Maximal displacement of spectrally negative branching Lévy processes

Christophe Profeta

▶ **To cite this version:**

Christophe Profeta. Maximal displacement of spectrally negative branching Lévy processes. 2022. hal-03736526v1

HAL Id: hal-03736526

https://hal.science/hal-03736526v1

Preprint submitted on 22 Jul 2022 (v1), last revised 14 Nov 2022 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

MAXIMAL DISPLACEMENT OF SPECTRALLY NEGATIVE BRANCHING LÉVY PROCESSES

CHRISTOPHE PROFETA

Abstract. We consider a branching Markov process in continuous time in which the particles evolve independently as spectrally negative Lévy processes. When the branching mechanism is critical or subcritical, the process will eventually die and we may define its overall maximum, i.e. the maximum location ever reached by a particule. The purpose of this paper is to give asymptotic estimates for the survival function of this maximum. In particular, we show that in the critical case the asymptotics is polynomial when the underlying Lévy process oscillates or drifts towards $+\infty$, and is exponential when it drifts towards $-\infty$.

1. Introduction

1.1. **Description of the model.** We consider a one-dimensional spectrally negative branching Lévy process in the sense of Kyprianou [10]. It is a continuous-time particle system in which individuals move according to independent spectrally negative Lévy processes, and split at exponential times into a random number of children.

More precisely, an initial ancestor begins its existence at the origin at time $t = 0$. It moves according to a spectrally negative Lévy process L up to an independent exponential random variable $\mathbf{e}$ of parameter 1. It then dies and splits into a random number of children with distribution $\boldsymbol{p} = (p_n)_{n\geq 0}$. Each of these children starts his life at the location of the ancestor and behaves independently of the others, following the same stochastic pattern as the ancestor : it moves according to L and branches at rate 1.

We assume that the offspring distribution $\boldsymbol{p}$ is non trivial, has expectation smaller or equal to one and admits moments of order at least 3 :

$$p_1 \neq 1, \qquad \mathbb{E}[\boldsymbol{p}] \leq 1, \qquad \mathbb{E}[\boldsymbol{p}^3] < +\infty.$$

As a consequence of the first two conditions, the branching process will almost surely die in finite time, and one may define its overall maximum $\mathbf{M}$, i.e. the maximum location ever reached by one particle. The purpose of this paper is to study the asymptotics of the survival function of $\mathbf{M}$.

The investigation of the maximal displacement of branching processes, or equivalently of their right-most particles, has already received a lot of attention in the literature. The emphasize has been put so far on the supercritical branching Brownian motion for which it is known that the survival function of the right-most particle is a travelling wave solution of the F-KPP equation, see Bransom [3].

Key words and phrases. Branching process ; Extreme values ; Spectrally negative Lévy process.

In our subcritical/critical set-up, this problem was first tackled by Fleischman & Sawyer [6] in the case of the branching Brownian motion, as a model of propagation of an allele mutation in a population. In this case, one may write an ODE satisfied by the survival function of $\mathbf{M}$, and the result follows by standard analytic methods. A generalization to symmetric stable Lévy processes was obtained by Lalley & Shao [12], using a pseudo-differential equation and a Feynman-Kac representation of the solution. More recently, the case of α-stable Lévy processes with positive jumps was solved in [15], using a different method based on integral equations. It was in particular proven that the asymptotics of the survival function of $\mathbf{M}$ is given as a power $-\alpha$ in the subcritical case, and $-\alpha/2$ in the critical case. We will see in the following theorems that the situation is different for spectrally negative stable Lévy processes.

It is finally noteworthy to point out that in the literature, the term "branching Lévy process" might refer to a construction more general than the one we just described. One may indeed encodes both the displacement of particules and the offspring reproduction into a general Lévy measure : we refer for instance to Bertoin & Mallein [5] or Mallein & Shi [14] for a study of such processes.

1.2. **Statement of the results.** For $\lambda \in \mathbb{C}$ such that $\Re(\lambda) \geq 0$, let us define the Laplace exponent $\Psi(\lambda) = \ln \mathbb{E}\left[e^{\lambda L_1}\right]$ of L by

$$\Psi(\lambda) = a\lambda + \frac{\eta^2}{2}\lambda^2 + \int_{-\infty}^{0} \left(e^{\lambda x} - 1 - \lambda x 1_{\{|x|<1\}}\right) \nu(dx)$$

where $a \in \mathbb{R}$ is the drift coefficient, $\eta \in \mathbb{R}$ the Gaussian coefficient and the Lévy measure ν satisfies $\int_{-\infty}^{0}(x^2 \wedge 1)\,\nu(dx) < +\infty$. We exclude the case where $-L$ is a subordinator (for which $\mathbf{M} = 0$ a.s.). As a consequence the function Ψ is strictly convex and tends to $+\infty$ as $\lambda \to +\infty$. This implies that for any $q \geq 0$, the equation $\Psi(\lambda) = q$ admits at most two solutions, and we denote by $\Phi(q)$ the largest one :

$$\Phi(q) = \sup\{\lambda \geq 0,\ \Psi(\lambda) = q\}.$$

The function Φ is well-known to be related to the maximum of L. Indeed, let us denote by S the running supremum of L :

$$S_t = \sup_{s \leq t} L_s, \qquad t \geq 0,$$

and let $\mathbf{e}$ be an exponential random variable with parameter 1 independent of L. Then, the random variable $S_{\mathbf{e}}$ is also exponentially distributed, see Bertoin [4, Corollary VII.2] :

$$\mathbb{P}\left(S_{\mathbf{e}} \geq x\right) = e^{-\Phi(1)x}. \tag{1.1}$$

We start with the subcritical case.

Theorem 1. *Assume that the branching process is subcritical, i.e.* $\mathbb{E}[\boldsymbol{p}] < 1$. *Then, there exists a finite constant* $\kappa > 0$ *such that*

$$\mathbb{P}\left(\mathbf{M} \geq x\right) \underset{x \to +\infty}{\sim} \kappa e^{-\Phi(1-\mathbb{E}[\boldsymbol{p}])x}.$$

Comparing this asymptotics with (1.1), we see that in the subcritical case the branching mechanism reduces the decay of the survival function of $\mathbf{M}$, although it remains exponential. Of course, when there are no branching, i.e. $\mathbb{E}[\boldsymbol{p}] = 0$, then $\mathbf{M} = S_{\mathbf{e}}$ a.s. and the equivalence is in fact an equality.

Before turning our attention to the critical case, we need to introduce the scale functions $W^{(q)}$, with $q \geq 0$, which are defined on $[0, +\infty)$ via their Laplace transforms by

$$\int_0^{+\infty} e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\Psi(\beta) - q} \qquad \text{for } \Re(\beta) > \Phi(q). \tag{1.2}$$

These functions are increasing, a.e. differentiable, and known to be related with the exit time problem for spectrally negative Lévy processes, see for instance Kuznetsov, Kyprianou & Rivero [9] or Hubalek & Kyprianou [7]. Analytically, the behavior at $+\infty$ of the scale function $W^{(0)} = W$ will explain the difference in the two cases of the following Theorem 2. Indeed, when $\Psi'(0^+) < 0$, the function $W^{(q)}$ has an exponential growth at $+\infty$ given for $q \geq 0$ by

$$W^{(q)}(x) \underset{x \to +\infty}{\sim} \frac{1}{\Psi'(\Phi(q))} e^{\Phi(q) x} \tag{1.3}$$

while in the case $\Psi'(0^+) \geq 0$, this asymptotics is rather subexponential.

Theorem 2. *Assume that the branching process is critical, i.e.* $\mathbb{E}[\boldsymbol{p}] = 1$.

(1) If $\Psi'(0^+) > 0$, *then :*

$$\mathbb{P}\left(\mathbf{M} \geq x\right) \underset{x \to +\infty}{\sim} \frac{2\Psi'(0^+)}{\sigma^2 \, x}.$$

(2) If $\Psi'(0^+) = 0$, *then there exist two positive constants* κ_1, κ_2 *such that for* x *large enough :*

$$\frac{\kappa_1}{xW(x)} \leq \mathbb{P}\left(\mathbf{M} \geq x\right) \leq \frac{\kappa_2}{xW(x)}.$$

(3) If $\Psi'(0^+) < 0$, *then there exists a finite constant* $\kappa > 0$ *such that :*

$$\mathbb{P}\left(\mathbf{M} \geq x\right) \underset{x \to +\infty}{\sim} \kappa e^{-\Phi(0) x}.$$

To better understand Theorem 2, it might be useful to recall the following facts on the large time behavior of the underlying Lévy process L, see Bertoin [4, Corollary VII.2] :

i) When $\Psi'(0^+) > 0$, the process L drifts a.s. towards $+\infty$. As a consequence, each particule tends to drift upward, and the branching mechanism allows to obtain a power decay instead of an exponential one. Note that the inequalities given in Point (2) remain valid in this case since $\lim_{x \to +\infty} W(x) = 1/\Psi'(0^+)$.

ii) When $\Psi'(0^+) = 0$, the process L oscillates, i.e. $\limsup_{t \to +\infty} L_t = -\liminf_{t \to +\infty} L_t = +\infty$. In this case, the fact that some particules may enjoy big deviations towards $+\infty$ also yields a power decay. This is typically the case of the branching Brownian motion.

iii) Finally, when $\Psi'(0^+) < 0$, the process L drifts a.s. towards $-\infty$. This case is then very similar to the subcritical case of Theorem 1, i.e. the branching mechanism only slightly reduces the tail of the asymptotics.

When $\Psi'(0^+) = 0$, as is usual, Theorem 2 may be refined by assuming a specific asymptotics of Ψ at 0.

Corollary 3. *Assume that* $\mathbb{E}[\boldsymbol{p}] = 1$ *and that* Ψ *is regularly varying at 0, i.e. that there exists a constant* $\alpha \in [1, 2]$ *and a slowly varying function* ℓ *such that*

$$\Psi(\lambda) \underset{\lambda \downarrow 0}{\sim} \lambda^{\alpha} \ell\left(\frac{1}{\lambda}\right). \tag{1.4}$$

Then,

$$\kappa_1 \frac{\ell(x)}{x^\alpha} \leq \mathbb{P}\left(\mathbf{M} \geq x\right) \leq \kappa_2 \frac{\ell(x)}{x^\alpha}$$

and there exists a sequence (x_n) *such that*

$$\mathbb{P}\left(\mathbf{M} \geq x_n\right) \underset{x_n \to +\infty}{\sim} \frac{2}{\sigma^2} \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \ell(x_n) x_n^{-\alpha}.$$

Remark 4. From Bertoin [4, Proposition VII.6], Assumption (1.4) is equivalent to the Spitzer's condition :

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \mathbb{P}(L_s \geq 0) ds = \frac{1}{\alpha}.$$

The simplest example is of course the α-stable spectrally negative Lévy process for which one may choose $\Psi(\lambda) = \lambda^\alpha$, hence $\Phi(q) = q^{1/\alpha}$.

1.3. **An integral equation.** To prove Theorems 1 and 2, we shall write down an integral equation which is similar to the stable case with positive jumps studied in [15] or to the centered branching random walk case studied in Lalley & Shao [13]. Let us set

$$u(x) = \mathbb{P}\left(\mathbf{M} \geq x\right) \quad \text{for } x \geq 0,$$

and $u(x) = 0$ for $x < 0$. The choice to take u null on the negative half-line will allow to work with Fourier transforms without need of renormalization.

Lemma 5. *The function* u *is a solution of the nonlinear integral equation :*

$$u(x) = \mathbb{E}[\boldsymbol{p}]\mathbb{E}\left[1_{\{L_{\mathbf{e}} < x\}} u(x - L_{\mathbf{e}})\right] - \frac{1}{2}\mathbb{E}\left[\boldsymbol{p}^2 - \boldsymbol{p}\right] \mathbb{E}\left[1_{\{L_{\mathbf{e}} < x\}} u^2(x - L_{\mathbf{e}})\right] + \mathbb{E}\left[1_{\{L_{\mathbf{e}} < x\}} \omega(x - L_{\mathbf{e}})\right] + R(x) \quad (1.5)$$

where the function ω *is such that*

$$\forall z \geq 0, \qquad 0 \leq \omega(z) \leq \mathbb{E}[\boldsymbol{p}^3] u^3(z), \quad (1.6)$$

and the remainder R *satisfies the bounds :*

$$|R(x)| \leq \begin{cases} K e^{-\Phi(1)x} & \text{if } x \geq 0, \\ K \mathbb{P}(L_{\mathbf{e}} \leq x) & \text{if } x < 0 \end{cases} \quad (1.7)$$

for some constant $K > 0$.

Proof. Let $x \in \mathbb{R}$ and recall that $\mathbf{e}$ is an exponential random variable of parameter 1 independent of L. We start by applying the Markov property at the first branching event :

$$\mathbb{P}(\mathbf{M} < x) = p_0 \mathbb{P}\left(S_{\mathbf{e}} < x\right) + \sum_{n=1}^{+\infty} p_n \, \mathbb{P}\left(S_{\mathbf{e}} < x, \; L_{\mathbf{e}} + \mathbf{M}^{(1)} < x, \ldots, L_{\mathbf{e}} + \mathbf{M}^{(n)} < x\right)$$

where the random variables $(\mathbf{M}^{(n)})_{n \in \mathbb{N}}$ are independent copies of $\mathbf{M}$, and are also independent of the pair $(L_{\mathbf{e}}, S_{\mathbf{e}})$. Using the formula $\mathbb{P}(\mathbf{M} \geq x) = u(x) + 1_{\{x<0\}}$ we thus obtain the integral equation :

$$1 - u(x) - 1_{\{x<0\}} = p_0 \mathbb{P}\left(S_{\mathbf{e}} < x\right) + \sum_{n=1}^{+\infty} p_n \, \mathbb{E}\left[1_{\{S_{\mathbf{e}} < x\}} \left(1 - u(x - L_{\mathbf{e}}) - 1_{\{x < L_{\mathbf{e}}\}}\right)^n\right]. \quad (1.8)$$

Note that developing the power n on the right-hand side, one may remove the indicator $1_{\{x<L_{\mathbf{e}}\}}$ since by definition $\{S_{\mathbf{e}} < x < L_{\mathbf{e}}\} = \emptyset$. Plugging the Taylor expansion with integral remainder

$$(1-u)^n = 1 - nu + \frac{n(n-1)}{2}u^2 - \frac{n(n-1)(n-2)}{6}u^3 \int_0^1 (1-ut)^{n-3}(1-t)^2 dt$$

in (1.8), we deduce after some simplifications that

$$u(x) = \mathbb{P}\left(S_{\mathbf{e}} \geq x\right) - 1_{\{x<0\}} + \mathbb{E}[\boldsymbol{p}]\mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}}u(x-L_{\mathbf{e}})\right] - \frac{1}{2}\mathbb{E}\left[\boldsymbol{p}^2 - \boldsymbol{p}\right]\mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}}u^2(x-L_{\mathbf{e}})\right] + \mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}}\omega(x-L_{\mathbf{e}})\right] \quad (1.9)$$

where the function ω equals :

$$\omega(z) = u^3(z)\sum_{n\geq 3} p_n \frac{n(n-1)(n-2)}{6}\int_0^1 (1-u(z)t)^{n-3}(1-t)^2 dt \;\leq\; \mathbb{E}[\boldsymbol{p}^3]u^3(z).$$

Looking at (1.9) and adding and substracting $1_{\{L_{\mathbf{e}}<x\}}$, one now obtains formula (1.5) with R given by

$$R(x) = \mathbb{P}\left(S_{\mathbf{e}} \geq x\right) - 1_{\{x<0\}} + \mathbb{E}\left[\left(1_{\{S_{\mathbf{e}}<x\}} - 1_{\{L_{\mathbf{e}}<x\}}\right)\left(\mathbb{E}[\boldsymbol{p}]u(x-L_{\mathbf{e}}) - \frac{1}{2}\mathbb{E}\left[\boldsymbol{p}^2-\boldsymbol{p}\right]u^2(x-L_{\mathbf{e}}) + \omega(x-L_{\mathbf{e}})\right)\right].$$

Finally, for $x<0$, we have using that $S_{\mathbf{e}} \geq 0$ a.s. and $u(z) \leq 1$:

$$|R(x)| \leq \mathbb{E}\left[1_{\{L_{\mathbf{e}}<x\}}\left(\mathbb{E}[\boldsymbol{p}] + \mathbb{E}\left[\boldsymbol{p}^2\right] + \mathbb{E}[\boldsymbol{p}^3]\right)\right] \leq K\mathbb{P}(L_{\mathbf{e}} < x)$$

while for $x>0$, using the explicit distribution of $S_{\mathbf{e}}$ given by (1.1) :

$$|R(x)| \leq \mathbb{P}(S_{\mathbf{e}} \geq x) + 2\mathbb{E}\left[1_{\{S_{\mathbf{e}}>x\}}\left(\mathbb{E}[\boldsymbol{p}] + \mathbb{E}\left[\boldsymbol{p}^2\right] + \mathbb{E}[\boldsymbol{p}^3]\right)\right] \leq Ke^{-\Phi(1)x}.$$

□

Starting from Lemma 5, the proofs of Theorems 1 and 2 both rely on the same three steps :

i) We first obtain some a priori estimates on u using Equation (1.5).
ii) We then use these estimates to write down a new integral equation satisfied by u.
iii) We finally compute the asymptotics of u using this new equation.

One of the key observation will be to notice that the three expectations on the right-hand side of Equation (1.5) are in fact convolution products. This will lead us to work with Laplace and Fourier transforms.

2. The subcritical case : proof of Theorem 1

We start with the subcritical case and first prove that the asymptotics of u must be at least exponential. In the following, we shall exclude the case $\mathbb{E}[\boldsymbol{p}] = 0$ for which $\mathbf{M} = S_{\mathbf{e}}$ a.s.

Lemma 6. *Assume that $\mathbb{E}[\boldsymbol{p}] < 1$. There exists two constants $C, \delta > 0$ such that*

$$\forall x \geq 0, \qquad u(x) \leq Ce^{-\delta x}.$$

Proof. Using that u is decreasing and bounded by 1, we first write

$$\mathbb{P}(L_{\mathbf{e}}>0)u^2(x) \leq \mathbb{E}\left[1_{\{L_{\mathbf{e}}>0\}}u^2(x-L_{\mathbf{e}})\right] \leq \mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}}u^2(x-L_{\mathbf{e}})\right] + \mathbb{P}(S_{\mathbf{e}}\geq x). \tag{2.1}$$

Of course, since L is not the opposite of a subordinator, we necessarily have $\mathbb{P}(L_{\mathbf{e}}>0)>0$. Going back to Equation (1.9), and using the bound on ω given in (1.6) as well as the obvious inequality $L_{\mathbf{e}} \leq S_{\mathbf{e}}$ a.s., we obtain for $x \geq 0$:

$$\begin{aligned}\frac{1}{2}\mathbb{E}\left[\boldsymbol{p}^2-\boldsymbol{p}\right]\mathbb{P}(L_{\mathbf{e}}>0)u^2(x) \leq \mathbb{E}[\boldsymbol{p}]\mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}}u(x-S_{\mathbf{e}})\right] - u(x)\\ + \mathbb{E}[\boldsymbol{p}^3]\mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}}u^3(x-S_{\mathbf{e}})\right] + \left(1+\mathbb{E}\left[\boldsymbol{p}^2\right]\right)\mathbb{P}(S_{\mathbf{e}}\geq x).\end{aligned} \tag{2.2}$$

Notice that we have implicitly used the fact that $\mathbb{E}\left[\boldsymbol{p}^2-\boldsymbol{p}\right]>0$ since $\boldsymbol{p}$ is integer-valued. We now integrate this relation on $[0,n]$ with $n\in\mathbb{N}$ and $\delta>0$:

$$\begin{aligned}&\int_0^n e^{\delta x}\left(\frac{1}{2}\mathbb{E}\left[\boldsymbol{p}^2-\boldsymbol{p}\right]\mathbb{P}(L_{\mathbf{e}}>0)u^2(x) - \mathbb{E}[\boldsymbol{p}^3]\mathbb{E}\left[1_{\{S_{\mathbf{e}}<x\}}u^3(x-S_{\mathbf{e}})\right]\right)dx\\ &\quad\leq \mathbb{E}[\boldsymbol{p}]\mathbb{E}\left[e^{\delta S_{\mathbf{e}}}\int_0^{(n-S_{\mathbf{e}})^+}e^{\delta x}u(x)dx\right] - \int_0^n e^{\delta x}u(x)dx + \left(1+\mathbb{E}\left[\boldsymbol{p}^2\right]\right)\int_0^n e^{\delta x}\mathbb{P}(S_{\mathbf{e}}\geq x)dx\\ &\quad\leq \left(\mathbb{E}[\boldsymbol{p}]\mathbb{E}\left[e^{\delta S_{\mathbf{e}}}\right]-1\right)\int_0^n e^{\delta x}u(x)dx + \left(1+\mathbb{E}\left[\boldsymbol{p}^2\right]\right)\int_0^n e^{\delta x}\mathbb{P}(S_{\mathbf{e}}\geq x)dx.\end{aligned}$$

Since $\mathbb{E}[\boldsymbol{p}]<1$, we may choose $\delta\in(0,\Phi(1))$ small enough such that $\mathbb{E}[\boldsymbol{p}]\mathbb{E}\left[e^{\delta S_{\mathbf{e}}}\right]<1$. Then, letting $n\to+\infty$, we obtain :

$$\limsup_{n\to+\infty}\int_0^n e^{\delta x}\left(\frac{1}{2}\mathbb{E}\left[\boldsymbol{p}^2-\boldsymbol{p}\right]\mathbb{P}(L_{\mathbf{e}}>0)u^2(x) - \mathbb{E}[\boldsymbol{p}^3]\mathbb{E}[e^{\delta S_{\mathbf{e}}}]u^3(x)\right)dx \leq \frac{1+\mathbb{E}[\boldsymbol{p}^2]}{\delta}\mathbb{E}[e^{\delta S_{\mathbf{e}}}].$$

Since u decreases to 0 as $x\to+\infty$, we deduce that

$$\int_0^{+\infty}e^{\delta x}u^2(x)dx<+\infty$$

and finally, for $x\geq 1$,

$$\int_0^{+\infty}e^{\delta z}u^2(z)dz \geq \int_{x-1}^{x}e^{\delta z}u^2(z)dz \geq e^{\delta(x-1)}u^2(x) \tag{2.3}$$

which implies the result. □

Thanks to Lemma 6, we may now take the Laplace transform of Formula (1.5) with $\lambda\in(0,\delta)$. To simplify the computation, we set

$$g(x)=\omega(x)-\frac{1}{2}\mathbb{E}\left[\boldsymbol{p}^2-\boldsymbol{p}\right]u^2(x).$$

Using the Fubini-Tonelli theorem to compute the convolution products on the right-hand side of (1.5), we obtain :

$$\int_{\mathbb{R}}e^{\lambda x}u(x)dx = \mathbb{E}[\boldsymbol{p}]\mathbb{E}\left[e^{\lambda L_{\mathbf{e}}}\right]\int_{\mathbb{R}}e^{\lambda x}u(x)dx + \mathbb{E}\left[e^{\lambda L_{\mathbf{e}}}\right]\int_{\mathbb{R}}e^{\lambda x}g(x)dx + \int_{\mathbb{R}}e^{\lambda x}R(x)dx.$$

By definition, the Laplace transform of the random variable $L_{\mathbf{e}}$ admits the expression

$$\mathbb{E}[e^{\lambda L_{\mathbf{e}}}] = \int_0^{+\infty}e^{-t}e^{t\Psi(\lambda)}dt = \frac{1}{1-\Psi(\lambda)}$$

which yields the formula

$$\int_{\mathbb{R}} e^{\lambda x} u(x)dx = \frac{1}{1-\mathbb{E}[\boldsymbol{p}]-\Psi(\lambda)} \int_{\mathbb{R}} e^{\lambda x}(g(x)+\mathbb{E}[\boldsymbol{p}]R(x))dx + \int_{\mathbb{R}} e^{\lambda x} R(x)dx. \tag{2.4}$$

Observe that this expression remains valid for $0 \leq \lambda < \Phi(1-\mathbb{E}[\boldsymbol{p}])$. We now prove that the fraction on the right-hand side may be written as a Laplace transform. Indeed

$$\begin{aligned}\frac{1}{1-\mathbb{E}[\boldsymbol{p}]-\Psi(\lambda)} &= \int_0^{+\infty} e^{-(1-\mathbb{E}[\boldsymbol{p}]-\Psi(\lambda))t} dt \\ &= \int_0^{+\infty} e^{-(1-\mathbb{E}[\boldsymbol{p}])t} \mathbb{E}\left[e^{\lambda L_t}\right] dt = \int_{\mathbb{R}} e^{\lambda z} \int_0^{+\infty} e^{-(1-\mathbb{E}[\boldsymbol{p}])t} \mathbb{P}(L_t \in dz)dt.\end{aligned}$$

From Kyprianou [11, Corollary 8.9], the q-potential measure of L is known to be absolutely continuous with density $\theta^{(q)}$ given for $q > 0$ by :

$$\int_0^{+\infty} e^{-qt}\mathbb{P}(L_t \in dz)dt = \theta^{(q)}(z)dz = \left(\Phi'(q)e^{-\Phi(q)z} - W^{(q)}(-z)\right) dz. \tag{2.5}$$

As a consequence, inverting Equation (2.4), we obtain a new integral equation for u :

$$u(x) = \int_{\mathbb{R}} \left(g(z)+\mathbb{E}[\boldsymbol{p}]R(z)\right) \theta^{(1-\mathbb{E}[\boldsymbol{p}])}(x-z)\, dz + R(x). \tag{2.6}$$

Finally letting $x \to +\infty$ and using the bound (1.7) on R as well as the asymptotics (1.3) of $W^{(q)}$ gives the result :

$$\lim_{x\to+\infty} e^{\Phi(1-\mathbb{E}[\boldsymbol{p}])x} u(x) = \Phi'(1-\mathbb{E}[\boldsymbol{p}]) \int_{\mathbb{R}} e^{\Phi(1-\mathbb{E}[\boldsymbol{p}])z} \left(g(z)+\mathbb{E}[\boldsymbol{p}]R(z)\right) dz = \kappa.$$

It seems however difficult to compute explicitly κ as it is given in terms of u. □

3. The Critical case : Proof of Theorem 2 when $\Psi'(0^+) \geq 0$

We now assume that $\mathbb{E}[\boldsymbol{p}] = 1$ and we recall that in this case $\sigma^2 = \mathbb{E}[\boldsymbol{p}^2 - \boldsymbol{p}]$ denotes the variance of $\boldsymbol{p}$. For the first part of the proof, we shall deal with Point (1) and Point (2) of Theorem 2 simultaneously.

3.1. **A priori estimates.**

As in the subcritical case, we start by proving some a priori estimates on the function u. These estimates will be necessary to justify some of the computation later.

Lemma 7. *For any $\delta > 0$, there exists a finite constant C_δ such that :*

$$\forall x > 0, \qquad u(x) \leq \frac{C_\delta}{x^{1-\delta}}.$$

Proof. We shall prove by induction that for every $n \in \mathbb{N}$, there exists a constant C_n such that

$$\forall x > 0, \qquad u(x) \leq C_n\, x^{\frac{1}{2n}-1}. \tag{3.1}$$

3.1.1. *The base case $n = 1$.* We start by taking the Laplace transform of Equation (2.2), whose right-hand side only involves the random variable $S_{\mathbf{e}}$. This yields with $\lambda > 0$:

$$\mathbb{P}(L_{\mathbf{e}} > 0)\frac{\sigma^2}{2}\int_0^{+\infty} e^{-\lambda z}u^2(z)dz - \mathbb{E}\left[e^{-\lambda S_{\mathbf{e}}}\right]\mathbb{E}[\boldsymbol{p}^3]\int_0^{+\infty} e^{-\lambda z}u^3(z)dz \leq \left(1+\mathbb{E}[\boldsymbol{p}^2]\right)\frac{1-\mathbb{E}\left[e^{-\lambda S_{\mathbf{e}}}\right]}{\lambda}$$

i.e. using the explicit distribution of $S_{\mathbf{e}}$

$$\int_0^{+\infty} e^{-\lambda z}\left(\mathbb{P}(L_{\mathbf{e}} > 0)\frac{\sigma^2}{2}u^2(z) - \mathbb{E}[\boldsymbol{p}^3]u^3(z)\right)dz \leq \frac{1+\mathbb{E}[\boldsymbol{p}^2]}{\lambda+\Phi(1)}.$$

As before, since u is decreasing and converges towards 0 as $x \to +\infty$, we deduce by letting $\lambda \downarrow 0$ that u^2 is integrable. Then a change of variables yields

$$u^2\left(\frac{1}{\lambda}\right)\int_0^1 e^{-z}dz \leq \lambda\int_0^{+\infty} e^{-\lambda z}u^2(z)dz \leq \lambda\int_0^{+\infty} u^2(z)dz.$$

Setting $x = 1/\lambda$, we finally conclude that there exists a finite constant C_1 such that :

$$u(x) \leq \frac{C_1}{\sqrt{x}}$$

which is (3.1) for $n = 1$.

3.1.2. *Induction step.* Fix $n \in \mathbb{N}$ and assume that Formula (3.1) holds true. Multiplying Equation (2.2) by x and taking the Laplace transform, we obtain after some simplifications using the decomposition $z = z - S_{\mathbf{e}} + S_{\mathbf{e}}$:

$$\begin{aligned}&\int_0^{+\infty} e^{-\lambda z}z\left(\mathbb{P}(L_{\mathbf{e}} > 0)\frac{\sigma^2}{2}u^2(z) - \mathbb{E}[\boldsymbol{p}^3]u^3(z)\right)dz\\ &\qquad\leq \mathbb{E}\left[S_{\mathbf{e}}e^{-\lambda S_{\mathbf{e}}}\right]\int_0^{+\infty} e^{-\lambda z}u(z)dz + \frac{1+\mathbb{E}[\boldsymbol{p}^2]}{(\lambda+\Phi(1))^2} + \mathbb{E}[\boldsymbol{p}^3]\mathbb{E}\left[S_{\mathbf{e}}e^{-\lambda S_{\mathbf{e}}}\right]\int_0^{+\infty} e^{-\lambda z}u^3(z)dz.\end{aligned}$$

Fix ε small enough and take $A > 0$ such that for any $x \geq A$

$$\mathbb{E}[\boldsymbol{p}^3]u(x) \leq \mathbb{P}(L_{\mathbf{e}} > 0)\frac{\sigma^2}{2} - \varepsilon.$$

A change of variables then yields

$$\begin{aligned}&\frac{\varepsilon}{\lambda^2}\int_{\lambda A}^{+\infty} e^{-x}xu^2\left(\frac{x}{\lambda}\right)dx\\ &\qquad\leq \frac{\mathbb{E}\left[S_{\mathbf{e}}\right]}{\lambda}\int_0^{+\infty} e^{-\lambda z}\left(\frac{z}{\lambda}\right)dz + \frac{1+\mathbb{E}[\boldsymbol{p}^2]}{(\lambda+\Phi(1))^2} + \mathbb{E}[\boldsymbol{p}^3]\mathbb{E}\left[S_{\mathbf{e}}\right]\int_0^{+\infty} u^3(z)dz + \mathbb{E}[\boldsymbol{p}^3]\int_0^A zu^3(z)dz\end{aligned}$$

i.e., for λ small enough such that $\lambda A \leq 1/2$,

$$\varepsilon u^2\left(\frac{1}{\lambda}\right)\int_{1/2}^1 e^{-x}xdx \leq \lambda\mathbb{E}\left[S_{\mathbf{e}}\right]\int_0^{+\infty} e^{-z}u\left(\frac{z}{\lambda}\right)dz + \lambda^2K$$

for some constant $K > 0$. Plugging the induction hypothesis in the right-hand side yields

$$\varepsilon u^2\left(\frac{1}{\lambda}\right)\int_{1/2}^1 e^{-x}xdx \leq \lambda^{2-\frac{1}{2^n}}C_n\mathbb{E}\left[S_{\mathbf{e}}\right]\int_0^{+\infty} e^{-z}z^{1-\frac{1}{2n}}dz + \lambda^2K$$

and replacing $x = 1/\lambda$ as before, we finally obtain the existence of a constant C_{n+1} such that :

$$u(x) \leq C_{n+1}\,x^{\frac{1}{2^{n+1}}-1}.$$

As a consequence, we deduce that (3.1) holds for every $n \in \mathbb{N}$, which proves Lemma 7 by monotony. □

3.2. **A new equation for u.** The purpose of this subsection is to prove the following new equation for the function u :

Proposition 8. *The function u is a solution of the integral equation :*

$$u(x) - R(x) = \int_0^{+\infty} \left(\frac{\sigma^2}{2} u^2(x+z) - \omega(x+z) - R(x+z) \right) W(z) dz.$$

In the following, we shall denote the Fourier transform of a measurable function f (provided it exists) by :

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}} e^{i\xi z} f(z) dz.$$

In particular, since u is decreasing and converges towards 0, we deduce from Abel's lemma that $\mathcal{F}(u)$ is well-defined (although u might not be integrable). Taking the Fourier transform of Equation (1.5), and applying the Fubini theorem on the left-hand side since u^2 is integrable from Lemma 7, we deduce that

$$\frac{\sigma^2}{2} \mathbb{E}[e^{i\xi L_{\mathbf{e}}}] \mathcal{F}(u^2)(\xi) = \int_{\mathbb{R}} e^{i\xi x} \int_{\mathbb{R}} u(x-z) \mathbb{P}(L_{\mathbf{e}} \in dz) dx - \mathcal{F}(u)(\xi) + \mathbb{E}[e^{i\xi L_{\mathbf{e}}}] \mathcal{F}(\omega)(\xi) + \mathcal{F}(R)(\xi). \quad (3.2)$$

To compute the convolution product on the right-hand side, we shall rely on the following lemma :

Lemma 9. *Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function and let $\varphi : [0, +\infty) \to [0, +\infty)$ be a measurable function, null on $(-\infty, 0)$ and such that*

$$\forall \xi \neq 0, \qquad \left| \int_0^{+\infty} e^{i\xi x} \varphi(x) dx \right| < +\infty.$$

Then, the Fourier transform of the convolution product is given by :

$$\int_{\mathbb{R}} e^{i\xi x} \left(\int_{\mathbb{R}} \varphi(x-z) f(z) dz \right) dx = \mathcal{F}(\varphi)(\xi) \times \mathcal{F}(f)(\xi), \qquad \xi \neq 0.$$

Proof. Of course, Lemma 9 is just a consequence of Fubini theorem if φ is integrable. But when φ is not integrable, as will be the case here, some care is needed. Fix $n > 0$. Applying the Fubini theorem and a change of variable, we first write

$$\int_{-n}^{n} e^{i\xi x} \int_0^{+\infty} f(x-z) \varphi(z) dz\, dx = \int_0^{+\infty} e^{i\xi z} \varphi(z) \int_{-n-z}^{n-z} e^{i\xi y} f(y) dy\, dz.$$

To avoid lengthy expressions, we shall proceed in two steps by cutting the last integral at 0. Integrating by parts, we first write:

$$\begin{aligned} &\mathcal{F}(\varphi)(\xi) \times \int_0^{+\infty} e^{i\xi y} f(y) dy - \int_0^{+\infty} e^{i\xi z} \varphi(z) \int_0^{n-z} e^{i\xi y} f(y) dy\, dz \\ &= \int_0^{+\infty} e^{i\xi z} \varphi(z) dz \int_n^{+\infty} e^{i\xi y} f(y) dy + \int_0^{+\infty} \left(\int_y^{+\infty} e^{i\xi z} \varphi(z) dz \right) e^{i\xi(n-y)} f(n-y) dy. \end{aligned}$$

The first term will converge to 0 as $n \to +\infty$. To study the limit of the second term, fix A large enough such that

$$\forall x \geq A, \qquad \left| \int_x^{+\infty} e^{i\xi z} \varphi(z) dz \right| \leq \varepsilon.$$

We then have :

$$\left|\int_0^A \left(\int_y^{+\infty} e^{i\xi z}\varphi(z)dz\right) e^{i\xi(n-y)} f(n-y)dy\right| + \left|\int_A^{+\infty} \left(\int_y^{+\infty} e^{i\xi z}\varphi(z)dz\right) e^{i\xi(n-y)} f(n-y)dy\right|$$
$$\leq \sup_{y\geq 0}\left|\int_y^{+\infty} e^{i\xi z}\varphi(z)dz\right| \int_0^A |f(n-y)|dy + \varepsilon\int_{\mathbb{R}} |f(x)|dx \xrightarrow[n\to+\infty]{} \varepsilon\int_{\mathbb{R}} |f(x)|dx.$$

A similar argument shows that

$$\lim_{n\to+\infty}\int_0^{+\infty} e^{i\xi z}\varphi(z)\int_{-n-z}^0 e^{i\xi y} f(y)dy\,dz = \mathcal{F}(\varphi)(\xi)\times\int_{-\infty}^0 e^{i\xi y} f(y)dy.$$

As a consequence, we have obtained that

$$\begin{aligned}\int_{\mathbb{R}} e^{i\xi x}\int_0^{+\infty} f(x-z)\varphi(z)dz\,dx &= \lim_{n\to+\infty}\int_{-n}^n e^{i\xi x}\int_0^{+\infty} f(x-z)\varphi(z)dz\,dx\\ &= \lim_{n\to+\infty}\int_0^{+\infty} e^{i\xi x}\varphi(x)\int_{-n-x}^{n-x} e^{i\xi z} f(z)dz\,dx\\ &= \mathcal{F}(\varphi)(\xi)\times\mathcal{F}(f)(\xi)\end{aligned}$$

which concludes the proof of Lemma 9. □

Recall now that the distribution of $L_{\mathbf{e}}$ is actually the 1-potential measure of L, i.e. from (2.5) the random variable $L_{\mathbf{e}}$ is absolutely continuous with density given by $\theta^{(1)}$. Applying Lemma 9 to Equation (3.2) with $f = \theta^{(1)}$ and $\varphi = u$, we thus obtain

$$\frac{\sigma^2}{2}\mathbb{E}[e^{i\xi L_{\mathbf{e}}}]\mathcal{F}(u^2)(\xi) = \mathbb{E}[e^{i\xi L_{\mathbf{e}}}]\mathcal{F}(u)(\xi) - \mathcal{F}(u)(\xi) + \mathbb{E}[e^{i\xi L_{\mathbf{e}}}]\mathcal{F}(\omega)(\xi) + \mathcal{F}(R)(\xi).$$

By definition, the characteristic function of $L_{\mathbf{e}}$ is given by

$$\mathbb{E}\left[e^{i\xi L_{\mathbf{e}}}\right] = \int_0^{+\infty} e^{-t}e^{t\Psi(i\xi)}dt = \frac{1}{1-\Psi(i\xi)}$$

which yields the equation

$$\mathcal{F}\left(\frac{\sigma^2}{2}u^2 - \omega - R\right)(\xi) = \Psi(i\xi)\mathcal{F}(u-R)(\xi).$$

Integrating by parts the right-hand side, we further have

$$\frac{i\xi}{\Psi(i\xi)}\mathcal{F}\left(\frac{\sigma^2}{2}u^2 - \omega - R\right)(\xi) = \mathcal{F}((R-u)')(\xi). \tag{3.3}$$

The next step consists in showing that the function $\xi \longrightarrow i\xi/\Psi(i\xi)$ is actually the Fourier transform of W'. Indeed, from [7], since $\Psi'(0^+) \geq 0$, the function W is concave on $(0,+\infty)$ hence it is absolutely continuous on any compact subset of $(0,+\infty)$. As a consequence, we may integrate by parts the definition of W given in (1.2) to obtain

$$\int_0^{+\infty} e^{-\beta x}W'(x)dx = \frac{\beta}{\Psi(\beta)}, \qquad \Re(\beta) > 0. \tag{3.4}$$

We now check that Formula (3.4) remains valid for $\beta = i\xi$ with $\xi \neq 0$. The function W' being positive and decreasing, let us denote by w_∞ its limit : $\lim_{x\to+\infty} W'(x) = \inf_{x\geq 0} W'(x) = w_\infty$. As a

consequence, from Abel's lemma, the Fourier transform of $W' - w_\infty$ is well-defined for $\xi \neq 0$. Let $\varepsilon > 0$ and take A large enough such that

$$\forall x \geq A, \qquad \left| \int_x^{+\infty} e^{-i\xi z}(W'(z) - w_\infty)dz \right| \leq \varepsilon.$$

Integrating by parts, we have for $h > 0$:

$$\begin{aligned}\left| \int_0^{+\infty} \left(e^{-hx} - 1\right) e^{-i\xi x}(W'(x) - w_\infty)dx \right| &= h \left| \int_0^{+\infty} e^{-hx} \int_x^{+\infty} e^{-i\xi z}(W'(z) - w_\infty)dz\, dx \right| \\ &\leq h \int_0^A \left| \int_x^{+\infty} e^{-i\xi z}(W'(z) - w_\infty)dz \right| dx + \varepsilon\end{aligned}$$

which proves that

$$\lim_{h\downarrow 0} \int_0^{+\infty} e^{-hx} e^{-i\xi x}(W'(x) - w_\infty)dx = \int_0^{+\infty} e^{-i\xi x}(W'(x) - w_\infty)dx.$$

Then, using the continuity of Ψ, we obtain still for $\xi \neq 0$:

$$\begin{aligned}\frac{i\xi}{\Psi(i\xi)} &= \lim_{h\downarrow 0} \frac{h + i\xi}{\Psi(h + i\xi)} \\ &= \lim_{h\downarrow 0} \int_0^{+\infty} e^{-hx} e^{-i\xi x}(W'(x) - w_\infty)dx + \int_0^{+\infty} e^{-hx} e^{-i\xi x} w_\infty dx \\ &= \int_0^{+\infty} e^{-i\xi x}(W'(x) - w_\infty)dx + \frac{w_\infty}{i\xi}.\end{aligned}$$

Plugging this last expression in (3.3) and computing the convolution product, using again Lemma 9 with $f = \frac{\sigma^2}{2}u^2 - \omega - R$ and $\varphi = W' - w_\infty$, one obtains :

$$\begin{aligned}\int_0^{+\infty} e^{-i\xi x} \int_0^{+\infty} \left(\frac{\sigma^2}{2}u^2(z + x) - \omega(z + x) - R(z + x)\right)(W'(z) - w_\infty)dz\, dx \\ = \mathcal{F}((R - u)')(\xi) - \frac{w_\infty}{i\xi}\mathcal{F}\left(\frac{\sigma^2}{2}u^2 - \omega - R\right)(\xi). \quad (3.5)\end{aligned}$$

We now check that the terms in w_∞ cancel. When $\Psi'(0^+) > 0$, we necessarily have $w_\infty = 0$ since the function W' is integrable as can be seen by letting $\beta \downarrow 0$ in Formula (3.4) and applying the monotone convergence theorem

$$\int_0^{\infty} W'(x)dx = \frac{1}{\Psi'(0^+)}.$$

When $\Psi'(0^+) = 0$, the cancelation will follow from the observation that

$$\mathcal{F}\left(\frac{\sigma^2}{2}u^2 - \omega - R\right)(0) = \int_{-\infty}^{+\infty} \left(\frac{\sigma^2}{2}u^2(z) - \omega(z) - R(z)\right) dz = 0.$$

Indeed, applying the dominated convergence theorem in (3.3) thanks to Lemma 7, we have

$$\lim_{\xi\downarrow 0} \mathcal{F}\left(\frac{\sigma^2}{2}u^2 - \omega - R\right)(\xi) = \lim_{\xi\downarrow 0} \frac{\Psi(i\xi)}{i\xi}\mathcal{F}\left((R - u)'\right)(\xi) = 0.$$

As a consequence, integrating by parts the last term of (3.5) and inverting the Fourier transform, we deduce that

$$\int_0^{+\infty} \left(\frac{\sigma^2}{2} u^2(z+x) - \omega(z+x) - R(z+x) \right) (W'(z) - w_\infty) dz$$
$$= (R-u)'(x) + w_\infty \int_{-\infty}^{x} \left(\frac{\sigma^2}{2} u^2(z) - \omega(z) - R(z) \right) dz$$

i.e.

$$\int_0^{+\infty} \left(\frac{\sigma^2}{2} u^2(z+x) - \omega(z+x) - R(z+x) \right) W'(z) dz = (R-u)'(x). \tag{3.6}$$

Finally, integrating both sides and using again the Fubini theorem :

$$\begin{aligned} u(x) - R(x) &= \int_x^{+\infty} \int_r^{+\infty} \left(\frac{\sigma^2}{2} u^2(z) - \omega(z) - R(z) \right) W'(z-r) dz dr \\ &= \int_x^{+\infty} \left(\frac{\sigma^2}{2} u^2(z) - \omega(z) - R(z) \right) W(z-x) dz \\ &= \int_0^{+\infty} \left(\frac{\sigma^2}{2} u^2(z+x) - \omega(z+x) - R(z+x) \right) W(z) dz \end{aligned}$$

which is the announced equation. □

3.3. **Study of the limit.** The last part of the proof now consists in studying the asymptotics of u, using the new equation :

$$u(x) - R(x) = x \int_0^{+\infty} \left(\frac{\sigma^2}{2} u^2(x(z+1)) - \omega(x(z+1)) - R(x(z+1)) \right) W(xz) dz. \tag{3.7}$$

Notice that by a coupling argument with the subcritical case obtained in Theorem 1, we have for any offspring distribution such that $\mathbb{E}[\boldsymbol{p}] < 1$:

$$\liminf_{x \to +\infty} e^{\Phi(1-\mathbb{E}[\boldsymbol{p}])x} u(x) > 0.$$

In particular, since $\Phi(0) = 0$ in the case $\Psi'(0^+) \geq 0$, this implies that for any $\delta > 0$,

$$\liminf_{x \to +\infty} e^{\delta x} u(x) = +\infty. \tag{3.8}$$

As a consequence, we deduce that for x large enough, there exists a constant $C > 0$ such that

$$\frac{\sigma^2}{2} u^2(x) - \omega(x) - R(x) \geq \frac{\sigma^2}{2} u^2(x) \left(1 - Cu(x) - Ce^{-(\Phi(1)-\delta)x} \right) \geq 0$$

which proves that the integrand on the right-hand side of (3.7) is positive for x large enough. Similarly, we deduce from (3.6) that the function $u - R$ is decreasing for x large enough.

Going back to Theorem 2, Point (2) is equivalent to showing that

$$0 < \liminf_{x \to +\infty} \gamma(x) \leq \limsup_{x \to +\infty} \gamma(x) < +\infty \tag{3.9}$$

where we have set, to simplify the notation :

$$\gamma(x) = xW(x)u(x).$$

3.3.1. *Computation of the upper bound.* We first prove that the limit superior of γ is finite. Let us take A large enough such that for any $x \geq A$, the quantity $\frac{\sigma^2}{2}u^2(x) - \omega(x) - R(x)$ is positive. We then decompose :

$$\begin{aligned} u(x) &\geq x \int_1^2 W(xz) \left(\frac{\sigma^2}{2} u^2(x(z+1)) - \omega(x(z+1)) - R(x(z+1)) \right) dz \\ &\geq \frac{\sigma^2}{2} x W(x) u^2(3x) - x W(x) \mathbb{E}[\boldsymbol{p}^3] u^3(2x) - x W(x) K e^{-2\Phi(1)x} \end{aligned}$$

for some constant $K > 0$ given by the bound (1.7) on R. Then, multiplying both sides by $xW(x)$ and taking the supremum on $[A, n]$, we deduce that

$$\sup_{[A,n]} \frac{\sigma^2}{18} \left(\frac{W(x)}{W(3x)} \right)^2 (\gamma(3x))^2 \leq \sup_{[A,n]} \gamma(x) + \frac{\mathbb{E}[\boldsymbol{p}^3] u(2A)}{4} \sup_{[A,n]} \left(\frac{W(x)}{W(2x)} \right)^2 (\gamma(2x))^2 + C$$

for some constant $C > 0$ independent of n. Furthermore, since W is positive, increasing and concave, we have the bounds

$$1 \geq \frac{W(x)}{W(2x)} \geq \frac{W(x)}{W(3x)} \geq \frac{1}{W(3x)} \left(\int_0^x W'(3y) dy + W(0) \right) = \frac{1}{3} + \frac{2}{3} \frac{W(0)}{W(3x)} \geq \frac{1}{3}.$$

Therefore

$$\frac{\sigma^2}{162} \sup_{[3A,3n]} (\gamma(x))^2 \leq \sup_{[A,3n]} \gamma(x) + \frac{\mathbb{E}[\boldsymbol{p}^3]}{4} u(2A) \sup_{[A,3n]} (\gamma(x))^2 + C$$

and dividing both sides

$$\left(\frac{\sigma^2}{162} - \frac{\mathbb{E}[\boldsymbol{p}^3]}{4} u(2A) \right) \sup_{[3A,3n]} \gamma(x) \leq \left(1 + \frac{\sup_{[A,3A]} \gamma(x) + C}{\sup_{[3A,3n]} \gamma(x)} \right) + \frac{\mathbb{E}[\boldsymbol{p}^3]}{4} u(2A) \frac{\sup_{[A,3A]} (\gamma(x))^2}{\sup_{[3A,3n]} \gamma(x)}.$$

Finally, by taking A large enough for the left-hand side to be positive and letting $n \to +\infty$, we conclude that

$$\sup_{x \geq 3A} \gamma(x) < +\infty$$

which implies that the limit superior is finite.

3.3.2. *Computation of the lower bound.* We now turn our attention to the limit inferior of γ. Fix $\delta > 0$ small enough. Using the limit superior as well as (3.8), we deduce that there exists $K > 0$ and $A > 0$ such that for every $z \geq A$, one has

$$\frac{\sigma^2}{2} u^2(z) - R(z) \leq K \left(\frac{1}{zW(z)} \right)^{1-\delta} (u(z) - R(z))^{1+\delta}.$$

Starting from (3.6), and recalling that the function $u - R$ is decreasing for x large enough, we first write the bound

$$(R(x) - u(x))' \leq K (u(x) - R(x))^{1+\delta} \int_0^{+\infty} \frac{x W'(xz)}{(x(1+z) W(x(1+z)))^{1-\delta}} dz$$

which implies, since $x \to xW(x)$ is increasing,

$$\frac{(R(x) - u(x))'}{(u(x) - R(x))^{1+\delta}} \leq K \left(\frac{W(x)}{(xW(x))^{1-\delta}} + x \int_1^{+\infty} \frac{W'(xz)}{(x(1+z) W(x(1+z)))^{1-\delta}} dz \right). \tag{3.10}$$

Furthermore, since W is concave and $z \geq 1$, we have the series of inequalities :

$$\frac{W'(xz)}{(x(1+z)W(x(1+z)))^{1-\delta}} \leq \frac{W(xz)}{xz}\frac{1}{(xzW(xz))^{1-\delta}} = \frac{(W(xz))^{\delta}}{(xz)^{2-\delta}} \leq \left(\frac{W(x)}{x}\right)^{\delta}\frac{1}{(xz)^{2-2\delta}}.$$

As a consequence, we deduce that the integral in (3.10) is bounded by

$$x\int_1^{+\infty}\frac{W'(xz)}{(x(1+z)W(x(1+z)))^{1-\delta}}dz \leq \left(\frac{W(x)}{x}\right)^{\delta}\int_x^{+\infty}\frac{1}{z^{2-2\delta}}dz = \frac{1}{1-2\delta}\frac{(W(x))^{\delta}}{x^{1-\delta}}.$$

Then, integrating (3.10) on $[A, y]$, the last inequality implies that there exists a constant $C > 0$ such that :

$$\frac{1}{(u(y)-R(y))^{\delta}} - \frac{1}{(u(A)-R(A))^{\delta}} \leq \delta C\int_A^y \frac{(W(x))^{\delta}}{x^{1-\delta}}dx \leq C(yW(y))^{\delta}$$

i.e.

$$u(y) - R(y) \geq \left(C(yW(y))^{\delta} + \frac{1}{(u(A)-R(A))^{\delta}}\right)^{-1/\delta}.$$

Finally, passing to the limit inferior, we conclude that

$$\liminf_{y\to+\infty}\gamma(y) = \liminf_{y\to+\infty} yW(y)u(y) \geq C^{-1/\delta} > 0$$

which proves Point (2) of Theorem 2.

3.4. **The case $\Psi'(0^+) > 0$.** We now prove Point (1) of Theorem 2. In this case, we may improve the previous inequality using the fact that W converges towards some positive value $W(\infty) = 1/\Psi'(0^+)$. Indeed, let us first write Equation (3.7) under the form :

$$u(x) = \frac{\sigma^2}{2}\int_0^{+\infty} u^2(x+z)W(x+z)dz + I(x) = \frac{\sigma^2}{2}\int_x^{+\infty} u^2(z)W(z)dz + I(x) \tag{3.11}$$

where the remainder I is given by :

$$I(x) = R(x) + \frac{\sigma^2}{2}\int_0^{+\infty} u^2(x+z)(W(z) - W(z+x))dz - \frac{\sigma^2}{2}\int_0^{+\infty}(\omega(z+x) + R(z+x))W(z)dz.$$

Differentiating and solving Equation (3.11), we deduce that

$$\begin{aligned} u(x) &= \frac{1}{1 + \frac{\sigma^2}{2}\int_0^x W(z)dz + \int_0^x \frac{I'(z)}{u^2(z)}dz} \\ &= \frac{1}{1 + \frac{\sigma^2}{2}\int_0^x W(z)dz + \frac{I(x)}{u^2(x)} - I(0) - 2\int_0^x \frac{I(z)u'(z)}{u^3(z)}dz}. \end{aligned} \tag{3.12}$$

Fix $\varepsilon > 0$ small enough. Using (3.9) as well as the bounds on ω and R, the function I is smaller than

$$\begin{aligned} |I(x)| &\leq K\left(\frac{1}{x^2} + \frac{1}{x}\int_0^{+\infty}\frac{1}{(1+z)^2}(W(x+zx) - W(zx))dz\right) \\ &\leq K\left(\frac{1}{x^2} + \frac{2W(\infty)}{x}\int_0^{\varepsilon}\frac{1}{(1+z)^2}dz + \frac{W(\infty) - W(\varepsilon x)}{x}\int_{\varepsilon}^{+\infty}\frac{1}{(1+z)^2}dz\right) \end{aligned}$$

for some constant $K > 0$. Furthermore, taking $\delta = 0$ in (3.10), we also deduce that $x \to x^2|u'(x)|$ is bounded for x large enough. As a consequence, for $x \geq 1$, there exists a constant $C > 0$ such that

$$\left| \int_1^x \frac{I(z)u'(z)}{u^3(z)} dz \right| \leq C \int_1^x |I(z)| z dz \leq CK \left(\ln(x) + 2W(\infty)\varepsilon x + \int_1^x (W(\infty) - W(\varepsilon z)) dz \right).$$

Going back to (3.12), this yields the bounds

$$\frac{1}{\frac{\sigma^2}{2}W(\infty) + 4KCW(\infty)\varepsilon} \leq \liminf_{x\to+\infty} xu(x) \leq \limsup_{x\to+\infty} xu(x) \leq \frac{1}{\frac{\sigma^2}{2}W(\infty) - 4KCW(\infty)\varepsilon}$$

and the result follows by letting $\varepsilon \downarrow 0$:

$$\lim_{x\to+\infty} xu(x) = \frac{1}{\frac{\sigma^2}{2}W(\infty)} = \frac{2}{\sigma^2}\Psi'(0^+).$$

□

3.5. **The regularly varying case.** We now assume that Assumption (1.4) holds :

$$\Psi(\lambda) \underset{\lambda\downarrow 0}{\sim} \lambda^\alpha \ell\left(\frac{1}{\lambda}\right)$$

where $\alpha \in [1, 2]$ and ℓ is a slowly varying function. Recalling that W is increasing, we deduce from (1.2) and the Tauberian theorem that W is regularly varying at $+\infty$:

$$W(x) \underset{x\to+\infty}{\sim} \frac{x^{\alpha-1}}{\Gamma(\alpha)\ell(x)}. \tag{3.13}$$

Let us define for $z \geq 0$ the function $b^{(x)}$ by

$$b^{(x)}(z) = \frac{W(x)W(xz)}{\left((1+z)W(x+xz)\right)^2}.$$

From the asymptotics (3.13), this function converges a.s. :

$$b^{(x)}(z) \xrightarrow[x\to+\infty]{} \frac{z^{\alpha-1}}{(1+z)^{2\alpha}}.$$

We now come back to Equation (3.7) which we rewrite :

$$\gamma(x) = \frac{\sigma^2}{2} \int_0^{+\infty} \gamma^2(x(1+z)) b^{(x)}(z) dz + I(x) \tag{3.14}$$

where the remainder $I(x)$ is given by

$$I(x) = xW(x)R(x) - \int_0^{+\infty} \big(x(1+z)W(x(1+z))\big)^2 \big(\omega(x(1+z)) + R(x(1+z))\big) b^{(x)}(z) dz.$$

Since W is increasing, the inequality $b^{(x)}(z) \leq 1/(1+z)^2$ together with the bounds on ω and R given in Lemma 5 allow to apply the dominated convergence theorem to obtain

$$\lim_{x\to+\infty} I(x) = 0.$$

Now, applying Fatou's lemma in Equation (3.14), we deduce that

$$\liminf_{x\to+\infty} \gamma(x) \geq \liminf_{x\to+\infty} \gamma^2(x) \frac{\sigma^2}{2} \int_0^{+\infty} \frac{z^{\alpha-1}}{(1+z)^{2\alpha}} dz$$

i.e.

$$\liminf_{x\to+\infty} \gamma(x) \leq \frac{2}{\sigma^2 B(\alpha,\alpha)}$$

where B denotes the Beta function. Similarly, applying the reverse Fatou lemma since γ is bounded, we conclude that

$$\frac{2}{\sigma^2 B(\alpha,\alpha)} \leq \limsup_{x\to+\infty} \gamma(x).$$

Finally, the existence of a sequence (x_n) such that

$$\lim_{n\to+\infty} \gamma(x_n) = \lim_{x\to+\infty} x_n W(x_n) u(x_n) = \frac{2}{\sigma^2 B(\alpha,\alpha)}$$

is a consequence of the continuity of γ. Now, Corollary 3 follows from the asymptotics of W given by (3.13) and the formula for the Beta function $B(\alpha,\alpha) = \Gamma^2(\alpha)/\Gamma(2\alpha)$. □

Remark 10. Notice that if we neglect the remainders, the equation

$$f(x) = \frac{\sigma^2}{2} \int_0^{+\infty} f^2(x(1+z)) \frac{z^{\alpha-1}}{(1+z)^{2\alpha}} dz$$

admits as solutions the functions

$$f_c(x) = \frac{x^\alpha}{\left(c + \left(\frac{\sigma^2}{2} B(\alpha,\alpha)\right)^{1/\alpha} x\right)^\alpha}$$

where c is any parameter in $[0,+\infty]$. This expression is in agreement with the explicit solution obtained in [6] for the Brownian case.

4. The critical case : proof of Theorem 2 when $\Psi'(0^+) < 0$

In this case, the underlying Lévy process drifts a.s. to $-\infty$, and we shall see that the asymptotics of $\mathbf{M}$ is no longer polynomial but exponential. Consequently, the proof will be similar to the subcritical case of Theorem 1. We start with the following a priori estimate :

Lemma 11. *Assume that $\Psi'(0^+) < 0$. Then, for any $\delta > 0$, there exists a finite constant C_δ such that*

$$\forall x \geq 0, \qquad u(x) \leq C_\delta \exp\left(-\frac{\Phi(0)-\delta}{2} x\right).$$

Proof. Fix $\lambda \in (0, \Phi(0))$ and notice that since Ψ is convex and $\Psi'(0^+) < 0$, we have $\mathbb{E}\left[e^{\lambda L_{\mathbf{e}}}\right] \leq 1$. Let $n \in \mathbb{N}$. Integrating Equation (1.5) against $\exp\left(\lambda x - \frac{1}{n} e^{\lambda x}\right)$, we obtain :

$$\begin{aligned}\int_0^{+\infty} \mathbb{E}\left[e^{\lambda(x+L_{\mathbf{e}})} \exp\left(-\frac{1}{n} e^{\lambda(x+L_{\mathbf{e}})}\right)\right] \left(\frac{\sigma^2}{2} u^2(x) - \mathbb{E}[\boldsymbol{p}^3] u^3(x)\right) dx - \int_0^{+\infty} e^{\lambda x} \exp\left(-\frac{1}{n} e^{\lambda x}\right) R(x) dx \\ \leq \int_0^{+\infty} \mathbb{E}\left[e^{\lambda(x+L_{\mathbf{e}})} \exp\left(-\frac{1}{n} e^{\lambda(x+L_{\mathbf{e}})}\right)\right] u(x) dx - \int_0^{+\infty} e^{\lambda x} \exp\left(-\frac{1}{n} e^{\lambda x}\right) u(x) dx.\end{aligned}$$

Integrating by parts and recalling that u is decreasing, the right-hand side is further equal to

$$\frac{n}{\lambda} \mathbb{E}\left[e^{-\frac{1}{n} e^{\lambda L_{\mathbf{e}}}} - 1\right] + \int_0^{+\infty} \mathbb{E}\left[e^{-\frac{1}{n} e^{\lambda x}} - e^{-\frac{1}{n} e^{\lambda x + \lambda L_{\mathbf{e}}}}\right] |u'(x)| dx.$$

Observe finally that this last integral is negative, since, from Jensen inequality,

$$\mathbb{E}\left[e^{-\frac{1}{n}e^{\lambda x}} - e^{-\frac{1}{n}e^{\lambda x+\lambda L_{\mathbf{e}}}}\right] \leq e^{-\frac{1}{n}e^{\lambda x}} - e^{-\frac{1}{n}e^{\lambda x}\mathbb{E}\left[e^{\lambda L_{\mathbf{e}}}\right]} \leq 0.$$

As a consequence, we deduce that

$$\limsup_{n\to+\infty} \int_0^{+\infty} \mathbb{E}\left[e^{\lambda(x+L_{\mathbf{e}})}\exp\left(-\frac{1}{n}e^{\lambda(x+L_{\mathbf{e}})}\right)\right]\left(\frac{\sigma^2}{2}u^2(x) - \mathbb{E}[\boldsymbol{p}^3]u^3(x)\right)dx \leq \int_0^{+\infty} e^{\lambda x}R(x)dx.$$

Applying the monotone convergence theorem, this implies that

$$\int_0^{+\infty} e^{\lambda x}u^2(x)dx < +\infty.$$

Lemma 11 now follows from the fact that u is decreasing, using the same argument as in (2.3). □

The remainder of the proof is now similar to the subcritical case of Section 2. Taking the Laplace transform of (1.5) with $\lambda \in (0, \Phi(0)/2)$, we obtain :

$$-\frac{1}{\Psi(\lambda)}\int_{\mathbb{R}} e^{\lambda x}\left(\frac{\sigma^2}{2}u^2(x) - \omega(x) - R(x)\right)dx = \int_{\mathbb{R}} e^{\lambda x}\left(u(x) - R(x)\right)dx. \tag{4.1}$$

We now observe that since $\Psi(\lambda) < 0$, the function $\lambda \longrightarrow -1/\Psi(\lambda)$ is the Laplace transform of the 0-potential of L :

$$-\frac{1}{\Psi(\lambda)} = \int_0^{+\infty} e^{t\Psi(\lambda)}dt = \int_0^{+\infty} \mathbb{E}\left[e^{\lambda L_t}\right]dt = \int_{\mathbb{R}} e^{\lambda z}\int_0^{+\infty} \mathbb{P}(L_t \in dz)dt.$$

Therefore, passing to the limit as $q \downarrow 0$ in (2.5)

$$\int_0^{+\infty} \mathbb{P}(L_t \in dz)dt = \left(\Phi'(0)e^{-\Phi(0)z} - W(-z)\right)dz$$

and plugging this last relation in (4.1), we obtain after inverting the Laplace transforms

$$\int_{\mathbb{R}}\left(\frac{\sigma^2}{2}u^2(z) - \omega(z) - R(z)\right)\left(\Phi'(0)e^{-\Phi(0)(x-z)} - W(z-x)\right)dz = u(x) - R(x).$$

Finally, passing to the limit as $x \to +\infty$, we conclude that

$$\lim_{x\to+\infty} e^{\Phi(0)x}u(x) = \Phi'(0)\int_{\mathbb{R}} e^{\Phi(0)z}\left(\frac{\sigma^2}{2}u^2(z) - \omega(z) - R(z)\right)dz = \kappa.$$

□

References

[1] S. Asmussen and H. Hering. *Branching processes*. Progress in Probability and Statistics, 3. Birkhäuser Boston, Inc., Boston, MA, 1983.

[2] K. .B. Athreya and P. E. Ney. *Branching processes*. Die Grundlehren der mathematischen Wissenschaften, Band 196. Springer-Verlag, New York-Heidelberg, 1972.

[3] M. D. Bramson. Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.*, **31** (5) 531–581, 1978.

[4] J. Bertoin. *Lévy Processes*. Cambridge Tracts in Mathematics 121. Cambridge Univ. Press, Cambridge, 1996.

[5] J. Bertoin and B. Mallein. Biggins' martingale convergence for branching Lévy processes. *Electron. Commun. Probab.* **23**, Paper No. 83, 2018.

[6] J. Fleischman and S. Sawyer. Maximum geographic range of a mutant allele considered as a subtype of a Brownian branching random field. *Proc. Natl. Acad. Sci. USA* **76** (2) 872–875, 1979.

[7] F. Hubalek and A.E. Kyprianou. Old and new examples of scale functions for spectrally negative Lévy processes. *Seminar on Stochastic Analysis, Random Fields and Applications VI*, 119–145, Progr. Probab. **63**, Birkhäuser/Springer Basel AG, Basel, 2011.
[8] J. Korevaar. *Tauberian theory. A century of developments.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 329. Springer-Verlag, Berlin, 2004.
[9] A. Kuznetsov, A.E. Kyprianou and V. Rivero. The theory of scale functions for spectrally negative Lévy processes. *Lévy Matters II*, 97–186, Lecture Notes in Math., **2061**, Lévy Matters, Springer, Heidelberg, 2012.
[10] A.E. Kyprianou. A note on branching Lévy processes. *Stochastic Process. Appl.* **82**, no.1, 1–14, 1999.
[11] A.E. Kyprianou. *Fluctuations of Lévy processes with applications. Introductory lectures.* Second edition. Universitext. Springer, Heidelberg, 2014.
[12] S. P. Lalley and Y. Shao. On the maximal displacement of critical branching random walk. *Probab. Theory Related Fields* **162**, no. 1-2, 71–96, 2015.
[13] S. P. Lalley and Y. Shao. Maximal displacement of critical branching symmetric stable processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **52**, no. 3, 1161–1177, 2016.
[14] B. Mallein and Q. Shi. A necessary and sufficient condition for the convergence of the derivative martingale in a branching Lévy process. Arxiv : 2105.07919, 2021.
[15] C. Profeta. Extreme values of critical and subcritical branching stable processes with positive jumps. Arxiv : 2109.04769, 2021.
[16] K.-I. Sato. *Lévy processes and infinitely divisible distributions.* Revised edition of the 1999 English translation. Cambridge Studies in Advanced Mathematics, 68. Cambridge University Press, Cambridge, 2013.

Université Paris-Saclay, CNRS, Univ Evry, Laboratoire de Mathématiques et Modélisation d'Evry, 91037, Evry-Courcouronnes, France. *Email* : `christophe.profeta@univ-evry.fr`