Computation of interaction coefficients of cylindrical inhomogeneities

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This document is a companion to [1], where an analytical integration is performed on the interaction coefficients of cylindrical inhomogeneities, in order to reduce the computation time.

Contents

1 Introduction

In [1], for each pair (α, β) of cylindrical inhomogeneities, interaction coefficients $T_{\alpha\beta}^{mn}$ must be computed for $0 \leq m, n \leq p$, where $p \in \mathbb{N}$. We recall that each cylindrical inhomogeneity is a circular cylinder whose radius is R and length is $2L$. Cylinder α is centered at $\mathbf{r}_\alpha$, and oriented by the unit-vector $\mathbf{n}_\alpha$. We note $\zeta_\alpha = \frac{1}{L}(\mathbf{x} - \mathbf{x}_\alpha) \cdot \mathbf{n}_\alpha$ and $\mathbf{r}_{\alpha\beta} = \mathbf{x}_\beta - \mathbf{x}_\alpha$. Then, $T_{\alpha\beta}^{mn}$ is given in terms of $U_{\alpha\beta}^{mn}$ and $\mathsf{V}_{\alpha\beta}^{mn}$ [see Eq (49) of the paper] by:

$$\mathsf{T}_{\alpha\beta}^{mn} = \sigma_0^{-1}\big(U_{\alpha\beta}^{mn}\mathsf{I}_3 - \mathsf{V}_{\alpha\beta}^{mn}\big), \tag{1}$$

where I_3 is the 3×3 identity matrix and σ_0 is the conductivity of the reference material. Besides, in [1], a multipole expansion is performed to evaluate the terms $U_{\alpha\beta}^{mn}$ and $\mathsf{V}_{\alpha\beta}^{mn}$, which are finally given by [see Eqs (63) and (64) of the paper]:

$$U_{\alpha\beta}^{mn} = \frac{L^{m+n+3}}{8}\frac{R^2}{L^2}\int_{-1}^{1}\int_{-1}^{1}\frac{\zeta_\alpha^m\,\zeta_\beta^n}{\|\mathbf{w}^0\|^3}\,\mathrm{d}\zeta_\alpha\,\mathrm{d}\zeta_\beta, \tag{2}$$

$$\big(\mathsf{V}_{\alpha\beta}^{mn}\big)_{ij} = \frac{3L^{m+n+3}}{8}\frac{R^2}{L^2}\int_{-1}^{1}\int_{-1}^{1}\frac{\zeta_\alpha^m\,\zeta_\beta^n w_i^0 w_j^0}{\|\mathbf{w}^0\|^5}\mathrm{d}\zeta_\alpha\,\mathrm{d}\zeta_\beta. \tag{3}$$

where

$$\mathbf{w}^0 = \mathbf{r}_{\alpha\beta} + L\zeta_\beta\mathbf{n}_\beta - L\zeta_\alpha\mathbf{n}_\alpha. \tag{4}$$

Their expressions involve double integrals on ζ_α and ζ_β, the longitudinal coordinates of cylinder α and cylinder β. In this document, we perform an analytical integration on ζ_α, in order to accelerate the computation.

2 Determination of $U_{\alpha\beta}^{mn}$ and $\mathsf{V}_{\alpha\beta}^{mn}$

First, we introduce the following integral:

$$I_K^{mn} = \int_{-1}^{1}\int_{-1}^{1} \frac{\zeta_\alpha^m\,\zeta_\beta^n}{\|\mathbf{w}^0/L\|^K}\,\mathrm{d}\zeta_\alpha\,\mathrm{d}\zeta_\beta \tag{5}$$

We then have:

$$U_{\alpha\beta}^{mn} = \frac{L^{m+n}}{8}\frac{R^2}{L^2} I_3^{mn} \tag{6}$$

and

$$\left(\mathsf{V}_{\alpha\beta}^{mn}\right)_{ij} = \frac{3L^{m+n}}{8}\frac{R^2}{L^2}\int_{-1}^{1}\int_{-1}^{1} \frac{\zeta_\alpha^m\,\zeta_\beta^n\,\left(\mathbf{w}_i^0/L\right)\left(\mathbf{w}_j^0/L\right)}{\|\mathbf{w}^0/L\|^5}\,\mathrm{d}\zeta_\alpha\,\mathrm{d}\zeta_\beta \tag{7}$$

so that

$$\begin{aligned}\mathsf{V}_{\alpha\beta}^{mn} =& \frac{3L^{m+n}}{8}\frac{R^2}{L^2}\Bigg(I_5^{mn}\frac{\mathsf{r}_{\alpha\beta}}{L}\frac{\mathsf{r}_{\alpha\beta}^{\mathrm{T}}}{L} + I_5^{m(n+2)}\mathsf{n}_\beta\mathsf{n}_\beta^{\mathrm{T}} + I_5^{(m+2)n}\mathsf{n}_\alpha\mathsf{n}_\alpha^{\mathrm{T}} \\ &+ I_5^{m(n+1)}\left(\frac{\mathsf{r}_{\alpha\beta}}{L}\mathsf{n}_\beta^{\mathrm{T}} + \mathsf{n}_\beta\frac{\mathsf{r}_{\alpha\beta}^{\mathrm{T}}}{L}\right) \\ &- I_5^{(m+1)n}\left(\frac{\mathsf{r}_{\alpha\beta}}{L}\mathsf{n}_\alpha^{\mathrm{T}} + \mathsf{n}_\alpha\frac{\mathsf{r}_{\alpha\beta}^{\mathrm{T}}}{L}\right) \\ &- I_5^{(m+1)(n+1)}\left(\mathsf{n}_\alpha\mathsf{n}_\beta^{\mathrm{T}} + \mathsf{n}_\beta\mathsf{n}_\alpha^{\mathrm{T}}\right)\Bigg)\end{aligned} \tag{8}$$

The integral I_K^{mn} is long to compute numerically. Besides, the analytical integration on the two variables involves heavy analytical expressions that would also be long to compute. Hence, we propose here to perform the integration upon ζ_α analytically, and the integration upon ζ_β numerically. We then note :

$$I_K^{mn} = \int_{-1}^{1} \zeta_\beta^m\, J_K^n(\zeta_\beta)\,\mathrm{d}\zeta_\beta \tag{9}$$

where

$$J_K^n(\zeta_\beta) = \int_{-1}^{1} \frac{\zeta_\alpha^n}{\|\mathbf{w}_0/L\|^K}\,\mathrm{d}\zeta_\alpha \tag{10}$$

We give the expression of J_K^n in next subsection. There are two cases. If $\mathbf{n}_\alpha$ and $\mathbf{r}_{\alpha\beta} + L\zeta_\beta\mathbf{n}_\beta$ are not colinear, J_K^n is given by (15), (17), (20) and (21). If $\mathbf{n}_\alpha$ and $\mathbf{r}_{\alpha\beta} + L\zeta_\beta\mathbf{n}_\beta$ are colinear, J_K^n is given by (32).

3 Determination of J_K^n

We recall that :

$$\|\mathbf{w}_0/L\|^K = \left(\frac{\mathbf{r}_{\alpha\beta}^2}{L^2} + \zeta_\alpha^2 + \zeta_\beta^2 - 2\zeta_\alpha \frac{\mathbf{r}_{\alpha\beta}}{L} \cdot \mathbf{n}_\alpha \right. \\ \left. + 2\zeta_\beta \frac{\mathbf{r}_{\alpha\beta}}{L} \cdot \mathbf{n}_\beta - 2\zeta_\alpha\zeta_\beta \mathbf{n}_\alpha \cdot \mathbf{n}_\beta \right)^{\frac{K}{2}} \tag{11}$$

Here, K is odd, then we note $K = 2k+1$. For each ζ_β, we put $c_0 = -\frac{\mathbf{r}_{\alpha\beta}}{L} \cdot \mathbf{n}_\alpha - \zeta_\beta \mathbf{n}_\alpha \cdot \mathbf{n}_\beta = -\left(\frac{\mathbf{r}_{\alpha\beta}}{L} + \zeta_\beta \mathbf{n}_\beta\right) \cdot \mathbf{n}_\alpha$ and $c_1^2 = \frac{\mathbf{r}_{\alpha\beta}^2}{L^2} + \zeta_\beta^2 + 2\zeta_\beta \frac{\mathbf{r}_{\alpha\beta}}{L} \cdot \mathbf{n}_\beta = \left(\frac{\mathbf{r}_{\alpha\beta}}{L} + \zeta_\beta \mathbf{n}_\beta\right)^2$. We notice that $c_0^2 \leq c_1^2$. We then put $\sqrt{c_1^2 - c_0^2} = c_2$ so that

$$J_K^n(\zeta_\beta) = \int_{-1}^{1} \frac{\zeta_\alpha^n \mathrm{d}\zeta_\alpha}{\left[(\zeta_\alpha + c_0)^2 + c_2^2\right]^{\frac{K}{2}}} \tag{12}$$

3.1 First case: $c_2^2 > 0$

In that case,

$$J_K^n(\zeta_\beta) = \int_{-1}^{1} \frac{\zeta_\alpha^n \mathrm{d}\zeta_\alpha}{c_2^K \left[\left(\frac{\zeta_\alpha + c_0}{c_2}\right)^2 + 1\right]^{\frac{K}{2}}} \tag{13}$$

We put $\sinh x = \frac{\zeta_\alpha + c_0}{c_2}$, and we note $y_1 = \mathrm{Argsh}\left(\frac{c_0 - 1}{c_2}\right)$ and $y_2 = \mathrm{Argsh}\left(\frac{c_0 + 1}{c_2}\right)$ then:

$$\begin{aligned} J_K^n(\zeta_\beta) &= \int_{y_1}^{y_2} \frac{(c_2 \sinh x - c_0)^n \, \mathrm{d}x}{c_2^{2k} \cosh^{2k} x} \\ &= \int_{y_1}^{y_2} \frac{1}{c_2^{2k} \cosh^{2k} x} \\ &\quad \times \sum_{p=0}^{n} \binom{n}{p} c_2^p (-1)^{n-p} c_0^{n-p} \sinh^p x \mathrm{d}x \end{aligned} \tag{14}$$

Then, for that first case, we have:

$$J_K^n(\zeta_\beta) = \sum_{p=0}^{n} \binom{n}{p} c_2^{p-2} (-1)^{n-p} c_0^{n-p} K_p(y_1, y_2) \tag{15}$$

where

$$K_p(y_1, y_2) = \int_{y_1}^{y_2} \frac{\sinh^p x \mathrm{d}x}{\cosh^{2k} x} \tag{16}$$

is below derived.

p is odd We put $p = 2q + 1$ and $u = \cosh x$:

$$\begin{aligned} K_p &= \int_{\cosh y_1}^{\cosh y_2} \frac{\left(u^2-1\right)^q \mathrm{d}u}{u^{2k}} \\ &= \int_{\cosh y_1}^{\cosh y_2} \sum_{t=0}^{q} \binom{q}{t} u^{2t-2k} (-1)^{q-t} \mathrm{d}u \\ &= \sum_{t=0}^{q} \frac{\binom{q}{t}(-1)^{q-t}}{2t-2k+1} \left[(\cosh y_2)^{2t-2k+1} - (\cosh y_1)^{2t-2k+1} \right] \end{aligned} \tag{17}$$

p is even We put $p = 2q$ and $u = \tanh x$:

$$K_p = \int_{\tanh y_1}^{\tanh y_2} \frac{u^{2q}}{(1-u^2)^{q+1-k}} \mathrm{d}u \tag{18}$$

We decompose into simple elements the integrand:

$$\frac{u^{2q}}{(1-u^2)^{q+1-k}} = P(u) + \sum_{l=1}^{q+1-k} \left(\frac{a_{l,p}}{(1-u)^l} + \frac{b_{l,p}}{(1+u)^l} \right) \tag{19}$$

so that:

$$K_p = k_p\left(\tanh y_2\right) - k_p\left(\tanh y_1\right) \tag{20}$$

with

$$\begin{aligned} k_p(u) = &\int P(u)\mathrm{d}u - a_{1,p} \ln\left(1-u\right) + b_{1,p} \ln\left(1+u\right) \\ &+ \sum_{l=2}^{q+1-k} \left[\frac{a_{l,p}}{l-1} \frac{1}{(1-u)^{l-1}} + \frac{b_{l,p}}{1-l} \frac{1}{(1+u)^{l-1}} \right] \end{aligned} \tag{21}$$

Determination of simple element decompositions We recall that here $p = 2q$ is even. The coefficients $a_{l,p}$ and $b_{l,p}$ are determined by the resolution of a linear system:

$$\begin{aligned} \frac{u^p}{(1-u^2)^{q+1-k}} &= \frac{(u^2-1+1)^{\frac{p}{2}}}{(1-u^2)^{q+1-k}} \\ &= \sum_{r=0}^{q} \binom{q}{r} \frac{(u^2-1)^r}{(1-u^2)^{q+1-k}} \\ &= \sum_{r=0}^{q} (-1)^r \binom{q}{r} \left(1-u^2\right)^{r-q-1+k} \\ &= \sum_{r=0}^{q-k} (-1)^r \binom{q}{r} \left(1-u^2\right)^{r-q-1+k} \\ &+ \sum_{r=q+1-k}^{q} (-1)^r \binom{q}{r} \sum_{s=0}^{r-q-1+k} \binom{r-q-1+k}{s} \left(-u^2\right)^s \end{aligned} \tag{22}$$

We then have:

$$P(u) = \sum_{r=q+1-k}^{q} (-1)^r \binom{q}{r} \sum_{s=0}^{r-q-1+k} \binom{r-q-1+k}{s} \left(-u^2\right)^s \tag{23}$$

And:

$$\begin{aligned} &\sum_{l=1}^{q+1-k} \left(\frac{a_{l,p}}{(1-u)^l} + \frac{b_{l,p}}{(1+u)^l} \right) \\ &= \sum_{r=0}^{q+1-k-1} (-1)^r \binom{q}{r} \left(1-u^2\right)^{r-q-1+k} \end{aligned} \tag{24}$$

So that:

$$\begin{aligned} &\sum_{r=0}^{q-k} (-1)^r \binom{q}{r} \left(1-u^2\right)^r \\ &= \sum_{l=1}^{q+1-k} \big(a_{l,p}(1+u)^{q+1-k}(1-u)^{q+1-k-l} \\ &\qquad + b_{l,p}(1-u)^{q+1-k}(1+u)^{q+1-k-l} \big) \\ &= \sum_{l=1}^{q+1-k} a_{l,p} \sum_{k_1=0}^{q+1-k} \sum_{k_2=0}^{q+1-k-l} \binom{q+1-k}{k_1} \binom{q+1-k-l}{k_2} (-1)^{k_2} u^{k_1+k_2} \\ &+ \sum_{l=1}^{q+1-k} b_{l,p} \sum_{k_1=0}^{q+1-k} \sum_{k_2=0}^{q+1-k-l} \binom{q+1-k}{k_1} \binom{q+1-k-l}{k_2} (-1)^{k_1} u^{k_1+k_2} \end{aligned} \tag{25}$$

The two members are polynomial in u, and the degree is $\leq 2q-2k+1$. On the left side, the coefficient of u^t is 0 for t odd, and $\sum_{r=0}^{q-k} (-1)^{r+\frac{t}{2}} \binom{q}{r} \binom{r}{\frac{t}{2}}$ for t even.
On the right side, for $0 \leq t \leq 2q-2k+1$, the coefficient of u^t is:

$$\begin{aligned} \sum_{l=1}^{q+1-k} \sum_{\substack{0 \leq k_1 \leq q+1-k \\ 0 \leq k_2 \leq \frac{p}{2}-l \\ k_1+k_2=t}} &\binom{q+1-k}{k_1} \binom{q+1-k-l}{k_2} \\ &\times \left(a_{l,p}(-1)^{k_2} + b_{l,p}(-1)^{k_1} \right) \end{aligned} \tag{26}$$

Then the linear system is $A_p \cdot X_p = B_p$ with:

$$\begin{aligned} (A_p)_{(t+1)l} &= \sum_{\substack{0 \leq k_1 \leq q+1-k \\ 0 \leq k_2 \leq q+1-k-l \\ k_1+k_2=t}} \binom{q+1-k}{k_1} \binom{q+1-k-l}{k_2} (-1)^{k_2} \\ (A_p)_{(t+1)(l+q+1-k)} &= \sum_{\substack{0 \leq k_1 \leq q+1-k \\ 0 \leq k_2 \leq q+1-k-l \\ k_1+k_2=t}} \binom{q+1-k}{k_1} \binom{q+1-k-l}{k_2} (-1)^{k_1} \end{aligned} \tag{27}$$

for $1 \leq l \leq q+1-k$ and $0 \leq t \leq 2q-2k+1$

and:

$$X_p = \begin{pmatrix} a_{1,p} \\ \cdot \\ \cdot \\ \cdot \\ a_{q+1-k,p} \\ b_{1,p} \\ \cdot \\ \cdot \\ \cdot \\ b_{q+1-k,p} \end{pmatrix}, \quad B_p = \begin{pmatrix} \sum_{r=0}^{q-k} (-1)^r \binom{q}{r} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \sum_{r=0}^{q-k} (-1)^{r+q-k} \binom{q}{r} \binom{r}{q-k} \\ 0 \end{pmatrix} \tag{28}$$

The linear system must be solved for each $0 \leq p \leq m$ even. The systems do not depend on ζ_β.

3.2 Second case: $c_2^2 = 0$

In that case,

$$J_K^n(\zeta_\beta) = \int_{-1}^{1} \frac{\zeta_\alpha^n \mathrm{d}\zeta_\alpha}{|\zeta_\alpha + c_0|^K} \tag{29}$$

In fact, the denominator is strictly positive (the axes of the cylinders do not intersect). Then $c_0 > 1$ or $c_0 < -1$. Hence:

$$J_K^n(\zeta_\beta) = \pm \int_{-1}^{1} \frac{\zeta_\alpha^n \mathrm{d}\zeta_\alpha}{(\zeta_\alpha + c_0)^K} \tag{30}$$

depending on the sign of c_0. We decompose into simple elements the integrand:

$$\frac{\zeta_\alpha^n}{(\zeta_\alpha + c_0)^K} = \sum_{p=0}^{n-K} \lambda_p \zeta_\alpha^p + \sum_{p=1}^{K} \frac{\mu_p}{(\zeta_\alpha + c_0)^p} \tag{31}$$

so that:

$$\begin{aligned} \pm J_K^n(\zeta_\beta) = & \sum_{p=0}^{n-K} \lambda_p \frac{1-(-1)^{p+1}}{p+1} + \mu_1 \ln \left| \frac{c_0+1}{c_0-1} \right| \\ & + \sum_{2}^{K} \frac{\mu_p}{1-p} \left[(c_0+1)^{1-p} - (c_0-1)^{1-p} \right] \end{aligned} \tag{32}$$

Determination of λ_p and μ_p We first make a division of polynomials: ζ_α^n divided by $(\zeta_\alpha + c_0)^K$. We have:

$$\begin{aligned} \frac{\zeta_\alpha^n}{(\zeta_\alpha + c_0)^K} &= \frac{(\zeta_\alpha + c_0 - c_0)^n}{(\zeta_\alpha + c_0)^K} \\ &= \sum_{r=0}^{n} \binom{n}{r} (-c_0)^{n-r} (\zeta_\alpha + c_0)^{r-K} \\ &= \sum_{r=0}^{K-1} \binom{n}{r} (-c_0)^{n-r} (\zeta_\alpha + c_0)^{r-K} \\ &\quad + \sum_{r=0}^{n-K} \binom{n}{r+K} (-c_0)^{n-r-K} (\zeta_\alpha + c_0)^r \end{aligned} \tag{33}$$

so that:

$$\begin{aligned}\mu_p &= \binom{n}{K-p}(-c_0)^{n-K+p} \\ \lambda_p &= \binom{n}{p+K}(-c_0)^{n-p-K}\sum_{s=0}^{p}\binom{p}{s}c_0^{p-s}\zeta_\alpha^s\end{aligned} \tag{34}$$

References

[1] A. Martin, S. Brisard and K. Sab, Assessment of the equivalent inclusion method for the numerical homogenization of fibrous composites, Journal of Computational physics (In preparation).