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Long strings and quasinormal winding modes

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ABSTRACT: We compute the path integral for a particle on the covering group of $SL(2, \mathbb{R})$ using a decomposition of the Lie algebra into adjoint orbits. We thus intuitively derive the Hilbert space of the particle on the group including discrete and continuous representations. Next, we perform a Lorentzian hyperbolic orbifold of the partition function and relate it to the Euclidean BTZ partition function. We use the particle model to inform further discussion of the spectral content of the one loop vacuum amplitude for strings on BTZ black hole backgrounds. We argue that the poles in the loop integrand code contributions of long string modes that wind the black hole. We moreover identify saddle point contributions of quasinormal winding modes.

KEYWORDS: Conformal Field Models in String Theory, AdS-CFT Correspondence, Black Holes in String Theory

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1 Introduction

String theory is a unitary theory of quantum gravity. The theory has black hole solutions. It would be of great interest to show that non-perturbative fundamental string theory exhibits a unitary evolution in the presence of a black hole. At present, we are far from attaining this unification grail of theoretical physics.

A more modest goal is to study perturbative scattering amplitudes on a black hole background. A prime candidate geometry to ponder is the asymptotically AdS_3 BTZ black hole [1, 2]. It has an asymptotic structure that allows for holography on the one hand, and for a deceptively simple algebraic world sheet description on the other. Indeed, the BTZ black hole background with Neveu-Schwarz Neveu-Schwarz flux is described by an orbifold of a Wess-Zumino-Witten model on the universal cover \tilde{G} of the group $G = SL(2, \mathbb{R})$ [2, 3]. As such, we have multiple tools at our disposal for studying strings on this black hole background. We may hope to make progress then in the modest study of perturbative string propagation on a fixed black hole geometry. To start with, we can attempt a description of the complete spectrum of fundamental strings propagating on the Lorentzian BTZ geometry. Steps towards this goal were taken in [3–11]. The determination of the full spectrum is a recognized and hard open problem.

In this paper, we approach this study in two distinct manners. Firstly, we consider the particle limit of the problem directly in the Lorentzian geometry. We study the partition function of a particle in a BTZ black hole background, arising as a hyperbolic orbifold of the covering group \tilde{G} . We recall how to derive the Hilbert space of a particle on a compact group from a path integral [12, 13]. We then show how to extend this reasoning to particles on the non-compact group $SL(2, \mathbb{R})$ as well as its universal cover \tilde{G} , exploiting the coadjoint orbit quantization results from [14]. This provides us with a direct derivation and interpretation of the Hilbert space for a particle on AdS_3 as well as the partition function twisted by global symmetry group elements. In a side step, we show that the elliptic orbifold of AdS_3 in the Lorentzian neatly connects with a particle partition function on Euclidean AdS_3 . Next, we analyze the Lorentzian hyperbolic orbifold of the BTZ partition function and stress the differences with the elliptic case. The main difference lies in having either a topologically trivial or non-trivial thermal circle direction.

In a second part of the paper, we revisit the partition function for strings on the Euclidean BTZ black hole background. While the partition function is partly understood [8–11, 15], we push our understanding further still. In particular, extending the intuition developed in finite temperature AdS_3 [15], we identify pole regions in the Euclidean BTZ partition sum that correspond to the contribution of long string winding modes that surround the black hole. We also identify short winding string contributions in a saddle point approximation, and associate them to quasinormal winding modes. We moreover are able to identify a bound on their $sl(2, \mathbb{R})$ spin from this analysis. The orbifold interpretation of both short and long string contributions will be clear, and the short string contribution allows for a rewriting in terms of a standard spectral sum. The differing form of the long string contribution (which we will obtain in (4.59)) on the other hand, will require further study.

The paper is partitioned as follows. It starts with the analysis of path integrals describing a particle on a group manifold in section 2. Using our knowledge of twisted partition functions on the group, we formulate the partition functions for particles on orbifolds of groups in section 3 and discuss how a Lorentzian orbifold partition sum can be analytically continued to a Euclidean partition function. Sections 2 and 3 are a sequel to [14]. In section 4, we identify the contribution of Lorentzian long strings to the Euclidean partition function of string theory on the BTZ hyperbolic orbifold of AdS_3 . We lay bare the multi-particle nature of the contribution explicitly and provide an argument and description of which poles in the partition function can be identified with physical long string contributions. We also identify quasinormal winding mode contributions. This second part is a sequel to [10]. Both parts make progress in the task to rhyme the Euclidean partition function with the Lorentzian spectrum of perturbative string theory in the NSNS BTZ black hole background. We conclude in section 5 by summarizing our main results and pointing out a few related open problems.

2 The path integral for a particle on a group

In this section, we recall how to derive the Hilbert space for a particle on a compact group directly from path integration [12, 13]. The technique is to show the equivalence of the path integral to an integration over left and right group elements that parameterize coadjoint orbits, then to apply coadjoint orbit theory [16]. We extend this transparent viewpoint on the derivation of the particle Hilbert space to a non-compact group. The main new feature is the split of the set of Lie algebra elements into intrinsically distinct classes.

2.1 The particle on a group

A particle on a group G has a phase space which is the cotangent bundle T^*G to the group. The wave-functions are taken to be quadratically integrable functions on the group G . The Hilbert space is $\mathcal{L}^2(G)$. The latter allows for a description in terms of left and right regular actions of the group on itself and as such decomposes in a direct sum of left and right irreducible representations. Firstly, we wish to make this canonical mathematical statement transparent from a path integral point of view.

To that end, we briefly review the approach of [13], which in turn is based on the path integral description of coadjoint orbits provided in [12]. The cotangent bundle T^*G of a group is a symplectic manifold. We define fields $t : L \rightarrow \mathfrak{g}^*$ which are maps from the world line L of the particle into the dual of the Lie algebra \mathfrak{g}^* as well as the group valued field $g : L \rightarrow G$. The canonical symplectic form Ω on the phase space T^*G is [13]:

$$\Omega = \frac{1}{2} \text{Tr}(dt dg g^{-1} + t dg g^{-1} dg g^{-1}) = d \left(\frac{1}{2} \text{Tr} t dg g^{-1} \right). \tag{2.1}$$

We can view the dual Lie algebra variable t as corresponding to momenta and the group variable g as coordinates. We quantize trajectories in the cotangent bundle with an action which is a primitive of the symplectic form Ω :

$$S = \frac{1}{2} \int_L * \text{Tr}(t dg g^{-1}), \tag{2.2}$$

where $*$ indicates the pullback to the world line L . The path integral is:¹

$$Z = \int [dt(\tau)][dg(\tau)] e^{iS[t,g]}. \tag{2.3}$$

The model has both a right and left global group invariance. It permits a left and right group $G_l \times G_r$ action and there are conserved charges corresponding to these group actions. Below, our first aim is to compute a partition function of the type:

$$Z(\gamma_j^{l,r}) = \int [dt(\tau)][dg(\tau)]_{\text{periodic}} e^{iS[t,g] + i \sum_j (\gamma_j^r H_r^j + \gamma_j^l H_l^j)}, \tag{2.4}$$

where we twist the partition function with the insertion of commuting left and right Hamiltonians $H_{l,r}^j$ that correspond to Cartan generators in the Lie algebras $\mathfrak{g}_l \oplus \mathfrak{g}_r$ of the symmetry group $G_l \times G_r$. The integration is over periodic maps such that the result has an interpretation as a trace. Following [13], we can reparameterize the dual Lie algebra element as $t = g_l t_d g_l^{-1}$ in terms of a Cartan element t_d and an element in the group g_l . We find the action:

$$S = \frac{1}{2} \int \text{Tr}(t_d dg_r g_r^{-1} + t_d g_l^{-1} dg_l), \tag{2.5}$$

where $g_r = g_l^{-1} g$. If we multiply g_l by a Cartan element, t_d is invariant, and we therefore have a redundant parameterization and a gauge symmetry [13]. The global symmetries act on g_r on the right and on g_l on the left. The local symmetry group conjugating t_d contains the Weyl subgroup. The idea in [13] is to read the resulting path integral as a product of path integrals over co-adjoint orbits, both of a type set by the Cartan Lie algebra element t_d . Co-adjoint orbit quantization [12, 16] then gives rise to the Hilbert space:

$$\mathcal{H} = \sum_i \mathcal{H}_i^l \otimes \mathcal{H}_i^r, \tag{2.6}$$

where we have a left-right correlation of the sum over representations i due to the fact that the Cartan element t_d appears in the action for the group valued field g_r as well as the field g_l . The path integrals are computed explicitly in Darboux variables for the classical compact simple groups in [12] using a classical configuration space counterpart to the Gelfand-Tsetlin basis for classical Lie algebras. Since there are differences in the treatment of the path integral compared to the ordinary quantization of co-adjoint orbits, and since the details in our treatment differ from those of [12, 13], we add a few pointers as to how this works for the compact group $G = \text{SU}(2)$ before turning to a generalization for non-compact groups. We refer the reader to e.g. [12, 13, 16] for the necessary background and the detailed derivation of the final formula.

¹We have the possibility of adding a quadratic Casimir Hamiltonian $H = \frac{1}{R^2} \text{Tr} t^2$. If we integrate over the variable t then, we obtain a particle model on the group with a second order kinetic term. This model has a Hamiltonian equal to the Laplacian on the group G .

2.2 The example of the group SU(2)

We concretely parameterize the path integral for the case of the simple compact group $G = \text{SU}(2)$. We parameterize the group element $g_l(\tau)$ as:

$$g_l = \begin{pmatrix} \cos \frac{\theta}{2} e^{i \frac{\phi+\psi}{2}} & i \sin \frac{\theta}{2} e^{i \frac{\phi-\psi}{2}} \\ i \sin \frac{\theta}{2} e^{i \frac{\psi-\phi}{2}} & \cos \frac{\theta}{2} e^{-i \frac{\phi+\psi}{2}} \end{pmatrix}. \quad (2.7)$$

The dual Lie algebra element t we diagonalize such that it becomes proportional to the σ_3 Pauli matrix:

$$t = g_l t_3 \sigma_3 g_l^{-1}. \quad (2.8)$$

In terms of these variables the measure on the (dual) Lie algebra becomes:

$$dt = t_3 dt_3 \wedge t_3 \sin \theta d\phi \wedge d\theta. \quad (2.9)$$

Thus, our path integral reads concretely:

$$Z(\dots) = \int dt_3 \int (t_3 \sin \theta d\phi \wedge d\theta) \int (t_3 dg_r) e^{\frac{i}{2} \int d\tau \text{Tr}(t_3 \sigma_3 g_l^{-1} \partial_\tau g_l) + \frac{i}{2} \int d\tau \text{Tr}(t_3 \sigma_3 \partial_\tau g_r g_r^{-1})} (\dots)$$

where the dots indicate possible operator insertions. We have split the measure into factors that are naturally associated to the left group element g_l and the right group element g_r . The integration over g_l and g_r are standard in co-adjoint orbit theory [12, 13, 16]. There are two important extra elements in the path integration though; firstly, the measure dg_r contains an integration over the variable ψ , as it is defined to be

$$g_r = \begin{pmatrix} \cos \frac{\theta_r}{2} e^{i \frac{\psi+\phi_r}{2}} & i \sin \frac{\theta_r}{2} e^{i \frac{\psi-\phi_r}{2}} \\ i \sin \frac{\theta_r}{2} e^{i \frac{\phi_r-\psi}{2}} & \cos \frac{\theta_r}{2} e^{-i \frac{\psi+\phi_r}{2}} \end{pmatrix}. \quad (2.10)$$

Note that ψ also appears in g_l (this happens via the field redefinition $g_r = g_l^{-1} g$). Second and more important is the time dependence of the (dual) Lie algebra element t_3 . To exhibit the consequences, we compute the action for the left and right group elements:²

$$S = \frac{1}{2} \text{Tr}(t_3 \sigma_3 g_l^{-1} \partial_\tau g_l) + \frac{1}{2} \text{Tr}(t_3 \sigma_3 \partial_\tau g_r g_r^{-1}) = t_3 \dot{\psi} + \frac{t_3}{2} (\dot{\phi} \cos \theta + \dot{\phi}_r \cos \theta_r). \quad (2.11)$$

The extra feature (compared to e.g. [12]) is due to the term proportional to ψ which in the presence of a time-dependent t_3 is significant.

We first perform the path integration on ψ . Due to the fact that ψ is periodic and because we take a periodic world line, we have a sum over winding configurations. (See e.g. [12] for similar details.) That sum discretizes the coefficient t_3 to take integer values. The path integral over ψ inserts a delta-function:

$$\text{PI}(\psi) = \sum_{n \in \mathbb{N}_0} \delta_{t_3, n}. \quad (2.12)$$

²In this calculation, for 2×2 Pauli matrix generators, the definition of the trace contains a factor of $-i$.

We plug this into the full path integral to find:

$$Z(\dots) = \sum_{n \in \mathbb{N}_0} \int \left(\frac{n}{2} \sin \theta d\phi \wedge d\theta \right) e^{i \frac{n}{2} \int d\tau \dot{\phi} \cos \theta} \int \left(\frac{n}{2} \sin \theta_r d\phi_r \wedge d\theta_r \right) e^{i \frac{n}{2} \int d\tau \dot{\phi}_r \cos \theta_r} (\dots).$$

We dropped the $n = 0$ term since it is zero and we used the \mathbb{Z}_2 Weyl symmetry to render n positive. After performing the co-adjoint orbit quantization with Hamiltonian insertions (2.4), we find the result of the path integral [12, 13]:

$$Z(h_l, h_r) = \sum_{j \in \frac{1}{2}\mathbb{N}} \chi_j(h_l) \chi_j(h_r), \tag{2.13}$$

where $n = 2j + 1$. Thus, we briefly reviewed and illustrated the general procedure in [13]. We wish to argue that a similar logic holds for a large class of non-compact groups.

2.3 A particle on a non-compact group

In this subsection, we again follow the logic outlined in [13], but apply it to the new domain of non-compact groups. Important differences arise. In this paper, we concentrate on the simple example of $G = \text{SL}(2, \mathbb{R})$ that illustrates some of these differences. We will see that the integration over Lie algebra orbits splits into non-trivially distinct cases — a difference that follows from the fact that the Killing metric on the Lie algebra has an indefinite signature.

2.3.1 The orbit integrals

Before we delve into the details, we describe our expectations. We again calculate the path integral in steps. We conjugate the (dual) Lie algebra element t which takes values in a Minkowski space $\mathfrak{sl}(2, \mathbb{R}) = \mathbb{R}^{2,1}$ into a Cartan subalgebra. We concentrate on the generic cases in which the Lie algebra element t is future time-like, past time-like or space-like. These cases will also be referred to as discrete plus, discrete minus and continuous. (See e.g. [14] for details.) The integral over the Lie algebra splits into a sum of three integrals accordingly. The integration over the left and right group elements will again build coupled left and right representation spaces. We know that the future and past time-like Lie algebra elements give rise to discrete plus and discrete minus representations under co-adjoint orbit quantization, and the space-like Lie algebra elements to continuous representations [17]. (See [14] for details.) We therefore expect a path integral result for periodic maps and with left and right Hamiltonian insertions:

$$Z(h_l, h_r) = \sum_{\substack{\alpha = \pm \\ j = 1, \frac{3}{2}, \dots}} \chi_{j,\alpha}(h_l) \chi_{j,\alpha}(h_r) + \sum_{\epsilon = 0, \frac{1}{2}} \int_c ds \rho(s) \chi_{s,\epsilon}(h_l) \chi_{s,\epsilon}(h_r), \tag{2.14}$$

where $\epsilon = 0, 1/2$ corresponds to representations that represent minus the identity in $\text{SL}(2, \mathbb{R})$ trivially or non-trivially. We aim to understand in a little more detail the summation over discrete representations and the integral over continuous representations from the path integral. To make our discussion efficient, we stress only the differences with the case of the group $G = \text{SU}(2)$, and for the co-adjoint orbit quantization we lean heavily on the results derived in [14].

2.3.2 The differing details

The most convenient parameterization of the group element g_l depends on whether the (dual) Lie algebra element t_d is time-like (elliptic) or space-like (hyperbolic). When the element is time-like, we choose the parameterization:

$$g_l = \begin{pmatrix} \cos t \cosh \rho + \cos \phi \sinh \rho & \sin t \cosh \rho - \sin \phi \sinh \rho \\ -\sin t \cosh \rho - \sin \phi \sinh \rho & \cos t \cosh \rho - \cos \phi \sinh \rho \end{pmatrix}. \quad (2.15)$$

We put

$$t_d = t_{\text{ell}} = t_0 \sigma_0 \quad (2.16)$$

where $\sigma_0 = i\sigma_2$ is proportional to a Pauli matrix. In this case we find the Lie algebra measure

$$dt = 2t_0 dt_0 \wedge t_0 \sinh 2\rho d(\phi + t) \wedge d\rho. \quad (2.17)$$

The treatment of the path integral over these types of Lie algebra elements goes much as in the $SU(2)$ case. The Lie algebra element t_0 becomes discretized upon path integration over the coordinate $\phi - t$, which appears in the dg_r measure. The integration over the other two coordinates $(\phi + t, \rho)$ corresponds to a co-adjoint orbit quantization for discrete plus and discrete minus orbits. The details can be found in [14] and references therein. This solves the discrete part of the path integral problem.

We still need to perform the part of the path integral where the Lie algebra element t diagonalizes to a hyperbolic element:

$$t_d = t_{\text{hyp}} = t_3 \sigma_3. \quad (2.18)$$

Because the Lie algebra element is hyperbolic, it is convenient to use a group element parameterization that includes a hyperbolic factor. On the other hand, as discussed in detail in [14], it is most direct in the co-adjoint orbit quantization procedure to parameterize the orbit in terms of an elliptic group factor. Thus, for instance, the group decomposition $SL(2, \mathbb{R}) = G = KNA$ where K is elliptic, N is parabolic and A is hyperbolic — an Iwasawa decomposition — is convenient in the detailed calculations. Thus, for this case, we parameterize the group element g_l as:

$$g_l = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^\rho & 0 \\ 0 & e^{-\rho} \end{pmatrix}. \quad (2.19)$$

For this parameterization, we find the action:

$$S_l = \frac{1}{2} \text{Tr}(t_3 \sigma_3 g_l^{-1} \partial_\tau g_l) = t_3 \partial_\tau \rho - t_3 x \dot{\theta}. \quad (2.20)$$

Because ρ is non-compact, the path integration over the field ρ does not lead to a discretization of t_3 . We integrate t_3 over a constant real line.³ We conclude that in the case of the group $G = SL(2, \mathbb{R})$, the quantization of the orbit label t_d only occurs for the discrete representations.

³There is a Weyl gauge equivalence that renders t_3 and $-t_3$ equivalent.

2.3.3 Result

Thus, we have understood the reason why the path integral splits, as well as the manner in which the discrete representations are summed over and the continuous representations are integrated over. This was our main objective. The rest of the derivation proceeds exactly along the lines in [12, 13], but now using the results for the co-adjoint orbit quantization in [14]. We record only the final result:

$$Z(h_l, h_r) = \sum_{j=1, \frac{3}{2}, \dots} (\chi_j^+(h_l)\chi_j^+(h_r) + \chi_j^-(h_l)\chi_j^-(h_r)) + \int \frac{ds}{2\pi} \sum_{\epsilon=0, \frac{1}{2}} \chi_{s,\epsilon}^c(h_l)\chi_{s,\epsilon}^c(h_r). \quad (2.21)$$

This is in accord with the standard Hilbert space decomposition of the space $\mathcal{L}^2(G)$ of quadratically integrable functions on the group.

For a more general non-compact group G , we expect that the final sum and integration are over the representations in the reduced dual \tilde{G}_r to the non-compact group G , namely the set of representations occurring in the decomposition of the space $\mathcal{L}^2(G)$ of quadratically integrable functions [17]. Extending our reasoning to covers of the group $SL(2, \mathbb{R})$ is possible along the lines of [14]. We will use the result in the following. It would certainly be interesting to determine more precisely the class of non-compact groups to which this path integral treatment of particle models can be generalized — our focus in this paper lies elsewhere.

3 A particle on the BTZ orbifold

We obtained the partition function of a particle on the non-compact group manifold \tilde{G} twisted by global symmetry group elements. In this section, we exploit the twisted partition function to obtain the partition function for a particle on the \mathbb{Z} orbifold of $\tilde{G} = AdS_3$ which is the geometry of the BTZ black hole. The generator of the BTZ \mathbb{Z} orbifold is hyperbolic [2]. We compare the BTZ orbifold to the case of a \mathbb{Z} orbifold of AdS_3 which is elliptic, namely the compactification of the time direction. The former is a Lorentzian counterpart to the Euclidean black hole while the latter is the counterpart to finite temperature Euclidean AdS_3 . We take the occasion to discuss to what extent Lorentzian partition sums can be related by direct analytic continuation to their Euclidean cousins.

3.1 The space-time interpretation

Our particle model will serve in part as a toy model for string theory. To that end it is useful to think of the particle theory with a quadratic Casimir Hamiltonian as a first quantized model arising from the introduction of an einbein on the world line [18]. This conceptual modification also allows us to compare the first quantized results to second quantized field theory results in AdS_3 or the black hole background. This modification entails two technical changes to the set-up. We introduce a mass m for our particle excitation as well as the constraint on the quadratic Casimir:

$$-j(j-1) + \frac{l^2 m^2}{4} = 0, \quad (3.1)$$

arising from the equation of motion of the einbein. Here, the curvature scale of the AdS_3 space-time is denoted l . Secondly, even after gauge fixing the einbein, we need to integrate over a modulus with the correctly normalized measure [18]. Note that for positive mass squared, there are no continuous representation solutions $j \in 1/2 + i\mathbb{R}_0$ to the on-shell constraint (3.1).⁴ The discrete plus D_j^+ modes are standard particle excitations in $AdS_3 = \tilde{G}$ and the discrete minus modes are their negative energy counterparts. The same on shell equation (3.1) holds for the BTZ orbifold, and the same conclusions, regarding the Casimir of on-shell modes, continue to hold.

3.2 The Euclidean partition function

In appendix A we take the twisted Lorentzian partition function for a particle on AdS_3 as a starting point and perform an elliptic \mathbb{Z} orbifold in order to compactify the time direction. We then compare the resulting Lorentzian partition function (with compactified time direction) to the one for a particle in thermal AdS_3 which is the Euclidean continuation of the geometry. The partition sums agree. This is no small feat since the on-shell AdS_3 modes are in the discrete representations while the unitary representations of the isometry group of the Euclidean model are necessarily continuous. Our goal in the rest of the section is to perform an analogous comparison exercise for the hyperbolic BTZ orbifold. The hyperbolic orbifold is a qualitatively different Lorentzian continuation of an equivalent Euclidean geometry.

To relate the Lorentzian hyperbolic orbifold to the Euclidean BTZ partition function for a point particle, we start by reviewing the latter. One way to obtain the partition function is as a relabelling of the thermal AdS_3 partition function. The one-loop result is [19]:

$$Z_{\text{BTZ}}^{\text{particle}} = \sum_{n=1}^{\infty} 2\sqrt{\pi k_b r_+} \int_0^{\infty} \frac{dt}{t^{3/2}} \frac{e^{-\frac{k_b n^2}{t}(2\pi r_+)^2} e^{-\frac{t}{4k_b}(m^2+1)}}{(4|\sin n\pi(r_- - ir_+)|^2)} \tag{3.2}$$

where we introduced the alternative notation $k_b = l^2/\alpha'$ for the cosmological constant scale which naturally arises from the string model to be discussed in the next section. We can expand the denominator using that $r_+ > 0$:

$$Z_{\text{BTZ}}^{\text{particle}} = \sum_{n=1}^{\infty} 2\sqrt{\pi k_b r_+} \int_0^{\infty} \frac{dt}{t^{3/2}} e^{-\frac{k_b n^2}{t}(2\pi r_+)^2} e^{-\frac{t}{4k_b}(m^2+1)} \sum_{r, \bar{r}=0}^{\infty} e^{-2\pi i n(r_- - ir_+)(r + \frac{1}{2})} e^{2\pi i n(r_- + ir_+)(\bar{r} + \frac{1}{2})}.$$

We can moreover borrow a technique from the analysis of thermal string spectrum in AdS_3 [15] to exhibit an integral over a continuous momentum:

$$\begin{aligned} Z_{\text{BTZ}}^{\text{particle}} &= \sum_{n=1}^{\infty} \int_0^{\infty} \frac{dt}{k_b \pi i n} \int_{-\infty}^{\infty} ds s e^{4\pi i r_+ n s - \frac{s^2 t}{k_b} - \frac{t}{4k_b}(m^2+1)} \sum_{r, \bar{r}=0}^{\infty} e^{-2\pi i n(r_- - ir_+)(r + \frac{1}{2})} e^{2\pi i n(r_- + ir_+)(\bar{r} + \frac{1}{2})} \\ &= \frac{1}{\pi i k_b} \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{+\infty} ds s \frac{k_b}{s^2 + \frac{m^2+1}{4}} e^{4\pi i r_+ n s} \sum_{r, \bar{r}=0}^{\infty} e^{-2\pi i n(r_- - ir_+)(r + \frac{1}{2})} e^{2\pi i n(r_- + ir_+)(\bar{r} + \frac{1}{2})}. \end{aligned}$$

⁴More precisely, only discrete modes will satisfy the Breitenlohner-Freedman bound.

If we close the contour in the upper half plane and pick up the pole at $j = \frac{1}{2} + is = \frac{1}{2} + \frac{1}{2}\sqrt{m^2 + 1}$ we find:

$$\begin{aligned}
 Z_{\text{BTZ}}^{\text{particle}} &= \sum_{n=1}^{\infty} \frac{1}{n} e^{-2\pi r_+ n \sqrt{m^2 + 1}} \sum_{r, \bar{r}=0}^{\infty} e^{-2\pi i n (r_- - ir_+) (r + \frac{1}{2})} e^{2\pi i n (r_- + ir_+) (\bar{r} + \frac{1}{2})} \\
 &= - \sum_{r, \bar{r}=0}^{\infty} \log \left(1 - e^{-2\pi r_+ (2j-1)} e^{-2\pi i (r_- - ir_+) (r + \frac{1}{2})} e^{2\pi i (r_- + ir_+) (\bar{r} + \frac{1}{2})} \right).
 \end{aligned}
 \tag{3.3}$$

It is well-known that this partition function is a \mathbb{Z} orbifold of the Euclidean H_3 model [19]. Moreover, the Euclidean one loop determinant can be rewritten as the partition function of a second quantized scalar field:

$$e^{Z_{\text{BTZ}}^{\text{particle}}} = \prod_{r, \bar{r}=0, 1, \dots} \frac{1}{(1 - e^{-2\pi i (r_- - ir_+) (j+r)} e^{2\pi i (r_- + ir_+) (j+\bar{r})})}.
 \tag{3.4}$$

Consider the full exponent of a given single particle excitation. It equals:

$$\text{single particle exponent} = -2\pi(r_+ + ir_-)(j + r) - 2\pi(r_+ - ir_-)(j + \bar{r}) = -2\pi i L,
 \tag{3.5}$$

where $j + r$ and $j + \bar{r}$ are eigenvalues of left/right-moving (elliptic in the Euclidean, hyperbolic in the Lorentzian) group generators. The total exponent is nothing but $-2\pi i$ times the particle angular momentum L . See e.g. [10] where this exponent was discussed in detail for the case of a fundamental string in the black hole background. In the one-loop particle amplitude the exponent n of the projection operator (in equation (3.2)) also plays the role of a multi-particle excitation number (e.g. in equation (3.3)) which allows us to exponentiate the first quantized formula into a neat second quantized quantum field theory result (3.4).

Remarkably, the same formula (3.4), after transformation, can be interpreted as the spectrum of single particle quasinormal modes on the black hole background with in-going boundary conditions at the horizon and a no-energy-loss requirement at infinity [25]. We have the equality [25, 27]:

$$Z_{\text{BTZ}}^{\text{particle}} = - \sum_{n>0, p \geq 0} \left(\log(1 - q^{n+p+j} \bar{q}^{p+j}) + \log(1 - q^{p+j} \bar{q}^{n+p+j}) \right) - \sum_{p \geq 0} \log(1 - (q\bar{q})^{p+j})
 \tag{3.6}$$

where $q = e^{-2\pi i (r_- - ir_+)}$. The exponents can be identified as the single particle quasinormal mode spectrum [29] when j is tuned such that the quasinormal mode energy coincides with a Matsubara frequency $\omega_{QN}(j) = \omega_{\text{Matsubara}}$ [25]. In other words, the poles of the partition function code the quasinormal mode frequencies and vice versa. The resulting spectrum coincides with the one obtained by direct calculation [29]:

$$E = \pm L - i(r_+ \mp r_-)(2j + 2p).
 \tag{3.7}$$

We have thus reviewed that there is both a Euclidean orbifold reading of the partition function in terms of a projection operator *and* an interpretation in terms of a quasinormal spectrum.

3.3 The BTZ Lorentzian hyperbolic orbifold

We wish to circle back now and interpret the Euclidean partition function result in terms of the Lorentzian representation theory we introduced previously. To that end, we note the intermediate result — see equation (3.3) —:

$$Z_{\text{BTZ}}^{\text{particle}} = \sum_{n=1}^{\infty} \frac{1}{n} e^{-2\pi r_+ n(2j-1)} \sum_{p, \bar{p}=0}^{\infty} e^{-2\pi i n(r_- - ir_+)(p+\frac{1}{2})} e^{2\pi i n(r_- + ir_+)(\bar{p}+\frac{1}{2})}. \quad (3.8)$$

Recall that the discrete character of a hyperbolic group element is [14]:

$$\chi_j^+(e^{t t_{\text{hyp}}}) = \frac{e^{-|2j-1||t|}}{|\sinh t|}. \quad (3.9)$$

We can also rewrite the result as

$$Z_{\text{BTZ}}^{\text{particle}} = \sum_{n>0} \frac{1}{n} \chi_j^+(e^{\pi n(r_+ + ir_-) t_{\text{hyp}}}) \chi_j^+(e^{\pi n(r_+ - ir_-) t_{\text{hyp}}}). \quad (3.10)$$

We sketch how this result arises from the direct calculation of the Lorentzian BTZ orbifold partition function.⁵ We obtain the partition function for a particle on the BTZ orbifold from the twisted partition function on the universal cover \tilde{G} . The twisted partition function on the cover is:

$$Z(h_l, h_r) = \int_{1/2}^{\infty} dj (\chi_j^+(h_l) \chi_j^+(h_r) + \chi_j^-(h_l) \chi_j^-(h_r)) + \int_0^{\infty} ds \int_0^1 d\epsilon \chi_{s,\epsilon}^c(h_l) \chi_{s,\epsilon}^c(h_r). \quad (3.11)$$

We wish to insert an orbifold projection operator in the trace which projects onto states that are invariant under the hyperbolic \mathbb{Z} orbifold. To that end, we sum over the insertion of a group generator $(h_l, h_r) = (e^{\pi(r_+ + ir_-) t_{\text{hyp}}}, e^{\pi(r_+ - ir_-) t_{\text{hyp}}})$ raised to an arbitrary power $n \in \mathbb{Z}$:

$$\begin{aligned} Z_{\text{BTZ}}^{\text{particle}} &= \sum_{n \in \mathbb{Z}} \int_{1/2}^{\infty} dj \left(\chi_j^+(e^{\pi n(r_+ + ir_-) t_{\text{hyp}}}) \chi_j^+(e^{\pi n(r_+ - ir_-) t_{\text{hyp}}}) \right. \\ &\quad \left. + \chi_j^-(e^{\pi n(r_+ + ir_-) t_{\text{hyp}}}) \chi_j^-(e^{\pi n(r_+ - ir_-) t_{\text{hyp}}}) \right) \\ &\quad + \int_0^{\infty} ds \int_0^1 d\epsilon \chi_{s,\epsilon}^c(e^{\pi n(r_+ + ir_-) t_{\text{hyp}}}) \chi_{s,\epsilon}^c(e^{\pi n(r_+ - ir_-) t_{\text{hyp}}}). \end{aligned} \quad (3.12)$$

The physical state condition (3.1) projects this partition function onto the spin j associated to the mass m of the state under consideration. In the calculation of the Euclidean partition function, it was the projection of the contour integral onto the pole that played the role of the on-shell projection. This is implemented by the Schwinger and momentum integral. The prefactor $1/n$ arises from these integrals in the same manner as in the Euclidean partition function calculation. By comparing the Lorentzian orbifold partition function (3.12) with the Euclidean partition function calculation, we again recognize that the multi-particle number n doubles as the order of the orbifold projection operator in the partition function (3.12). It is important in making this match that the discrete character is a function of the smallest hyperbolic eigenvalue, thus rendering the character even in n .⁶

⁵In doing so, we will assume that ir_- is real which is true after analytic continuation to the Lorentzian.

⁶The background independent term $n = 0$ can again be ignored.

3.3.1 Summary

We conclude that the Euclidean particle partition function on the BTZ black hole affords a reading as an on-shell hyperbolic Lorentzian orbifold with respect to the angular momentum operator L . Simultaneously, it can be read as the result of tracing over the black hole quasinormal modes. In the next section, we generalize this treatment to the case of the Euclidean fundamental string partition function.

4 The string spectrum on a black hole

In this section, we revisit the fundamental string spectrum on the BTZ black hole background with NSNS flux [3–11]. We improve on the analysis of the world sheet partition function [7–11] and acquire a tentative but conjecturally rather complete picture of its spectral content. All along the analysis it is important to keep the lessons in mind that we take away from the particle partition functions.

The interpretation of the string partition function poses considerable extra hurdles. The boundary circles of the Euclidean geometry allow for winding modes and we need to disentangle the associated novelties. Because the thermal circle in the AdS_3 background is topologically non-trivial while in the black hole background it is topologically trivial, the analytic continuation to the Lorentzian proceeds quite differently in regard to the winding modes. Thus, while our analysis certainly draws upon the knowledge acquired in the context of thermal AdS_3 [15], it follows a very different logic and this is reflected in the course of the calculation.

4.1 The BTZ partition function

Starting from the one loop string amplitude for thermal AdS_3 [15], one obtains the corresponding result for the Euclidean BTZ black hole:

$$Z = \int_F \frac{d^2\tau}{2\tau_2} Z_{\text{BTZ}} Z_{gh} Z_{\text{int}} \quad (4.1)$$

$$Z_{\text{BTZ}} = \frac{2r_+ \sqrt{k_b - 2}}{\sqrt{\tau_2}} \sum_{m,w} \frac{e^{-\pi \frac{k_b}{\tau_2} r_+^2 |m - w\tau|^2 + \frac{2\pi}{\tau_2} \text{Im}(\bar{U}_{m,w})^2}}{|\theta_1(\bar{U}_{m,w}, \tau)|^2}. \quad (4.2)$$

We have defined ghost non-zero modes and internal partition function factors $Z_{gh} = |\eta(\tau)|^4$ and Z_{int} . The internal conformal field theory is a compact conformal field theory with central charge $c_{\text{int}} = 26 - \left(3 + \frac{6}{k_b - 2}\right)$ where k_b is the level of the \tilde{G} Wess-Zumino-Witten model. We have also denoted the holonomy on which the path integral depends as

$$\bar{U}_{m,w} = (r_- - ir_+)(m - w\tau). \quad (4.3)$$

For the notation and further details we refer to [10]. In brief, the formula is obtained by mapping the inverse temperature β and the fugacity $\beta\mu$ for the angular momentum on the thermal AdS_3 side [15] to the outer and inner horizon radius of the black hole [20, 21]:

$$\left(\frac{\beta}{2\pi}, \frac{\beta\mu}{2\pi}\right) \leftrightarrow (r_+, r_-). \quad (4.4)$$

The path integral in the Euclidean geometries are identical because the geometries are related by an S-duality transformation on the boundary torus [21].

One open question about this partition function is whether one can rewrite it as a sum of logarithms which is necessary to find an interpretation of the spectrum in terms of single particle states. In this section we give a detailed answer to this question in two parts: first of all we exploit the methods of [15] and perform a saddle point approximation to the partition function. It is the discrete representations of $\mathfrak{sl}(2, \mathbb{R})$ that contribute to the saddle points and we find a stringy generalization of the particle calculation performed in the previous section, leading to an interpretation of this part of the spectrum in terms of quasinormal winding modes. We also provide a space-time description of these modes. In a second part, we analyze the partition function near the poles of the integrand and interpret the contribution near the poles as arising from multiply wound long strings. This contribution lies in the irreducible continuous representations of $\mathfrak{sl}(2, \mathbb{R})$ and it has a distinct dependence on the boundary modular parameter.

4.1.1 Affine character reading

As a preface to expanding and contracting the formula (4.2) for the partition function, we attempt to read it at face value. We identify the one loop vacuum amplitude as a sum over images of a product of hyperbolic affine characters. To that end we conjecture the existence of hyperbolic affine characters — they are generalizations of $\mathfrak{sl}(2, \mathbb{R})$ hyperbolic characters —

$$\hat{\chi}(\tau, t) = \frac{\cos 2ts}{|\sinh t|} \frac{1}{\prod_n (1 - e^{2tq^n})(1 - e^{-2tq^n})(1 - q^n)}. \tag{4.5}$$

In appendix B we provide first arguments for why these proposed characters are sensible, based on the decomposition of the tensor product of infinite and finite dimensional representations of $\mathfrak{sl}(2, \mathbb{R})$. Given this character, we identify its denominator as proportional to the θ_1 factor in the partition function where we make the identification $i\pi\bar{U}_{m,w} = t$. As in the particle case, we surmise that we are summing over orbifold images labelled by m . Moreover, as always in conformal field theory orbifolds, we have introduced twisted sectors labelled by the winding number w . There are however important issues that remain to be understood: first of all, we need to identify the correct spectrum of momenta s that is integrated over in the partition function and whether there are discrete representation characters that contribute in the Lorentzian theory; secondly we need to determine which part of the world sheet spectrum contributes on-shell in space-time. We answer these questions in the next two subsections.

4.2 The saddle points and the quasinormal winding modes

In this subsection, we mould the partition function into a form in which we can perform a telling saddle point analysis. It will allow us to identify the contribution of quasinormal winding modes to the one-loop amplitude.

We first introduce the radial momentum s -integral as in [10]:

$$\int_{-\infty}^{+\infty} ds e^{-4\pi\tau_2 \frac{s^2}{k_b-2}} e^{-4\pi i s \text{Im}(\bar{U}_{m,w})} = \frac{1}{2} \frac{\sqrt{k_b-2}}{\sqrt{\tau_2}} e^{-\frac{\pi(k_b-2)}{\tau_2} \text{Im}(\bar{U}_{m,w})^2}. \tag{4.6}$$

This allows one to rewrite the BTZ factor of the one loop integrand as:

$$Z_{\text{BTZ}} = 4r_+ \sum_{m,w} \int_{-\infty}^{+\infty} ds e^{-4\pi\tau_2 \frac{s^2}{k_b-2}} e^{-4\pi i s \text{Im}(\bar{U}_{m,w})} \frac{e^{-\pi \frac{k_b}{\tau_2} r_+^2 |m-w\tau|^2 + \frac{k_b\pi}{\tau_2} \text{Im}(\bar{U}_{m,w})^2}}{|\theta_1(\bar{U}_{m,w}, \tau)|^2}. \quad (4.7)$$

We follow the route taken in [22] that makes it possible to expand the θ -function in the denominator. For this purpose we introduce the delta-function:

$$1 = \tau_2 \int_0^1 d^2 s \sum_{v,w'} \delta^2(\bar{U}_{m,w} - (s_1 + w')\tau + s_2 + v). \quad (4.8)$$

We now implement a trick, inspired by (but not identical to) the unfolding of thermal partition functions in string theory [24], which uses the modular invariance of all factors in the integrand of the integral over the fundamental domain. Given the invariance, each factor is identical in each copy of the fundamental domain. Moreover, under modular transformations, the pair of integers (v, w') transforms as a doublet. For this particular delta-function factor, instead of integrating a modular invariant expression over a single fundamental domain, we integrate the vectors $(v, 0)$ over all copies of the fundamental domain in the strip. This restores the full doublet when transformed back to the fundamental domain. These integrations are therefore identical and because all other factors in the integrand are modular invariant, they go along for the ride. Thus, we set $w' = 0$ from now on, departing strongly from the analysis of thermal partition functions in AdS_3 , and unfold the Schwinger τ_2 -integral, such that we integrate over the strip in the τ -plane, with $|\tau_1| < \frac{1}{2}$ and $0 < \tau_2 < \infty$. We follow this up by introducing the integral representation for the δ function [22]:

$$\delta^2(\bar{U}_{m,w} - s_1\tau + s_2 + v) = \int d^2 \lambda_i e^{2\pi i \lambda_1 (r_-(m-w\tau_1) - r_+ w \tau_2 - s_1 \tau_1 + s_2 + v)} e^{2\pi i \lambda_2 (-r_+(m-w\tau_1) - r_- w \tau_2 - s_1 \tau_2)}. \quad (4.9)$$

By making use of the elliptic properties of the θ -function and after simplification we find that

$$Z_{\text{BTZ}} = 4r_+ \tau_2 \int_0^1 d^2 s \sum_{m,w,v} \int d^2 \lambda_i \int_{-\infty}^{+\infty} ds \frac{e^{-4\pi\tau_2 \frac{s^2}{k_b-2}} e^{-4\pi i s \text{Im}(\bar{U}_{m,w})}}{|\theta_1(s_1\tau - s_2, \tau)|^2} e^{2\pi i \lambda_2 (-r_+(m-w\tau_1) - r_- w \tau_2 - s_1 \tau_2)} \\ \times e^{2\pi k_b m w r_+ r_- + \pi k_b w^2 (-2r_+ r_- \tau_1 + r_-^2 \tau_2 - r_+^2 \tau_2)} e^{2\pi i \lambda_1 (r_-(m-w\tau_1) - r_+ w \tau_2 - s_1 \tau_1 + s_2 + v)}. \quad (4.10)$$

Given the range of the holonomy s_1 one can expand the θ -functions:

$$\frac{1}{|\theta_1(\nu, \tau)|^2} = \frac{\sqrt{z\bar{z}}}{|\eta(\tau)|^6} \sum_{r,\bar{r} \in \mathbb{Z}} z^r \bar{z}^{\bar{r}} S_r S_{\bar{r}}, \quad (4.11)$$

where $z = e^{2\pi i \nu}$ is the fugacity and the series S_r is given by $S_r = \sum_{n=0}^{\infty} (-1)^n q^{\frac{n}{2}(n+2r+1)}$. We list further steps performed in detail in [22]: i) we perform the sum over v , which leads to a Dirac comb for λ_1 . ii) The integral over s_2 leads to the constraint $\lambda_1 = r - \bar{r} \in \mathbb{Z}$. iii)

These two combine and the result is a trivial integral over λ_1 . Taking these three steps we obtain

$$Z_{\text{BTZ}} = 4r_+ \tau_2 \int_0^1 ds_1 \sum_{m,w} \int d\lambda_2 \int_{-\infty}^{+\infty} ds e^{-4\pi\tau_2 \frac{s^2}{k_b-2}} e^{-4\pi i s \text{Im}(\bar{U}_{m,w})} e^{2\pi k_b m w r_+ r_- + \pi k_b w^2 (-2r_+ r_- \tau_1 + r_-^2 \tau_2 - r_+^2 \tau_2)} \\ \times \sum_{r,\bar{r}} e^{2\pi i (r-\bar{r})(r_-(m-w\tau_1)-r_+ w \tau_2)} e^{2\pi i \lambda_2 (-r_+(m-w\tau_1)-r_- w \tau_2 - s_1 \tau_2)} \frac{1}{|\eta|^6} e^{-2\pi s_1 \tau_2 (r+\bar{r}+1)} S_r S_{\bar{r}}. \quad (4.12)$$

We use the value of the imaginary part of the holonomy $\text{Im} U_{m,w} = -(r_+(m-w\tau_1)+r_- w \tau_2)$ and the primaries plus oscillators expansion

$$\frac{1}{|\eta|^6} S_r S_{\bar{r}} Z_{gh} Z_{\text{int}} = (q\bar{q})^{\frac{1}{4(k-2)}-1} \sum_{h,N} d_{h,N} q^{h+N} \bar{q}^{\bar{h}+\bar{N}}, \quad (4.13)$$

to further simplify the worldsheet partition function Z_{BTZ} . We suppress the summation over the anti-holomorphic indices and labels $(\bar{r}, \bar{h}, \bar{N})$ on the degeneracies to avoid clutter. Substituting the resulting expression into the one loop string amplitude, we collect terms in τ_1 and τ_2 in the exponent to obtain

$$Z = 4r_+ \int d^2\tau \int_0^1 ds_1 \sum_{m,w,r,h,N} \int d\lambda_2 \int_{-\infty}^{+\infty} ds d_{h,N} e^{2\pi i m (r_-(r-\bar{r})+r_+(2s-\lambda_2)-ik_b w r_+ r_-)} \\ \times e^{2\pi i \tau_1 (h+N-\bar{h}-\bar{N}-w(r_-(r-\bar{r})+r_+(2s-\lambda_2)-ik_b w r_+ r_-))} \\ \times e^{-2\pi\tau_2 \left(-2+h+N+\bar{h}+\bar{N} + \frac{2s^2+\frac{1}{2}}{(k_b-2)} + \frac{k_b w^2}{2} (r_+^2 - r_-^2) + s_1 (r+\bar{r}+1+i\lambda_2) + i w r_+ (r-\bar{r}) - i w r_- (2s-\lambda_2) \right)}. \quad (4.14)$$

We perform the τ_1 -integral by solving the level matching constraint for \bar{L}_0 in terms of L_0 first and perform the radial momentum s -integral next to obtain

$$Z = 2r_+ \sqrt{k_b-2} \int \frac{d\tau_2}{2\sqrt{\tau_2}} \int_0^1 ds_1 \sum_{m,w,r,h,N} \int d\lambda_2 d_{h,N} e^{2\pi i m (r_-(r-\bar{r})-r_+ \lambda_2 - ik_b w r_+ r_-)} \\ \times e^{-2\pi\tau_2 \left(-2+2h+2N + \frac{1}{2(k_b-2)} + \frac{k_b w^2}{2} (r_+ + ir_-)^2 + s_1 (r+\bar{r}+1+i\lambda_2) + i w (r_+ + ir_-)(r-\bar{r}) + (r_+ + ir_-) w \lambda_2 \right)} \\ \times e^{-\frac{\pi(k_b-2)}{\tau_2} (m r_+ + w(r_- - ir_+) \tau_2)^2}. \quad (4.15)$$

The λ_2 -integral can be trivially done to find the δ -function constraint that allows one to perform the s_1 -integral:

$$s_1 \tau_2 = -m r_+ + i w \tau_2 (r_+ + ir_-). \quad (4.16)$$

At this point it is clear that if we are to find a real solution for s_1 , we need to perform an analytic continuation, and consider the combination $i(r_+ + ir_-)$ to be real. Crucially, in the sequel we consider the continuation for which $i w (r_+ + ir_-)$ is negative. In that case, m is necessarily negative. Furthermore it is important to keep in mind that this constrains the range of the parameter τ_2 as the solution for s_1 has to be in the open interval $(0, 1)$.

$$0 < -\frac{m r_+}{\tau_2} + i w (r_+ + ir_-) < 1. \quad (4.17)$$

Thus, the s_1 and λ_2 -integrals can be done and we obtain

$$Z = 2r_+ \sqrt{k_b - 2} \int \frac{d\tau_2}{\tau_2^{\frac{3}{2}}} \sum_{w,r,h,N} \sum_{m>0} d_{h,N} e^{2\pi i m \left(-i(r_+ + ir_-)(r + \frac{1}{2}) - i(r_+ - ir_-)(\bar{r} + \frac{1}{2}) - 2ir_+ r_- w + (k_b - 2)r_+^2 w \right)}$$

$$\times e^{-2\pi\tau_2 \left(-2 + 2h + 2N + \frac{1}{2(k_b - 2)} + iw(1 + 2r)(r_+ + ir_-) + w^2(r_+ + ir_-)^2 \right)} e^{-\frac{\pi(k_b - 2)m^2 r_+^2}{\tau_2}},$$

where we have restricted the sum to just $m < 0$, and multiplied by a factor of two. After these preparations, since we have not found a method yet to continue exactly, we perform a saddle point approximation to the τ_2 -integral. The saddle point is at

$$(\tau_2)_{\text{saddle}} = - \frac{(k_b - 2)mr_+}{\sqrt{1 + 4(k_b - 2) \left(-1 + h + N + i\frac{w}{2}(r_+ + ir_-)(1 + 2r) + \frac{w^2}{2}(r_+ + ir_-)^2 \right)}}. \quad (4.18)$$

We denote the argument of the square root as

$$A = 1 + 4(k_b - 2) \left(-1 + h + N + i\frac{w}{2}(r_+ + ir_-)(1 + 2r) + \frac{w^2}{2}(r_+ + ir_-)^2 \right). \quad (4.19)$$

We shall soon relate this argument to the spin quantum number j . Evaluating the partition function on the saddle point, and performing the resulting Gaussian integral, we find that

$$Z_{\text{saddle}} = - \sum_{w,r,h,N} d_{h,N} \sum_{m<0} \frac{1}{m} e^{2\pi i m \left(-i(r_+ + ir_-)(r + \frac{1}{2}) - i(r_+ - ir_-)(\bar{r} + \frac{1}{2}) - 2ir_+ r_- w + (k_b - 2)r_+^2 w - ir_+ \sqrt{A} \right)}$$

$$= - \sum_{w,r,h,N} d_{h,N} \log \left(1 - e^{-2\pi i \left(-i(r_+ + ir_-)(r + \frac{1}{2}) - i(r_+ - ir_-)(\bar{r} + \frac{1}{2}) - 2ir_+ r_- w + (k_b - 2)r_+^2 w - ir_+ \sqrt{A} \right)} \right).$$

We have obtained a logarithm that is the result of resummation over the multiple short string contributions to the one loop amplitude.

Let us interpret this result by exploiting properties of the worldsheet conformal field theory that describes the BTZ black hole background [3–11]. The first details to recall are the worldsheet conformal dimensions of the short strings in the Euclidean theory. These are [10]:

$$L_0 = -\frac{j(j-1)}{k_b-2} + i(j+r)(r_+ + ir_-)w + \frac{k_b w^2}{4}(r_+ + ir_-)^2 + h + N$$

$$\bar{L}_0 = -\frac{j(j-1)}{k_b-2} - i(j+\bar{r})(r_+ - ir_-)w + \frac{k_b w^2}{4}(r_+ - ir_-)^2 + \bar{h} + \bar{N}. \quad (4.20)$$

We imposed the level matching condition $L_0 = \bar{L}_0$ while integrating over τ_1 (and requiring translation invariance in the world sheet coordinate σ) and the on-shell conditions therefore reduce to $L_0 = 1$. Solving for the spin j , we find the relation

$$2j - 1 - i(k_b - 2)w(r_+ + ir_-) = \sqrt{A}. \quad (4.21)$$

This relation allows us to interpret the exponent that appears in the partition function for the saddle point contribution intuitively. We note that with our analytic continuation, both the square root as well as the spin j are real. Substituting for A , we obtain

$$Z_{\text{saddle}} = - \sum_{r,h,N,w \geq 0} d_{h,N} \log \left(1 - e^{-2\pi i(-i(r_+ + ir_-)(j+r) - i(r_+ - ir_-)(j+\bar{r}) - ik_b r_+ r_- w)} \right). \tag{4.22}$$

We recognize the exponent as the eigenvalue of the operator $-2\pi i L_{\text{string}}$, where L_{string} is the quantum angular momentum operator [4] in the Euclidean theory

$$\begin{aligned} L_{\text{string}} &= \frac{1}{w}(h + N - \bar{h} - \bar{N}) \\ &= -i(r_+ + ir_-)(j + r) - i(r_+ - ir_-)(j + \bar{r}) - ik_b w r_+ r_-. \end{aligned} \tag{4.23}$$

Thus, the full saddle point contribution can be given the interpretation as the trace:

$$Z_{\text{saddle}} = - \sum_{m < 0} \frac{1}{m} \text{Tr}_{\text{discrete saddle}} e^{2\pi i m L_{\text{string}}}. \tag{4.24}$$

The ‘‘multi-particle’’ number m is the dual of the angular momentum (as in [10]). We recognized this as the logarithm of the single string trace:

$$Z_{\text{single string}} = \text{Tr}_{\text{discrete saddle}} e^{-2\pi i L_{\text{string}}}. \tag{4.25}$$

This trace formula is a stringy generalization of the particle formula (3.5) in section 3.

4.2.1 The bound on the spin

As we have seen in the inequality (4.17) the constraint that $0 < s_1 < 1$ translates to a bound on τ_2 . Combining the saddle point equation with the expression (4.21) for the square root in terms of the spin one can write

$$-\frac{mr_+}{(\tau_2)_{\text{saddle}}} + iw(r_+ + ir_-) = \frac{2j - 1}{k_b - 2}. \tag{4.26}$$

Substituting this into the constraint (4.17) we find the bound on spin:

$$\frac{1}{2} < j < \frac{k_b}{2}. \tag{4.27}$$

We note that there is an intuitive interpretation of this bound. In the context of a $su(2)$ Wess-Zumino-Witten model for instance, we can argue that a spin j that reaches the level $k_b/2$ will create a wiggle in space-time which is of the string scale and equivalent to an oscillator mode. Thus, the bound on the spin is interpreted as a rewriting of a higher spin excitation as a string oscillation. The same reasoning holds in $sl(2, \mathbb{R})$. When the spin j reaches the curvature scale squared $k_b/2$, angular momentum turns into string scale oscillations, also in non-compact, locally AdS_3 spaces. For global AdS_3 , this bound was indeed found to hold in [15, 23]. The physical phenomenon that we described is independent of the global structure of the space-time and thus we may expect the bound on spin to be independent of the background parameters (r_+, r_-) of the asymptotically AdS_3 geometry. Our saddle point analysis confirms that this is so.

4.2.2 The quasinormal winding modes

We have derived through analytic continuation and saddle point evaluation of the one loop vacuum amplitude a contribution of discrete single string modes. It is natural to ask for a more direct space-time description of this contribution. We propose to use this contribution to identify space-time frequencies of quasinormal winding modes following the method of [25] which consists in exploiting poles of the partition function to determine quasinormal frequencies. We consider the contribution to the partition function from a primary twisted string, fixing the winding w and the internal conformal dimension h . We concentrate on the leading terms in the descendant degeneracy function $S_r(q)$, which is a constant equal to 1 for $r \geq 0$ (and subleading for $r < 0$), and similarly for the quantum number \bar{r} . In other words, we concentrate on primaries. Thus, we restrict to $r \geq 0$ and $\bar{r} \geq 0$ and start out with the expression

$$Z_{\text{saddle}}^{(w)} = \sum_{|m|=1}^{\infty} \sum_{r, \bar{r} \geq 0} \frac{1}{|m|} e^{-2\pi|m| \left((r_+ + ir_-) \left(j + r - i \frac{k_b w (r_+ + ir_-)}{4} \right) + (r_+ - ir_-) \left(j + \bar{r} + i \frac{k_b w (r_+ - ir_-)}{4} \right) \right)}. \quad (4.28)$$

The expansion of the logarithm can then be rewritten, following [25–27] as — see also section 3 —:

$$\begin{aligned} Z_{\text{saddle}}^{(w)} = & - \sum_{n>0, p \geq 0} \log \left(1 - q^{n+p+j+\frac{k_b w r_-}{2}} \bar{q}^{p+j+\frac{k_b w r_-}{2}} \right) - \sum_{n>0, p \geq 0} \log \left(1 - q^{p+j+\frac{k_b w r_-}{2}} \bar{q}^{n+p+j+\frac{k_b w r_-}{2}} \right) + \\ & - \sum_{p \geq 0} \log \left(1 - (q\bar{q})^{p+j+\frac{k_b w r_-}{2}} \right). \end{aligned} \quad (4.29)$$

Here we have defined $q = e^{-2\pi(r_+ + ir_-)}$ and $\bar{q} = e^{-2\pi(r_+ - ir_-)}$. By the logic of [25], which we follow in appendix C, identifying the poles of $e^{Z_{\text{saddle}}}$ with the locations of the quasinormal modes, we propose that there are winding quasinormal modes at frequencies:

$$E_{\text{string}} = \mp L_{\text{string}} - ik_b w r_+ r_- - i(r_+ \pm ir_-)(2p + 2j) \quad (4.30)$$

$$E_{\text{string}} = \mp L_{\text{string}} - ik_b w r_+ r_- + i(r_+ \pm ir_-)(2p + 2j + 2k_b w r_-). \quad (4.31)$$

It will be interesting to see whether this treatment of primaries only holds up to closer scrutiny. Unfortunately, we are unable to provide a similar calculation for descendants at this stage. An independent and more direct argument for the validity of part of the quasinormal winding spectrum is provided in the next subsection.

4.2.3 A space-time description of the quasinormal winding strings

One way to describe the string modes we identified above is in terms of a curved doubled field theory, namely a doubled field theory [28] adapted to our curved, orbifolded group manifold. Consider generalized scalar wave-functions Φ_{double} of string states with a dependence on the coordinates (t, ϕ, r) of the BTZ metric and on top of that a winding dependence on a dual coordinate w_D , $\Phi_{\text{double}} = \Phi(t, \phi, r) e^{i w w_D}$. We think of the left

and right-moving Virasoro constraints as imposing generalized wave operator equations (involving derivatives $\partial_{\phi,t,r}$ but also ∂_{w_D}) on the Lorentzian wave-functions Φ_{double} :

$$\begin{aligned} L_0 - 1 &= \square + h + N - 1 - w(r_+ - r_-)J - \frac{k_b}{4}(r_+ - r_-)^2 w^2 \\ \bar{L}_0 - 1 &= \square + \bar{h} + \bar{N} - 1 - w(r_+ + r_-)\bar{J} - \frac{k_b}{4}(r_+ + r_-)^2 w^2, \end{aligned} \quad (4.32)$$

where we denoted the string zero modes of the hyperbolic currents by J and \bar{J} — see [4, 10] — and the wave-operator \square equals the quadratic Casimir of the representation in which the ordinary wave-function Φ transforms. The wave-function Φ codes the dependence on the time t and angular coordinate ϕ of the string through $(E_{\text{string}}, L_{\text{string}})$ where L_{string} is related to the $-i\partial_\phi$ eigenvalue L_ϕ through [4]

$$L_{\text{string}} = L_\phi + k_b w r_+ r_-. \quad (4.33)$$

The time translation charge E_{string} and angular charge L_ϕ are the ones that appear in the quadratic wave-operator \square that involves time and angular derivatives. The current zero modes are related to the string energy and angular momentum by the relations [4]:⁷

$$\begin{aligned} (r_+ - r_-)J &= \frac{E_{\text{string}} - L_{\text{string}} + k_b w r_+ r_-}{2} \\ (r_+ + r_-)\bar{J} &= \frac{E_{\text{string}} + L_{\text{string}} - k_b w r_+ r_-}{2}. \end{aligned} \quad (4.34)$$

A given wave-function Φ_{double} will satisfy the physical state conditions if on the one hand the wave-function Φ satisfies the ordinary wave-equation $\square\Phi = -j(j-1)/k$, and on the other hand, the string energy and angular momentum are chosen such that the on-shell conditions $L_0 = 1 = \bar{L}_0$ are satisfied.

We choose the boundary conditions on the wave-function Φ in a discrete representation j to be in-going at the horizon and without energy loss at infinity. The modes satisfying these boundary conditions are the quasinormal modes. The analysis of these wave-functions in the BTZ black hole background is well-known [29]. There is a spectrum of quasinormal modes that satisfy one of the two equations [29]:

$$iJ = j + p, \quad -i\bar{J} = j + \bar{p} \quad (4.35)$$

where $p \geq 0$ or $\bar{p} \geq 0$ is a positive integer. We have used here that it is the string eigenvalues that determine the time and angular dependence of the wave-functions. For instance, if we assume the first equation in (4.35) is satisfied, we find the quasinormal frequencies:

$$E_{\text{string}} = L_{\text{string}} - k_b w r_+ r_- - i(r_+ - r_-)(2j + 2p), \quad (4.36)$$

where the string angular momentum L_{string} is quantized. This is a stringy modification of the particle quasinormal mode spectrum (3.7). We note that these modes, derived from a space-time perspective, match a tower (4.30) of quasinormal modes that we found through

⁷Our notation matches [10].

the world sheet approach (after analytic continuation to the Lorentzian). To reproduce the remaining towers is a challenge, since one needs to model the spin and angular momentum of the excitations more accurately in the space-time wave-function Φ_{double} . This requires further development of the curved doubled field theory that we introduced.

4.3 The long string poles

In this second part, we treat the piece of the partition function that is missed by the saddle point analysis of the previous section. These are the divergent contributions that arise from the poles of the θ_1 function in the denominator.⁸ Such a pole was analyzed in [10]. Our analysis will be more complete.

We start over with the one loop string amplitude (4.1):

$$Z = 2r_+ \sqrt{k_b - 2} \int_F \frac{d^2\tau}{\tau_2^{3/2}} \sum_{m,w} \frac{e^{-\pi \frac{k_b}{\tau_2} r_+^2 |m-w\tau|^2 + \frac{2\pi}{\tau_2} \text{Im}(\bar{U}_{m,w})^2}}{|\theta_1(\bar{U}_{m,w}, \tau)|^2} |\eta(\tau)|^4 Z_{\text{int}}. \quad (4.37)$$

We are interested in the behaviour of the partition function near its singularities, which occur at the zeroes of the θ -function. We use the product form of the θ -function

$$\frac{\eta^2(\tau)}{\theta_1(\bar{U}_{m,w}, \tau)} = \frac{q^{\frac{1}{12}}}{2q^{\frac{1}{8}} \sin \pi \bar{U}_{m,w}} \prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 - e^{2\pi i \bar{U}_{m,w}} q^n)(1 - e^{-2\pi i \bar{U}_{m,w}} q^n)}. \quad (4.38)$$

We have poles at those values of τ for which:

$$\bar{U}_{m,w} + \tau_{\text{pole}} \ell + p = 0, \quad \text{or} \quad \tau_{\text{pole}} = \frac{p + m(r_- - ir_+)}{w(r_- - ir_+) - \ell} \quad (4.39)$$

where p and ℓ are integers. The set of poles again transforms as a doublet under $\text{SL}(2, \mathbb{Z})$ modular transformations and by the same unfolding logic as before, we can concentrate on the poles with $\ell = 0$ in the strip. For the case that $\ell = 0$, we see that it is the zero of the sine-function that leads to singularities in the partition function. We can approximate this factor by

$$\sin \pi \bar{U}_{m,w} \sim (-1)^p \pi (r_- - ir_+) w (\tau - \tau_{\text{pole}}), \quad (4.40)$$

and the poles in the τ -plane are located at

$$\tau_{\text{pole}} = \left(\frac{m}{w} + \frac{pr_-}{w(r_-^2 + r_+^2)} \right) + i \frac{pr_+}{w(r_-^2 + r_+^2)}. \quad (4.41)$$

We concentrate on the modes with strictly positive winding w , such that for $\tau_2 > 0$, we only have poles with $p \geq 1$. The contribution from this pole to the τ -integral can be evaluated and we find a logarithmic divergence:

$$\frac{1}{\pi^2 w^2 (r_+^2 + r_-^2)} \int \frac{d^2\tau}{|\tau - \tau_{\text{pole}}|^2} = \frac{2}{\pi w^2 (r_+^2 + r_-^2)} \log \epsilon. \quad (4.42)$$

⁸These were missed because we assumed s_1 to be in the open interval $0 < s_1 < 1$ when performing the expansion of the θ_1 function.

Here, we have defined $\tau - \tau_{\text{pole}} = \epsilon$. We now evaluate the leading term in the partition function near the pole in the τ -plane, that is the coefficient of the logarithmically divergent term. For starters we see that the non-trivial exponent simplifies drastically

$$-\pi \frac{k_b}{\tau_2} r_+^2 |m - w\tau|^2 + \frac{2\pi}{\tau_2} \text{Im}(\bar{U}_{m,w})^2 = -\pi k_b p r_+ w. \quad (4.43)$$

The contribution from the ghost, internal conformal field theory and rest of the θ -function can again be succinctly expressed in a q -expansion:

$$(q\bar{q})^{\frac{1}{4(k-2)}-1} \sum_{h,N} d_{h,N} q^{h+N} \bar{q}^{\bar{h}+\bar{N}}. \quad (4.44)$$

The overall exponent is fixed by the central charges of the world sheet conformal field theories and $d_{h,N}$ denotes the degeneracies of states with the given quantum numbers. In this expression we substitute $\tau = \tau_{\text{pole}}$ to obtain the leading contribution to the partition function. Collecting the various pieces, we find that

$$\begin{aligned} Z_{\text{pole}} &= \frac{1}{\pi} \log \epsilon \sum_{m,w,p} \sum_{h,N} d_{h,N} \sqrt{\frac{(k_b - 2)(r_+^2 + r_-^2)}{p^3 r_+ w}} e^{2\pi i \frac{m}{w} (h+N-\bar{h}-\bar{N})} \\ &\times e^{-2\pi p \left(\frac{k_b w r_+}{2} + \frac{1}{w(r_+ + ir_-)} \left(h+N-1 + \frac{1}{4(k_b-2)} \right) + \frac{1}{w(r_+ - ir_-)} \left(\bar{h}+\bar{N}-1 + \frac{1}{4(k_b-2)} \right) \right)}. \end{aligned} \quad (4.45)$$

We now introduce the s -integral:

$$\int_{-\infty}^{\infty} ds e^{-\frac{2\pi p}{w} \frac{2r_+}{r_+^2 + r_-^2} \frac{s^2}{k_b - 2}} = \frac{1}{2} \sqrt{\frac{w(r_+^2 + r_-^2)(k_b - 2)}{p r_+}}. \quad (4.46)$$

The pole contribution is thereby rewritten as

$$\begin{aligned} Z_{\text{pole}} &= \frac{2}{\pi} \log \epsilon \int_{-\infty}^{\infty} ds \sum_{m,w,p} \sum_{h,N} d_{h,N} \frac{1}{p w} e^{2\pi i \frac{m}{w} (h+N-\bar{h}-\bar{N})} \\ &\times e^{-2\pi p \left(\frac{k_b w r_+}{2} + \frac{1}{w(r_+ + ir_-)} \left(\frac{s^2 + \frac{1}{4}}{k_b - 2} + h+N-1 \right) + \frac{1}{w(r_+ - ir_-)} \left(\frac{s^2 + \frac{1}{4}}{k_b - 2} + \bar{h}+\bar{N}-1 \right) \right)}. \end{aligned} \quad (4.47)$$

The sum over the integer m , where $-1/2 < m/w \leq 1/2$ for the pole to lie in the strip, implies that $h+N-\bar{h}-\bar{N}$ is an integer multiple of w . This exponential factor corresponds to the insertion of the operator $e^{2\pi i m L}$ in the amplitude. The sum over m imposes the angular momentum constraint and adds a factor of w , which implies that the pole contribution takes the logarithmic form

$$\begin{aligned} Z_{\text{pole}} &= \frac{2}{\pi} \log \epsilon \int_{-\infty}^{\infty} ds \sum_{w,h,N} d_{h,N} \sum_{p=1}^{\infty} \frac{1}{p} e^{-2\pi p \left(\frac{k_b w r_+}{2} + \frac{1}{w(r_+ + ir_-)} \left(\frac{s^2 + \frac{1}{4}}{k_b - 2} + h+N-1 \right) + \frac{1}{w(r_+ - ir_-)} \left(\frac{s^2 + \frac{1}{4}}{k_b - 2} + \bar{h}+\bar{N}-1 \right) \right)} \\ &= \frac{2}{\pi} \log \epsilon \int_{-\infty}^{\infty} ds \sum_{w,h,N} d_{h,N} \log(1 - e^{-2\pi f(s)}). \end{aligned} \quad (4.48)$$

In order to write the second equality, we have recognized the sum over the positive integers p as a sum over multi-long-string contributions that generates a logarithm. We identify the exponent $f(s)$ associated to a single long string with radial momentum s as

$$f(s) = \frac{k_b w(r_+ + ir_-)}{4} + \frac{1}{w(r_+ + ir_-)} \left(\frac{s^2 + \frac{1}{4}}{k_b - 2} + h + N - 1 \right) + \frac{k_b w(r_+ - ir_-)}{4} + \frac{1}{w(r_+ - ir_-)} \left(\frac{s^2 + \frac{1}{4}}{k_b - 2} + \bar{h} + \bar{N} - 1 \right), \quad (4.49)$$

where we need to impose the level matching condition $N + h - \bar{N} - \bar{h} = wL_{\text{string}}$, with L_{string} being the angular momentum. We recall that for the long strings the left and right worldsheet conformal dimensions are [10]:

$$L_0 = \frac{s^2 + \frac{1}{4}}{k_b - 2} + h + N + w(r_+ + ir_-)(iJ) + \frac{k_b w^2}{4}(r_+ + ir_-)^2 \quad (4.50)$$

$$\bar{L}_0 = \frac{s^2 + \frac{1}{4}}{k_b - 2} + \bar{h} + \bar{N} + w(r_+ - ir_-)(-i\bar{J}) + \frac{k_b w^2}{4}(r_+ - ir_-)^2, \quad (4.51)$$

where (iJ) and $(-i\bar{J})$ are identified with the zero modes of the worldsheet currents in the w -twisted sector. Solving for the on-shell conditions $L_0 = \bar{L}_0 = 1$, we can solve for these zero modes in terms of the internal dimensions and the twist w . We thereby recognize the combination that appears in the exponent in (4.48) as

$$-2\pi f(s) = 2\pi(iJ + (-i\bar{J})). \quad (4.52)$$

The current zero modes are related to the Euclidean energy and angular momentum of the twisted string [4, 10]:

$$E_{\text{string}} = (r_+ + ir_-)(iJ) + (r_+ - ir_-)(-i\bar{J}) \quad (4.53)$$

$$L_{\text{string}} = -(r_+ + ir_-)(iJ) + (r_+ - ir_-)(-i\bar{J}) - ik_b w r_+ r_-. \quad (4.54)$$

While these energy and angular momentum generators were initially discussed classically in [4, 10], we stress that these expressions are quantum-mechanically exact [3–7]. Indeed, there are unique affine (purely holomorphic or purely anti-holomorphic) currents whose zero modes generate time and angular translations in the quantum mechanical Wess-Zumino-Witten model. In detail, we follow the quantum mechanically exact expressions derived in [4].⁹ We thus have the exponent

$$-2\pi f(s) = \frac{2\pi r_+}{r_+^2 + r_-^2} E_{\text{string}} + \frac{2\pi ir_-}{r_+^2 + r_-^2} \left(L_{\text{string}} - \frac{k_b w}{2} J_{BH} \right). \quad (4.55)$$

⁹The classical expressions are protected by holomorphy and even the level does not renormalize in the quantum current algebra. The horizon radii are also protected from renormalization in this Wess-Zumino-Witten model. A potential total derivative term can be eliminated by demanding that the current have a single, either holomorphic or anti-holomorphic component. These are the currents that are to be used in expressions (4.53) and (4.54).

We recall the definition of the boundary modular parameter for the BTZ black hole. It is defined as the S-dual of the boundary modular parameter of the thermal AdS_3 background, which is given by [15]

$$\tau_{AdS_3} = \frac{1}{2\pi}(\beta\mu + i\beta). \tag{4.56}$$

Here β is the inverse temperature and μ is the chemical potential for angular momentum in the thermal AdS_3 space-time [15]. In order to write down the partition function of the Euclidean BTZ background, recall that we made the substitutions

$$\left(\frac{\beta}{2\pi}, \frac{\beta\mu}{2\pi}\right) = (r_+, r_-). \tag{4.57}$$

Thus, the boundary modular parameter corresponding to the Euclidean BTZ background is given by

$$\tau_{BTZ} = -\frac{1}{\tau_{AdS_3}} = -\frac{1}{r_- + ir_+} = -\frac{r_- - ir_+}{r_+^2 + r_-^2}. \tag{4.58}$$

Putting all this together, one can rewrite (4.55) in terms of the boundary modular parameter of the Euclidean BTZ background and interpret the pole contribution to the partition function as the trace over the multiply wound long strings:

$$Z_{\text{pole}} = \frac{2}{\pi} \log \epsilon \text{Tr}_{\text{multi-long-strings in BTZ}} \left[q_{\text{BTZ}}^{\mathcal{L}_0} \bar{q}_{\text{BTZ}}^{\bar{\mathcal{L}}_0} \right], \tag{4.59}$$

where $q_{\text{BTZ}} = e^{2\pi i \tau_{\text{BTZ}}}$, and we have defined

$$\mathcal{L}_0 = -\frac{E_{\text{string}} - \tilde{L}_{\text{string}}}{2}, \quad \bar{\mathcal{L}}_0 = -\frac{E_{\text{string}} + \tilde{L}_{\text{string}}}{2}, \tag{4.60}$$

with $\tilde{L}_{\text{string}} = L_{\text{string}} - \frac{k_b w}{2} J_{BH}$.

It is an interesting challenge to interpret our final formula (4.59), for multiple reasons. Firstly, the trace has Hamiltonian insertions \mathcal{L}_0 that have the opposite sign relation to the energy of states. We recall that long strings in the BTZ background have negative energy, as was confirmed by a probe string calculation in [10]. Our definition of the operator $\mathcal{L}_0 + \bar{\mathcal{L}}_0$ in (4.60) leads it to have positive eigenvalues as one may expect of a boundary scaling operator. Secondly, we note that there is no manifest invariance under the T-transformation $\tau_{\text{BTZ}} \rightarrow \tau_{\text{BTZ}} + 1$ in our final result. We explain in appendix D that this invariance is lacking when we concentrate on the contribution of a single BTZ black hole background and that it can be restored by summing over contributions from a class of background geometries.

A naive argument would run that the long strings reach infinity and we may therefore expect to be able to write the pole contribution Z_{pole} to the one-loop integrand as a trace in the boundary conformal field theory with modular parameter τ_{BTZ} . We note that our final expression takes this form with both a modification in the sign of the energy operator and in the twist operator L_{string} which is shifted by a contribution proportional to the winding and the black hole angular momentum.¹⁰ However, we note that a winding string

¹⁰At zero angular momentum J_{BH} or zero winding, the second point is moot.

reaching asymptotic infinity has a drastic influence on the boundary conformal field theory — it even changes the central charge of the boundary conformal field theory. Thus, it is intriguing that we reach a final expression (4.59) that is close to an ordinary conformal field theory trace.

Finally, we wish to provide an overview of the aspects in which the short and long string contributions are alike and different. Firstly, we want to stress that the expressions (4.24) and (4.47) express respectively the short string and the long string contributions in terms of the trace of an orbifold projection operator $e^{2\pi imL}$. Indeed, the full partition function can be written as such a trace and so can its individual contributions [10]. Secondly, we have reshaped the short string contribution (4.28) to interpret it in terms of a trace over modes with particular space-time on-shell energies. We have manipulated the long string contribution (4.48) in order to obtain a seemingly different interpretation in terms of on-shell space-time modes. It would certainly be interesting to understand this distinction better.

5 Conclusions

In this paper, we studied the path integral derivation of the Hilbert space of a particle on the non-compact universal cover \tilde{G} of $SL(2, \mathbb{R})$. The group \tilde{G} doubles as the AdS_3 space-time geometry and we described how to twist and orbifold the particle partition function on the group, giving a direct algebraic handle on the particle partition sum in thermal AdS_3 and BTZ black hole backgrounds. Indeed, we were able to describe these partition sums via an elliptic and hyperbolic orbifold procedure respectively. We thus showed that the BTZ particle partition function affords not only a Euclidean but also a Lorentzian hyperbolic orbifold interpretation, as well as a description in terms of a sum over quasinormal modes.

We exploited the lessons we learnt in the context of the first quantized particle to advance our understanding of the string theory spectrum in the BTZ black hole background with NSNS flux. We wrote the one loop vacuum amplitude as a sum over multi-string contributions corresponding to a single string spectrum that we were able to identify. Firstly, we found discrete winding quasinormal modes in a saddle point approximation to the partition function and described their interpretation in space-time. Secondly, we identified divergent pole contributions that correspond to winding long strings in the spectrum of string theory on the black hole background. We thus confirmed the identification of contributions from [10] and put them on a considerably firmer footing.

While we believe that our results constitute a significant step forward, more work is needed. While hyperbolic characters of groups with $sl(2, \mathbb{R})$ algebra are well-understood, the hyperbolic characters of the affine $sl(2, \mathbb{R})$ algebra deserve further study. These can serve the Lorentzian and algebraic interpretation of the partition function well. Indeed, we have made several approximations in our claim that we identified the two main contributions to the Euclidean partition function, namely an approximation of the integrand near poles and the saddle point approximation to the discrete contribution. While these contributions are understood to be neatly complementary, they are not as closely intertwined yet as in their (thermal) AdS_3 counterpart. A better understanding of the state space origin of

affine hyperbolic characters could help in closing this gap. Secondly, the deceptively simple long string pole contribution poses the challenge to provide a transparent interpretation.

More generally, we hope that our understanding of a quantum theory of gravity on black hole backgrounds, both in terms of the perturbative and non-perturbative properties of the spectrum and interactions may advance further, whether through speculative conjectures or laying bricks one by one.

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A An elliptic orbifold of AdS_3

We perform an elliptic orbifold on the partition function for a particle on $AdS_3 = \tilde{G}$. This is a group theoretic procedure to compactify time. Naive analytic continuation relates the closed time AdS_3 manifold to Euclidean thermal AdS_3 . One may therefore expect that the partition functions on these backgrounds agree. In this case, in which the (thermal) circle is topologically non-trivial, the analytic continuation proceeds smoothly.

A.1 Compactifying time

We derive the partition function for a particle on a closed time AdS_3 manifold with an arbitrary radius as an orbifold of $AdS_3 = \tilde{G}$. We start with the twisted covering group partition function:

$$Z(h_l, h_r) = \int_{1/2}^{\infty} dj (\chi_j^+(h_l)\chi_j^+(h_r) + \chi_j^-(h_l)\chi_j^-(h_r)) + \int_0^{\infty} ds \int_0^1 d\epsilon \chi_{s,\epsilon}^c(h_l)\chi_{s,\epsilon}^c(h_r), \tag{A.1}$$

with left and right Cartan twists (h_l, h_r) . To compactify the Lorentzian AdS_3 time direction, we choose to sum over powers of the elliptic projection operator generator $(h_l, h_r) = (e^{i\beta_L^{\text{Lor}} t_{\text{ell}}}, e^{i\beta_R^{\text{Lor}} t_{\text{ell}}})$. We allow for a different left-right elliptic twist to not only compactify time, but also introduce a fugacity for the AdS_3 angular momentum. We find the orbifold partition function:

$$Z_{\text{El.O. } AdS_3}^{\text{particle}} = \sum_{n \in \mathbb{Z}} \int_{1/2}^{\infty} dj \left(\chi_j^+ \left(e^{in\beta_L^{\text{Lor}} t_{\text{ell}}} \right) \chi_j^+ \left(e^{in\beta_R^{\text{Lor}} t_{\text{ell}}} \right) + \chi_j^- \left(e^{in\beta_L^{\text{Lor}} t_{\text{ell}}} \right) \chi_j^- \left(e^{in\beta_R^{\text{Lor}} t_{\text{ell}}} \right) \right). \tag{A.2}$$

We used the expressions for the $\mathfrak{sl}(2, \mathbb{R})$ characters evaluated on elliptic group elements (see e.g. [14]). The continuous characters are zero up to a delta-function contribution which we

ignore. To obtain a more explicit expression for the partition sum, we can write out the discrete characters for elliptic group elements:

$$Z_{\text{El.O. } AdS_3}^{\text{particle}} = \sum_{n \in \mathbb{Z}} \int_{1/2}^{\infty} dj \left\{ \frac{e^{inj\beta_L^{\text{Lor}}}}{1 - e^{in\beta_L^{\text{Lor}}}} \frac{e^{inj\beta_R^{\text{Lor}}}}{1 - e^{in\beta_R^{\text{Lor}}}} + \frac{e^{-inj\beta_L^{\text{Lor}}}}{1 - e^{-in\beta_L^{\text{Lor}}}} \frac{e^{-inj\beta_R^{\text{Lor}}}}{1 - e^{-in\beta_R^{\text{Lor}}}} \right\}. \quad (\text{A.3})$$

If we view our model as a first quantized particle model on the group manifold and introduce an einbein modulus in the one-loop integral, we find:

$$Z_{\text{1-loop}}^{\text{particle}} = \int_0^{\infty} \frac{dt}{2t} \sum_{n \in \mathbb{Z}} \int_{1/2}^{\infty} dj \left\{ \frac{e^{inj\beta_L^{\text{Lor}}}}{1 - e^{in\beta_L^{\text{Lor}}}} \frac{e^{inj\beta_R^{\text{Lor}}}}{1 - e^{in\beta_R^{\text{Lor}}}} + \frac{e^{-inj\beta_L^{\text{Lor}}}}{1 - e^{-in\beta_L^{\text{Lor}}}} \frac{e^{-inj\beta_R^{\text{Lor}}}}{1 - e^{-in\beta_R^{\text{Lor}}}} \right\} e^{-itj(j-1) + it\frac{t^2 m^2}{4}}. \quad (\text{A.4})$$

To relate the Lorentzian orbifold result to the Euclidean thermal partition function, we rewrite the expression in terms of the quantity $j - 1/2$:

$$Z^{\text{particle}} = \int_0^{\infty} \frac{dt}{2t} \sum_{n \in \mathbb{Z}} \int_{1/2}^{\infty} dj \left\{ \frac{e^{in(j-\frac{1}{2})\beta_L^{\text{Lor}}}}{-2i \sin(n\beta_L^{\text{Lor}})} \frac{e^{in(j-\frac{1}{2})\beta_R^{\text{Lor}}}}{-2i \sin(n\beta_R^{\text{Lor}})} + \frac{e^{-in(j-\frac{1}{2})\beta_L^{\text{Lor}}}}{2i \sin(n\beta_L^{\text{Lor}})} \frac{e^{-in(j-\frac{1}{2})\beta_R^{\text{Lor}}}}{2i \sin(n\beta_R^{\text{Lor}})} \right\} e^{-it(j-\frac{1}{2})^2 + i\frac{t}{4} + it\frac{t^2 m^2}{4}}. \quad (\text{A.5})$$

The $n = 0$ term corresponds to a $\beta_{L,R}^{\text{Lor}}$ independent divergence associated to a renormalization of the cosmological constant — we set $n \neq 0$ from now on. Moreover, as an expression of $j - 1/2$ the integrand is even (for each value of n), and we can unfold the $j - 1/2$ integral into an integral from $-\infty$ to $+\infty$. We find, with $j = 1/2 + p$:

$$Z^{\text{particle}} = - \int_0^{\infty} \frac{dt}{8t} \sum_{n \neq 0} \int_{-\infty}^{\infty} dp \frac{e^{inp\beta_L^{\text{Lor}}}}{\sin(n\beta_L^{\text{Lor}})} \frac{e^{inp\beta_R^{\text{Lor}}}}{\sin(n\beta_R^{\text{Lor}})} e^{-itp^2 + i\frac{t}{4} + it\frac{t^2 m^2}{4}}. \quad (\text{A.6})$$

After performing the Gaussian p integral and analytically continuing in the world sheet time t , we recognize the Euclidean heat kernel expression — see e.g. [19] or equation (3.2) — up to an overall constant of which we did not keep track.

A.2 Summary

We have travelled a smooth path from the Lorentzian orbifold to the Euclidean partition function. Note that this happened despite the fact that in the Lorentzian, all on-shell states are in discrete representations, while in the Euclidean, all unitary representations are in the continuous representations. In the bulk of the paper, we examine a hyperbolic orbifold in which the thermal (Lorentzian time) circle is topologically trivial.

B Hyperbolic affine characters

In this short appendix, we make a simple point about the proposed hyperbolic affine characters. Firstly, we recall that discrete representations of $\mathfrak{sl}(2, \mathbb{R})$ with lowest weight state characterized by j have a hyperbolic character — see e.g. [14] —:

$$\chi_j^+(e^{t t_{\text{hyp}}}) = \frac{e^{-(2j-1)t}}{|\sinh t|}. \quad (\text{B.1})$$

Strictly speaking, this is a distribution on the space of functions on the group. We propose that a hyperbolic character of the corresponding Verma current algebra module is given by:

$$\hat{\chi}_j^+(e^{t t_{\text{hyp}}}) = \frac{e^{-(2j-\frac{1}{2})t}}{|\sinh t|} \frac{1}{\prod_{n=1}^{\infty} (1-q^n)(1-e^{2t}q^n)(1-e^{-2t}q^n)}. \quad (\text{B.2})$$

The logic is simple. We only illustrate it at the first level and hope the sequel will be clear. Firstly, consider the lowest weight representation of the current algebra built on the primary representation of spin j . Consider it in an elliptic basis. Next, contemplate the first level descendants of the representation, corresponding to the coefficient of the power q in the expansion of the character. In the elliptic basis, it is clear how to decompose the tensor product of the discrete lowest weight representation D_j^+ of the current algebra primary and the adjoint representation of $\mathfrak{sl}(2, \mathbb{R})$ made up of the current components $J_{-1}^{\pm,3}$. We obtain representations $D_j^+ \otimes \text{adj} = D_{j-1}^+ \oplus D_j^+ \oplus D_{j+1}^+$. This statement is basis independent. Therefore the character of the representation of the global $\mathfrak{sl}(2, \mathbb{R})$ representation at this level is necessarily $\chi_{j-1}^+ + \chi_j^+ + \chi_{j+1}^+$. For a hyperbolic group element, this character evaluates to the coefficient of q in the proposed affine character (B.2). In the case of the affine continuous character used in the bulk of the paper, the same derivation holds true. The resulting characters are characters of (often non-unitary) continuous representations. The result of this reasoning is stated in equation (4.5).

C Derivation of the quasinormal winding frequencies

In this appendix, we start from the partition sum (C.1) and use the method of identifying poles in the Matsubara frequency dependence to surmise quasinormal winding mode frequencies [25]. The technical details mostly follow [26, 27] but there are some differences due to the shift of the angular momentum of the string by a term proportional to the winding and black hole angular momentum.

We begin with the saddle point contribution (4.28) to the partition function from the sector with winding w and rewrite it:

$$\begin{aligned} Z_{\text{saddle}}^{(w)} &= \sum_{|m|=1}^{\infty} \sum_{r, \bar{r} \geq 0} \frac{1}{|m|} e^{-2\pi|m| \left((r_+ + ir_-) \left(j + r + \frac{k_b w r_-}{2} \right) + (r_+ - ir_-) \left(j + \bar{r} + \frac{k_b w r_-}{2} \right) \right)} \\ &= \sum_{|m|=1}^{\infty} \frac{1}{|m|} (q\bar{q})^{|m|} \left(j + \frac{k_b w r_-}{2} \right) \frac{1}{(1 - q^{|m|})(1 - \bar{q}^{|m|})}. \end{aligned} \quad (\text{C.1})$$

In the second line we have defined the space-time nomes $q = e^{-2\pi(r_+ + ir_+)}$ and $\bar{q} = e^{-2\pi(r_+ - ir_-)}$. By manipulating the summation as in [27], we can rewrite the expression as a sum of three logarithms:

$$\begin{aligned}
 Z_{\text{saddle}}^{(w)} = & - \sum_{n>0, p \geq 0} \log \left(1 - q^{n+p+j+\frac{k_b wr_-}{2}} \bar{q}^{p+j+\frac{k_b wr_-}{2}} \right) - \sum_{n>0, p \geq 0} \log \left(1 - q^{p+j+\frac{k_b wr_-}{2}} \bar{q}^{n+p+j+\frac{k_b wr_-}{2}} \right) + \\
 & - \sum_{p \geq 0} \log \left(1 - (q\bar{q})^{p+j+\frac{k_b wr_-}{2}} \right). \tag{C.2}
 \end{aligned}$$

We exponentiate to rewrite:

$$\exp(-2Z_{\text{saddle}}^{(w)}) = \prod_{n>0, p \geq 0} (1 - q^{n+p+H} \bar{q}^{p+H})^2 \prod_{n>0, p \geq 0} (1 - q^{p+H} \bar{q}^{n+p+H})^2 \prod_{p \geq 0} (1 - (q\bar{q})^{p+H})^2.$$

We have defined an effective boundary conformal dimension $H = j + \frac{k_b wr_-}{2}$. We shall denote the exponential by $P_1 P_2 P_3$, where each of the P_i corresponds to one of the three infinite products. We now show how this can be equivalently written in a product form in terms of (winding) quasinormal modes. We define $2\pi T_R = (r_+ + ir_-)$ and $2\pi T_L = (r_+ - ir_-)$. We shall consider the first two factors in tandem, and, following [25–27], it is possible to obtain the expression:

$$\begin{aligned}
 P_1 P_2 = & \prod_{n>0, p \geq 0, \ell} \left[\left(\frac{2T_R}{T_R + T_L} \ell + \frac{4\pi i T_L T_R n}{T_R + T_L} + 2\pi i T_R (2p + 2H) \right) \right. \\
 & \left. \times \left(-\frac{2T_L}{T_R + T_L} \ell + \frac{4\pi i T_L T_R n}{T_R + T_L} + 2\pi i T_L (2p + 2H) \right) \right] \\
 & \times \prod_{n<0, p \geq 0, \ell} \left[\left(\frac{2T_R}{T_R + T_L} \ell + \frac{4\pi i T_L T_R n}{T_R + T_L} - 2\pi i T_R (2p + 2H) \right) \right. \\
 & \left. \times \left(-\frac{2T_L}{T_R + T_L} \ell + \frac{4\pi i T_L T_R n}{T_R + T_L} - 2\pi i T_L (2p + 2H) \right) \right]. \tag{C.3}
 \end{aligned}$$

We now recall the Matsubara frequencies [25–27]

$$\omega_n = \frac{T_R - T_L}{T_R + T_L} \ell_\Phi + \frac{4\pi i T_L T_R}{T_R + T_L} n, \tag{C.4}$$

where the angular momentum l_Φ associated to translation in the angular coordinate ϕ in the BTZ black hole background equals

$$\ell_\Phi = \ell + ik_b wr_+ r_-,$$

and we have identified $L_{\text{string}} = \ell \in \mathbb{Z}$ to be the quantized string angular momentum. The quantum number n is the quantized Euclidean time momentum. Then, re-expressing all

quantities in terms of the horizon radii r_{\pm} , we obtain

$$\begin{aligned}
 P_1 P_2 = & \prod_{n>0, p \geq 0, \ell} [(\omega_n + \ell + ik_b w r_+ r_- + i(r_+ + ir_-)(2p + 2j)) \\
 & \times (\omega_n - \ell + ik_b w r_+ r_- + i(r_+ - ir_-)(2p + 2j))] \\
 & \times \prod_{n<0, p \geq 0, \ell} [(\omega_n + \ell + ik_b w r_+ r_- - i(r_+ + ir_-)(2p + 2j + 2k_b w r_-)) \\
 & \times (\omega_n - \ell + ik_b w r_+ r_- - i(r_+ - ir_-)(2p + 2j + 2k_b w r_-))] .
 \end{aligned} \tag{C.5}$$

Let us now consider the remaining factor

$$P_3 = \prod_{p \geq 0} (1 - (q\bar{q})^{p+H})^2 . \tag{C.6}$$

By following the same sort of logic one can check that we simply obtain the $n = 0$ factor in the first product in (C.5), with $\omega_0 = \frac{T_R - T_L}{T_R + T_L} \ell_{\Phi}$. From the product of P_1 , P_2 and P_3 one can then read off the quasinormal modes by conjecturally generalizing the principle of [25] that the quasinormal frequencies arise as zeroes of this expression as a function of the Matsubara frequencies ω_n . For frequencies $\omega_{n \geq 0}$, we have energies $E_{\text{string}} = \omega$, with

$$\begin{aligned}
 \omega &= -\ell - ik_b w r_+ r_- - i(r_+ + ir_-)(2p + 2j) \\
 \omega &= +\ell - ik_b w r_+ r_- - i(r_+ - ir_-)(2p + 2j) ,
 \end{aligned} \tag{C.7}$$

while for frequencies with $\omega_{n < 0}$, we have the modes

$$\begin{aligned}
 \omega &= -\ell - ik_b w r_+ r_- + i(r_+ + ir_-)(2p + 2j + 2k_b w r_-) \\
 \omega &= +\ell - ik_b w r_+ r_- + i(r_+ - ir_-)(2p + 2j + 2k_b w r_-) .
 \end{aligned} \tag{C.8}$$

In Lorentzian conventions, for frequencies $\omega_{n \geq 0}$, we have

$$\begin{aligned}
 \omega &= -\ell + k_b w r_+ r_- - i(r_+ - r_-)(2p + 2j) \\
 \omega &= +\ell + k_b w r_+ r_- - i(r_+ + r_-)(2p + 2j) ,
 \end{aligned} \tag{C.9}$$

while for frequencies $\omega_{n < 0}$, we have the modes

$$\begin{aligned}
 \omega &= -\ell + 2k_b w r_-^2 - k_b w r_- r_+ + i(r_+ - r_-)(2p + 2j) \\
 \omega &= +\ell - 2k_b w r_-^2 - k_b w r_- r_+ + i(r_+ + r_-)(2p + 2j) .
 \end{aligned} \tag{C.10}$$

These are the winding quasinormal mode frequencies quoted in equations (4.30) and (4.31). We wish to stress that the assumption that the zeroes in the Matsubara frequencies indeed also give the frequencies of fundamental string winding modes could benefit from further corroboration. We provide an independent argument for one of these towers of modes in the bulk of the paper.

Symmetries. We perform a mild check on our proposed spectrum. A \mathbb{Z}_2 symmetry of the world sheet action in the BTZ background consists in the combined action of world sheet parity, $r_- \leftrightarrow -r_-$ as well as the flip $\phi \leftrightarrow -\phi$ of the BTZ angular coordinate ϕ . The first operation combined with the third leaves the B-field invariant while the second combined with the third leaves the metric invariant. Moreover, the first combined with the second action leaves the orbifold group action invariant. On the quantum number $L_{\text{string}} = \ell$ this symmetry acts as $L_{\text{string}} \leftrightarrow -L_{\text{string}}$. When we act on our spectrum, we note that $(\ell, r_-) \leftrightarrow (-\ell, -r_-)$ leaves the set of modes indexed by the integer w invariant.

D Modular equivalence classes

The solid torus corresponding to Euclidean thermal AdS_3 or Euclidean BTZ has a boundary torus with modular parameter $\tau_{\text{s.t.}}$. As discussed in pedagogical detail in [30], the fact that the non-contractible cycle in the solid torus is ambiguous (up to the addition of the contractible cycle) makes for the more accurate statement that a given bulk geometry corresponds to an equivalence class of boundary modular parameters $\tau_{\text{s.t.}}$. In turn, this implies that the path integral partition function must be invariant under the equivalence relation.

For the BTZ path integral (4.2) under study in the bulk of the paper this implies that it must be invariant under the transformation:

$$r_- \rightarrow r_- + 1. \tag{D.1}$$

Using the elliptic property of the θ_1 function:

$$|\theta_1(\bar{U}_{m,w} + m - w\tau, \tau)|^2 = |q|^{-w^2} e^{-4\pi\text{Im}(\bar{U}_{m,w})w}, \tag{D.2}$$

it is straightforward to verify that the path integral (4.2) is indeed invariant under the transformation (D.1).

We note a crucial consequence of this invariance. For the thermal AdS_3 partition function, the invariance (D.1) translates into the invariance of the partition function under the transformation $\beta\mu \rightarrow \beta\mu + 2\pi$ where $\beta\mu$ functions as the angular chemical potential. Thus, the partition function is periodic in the angular fugacity due to invariance under the T transformation $\tau_{\text{s.t.}} \rightarrow \tau_{\text{s.t.}} + 1$. For the Euclidean BTZ geometry however, the periodicity in the inner horizon radius r_- implies periodicity in the real part of $-1/\tau_{\text{s.t.}}$. Once more this is a consequence of the thermal circle being topologically non-trivial for thermal AdS_3 and topologically trivial for BTZ. To restore periodicity in the real part of $\tau_{\text{s.t.}}$ for the BTZ partition function, we may sum over bulk geometries that differ by the twist $\tau_{\text{s.t.}} \rightarrow \tau_{\text{s.t.}} + 1$.

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