrow \mathcal{J}^{p+1} \longrightarrow \mathcal{J}^p \xrightarrow{\pi_p} \operatorname{gr}_p\mathcal{O} \longrightarrow 0. \tag{2}$$

A supermanifold $(M, \mathcal{O})$ is split if and only if there exists an isomorphism of superalgebra sheaves $h : \operatorname{gr}\mathcal{O} \to \mathcal{O}$, whose restriction $h_p : \operatorname{gr}_p\mathcal{O} \to \mathcal{J}^p$ splits the sequence (2), i.e., satisfies the condition $\pi_p \circ h_p = \mathrm{id}$. In general, this splitting exists in a neighborhood of any point in M. It can be given by means of local coordinates.

1.2 The tangent sheaf For an arbitrary supermanifold $(M, \mathcal{O})$ denote by $\mathcal{T} := \mathcal{D}er\ \mathcal{O}$ its *tangent sheaf* (or the sheaf of vector fields). It is the sheaf of derivations (over $\mathbb{C}$) of the structure sheaf $\mathcal{O}$. Note that the tangent sheaf is a sheaf of $\mathbb{Z}/2$-graded left $\mathcal{O}$-modules, and also a sheaf of Lie superalgebras. The sections of the tangent sheaf are called *holomorphic vector fields* on $(M, \mathcal{O})$. They form the Lie superalgebra $\mathfrak{v}(M, \mathcal{O})$ of vector fields on $(M, \mathcal{O})$.

Note that the filtration (1) determines the filtration of the tangent sheaf

$$\mathcal{T} = \mathcal{T}_{(-1)} \supset \mathcal{T}_{(0)} \supset \cdots \supset \mathcal{T}_{(n)} \supset \mathcal{T}_{(n+1)} = 0, \tag{3}$$

where

$$\mathcal{T}_{(p)} = \{v \in \mathcal{T} \mid v(\mathcal{O}) \subset \mathcal{J}^p,\ v(\mathcal{J}) \subset \mathcal{J}^{p+1}\},\ p \geq 0.$$

Since $(M, \mathcal{O}_{\mathrm{gr}})$ is split, its tangent sheaf $\mathcal{T}_{\mathrm{gr}}$ is a $\mathbb{Z}$-graded sheaf of Lie superalgebras

$$\mathcal{T}_{\mathrm{gr}} = \bigoplus_{-1 \leq p \leq n} (\mathcal{T}_{\mathrm{gr}})_p,$$

where

$$(\mathcal{T}_{\mathrm{gr}})_p := \mathcal{D}er_p\mathcal{O}_{\mathrm{gr}} = \{v \in \mathcal{T}_{\mathrm{gr}} \mid v((\mathcal{O}_{\mathrm{gr}})_q) \subset (\mathcal{O}_{\mathrm{gr}})_{q+p},\ q \in \mathbb{Z}\}.$$

This grading is compatible with the $\mathbb{Z}/2$-grading. The Lie superalgebra $\mathfrak{v}(M, \mathcal{O}_{\mathrm{gr}})$ of vector fields is a graded algebra with the $\mathbb{Z}$-grading compatible with the $\mathbb{Z}/2$-grading.

Since $\mathcal{F} \subset \mathcal{O}_{\mathrm{gr}}$, the tangent sheaf $\mathcal{T}_{\mathrm{gr}}$ is a $\mathbb{Z}$-graded analytic sheaf on M. This sheaf is locally free (see [O3]), and hence it is the sheaf of holomorphic sections of a $\mathbb{Z}$-graded holomorphic vector bundle $\mathbf{ST}$ over M (the *supertangent bundle*).

1.3 Sheaves of automorphisms and the classification theorem Let $(M, \mathcal{O})$ be an complex supermanifold. Denote by $\operatorname{Aut}(M, \mathcal{O})$ the group of automorphisms of $(M, \mathcal{O})$. By definition, $F \in \operatorname{Aut}(M, \mathcal{O})$ is a pair (f, φ), where $f : M \to M$ belongs to group $\operatorname{Bih} M$ of biholomorphic transformations of the manifold M and φ is an automorphism of the superalgebra sheaf $\mathcal{O}$ over f. Denote by $\mathcal{A}ut\,\mathcal{O}$ the sheaf of automorphisms of the structure sheaf $\mathcal{O}$ (mapping every stalk $\mathcal{O}_x$, where $x \in M$, onto itself). Moreover, for any $F = (f, \varphi) \in \operatorname{Aut}(M, \mathcal{O})$ the map $\operatorname{Int} F : a \mapsto \varphi \circ a \circ \varphi^{-1}$ is an automorphism of the group sheaf $\mathcal{A}ut\ \mathcal{O}$. Hence, we get the action Int of the group $\operatorname{Aut}(M, \mathcal{O})$ on $\mathcal{A}ut\ \mathcal{O}$ by automorphisms. The subsheaf

$$\mathcal{A}ut_{(2)}\mathcal{O} = \{a \in \mathcal{A}ut\,\mathcal{O} \mid a(f) - f \in \mathcal{J}^2,\ f \in \mathcal{O}\} \tag{4}$$

is invariant under this action.

Let $\mathbf{E}$ be a holomorphic vector bundle over $(M,\mathcal{F})$ and $\operatorname{Aut}\mathbf{E}$ the group of its automorphisms. Clearly, any element of this group gives rise to an automorphism of the split supermanifold $(M,\mathcal{O}_{\rm gr})$ corresponding to $\mathbf{E}$ which preserves the $\mathbb{Z}$-grading of the structure sheaf. Hence, we can identify $\operatorname{Aut}\mathbf{E}$ with a subgroup of $\operatorname{Aut}(M,\mathcal{O}_{\rm gr})$, consisting of automorphisms that preserve the $\mathbb{Z}$-grading. So the $\operatorname{Aut}(M,\mathcal{O}_{\rm gr})$-sheaves $\mathcal{A}ut\,\mathcal{O}_{\rm gr}$ and $\mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr}$ are also $\operatorname{Aut}\mathbf{E}$-sheaves.

1.3.1. Theorem ([G]). *Any supermanifold $(M,\mathcal{O})$ corresponds to an element of the set of 1-cohomology $H^1(M,\mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})$. This correspondence gives rise to a bijection between the isomorphism classes of supermanifolds satisfying the above condition, and the orbits of the group $\operatorname{Aut}\mathbf{E}$ on $H^1(M,\mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})$ under the natural $\operatorname{Aut}\mathbf{E}$-action.*

Let us describe the correspondence mentioned in Theorem 1.3.1. Let $(M,\mathcal{O})$ be a supermanifold with retract $(M,\mathcal{O}_{\rm gr})$. Then, we can choose an open cover $\mathfrak{U}=(U_i)_{i\in I}$ of M such that there exist isomorphisms $h_i : \mathcal{O}_{\rm gr}|_{U_i}\to\mathcal{O}|_{U_i}$, where $i\in I$, with conditions $\pi_p\circ(h_i)_p=\mathrm{id}$ on $(\mathcal{O}_{\rm gr})_p|_{U_i}$ (see (2)). Setting $z_{ij}=h_i^{-1}h_j$, we get a 1-cocycle $z=(z_{ij})\in Z^1(\mathfrak{U},\mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})$. Its class $\zeta\in H^1(M,\mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})$ does not depend on the choice of h_i and corresponds to $(M,\mathcal{O})$.

1.4 A non-abelian complex Recall the construction of a non-abelian complex (see [O3], [O2]) which allows to express $H^1(M,\mathcal{A}ut_{(2)}\mathcal{O}_{\rm gr})$ in terms of differential forms. Let $\Phi^{p,q}$ be the sheaf of smooth differential (p,q)-forms on M. First, we construct the Dolbeault–Serre resolution of the sheaf $\mathcal{O}_{gr}$:

$$\widehat{\Phi}:=\bigoplus_{p,q\geq 0}\widehat{\Phi}^{p,q},\quad \widehat{\Phi}^{p,q}:=\Phi^{0,q}\otimes(\mathcal{O}_{gr})_p,$$

$$\overline{\partial}(\varphi\otimes u)=(\overline{\partial}\varphi)\otimes u\ \text{ for any }\ \varphi\in\Phi^{0,q},\ u\in(\mathcal{O}_{gr})_p.$$

Then, regarding $\widehat{\Phi}$ as a sheaf of graded superalgebras with respect to the total degree $p+q$, we get a sheaf of graded Lie superalgebras $\widehat{\mathcal{T}}=\mathcal{D}er\,\widehat{\Phi}$. The sheaf $\widehat{\mathcal{T}}$ has the derivation $\overline{D}=\mathrm{ad}_{\overline{\partial}}$ of degree 1 (and of bidegree $(0,1)$). Denote

$$\mathcal{S}=\{u\in\widehat{\mathcal{T}}\mid u(\overline{f})=u(d\overline{f})=0\text{ for any }f\in\mathcal{F}\}.$$

This is a subsheaf of bigraded subalgebras, and $\overline{D}(\mathcal{S})\subset\mathcal{S}$. As it was shown in [O3], the subsheaf $\mathcal{S}_{p,q}$ is naturally identified with the sheaf $\Phi^{0,q}\otimes(\mathcal{T}_{\rm gr})_p$ of $(0,q)$-forms with values in the vector bundle $\mathbf{ST}_p$, and $\overline{D}:\mathcal{S}_{p,q}\to\mathcal{S}_{p,q+1}$ goes over to the operator $\overline{\partial}:\varphi\otimes v\mapsto\overline{\partial}\varphi\otimes v$, where $\varphi\in\Phi^{0,q}$, and $v\in(\mathcal{T}_{\rm gr})_p$. Hence, the sequence

$$0\longrightarrow\mathcal{T}_{gr}\xrightarrow{\ i\ }\mathcal{S}_{*,0}\xrightarrow{\ \overline{D}\ }\mathcal{S}_{*,1}\xrightarrow{\ \overline{D}\ }\dots,\tag{5}$$

where i is a natural inclusion, is identified with the Dolbeault–Serre resolution of $\mathcal{T}_{gr}$. Set $S_{p,q}:=\Gamma(M,\mathcal{S}_{p,q})$ and $S:=\bigoplus_{p,q\geq 0}S_{p,q}$. Then, the bigraded Lie superalgebras $H^*(M,\mathcal{T})$ and $H(S,\overline{D})$ are isomorphic.

The desired non-abelian complex is the non-linear complex associated to the differential bigraded Lie superalgebra $(S,\overline{D})$. More precisely, denote by $\mathcal{F}^\infty$ the sheaf of differentiable complex-valued functions on M. Consider the sheaves $\mathcal{O}^\infty_{\rm gr}:=\mathcal{F}^\infty\otimes\mathcal{O}_{\rm gr}$ and the group

$$\mathrm{PAut}_{(2)}\mathcal{O}^\infty_{\rm gr}:=\{a\in\operatorname{Aut}\mathcal{O}^\infty_{\rm gr}\mid a(u)-u\in\bigoplus_{k\geq 2}(\mathcal{O}^\infty_{\rm gr})_k,\ u\in\mathcal{O}^\infty_{\rm gr}\}.$$

The non-abelian complex is the triple $K = (K^0, K^1, K^2)$, where

$$K^0 := \mathrm{PAut}_{(2)}\mathcal{O}^\infty_{\mathrm{gr}}, \quad K^q := \bigoplus_{k\geq 1} S_{2k,q} \ \text{ for } \ q = 1, 2,$$

with the coboundary operators $\delta_i : K^i \to K^{i+1}$ for $i = 0, 1$, given by

$$\delta_0(a) = \overline{\partial} - a\overline{\partial}a^{-1} \ \text{ for any } \ a \in K^0,$$
$$\delta_1(u) = \overline{D}u - \tfrac{1}{2}[u, u] = -\tfrac{1}{2}[u - \overline{\partial}, u - \overline{\partial}] \ \text{ for any } \ u \in K^1.$$

The gauge action ρ of K^0 on K^1 is given by

$$\rho(a)(u) = a(u - \overline{\partial})a^{-1} + \overline{\partial} \ \text{ for any } \ a \in K^0, u \in K^1.$$

Define $Z^1(K) := \{u \in K^1 \mid \delta_1 u = 0\}$ and $H^1(K) := Z^1(K)/\rho$. In [O3], it is proved that there is an isomorphism of pointed sets

$$\mu : \ H^1(K) \longrightarrow H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{gr}).$$

To describe this isomorphism, take a cocycle $w \in Z^1(K_{(1)})$ and an open cover $\mathfrak{U} = (U_i)$ on M such that $w = \delta_0(a_i)$, where $a_i \in \Gamma(U_i, \mathcal{A}ut_{(2)}\mathcal{O}^\infty_{\mathrm{gr}})$. Then, we get the Čech cocycle $z = (z_{ij}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$, where $z_{ij} = a_i^{-1}a_j$. We have $\mu(\omega) = \zeta$, where $\zeta \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$ and $\omega \in H^1(K)$ are the cohomology classes of the cocycles w and z, respectively.

Note that the group $\mathrm{Aut}\,\mathbf{E}$ acts on the complex K and on $H^1(K)$ in a natural way.

Using eq. (5), we can also construct a fine resolution of the tangent sheaf of any supermanifold with retract $(M, \mathcal{O}_{gr})$. Consider the supermanifold $(M, \mathcal{O})$ with retract $(M, \mathcal{O}_{gr})$ that corresponds to the cohomology classes ω and ζ of cocycles $w \in Z^1(K)$ and $z = (z_{ij})$, as above. Twisting eq. (5) by z, we get the fine resolution

$$0 \longrightarrow \mathcal{T}^{\mathrm{Int}z}_{\mathrm{gr}} \xrightarrow{\ i\ } \mathcal{S}^{\mathrm{Int}z}_{*,0} \xrightarrow{\ \overline{D}\ } \mathcal{S}^{\mathrm{Int}z}_{*,1} \xrightarrow{\ \overline{D}\ } \dots . \tag{6}$$

Here any $v \in \mathcal{T}^{\mathrm{Int}\,z}_{\mathrm{gr}}$ is a family $v = (v^i)$, where $v^i \in \Gamma(U_i, \mathcal{T}_{\mathrm{gr}})$ and $v^i = z_{ij} \circ v^j \circ z_{ij}^{-1}$ in $U_i \cap U_j$. In the same way we express the sections of the sheaves $\mathcal{S}^{\mathrm{Int}z}_{*,q}$.

The correspondence $(v^i) \mapsto (h_i \circ v^i \circ h_i^{-1})$ gives an isomorphism $\mathcal{T}^{\mathrm{Int}z}_{\mathrm{gr}} \simeq \mathcal{T}$, and the correspondence $(v^i) \mapsto (a_i \circ v^i \circ a_i^{-1})$ gives an isomorphism $\mathcal{S}^{\mathrm{Int}z}_{*,p} \simeq \mathcal{S}_{*,p}$ for $p \geq 0$. Then, eq. (6) gives the following fine resolution of $\mathcal{T} = \mathcal{D}er\,\mathcal{O}$:

$$0 \longrightarrow \mathcal{T} \xrightarrow{\ \tau\ } \mathcal{S}_{*,0} \xrightarrow{\ \overline{D}^w\ } \mathcal{S}_{*,1} \xrightarrow{\ \overline{D}^w\ } \dots , \tag{7}$$

where $\overline{D}^w := \overline{D} - \mathrm{ad}_w = \mathrm{ad}_{\overline{\partial} - w}$. Considering global sections, we get a complex $(S, \overline{D}^w)$ for calculating cohomology with values in the sheaf $\mathcal{T}$.

We give an explicit expression of τ. As we have seen in Subsection 1.3, the cocycle $z = (z_{ij}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$ of the cover $\mathfrak{U}$ can be represented in the form $z_{ij} = h_i^{-1}h_j$. But it can also be represented in the form $z_{ij} = a_i^{-1}a_j$. Then, in $U_i \cap U_j$ we have $h_i^{-1}h_j = a_i^{-1}a_j$, and $\varrho = a_i h_i^{-1} = a_j h_j^{-1}$ is an injective homomorphism $\mathcal{O} \to \mathcal{O}^\infty_{\mathrm{gr}}$. It follows that $\tau : \mathcal{T} = \mathcal{D}er\,\mathcal{O} \to \mathcal{S}_{*,0}$ is expressed by the formula $v \mapsto \varrho v \varrho^{-1}$.

1.4.1. Theorem. *The mapping $\tau : v \mapsto \varrho v \varrho^{-1}$ is an isomorphism of the graded Lie superalgebra $H^*(M, \mathcal{T})$ onto $H^*(\mathcal{S}, \overline{D}^w)$.*

In particular, we get the isomorphism $\tau : \mathfrak{v}(M, \mathcal{O}) \to \mathrm{K}er\,\overline{D}^w \subset S_{*,0}$.

1.5 An application of the Hodge theory Suppose that M is compact. Then, we can develop the standard Hodge theory in the complex $(S, \overline{D})$ regarding it as the complex of $(0, *)$-forms with values in the bundle $\mathbf{ST}$, see [O3]. Endow M and $\mathbf{E}$ with smooth Hermitian metrics and consider the corresponding Hermitian metric on $\mathbf{ST}$. Denote by $\overline{D}^*$ the operator conjugate to $\overline{D}$ and by $\Box := [\overline{D}, \overline{D}^*]$ the Beltrami–Laplace operator. Their bidegrees are $(0, -1)$ and $(0, 0)$, respectively. Then, we have the orthogonal decomposition

$$S = \mathbf{H} \oplus \overline{D}S \oplus \overline{D}^* S, \tag{8}$$

where $\mathbf{H} = \operatorname{Ker} \Box$ is the bigraded subspace of harmonic elements. Moreover,

$$\mathrm{id} = H + \Box G = H + \overline{D}\,\overline{D}^* G + \overline{D}^*\,\overline{D}\, G,$$

where H is the projection onto $\mathbf{H}$ in eq. (8) and G is the Green operator. It is well known that

$$\mathbf{H}_{p,q} \simeq H^{p,q}(S, \overline{D}) \simeq H^q(M, (\mathcal{T}_{\mathrm{gr}})_p) \quad \text{for any} \quad p, q \geq 0. \tag{9}$$

Consider now the nonlinear complex K. Denote

$$\mathbf{H}_{(1)} := \bigoplus_{p \geq 1} \mathbf{H}_{2p,1},$$
$$L_1 := \operatorname{Ker} \overline{D}^* \cap K^1, \quad \mathbf{K} := Z^1(K) \cap L_1,$$

and define also the subset $\mathbf{K}_0 \subset K^1$ consisting of the u such that

$$u - \frac{1}{2}\overline{D}^* G[u, u] = Hu. \tag{10}$$

1.5.1. Theorem ([O3]). *We have $\mathbf{K} \subset \mathbf{K}_0 \subset L_1$. The mapping $H : \mathbf{K}_0 \to \mathbf{H}_{(1)}$ is a bijection and maps $\mathbf{K}$ onto the connected algebraic subset $\mathbf{V} \subset \mathbf{H}_{(1)} \simeq \bigoplus_{p \geq 1} H^1(M, (\mathcal{T}_{\mathrm{gr}})_{2p})$ given by the equation*

$$H[\varphi(h), \varphi(h)] = 0,$$

where $\varphi : \mathbf{H}_{(1)} \to L_1$ is inverse to H.

The natural mapping $\mathbf{K} \to H^1(K) \simeq H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$ is onto.

The set $\mathbf{K}$ is an analogue of the *Kuranishi family* of complex structures on a compact manifold. By Theorem 1.5.1 we can see that this family cuts every cohomology class, and hence it can be used for classification of supermanifolds with retract $(M, \mathcal{O}_{\mathrm{gr}})$.

1.6 Actions on supermanifolds Let $(M, \mathcal{O})$ be an arbitrary supermanifold. An *action* of a (real or complex) Lie group G on $(M, \mathcal{O})$ is a homomorphism $\Psi : G \to \operatorname{Aut}(M, \mathcal{O})$. For any $g \in G$ we have $\Psi(g) = (f(g), \psi(g))$, where $f : g \mapsto f(g) \in \operatorname{Bih} M$ is an (analytic) action of the group G on the complex manifold M and $\psi(g)$ is an automorphism of the sheaf $\mathcal{O}$ over $f(g)$.

Let $\mathbf{E}$ be a holomorphic vector bundle over a complex manifold M and G a Lie group. Suppose that $\mathbf{E}$ has a structure of the G-bundle, i.e., a homomorphism $\Phi : G \to \operatorname{Aut} \mathbf{E}$ satisfying the natural conditions of analiticity is given. Using the inclusion $\operatorname{Aut} \mathbf{E} \subset \operatorname{Aut}(M, \mathcal{O})$, we may consider Φ as an action on the split supermanifold $(M, \mathcal{O}_{gr})$ corresponding to the bundle $\mathbf{E}$. This action is

$\mathbb{Z}$-graded, i.e., all $\varphi(g)$, where $g \in G$, preserve the $\mathbb{Z}$-grading of the structure sheaf. Conversely, any $\mathbb{Z}$-graded action of the group G on $(M, \mathcal{O}_{gr})$ extends an action on the vector bundle $\mathbf{E}$.

Let again $(M, \mathcal{O})$ be an arbitrary complex supermanifold, $(M, \mathcal{O}_{gr})$ its retract and $\mathbf{E}$ the corresponding vector bundle.

If $F = (f, \psi) \in \mathrm{Aut}(M, \mathcal{O})$, then the automorphism ψ of $\mathcal{O}$ over f preserves a filtration (1), and hence determines an automorphism φ of the $\mathbb{Z}$-graded sheaf $\mathcal{O}_{gr}$ over f. Here, φ is uniquely determined by the relation $\pi_p \circ \psi = \varphi \circ \pi_p$ on $\mathcal{J}^p$.

Define $\overline{F} = (f, \varphi) \in \mathrm{Aut}(M, \mathcal{O}_{gr})$ for every $F = (f, \psi) \in \mathrm{Aut}(M, \mathcal{O})$. Thus. we get a homomorphism $\mathrm{Aut}(M, \mathcal{O}) \to \mathrm{Aut}(M, \mathbf{E})$. It follows that any action $\Psi : G \to \mathrm{Aut}(M, \mathcal{O})$ induces a $\mathbb{Z}$-graded action $\Phi : G \to \mathrm{Aut}(M, \mathcal{O}_{gr})$. In this case, we say that the action Φ *lifts to the action* Ψ *on* $(M, \mathcal{O})$.

There is the following lifting criterion:

1.6.1. Theorem ([O2]). *Let G be a compact Lie group and suppose an analytic $\mathbb{Z}$-graded action Ψ of G on a split supermanifold $(M, \mathcal{O}_{gr})$ be given. Let $(M, \mathcal{O})$ be the supermanifold corresponding to a given class $\zeta \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{gr})$ by Theorem 1.3.1. Then, the following conditions are equivalent:*

(i) *the action Ψ lifts to $(M, \mathcal{O})$;*

(ii) *the class ζ contains a G-invariant cocycle $z \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\mathcal{O}_{gr})$ where $\mathfrak{U}$ is an open G-cover of M;*

(iii) *the class $\mu_1^{-1}(\zeta) \in H^1(K)$ (see Theorem 1.3.1) contains a G-invariant cocycle.*

Now we give definitions of homogeneous and $\overline{0}$-homogeneous supermanifolds. Let $(M, \mathcal{O})$ be a complex supermanifold. For any $x \in M$ we can define the *tangent space* $T_x(M, \mathcal{O}) := (m_x/m_x^2)^*$, where m_x is the maximal ideal of the local superalgebra $\mathcal{O}_x$.

There is a natural even linear mapping $\mathrm{ev}_x : \mathfrak{v}(M, \mathcal{O}) \to T_x(M, \mathcal{O})$. Namely, every $v \in \mathfrak{v}(M, \mathcal{O})$ determines a linear mapping $m_x \to \mathcal{O}_x$ with $v(m_x^2) \subset m_x$, and hence a linear mapping

$$m_x/m_x^2 \to \mathcal{O}_x/m_x = \mathbb{C},$$

i.e., an element $\mathrm{ev}_x(v) \in (m_x/m_x^2)^*$.

The subalgebra $\mathfrak{g} \subset \mathfrak{v}(M, \mathcal{O})$ is called *transitive* if $\mathrm{ev}_x : \mathfrak{g} \to T_x(M, \mathcal{O})$ is surjective for all $x \in M$ and if $\mathrm{ev}_x : \mathfrak{g}_{\overline{0}} \to T_x(M, \mathcal{O})_{\overline{0}} = T_x(M)$ is surjective for all $x \in M$, then it is called $\overline{0}$-*transitive*. A supermanifold $(M, \mathcal{O})$ is called *homogeneous* ($\overline{0}$-*homogeneous*) if there is a transitive ($\overline{0}$-transitive) subalgebra $\mathfrak{g} \subset \mathfrak{v}(M, \mathcal{O})$ of finite dimension. In the case when M is a compact we can replace $\mathfrak{g}$ by $\mathfrak{v}(M, \mathcal{O})$.

1.6.2. Theorem ([OP]). *If a supermanifold $(M, \mathcal{O})$ is homogeneous ($\overline{0}$-homogeneous), then $(M, \mathrm{gr}\,\mathcal{O})$ is homogeneous ($\overline{0}$-homogeneous).*

2 Supermanifolds associated with the complex torus

2.1 Complex tori Let $\Gamma \subset \mathbb{C}^m$ be a discrete subgroup of rank $2m$. Then, the manifold $T = \mathbb{C}^m/\Gamma$ is a *complex torus* of dimension m. Note that T is a compact complex commutative Lie group. There is a local coordinate system in a neighborhood of any point of the manifold T formed by the standard coordinates $z_1, \ldots, z_m$ in $\mathbb{C}^m$. Let us denote these coordinates on T also by $z_1 \ldots, z_m$. The differential forms $dz_1, \ldots, dz_m$ are defined on T globally, since they are not changed if we add a complex number to the variable. Using duality between differential forms and

vector fields, we get the vector fields $\partial_{z_1}, \dots, \partial_{z_m}$ which are defined globally, too. The tangent and the cotangent bundles over T are trivial, and dz_i, ∂_{z_i} are basis sections of these vector bundles.

2.1.1. Proposition ([GH]). *Let M be a compact Kähler manifold. A form $\alpha \in \Gamma(M, \Phi^{0,q})$ is harmonic if and only if α is an antiholomorphic form, i.e., $\partial\alpha = 0$.*

It is well known that $T = \mathbb{C}^m/\Gamma$ is a compact Kähler manifold with flat metrics induced by Hermitian metrics in $\mathbb{C}^m$ (see [GH]). We can represent any $\alpha \in \Gamma(M, \Phi^{0,q})$ in a form

$$\alpha = \sum_{1 \le i_1 < \dots < i_q \le m} a_{i_1 \dots i_q}(z, \overline{z}) d\overline{z}_{i_1} \dots d\overline{z}_{i_q}, \tag{11}$$

where $a_{i_1 \dots i_q}(z, \overline{z})$ are smooth global defined functions, $z = (z_1, \dots, z_m)$, $\overline{z} = (\overline{z}_1, \dots, \overline{z}_m)$. Since any antiholomorphic function on T is constant, we have

2.1.2. Proposition. *A form $\alpha \in \Gamma(M, \Phi^{0,q})$ is harmonic if and only if*

$$\alpha = \sum_{1 \le i_1 < \dots < i_q \le m} a_{i_1 \dots i_q} d\overline{z}_{i_1} \dots d\overline{z}_{i_q}, \quad \text{where } a_{i_1 \dots i_q} \in \mathbb{C}.$$

A form on T is called *T-invariant* if it is invariant under the action of the group T on itself by translations.

2.1.3. Proposition. *The spaces of harmonic and T-invariant $(0, q)$-forms on T coincide.*

Proof. Clearly, the forms $d\overline{z}_{i_1} \dots d\overline{z}_{i_q}$ are T-invariant. It follows that the form (11) is T-invariant if and only if $a_{i_1 \dots i_q} \in \mathbb{C}$. Then, we apply Proposition 2.1.2. □

2.2 Supermanifolds corresponding to the trivial bundle over the complex torus Let $\mathbf{E} = T \times \mathbb{C}^n$ be a trivial holomorphic vector bundle of rank n over T and $\xi_1, \dots, \xi_n$ be the standard basis of $\mathbb{C}^n$. Denote by $T^{m|n} = (T, \mathcal{O}_{\mathrm{gr}})$ the split supermanifold corresponding to the bundle $\mathbf{E}$. The structure sheaf $\mathcal{O}_{\mathrm{gr}}$ has the form $\mathcal{F} \otimes \bigwedge(\xi_1, \dots, \xi_n)$. The local coordinates $z_1, \dots, z_m$ on T are even coordinates on $T^{m|n}$, and $\xi_1, \dots, \xi_n$ are odd ones.

Consider the tangent sheaf $\mathcal{T}_{gr} = \mathcal{D}er\, \mathcal{O}_{gr}$. This sheaf is free over $\mathcal{F}$, or, equivalently, the bundle $\mathbf{ST}$ is trivial, and the basis of its sections is

$$\begin{gathered} \xi_{i_1} \dots \xi_{i_k} \partial_{z_j}, \ \xi_{i_1} \dots \xi_{i_k} \partial_{\xi_l}, \quad \text{where} \\ 1 \le i_1 < \dots < i_k \le n, \quad j = 1, \dots, m, \ l = 1, \dots, n. \end{gathered} \tag{12}$$

Hence, the elements of $S_{p,q}$ have the form

$$\alpha = \sum_{i_1 < \dots < i_q} \Big(\sum_{\substack{j_1 < \dots < j_p \\ i = 1, \dots, m}} a^{i, i_1, \dots, i_q}_{j_1, \dots, j_p}(z, \overline{z}) \xi_{j_1} \dots \xi_{j_p} \partial_{z_i} + \sum_{\substack{j_1 < \dots < j_{p+1} \\ j = 1, \dots, n}} b^{j, i_1, \dots, i_q}_{j_1, \dots, j_{p+1}}(z, \overline{z}) \xi_{j_1} \dots \xi_{j_{p+1}} \partial_{\xi_j} \Big) d\overline{z}_{i_1} \dots d\overline{z}_{i_q},$$

where $a^{i, i_1, \dots, i_q}_{j_1, \dots, j_p}(z, \overline{z})$ and $b^{j, i_1, \dots, i_q}_{j_1, \dots, j_{p+1}}(z, \overline{z})$ are smooth globally defined functions on T. Thus, from Proposition 2.1.2 we get

2.2.1. Proposition. *The form $\alpha \in S_{p,q}$ is harmonic if and only if*

$$\alpha = \sum_{i_1 < \dots < i_q} \Big(\sum_{\substack{j_1 < \dots < j_p \\ i = 1, \dots, m}} a^{i, i_1, \dots, i_q}_{j_1, \dots, j_p} \xi_{j_1} \dots \xi_{j_p} \partial_{z_i} + \sum_{\substack{j_1 < \dots < j_{p+1} \\ j = 1, \dots, n}} b^{j, i_1, \dots, i_q}_{j_1, \dots, j_{p+1}} \xi_{j_1} \dots \xi_{j_{p+1}} \partial_{\xi_j} \Big) d\overline{z}_{i_1} \dots d\overline{z}_{i_q}, \tag{13}$$

where $a^{i, i_1, \dots, i_q}_{j_1, \dots, j_p}$, $b^{j, i_1, \dots, i_q}_{j_1, \dots, j_{p+1}} \in \mathbb{C}$.

2.2.1a. Corollary. *If $\alpha, \beta \in \mathbf{H}$, then $[\alpha, \beta] \in \mathbf{H}$.*

Assigning to every cohomology class from $H^q(M, (\mathcal{T}_{\mathrm{gr}})_p)$ the correspondent harmonic form from $\mathbf{H}_{p,q}$ (see (9)), we get an isomorphism of graded Lie superalgebras $H(M, \mathcal{T}_{\mathrm{gr}})$ onto the subalgebra $\mathbf{H} \subset S$.

Since $\xi_{j_1} \dots \xi_{j_p} \partial_{z_i}$ and $\xi_{j_1} \dots \xi_{j_{p+1}} \partial_{\xi_j}$ are T-invariant, from Proposition 2.1.3 we get

2.2.2. Proposition. *Any harmonic form from S is T-invariant, and the other way round.*

2.2.3. Theorem. *We have $K_0 = \mathbf{H}_{(1)}$ and*

$$\mathbf{K} = \mathbf{V} = \{w \in \mathbf{H}_{(1)} \mid [w, w] = 0\}.$$

Proof. Take $w \in \mathbf{K}_0$ and denote $h = Hw$. We write $h = \sum\limits_{k \geq 1} h_{2k}$, and $w = \sum\limits_{k \geq 1} w_{2k}$, where $h_{2k} \in \mathbf{H}_{2k,1}$, $w_{2k} \in \mathrm{S}_{2k,1}$. From (10) we get the following equations:

$$\begin{aligned}
&w_2 = h_2,\\
&w_4 - \tfrac{1}{2}\overline{D}^{\,*} G[w_2, w_2] = h_4,\\
&\dots\\
&w_{2k} - \tfrac{1}{2}\overline{D}^{\,*} G \sum_{1 \leq s \leq k-1} [w_{2s}, w_{2(k-s)}] = h_{2k},\\
&\dots
\end{aligned}$$

We prove that $w_{2k} = h_{2k}$ by induction on k. For $k = 1$ this follows from the first equation. Suppose that $w_{2i} = h_{2i}$ for $1 \leq i \leq k-1$. By Corollary 2.2.1a we see that

$$h' = \sum_{1 \leq s \leq k-1} [w_{2s}, w_{2(k-s)}] \in \mathbf{H}.$$

Since $\overline{D}^{\,*}$ and G commute and $\overline{D}^{\,*} h' = 0$, we get $w_{2k} = h_{2k}$.

So we have proved that $w = h \in \mathbf{H}_{(1)}$. By Theorem 3, $\mathbf{K}_0 = \mathbf{H}_{(1)}$, and $\varphi = \mathrm{id}$. Therefore, $\mathbf{K} = \mathbf{V} = \{w \in \mathbf{H}_{(1)} \mid [w, w = 0\}$. □

2.3 Lie superalgebras of vector fields on supermanifolds with retract $T^{m|n}$ Consider holomorphic vector fields on the split supermanifold $T^{m|n}$. Clearly, any $v \in \mathfrak{v}_p(T^{m|n})$ is a linear combination of the fields

$$\begin{aligned}
&\xi_{j_1} \dots \xi_{j_p} \partial_{z_i} \text{ for } j_1 < \dots < j_p \text{ and } i = 1, \dots, m,\\
&\xi_{j_1} \dots \xi_{j_{p+1}} \partial_{\xi_j} \text{ for } j_1 < \dots < j_{p+1} \text{ and } j = 1, \dots, n \text{ (see (12))}
\end{aligned}$$

with holomorphic coefficients. Since any holomorphic function on T is constant,

$$v = \sum_{1 \leq i \leq m} \sum_{j_1 < \dots < j_p} a^i_{j_1,\dots,j_p} \xi_{j_1} \dots \xi_{j_p} \partial_{z_i} + \sum_{1 \leq j \leq n} \sum_{j_1 < \dots < j_{p+1}} b^j_{j_1,\dots,j_{p+1}} \xi_{j_1} \dots \xi_{j_{p+1}} \partial_{\xi_j}, \tag{14}$$

where $a^i_{j_1,\dots,j_p}$, $b^j_{j_1,\dots,j_{p+1}} \in \mathbb{C}$. Since

$$\mathfrak{v}(T^{m|n}) = \mathbf{H}_{p,0} = \{v \in S_{p,0} \mid \overline{D} v = 0\},$$

we see that (14) is a special case of the formula (13).

By Theorem 1.5.1, any supermanifold with retract $T^{m|n}$ can be described by a form from the Kuranishi family $\mathbf{K}$. By Theorem 2.2.3, $\mathbf{K} = \mathbf{V}$ consists of harmonic elements.

2.3.1. Theorem. *Let $(T,\mathcal{O})$ be a supermanifold with retract $T^{m|n}$ given by an element $w\in\mathbf{V}$. Then, the mapping τ from (1.7) determines an isomorphism*

$$\mathfrak{v}(T,\mathcal{O})\to\{v\in S_{*,0}\mid \overline{D}v=[w,v]=0\}=\{v\in\mathfrak{v}_p(T^{m|n})\mid [w,v]=0\}.$$

Proof. We can write $w=w_2+w_4+\dots$, where $w_{2k}\in\mathbf{H}_{2k,1}$. Let $v\in S_{*,0}$ and $\overline{D}v=[w,v]$. Then, $v=v_{-1}+v_0+v_1+\dots$, where $v_i\in S_{i,0}$. The equation $\overline{D}v=[w,v]$ gives the finite system of equations:

$$\begin{aligned}
&\overline{D}v_{-1}=0,\\
&\overline{D}v_0=0,\\
&\overline{D}v_1=[w_2,v_{-1}],\\
&\overline{D}v_2=[w_2,v_0],\\
&\overline{D}v_3=[w_2,v_1]+[w_4,v_{-1}],\\
&\overline{D}v_4=[w_2,v_2]+[w_4,v_0],\\
&\dots
\end{aligned}\tag{15}$$

Let us prove that $\overline{D}v_k=0$ for $k=-1,0,\dots$, by induction on k. For $k=-1,0$ this follows from the first and the second equations of (15). Suppose that $\overline{D}v_{-1}=\dots=\overline{D}v_{k-1}=0$. Then, using system (15), we see that $\overline{D}v_k$ is a sum of commutators of the fields $v_{-1},\dots,v_{k-2}$ with the forms w_{2l}. Since $w_{2l}\in\mathbf{H}_{2l,1}$, Corollary 2.2.1a shows that $\overline{D}v_k\in\mathbf{H}_{p,1}$. Hence $\overline{D}v_k=0$.

Thus, we proved that the kernel of $\overline{D}_w=\overline{D}-\operatorname{ad}w$ in $S_{*,0}$ coincides with the subalgebra $\{v\in\mathfrak{v}_p(T^{m|n})\mid[w,v]=0\}\subset\mathfrak{v}_p(T^{m|n})$. Now our statement follows from Theorem 1.4.1. □

2.4 Homogeneous supermanifolds with retract $T^{m|n}$

2.4.1. Theorem. *Any supermanifold $(T,\mathcal{O})$ with retract $T^{m|n}$ is $\overline{0}$-homogeneous. It is homogeneous if and only if $(T,\mathcal{O})\simeq T^{m|n}$.*

Proof. Let $(T,\mathcal{O})$ be an arbitrary supermanifold with retract $T^{m|n}$. From Theorem 2.2.3 we see that it is determined by a harmonic form $w\in\mathbf{V}$. By Proposition 2.2.2 w is invariant under the natural action of the group T. Hence, by Theorem 1.6.1 the action of the group T on $T^{m|n}$ lifts to an action on $(T,\mathcal{O})$. So $(T,\mathcal{O})$ is $\overline{0}$-homogeneous.

From eq. (12) we see that the Lie superalgebra $\mathfrak{v}(T^{m|n})$ is transitive. Then, $T^{m|n}$ is homogeneous. Let $(T,\mathcal{O})$ be the supermanifold with retract $T^{m|n}$ determined by a cocycle $w\in\mathbf{V}\subset\mathbf{H}_{(1)}$. Let $(T,\mathcal{O})$ be homogeneous. Take a point $x_0\in T$, and denote by $\xi_1,\dots,\xi_n$ the odd local coordinates in the neighborhood U of x_0 which correspond to the coordinates $\xi_1,\dots,\xi_n$ on $T^{m|n}$ by the local spliting $h_U:\mathcal{O}_{\mathrm{gr}}|_U\to\mathcal{O}|_U$. Since $\mathrm{ev}_{x_0}:\mathfrak{v}(T,\mathcal{O})_{\overline{1}}\to T_{x_0}(T,\mathcal{O})_{\overline{1}}$ is surjective, then for any j such that $1\le j\le n$, there exists a field $v_j\in\mathfrak{v}(T,\mathcal{O})_{\overline{1}}$ such that $v_j=\partial_{\xi_j}+v'_j$ in U, where $(v'_j)_{x_0}\in m_{x_0}\mathcal{T}_{x_0}$. We can assume that the neighborhood $U=U_i$ is included into the cover (U_i) which we used in the description of τ in Subsection 1.4. As was shown in Subsection 1.4, we have

$$\tau(v_j)=a_i(h_i^{-1}\partial_{\xi_j}h_i)a_i^{-1}+(a_ih_i^{-1})v'_j(h_ia_i^{-1}).$$

The sheaf $\mathcal{S}_{*,0}=\mathcal{T}^\infty_{\mathrm{gr}}=\mathcal{D}er\,\mathcal{O}^\infty_{\mathrm{gr}}$ has a filtration similar to (3):

$$\mathcal{T}^\infty_{\mathrm{gr}}=\mathcal{T}^\infty_{\mathrm{gr}(-1)}\subset\mathcal{T}^\infty_{\mathrm{gr}(0)}\subset\mathcal{T}^\infty_{\mathrm{gr}(1)}\subset\dots$$

Since $a_i\in\Gamma(U_i,\mathcal{A}ut_{(2)}\mathcal{O}^\infty_{\mathrm{gr}})$, we have

$$a_i(h_i^{-1}\partial_{\xi_j}h_i)a_i^{-1}=a_i\partial_{\xi_j}a_i^{-1}=\partial_{\xi_j}+u_j,$$

where $u_j \in \Gamma(U_i, \mathcal{T}^\infty_{\mathrm{gr}(1)})$. Clearly,

$$(a_i h_i^{-1}) v_j'(h_i a_i^{-1}) = v_j''$$

satisfies $(v_j'')_{x_0} \in m_x^\infty(\mathcal{T}^\infty_{\mathrm{gr}})_{x_0}$, where $m^\infty_{x_0}$ is the maximal ideal of $(\mathcal{O}^\infty_{\mathrm{gr}})_{x_0}$. Hence in U_i we have

$$\tau(v_j) = \partial_{\xi_j} + u_j + v_j'',$$

where $(u_j + v_j'')_{x_0} \in m^\infty_{x_0}(\mathcal{T}^\infty_{\mathrm{gr}})_{x_0}$. By Theorem 2.3.1 $\tau(v_j) \in \mathfrak{v}(T^{m|n})_{\overline{1}}$.

So, if $\tilde{v}_j = \tau(v_j) - \partial_{\xi_j} \in \mathfrak{v}(T^{m|n})_{\overline{1}}$, then $\tilde{v}_{x_0} \in m_{x_0}(\mathcal{T}_{\mathrm{gr}})_{x_0}$. Therefore, $\tilde{v}_j \in \bigoplus_{k\geq 0} \mathfrak{v}(T^{m|n})_{2k+1}$.

By Theorem 2.3.1

$$[w, \partial_{\xi_j} + \tilde{v}_j] = 0 \quad \text{for} \quad j = 1, \ldots, n. \tag{16}$$

We write w as $w = w_2 + w_4 + \ldots$, where $w_{2k} \in \mathbf{H}_{2k,1}$, and prove that $w_{2k} = 0$ for $k = 1, 2, \ldots$, by induction on k.

Considering the component of degree 1 of the left part of formula (16), we see that $[w_2, \partial_{\xi_j}] = 0$. Hence,

$$[w_2, \partial_{\xi_j}](z_r) = \partial_{\xi_j}(w_2(z_r)) = 0 \quad \text{for } r = 1, \ldots, m,$$
$$[w_2, \partial_{\xi_j}](\xi_s) = \partial_{\xi_j}(w_2(\xi_s)) = 0 \quad \text{for } s = 1, \ldots, n.$$

From (13) we see that $w_2 = 0$.

Suppose we have proved that $w_2 = w_4 = \ldots = w_{2k-2} = 0$. Considering the component of degree $2k-1$ of the left part of the formula (16), we get $[w_{2k}, \partial_{\xi_j}] = 0$ for all $j = 1, \ldots, n$. As above, we can prove that $w_{2k} = 0$.

Thus, $w = 0$. Hence, $(T, \mathcal{O}) \simeq T^{m|n}$. □

Acknowledgements I am grateful to Prof. A.L. Onishchik for constant attention to this work and to Erwin Schrödinger International Institute for Mathematical Physics for the hospitality during my visit to Vienna in December 2002. This work was supported in part by the Russian Foundation for Basic Research (Grant 01-01-00709). I am thankful to the referee for a clarifying comment and to D. Leites for helping me to answer this comment by adding the appropriate reference [BGLS*].

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